TO: Users of the ACTEX Review Seminar on DVD for SOA Exam MLC

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Dear Students,

Thank you for purchasing the DVD recording of the ACTEX Review Seminar for SOA Exam M, Life Contingencies segment (MLC). This version is intended for the exam offered in May 2007 and thereafter. Please be aware that the DVD seminar does ***not*** deal with the Financial Economics segment (MFE) of Exam M.

The purpose of this memo is to provide you with an orientation to this seminar, which is devoted to a review of life contingencies plus the related topics of multi-state models and the Poisson process. A 6-page summary of the topics covered in the seminar is attached to this cover memo. Although the seminar is organized independently of any particular textbook, the summary of topics shows where each topic is covered in the textbook *Models for Quantifying Risk* (Second Edition), by Cunningham, Herzog, and London.

In order to be ready to write the MLC exam, you need to accomplish three stages of preparation:

This first stage is to obtain an understanding of all the underlying mathematical theory, including a mastery of standardized international actuarial notation. This DVD review seminar is designed to enable you to obtain that understanding.

The second stage is to prepare, and then master, a complete list of all formulas and relationships (of which there are many) needed for the exam. For those who do not wish to prepare their own lists, a very complete set of formulas (364 in total) is available from ACTEX in the form of study flash cards.

The third stage is to do a large number of exam-type practice questions, beginning with the end-of-chapter exercises is the textbook. In addition, you should purchase one (or more) of the several exam-prep study guides that have been prepared for that purpose. A relatively small number of sample problems (34) are worked out for the group in this DVD. A copy of these questions, with detailed solutions, is attached to this cover memo for your reference.

Good luck to you on your exam

Summary of Topics

A. Parametric Survival Models

1. Age-at-Failure Random Variable X (3.1)
   a. CDF
   b. SDF
   c. PDF
   d. HRF
   e. CHF
   f. Moments
   g. Actuarial notation and terminology

2. Examples (3.2)
   a. Uniform
   b. Exponential
   c. Gompertz
   d. Makeham
   e. Weibull
   f. Others via transformation

3. Time-to-Failure Random Variable T_x (3.3)
   a. SDF
   b. CDF
   c. PDF
   d. HRF
   e. Moments
   f. Discrete counterpart K_x
   g. Curtate duration $K(x)$

4. Central Rate (3.4)

5. Select Models (3.5)

B. The Life Table

1. Definition (4.1)

2. Traditional Form (4.2)
   a. ℓ_x
   b. d_x and ${}_n d_x$
   c. p_x and ${}_n p_x$
   d. q_x and ${}_n q_x$

3. Other Functions (4.3)
   a. μ_x and μ_{x+t}
   b. PDF and moments of X

c. $_{n|m}q_x$
d. PDF and moments of T_x
e. Temporary expectation
f. Curtate expectation
g. Temporary curtate expectation
h. Central rate

4. Non-Integral Ages (4.5)
   a. Linear assumption
   b. Exponential assumption
   c. Hyperbolic assumption

5. Select Tables (4.6)

C. Contingent Payment Models

1. Discrete Models (5.1)
   a. Z_x, and its moments
   b. $Z^1_{x:\overline{n}|}$ and its moments
   c. $_{n|}Z_x$, and its moments; covariance with $Z^1_{x:\overline{n}|}$
   d. $Z_{x:\overset{1}{\overline{n}|}}$ and its moments; covariance with $Z^1_{x:\overline{n}|}$
   e. $Z_{x:\overline{n}|}$, and its moments

2. Group Deterministic Interpretation (5.2)

3. Continuous Models (5.3)
   a. $\bar{Z}_x$, and its moments
   b. $\bar{Z}^1_{x:\overline{n}|}$ and $_{n|}\bar{Z}_x$, and their moments
   c. $\bar{Z}_{x:\overline{n}|}$
   d. $Z^{(m)}_x$, and its moments
   e. Evaluation under exponential distribution
   f. Evaluation under uniform distribution

4. Varying Payments (5.4)
   a. B_x in general
   b. $(IA)_x$
   c. $(IA)^1_{x:\overline{n}|}$ and $(DA)^1_{x:\overline{n}|}$
   d. $(\overline{IA})_x$, $(\overline{IA})^1_{x:\overline{n}|}$ and $(\overline{DA})^1_{x:\overline{n}|}$
   e. $(I\bar{A})_x$, $(I\bar{A})^1_{x:\overline{n}|}$, and $(D\bar{A})^1_{x:\overline{n}|}$,

5. Approximation from Life Table (5.5)
   a. Continuous models
   b. m^{thly} models

D. Contingent Annuity Models

1. Whole Life (6.1)
    a. Immediate
    b. Due
    c. Continuous

2. Temporary (6.2)
    a. Immediate
    b. Due
    c. Continuous

3. Deferred (6.3)
    a. Immediate
    b. Due
    c. Continuous

4. m^{thly} Payments (6.4)
    a. Immediate
    b. Due
    c. Random variables
    d. Approximation from life table

5. Non-Level Payments (6.5)
    a. Immediate
    b. Due
    c. Continuous

E. Funding Plans

1. Annual Payment Funding (7.1)
    a. Discrete payment models
    b. Continuous payment models
    c. Non-level funding

2. Random Variable Analysis (7.2)
    d. Present value of loss random variable
    e. Expected value
    f. Variance

3. Continuous Payment Funding (7.3)
    a. Discrete payment models
    b. Continuous payment models

4. m^{thly} Payment Funding (7.4)
    a. Discrete payment models
    b. Continuous payment models

5. Incorporation of Expenses (7.5)

F. Reserves

1. Annual Payment Funding (8.1)
   a. Prospective method
   b. Retrospective method
   c. Additional expressions
   d. Random variable analysis
   e. Continuous payment models
   f. Contingent annuity model

2. Recursive Relationships (8.2)
   a. Group deterministic analysis
   b. Random variable analysis – cash basis
   c. Random variable analysis – accrued basis

3. Continuous Payment Funding (8.3)
   a. Discrete payment models
   b. Continuous payment models
   c. Random variable analysis

4. m^{thly} Payment Funding (8.4)

5. Incorporation of Expenses (8.5)

6. Fractional Duration Reserves (8.6)

7. Non-Level Benefits and Premiums (8.7)
   a. Discrete models
   b. Continuous models

G. Multi-Life Models

1. Joint-Life Model (9.1)
   a. Random variable T_{xy}
   b. SDF
   c. CDF
   d. PDF
   e. HRF
   f. Conditional probabilities
   g. Moments

2. Last-Survivor Model (9.2)
   a. Random variable $T_{\overline{xy}}$
   b. CDF
   c. SDF
   d. PDF
   e. HRF
   f. Moments
   g. Relationships of T_{xy} and $T_{\overline{xy}}$

3. Contingent Probability Functions (9.3)

4. Contingent Multi-Life Contracts (9.4)
   a. Contingent payment models
   b. Contingent annuity models
   c. Premiums and reserves
   d. Reversionary annuities
   e. Contingent insurance functions

5. Random Variable Analysis (9.5)
   a. Marginal distributions
   b. Covariance
   c. Joint functions
   d. Joint-life
   e. Last-survivor

6. Common Shock (9.6)

H. Multiple-Decrement Models

1. Discrete Models (10.1)
   a. Multiple-decrement tables
   b. Random variable analysis

2. Competing Risks (10.2)

3. Continuous Models (10.3)

4. Uniform Distribution (10.4)
   a. Multiple-decrement context
   b. Single-decrement context

5. Actuarial Present Value (10.5)

6. Asset Shares (10.6)

I. Poisson Process

1. Properties (11.3.1-11.3.3)

2. Mixture Process (11.3.4)

3. Nonstationary Process (11.3.5)

4. Compound Poisson Process (14.1)

J. Multi-State Models

1. Review of Markov Chains (Appendix A)
2. Homogeneous Multi-State Process (10.7.1)
3. Nonhomogeneous Multi-State Process (10.7.2)
4. Daniel's Notation (SN: M-24-05)

Practice Questions

1. Let a survival distribution be defined by $S_X(x) = ax^2 + b$, for $0 < x \le k$. If the expected value of X is 60, find the median of X.

2. Given that $\mu_x = kx$, for all $x > 0$, and ${}_{10}p_{35} = .81$, find the value of ${}_{20}p_{40}$.

3. If X has a uniform distribution over $(0, \omega)$, show that $\mu_x = \frac{m_x}{1+.50m_x}$, for $x \le \omega - 1$.

4. Given that $\mathring{e}_0 = 25$ and $\ell_x = \omega - x$, for $0 \le x \le \omega$, find the value of $Var(T_{10})$, where T_{10} denotes the future lifetime random variable for an entity known to exist at age 10.

5. Given the UDD assumption and the values $\mu_{80.5} = .0202$, $\mu_{81.5} = .0408$, and $\mu_{82.5} = .0619$, find the value of ${}_2q_{80.5}$.

6. Let μ_{x+k} denote a constant force of mortality for the age interval $(x+k, x+k+1)$. Find the value of $\mathring{e}_{x:\overline{3}|}$, the expected number of years to be lived over the next three years by a life age x, given the following data:

k	$e^{-\mu_{x+k}}$	$\frac{1-e^{-\mu_{x+k}}}{\mu_{x+k}}$
0	.9512	.9754
1	.9493	.9744
2	.9465	.9730

7. Given the following excerpt from a select and ultimate table with a two-year select period, and assuming UDD between integral ages, find the value of ${}_{.90}q_{[60]+.60}$.

x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
60	80,625	79,954	78,839	62
61	79,137	78,402	77,252	63
62	77,575	76,770	75,578	64

8. Calculate the value of $Var(Z_{51})$, given the following values:

$$A_{51} - A_{50} = .004 \qquad i = .02$$
$$^2A_{51} - {}^2A_{50} = .005 \qquad p_{50} = .98$$

9. Let the age-at-failure random variable X have a uniform distribution with $\omega = 110$. Let $f_Z(z)$ denote the PDF of the random variable $\bar{Z}_{40}$. Calculate the value of $f_Z(.80)$, given also that $\delta = .05$.

10. (a) Show that $(IA)_x \;=\; A_x + {}_1E_x \cdot (IA)_{x+1}$.

(b) Calculate the value of $(IA)_{36}$, given the following values:

$$(IA)_{35} = 3.711 \qquad A^{\;\,1}_{35:\overline{1|}} = .9434$$
$$A_{35} = .1300 \qquad p_{35} = .9964$$

11. Assuming failures are uniformly distributed over each interval $(x, x+1)$, calculate the value of ${}^2\bar{A}^{1}_{x:\overline{2|}}$ given the values $i = .12$, $q_x = .10$, and $q_{x+1} = .20$.

12. Find the value of $\bar{A}_x$, given the values $\bar{a}_x = 10$, ${}^2\bar{a}_x = 7.375$, and $Var(\bar{a}_{\overline{T_x|}}) = 50$.

13. Let S denote the number of annuity payments actually made under a unit 5-year deferred whole life annuity-due. Find the value of $Pr(S > {}_{5|}\ddot{a}_x)$, given the following values:

$$\ddot{a}_{x:\overline{5|}} = 4.542 \qquad i = .04 \qquad \mu_{x+t} = .01, \quad \text{for all } t$$

14. Show that, under the UDD assumption $\bar{A}_x \;=\; \dfrac{i}{\delta} - \dfrac{(i-d)\ddot{a}_x}{\delta}$.

15. Calculate the probability that the present value of payments actually made under a unit 3-year temporary increasing annuity-due will exceed the APV of the annuity contract, given the following values:

$$p_x = .80 \qquad p_{x+1} = .75 \qquad p_{x+2} = .50 \qquad v = .90$$

16. Calculate the value of $1000P(\bar{A}_{x:\overline{n|}})$, assuming UDD over each interval $(x, x+1)$ and the following values:

$$\bar{A}_{x:\overline{n|}} = .804 \qquad {}_nE_x = .600 \qquad i = .04$$

17. For a unit whole life insurance, let L_x denote the present value of loss random variable when the premium is chosen such that $E[L_x] = 0$, and let L_x^* denote the present value of loss random variable when the premium is chosen such that $E[L_x^*] \;=\; -.20$. Given that $Var(L_x) = .30$, find the value of $Var(L_x^*)$.

18. If the force of interest is δ and the force of failure is $\lambda(x)=\lambda$ for all x, show that $Var[\bar{L}(\bar{A}_x)]=\frac{\lambda}{\lambda+2\delta}$, with continuous premium rate determined by the equivalence principle.

19. Consider a 20-pay unit discrete whole life insurance, with expense factors of a flat amount .02 each year, plus an additional .05 in the first year only, plus 3% of each premium paid. Find the gross annual premium for this contract, given the values $\ddot{a}_x = 20$, $\ddot{a}_{x:\overline{20|}}=10$, and $d=.04$.

20. Calculate the value of $1000({}_2V_{x:\overline{3|}} - {}_1V_{x:\overline{3|}})$, given the following values:

$$P_{x:\overline{3|}}=.33251 \qquad i=.06 \qquad \ell_x=100 \qquad \ell_{x+1}=90$$

21. A 2-year term insurance of amount 400 is issued to (x), with benefit premium determined by the equivalence principle. Find the probability that the loss at issue is less than 190, given the values $P^1_{x:\overline{2|}}=.185825$, ${}_1V^1_{x:\overline{2|}}=.04145$, and $i=.10$.

22. A 10-pay whole life contract of amount 1000 is issued to (x). The net annual premium is 32.88 and the benefit reserve at the end of year 9 is 322.87. Given that $i=.06.$ and $q_{x+9}=.01262$, find the value of P_{x+10}.

23. Let Λ_{11} denote the accrued cost random variable in the 11^{th} year for a discrete whole life insurance of amount 1000 issued to (40). Calculate the value of $Var(\Lambda_{11}|\,K_{40}>10)$, given the following values:

$$i=.06, \qquad \ddot{a}_{40}=14.8166 \qquad \ddot{a}_{50}=13.2669 \qquad \ddot{a}_{51}=13.0803$$

24. Let ${}_0\bar{L}(\bar{A}_x)$ denote the present value of loss at issue for a fully continuous whole life contract issued to (x). Find the value of ${}_{20}\bar{V}(\bar{A}_x)$. given the following values:

$$Var[{}_0\bar{L}(\bar{A}_x)]=.20 \qquad {}^2\bar{A}_x=.30 \qquad \bar{A}_{x+20}=.70$$

25. A 2-year endowment contract issued to (x) has a failure benefit of 1000 plus the reserve at the end of the year of failure and a pure endowment benefit of 1000. Given that $i=.10$, $q_x=.10$, and $q_{x+1}=.11$, calculate the net level benefit premium.

26. Let T_x and T_y be independent future lifetime random variables. Given $q_x=.080$, $q_y=.004$, ${}_tp_x=1-t^2\cdot q_x$, and ${}_tp_y=1-t^2\cdot q_y$, both for $0\le t\le 1$, evaluate the PDF of T_{xy} at $t=.50$.

27. Let T_{80} and T_{85} be independent random variables with uniform distributions with $\omega = 100$. Find the probability that the second failure occurs within five years from now.

28. A discrete unit benefit contingent contract is issued to the last-survivor status $(\overline{xx})$, where the two future lifetime random variables T_x are independent. The contract is funded by discrete net annual premiums, which are reduced by 25% after the first failure. Find the value of the initial net annual premium, under the equivalence principle, given the following values:

$$A_x = .40 \qquad A_{xx} = .55 \qquad \ddot{a}_x = 10.00$$

29. The APV for a last-survivor whole life insurance on $(\overline{xy})$, with unit benefit paid at the instant of failure of the status, was calculated assuming independent future lifetimes for (x) and (y) with constant hazard rate .06 for each. It is now discovered that although the total hazard rate of .06 is correct, the two lifetimes are not independent since each includes a common shock hazard factor with constant force .02. The force of interest used in the calculation is $\delta = .05$. Calculate the increase in the APV that results from recognition of the common shock element.

30. The career of a 50-year-old Professor of Actuarial Science is subject to two decrements. Decrement 1 is mortality, which is governed by a uniform survival distribution with $\omega = 100$, and Decrement 2 is leaving academic employment, which is governed by the HRF $\mu_y^{(2)} = .05$, for all $y \geq 50$. Find the probability that this professor remains in academic employment for at least five years but less than ten years.

31. Find the value of $p_x'^{(1)}$, given $q_x^{(1)} = .48$, $q_x^{(2)} = .32$, $q_x^{(3)} = .16$, and each decrement is uniformly distributed over $(x, x+1)$ in the multiple-decrement context.

32. Decrement 1 is uniformly distributed over the year of age in its associated single-decrement table with $q_x'^{(1)} = .100$. Decrement 2 always occurs at age $x + .70$ in its associated single-decrement table with $q_x'^{(2)} = .125$. Find the value of $q_x^{(2)}$.

33. Events occur according to a Poisson process with rate 2 per day.

(a) What is the expected waiting time until the tenth event occurs?
(b) What is the probability that the 11^{th} event will occur more than two days after the 10^{th} event?

34. During a certain type of epidemic, infections occur in a population at a Poisson rate of 20 per day, but are not immediately identified. The time elapsed from onset of the infection to its identification is an exponential random variable with mean of 7 days. Find the expected number of identified infections over a 10-day period.

Solutions to Practice Questions

1. We have $S(0)=b=1$ and $S(k)=ak^2+1=0$. Thus $a=\frac{-1}{k^2}$, so $S(x)=1-\frac{x^2}{k^2}$. Then

$$E[X] = \int_0^k S(y)\,dy = k-\frac{k^3}{3k^2} = k-\frac{1}{3}k = \frac{2}{3}k = 60,$$

so $k=90$, and thus $a=\frac{-1}{8100}$. Finally, $S(m)=1-\frac{m^2}{8100}=\frac{1}{2}$, which leads to $m^2=4050$, and $m=45\sqrt{2}$.

2. Recall that

$$_tp_x = e^{-\int_0^t \mu_{x+r}\,dr} = e^{-\int_0^{x+t} \mu_y\,dy}.$$

Here

$$_{10}p_{35} = .81 = e^{-\int_{35}^{45} ky\,dy}$$

$$= e^{-\left[\frac{1}{2}k(45)^2-\frac{1}{2}k(35)^2\right]} = e^{-400k},$$

and

$$_{20}p_{40} = e^{-\int_{40}^{60} ky\,dy}$$

$$= e^{-\left[\frac{1}{2}k(60)^2-\frac{1}{2}k(40)^2\right]}$$

$$= e^{-1000k}$$

$$= \left(e^{-400k}\right)^{2.5} = (.81)^{2.5} = .59049.$$

3. If X is uniform, then $\mu_x=\frac{1}{\omega-x}$ and

$$_tp_x = \frac{S(x+t)}{S(x)} = 1-\frac{t}{\omega-x},$$

so that $_tp_x\cdot\mu_{x+t} = \frac{1}{\omega-x}$. From Equation (3.62) we have

$$m_x = \frac{\int_0^1 {}_tp_x\mu_{x+t}\,dt}{\int_0^1 {}_tp_x\,dt}$$

$$= \frac{\int_0^1 \frac{1}{\omega-x}\,dt}{\int_0^1 \frac{\omega-x-t}{\omega-x}\,dt} = \frac{\int_0^1 dt}{\int_0^1(\omega-x-t)\,dt} = \frac{1}{\omega-x-.50}.$$

Then

$$\frac{m_x}{1+.50m_x} = \frac{1}{\frac{1}{m_x}+.50} = \frac{1}{\omega-x-.50+.50} = \frac{1}{\omega-x} = \mu_x,$$

as required.

4. The survival model is uniform. The fact that $\overset{o}{e}_0 = 25$ tells us that $\omega = 50.$ Then T_{10} is uniform over $(0, 40),$ so its variance is $\frac{(40)^2}{12} = 133.3\dot{3}.$

5. First we use $\mu_{80.5} = \frac{q_{80}}{1-.50q_{80}} = .0202$ to solve for $q_{80} = .02.$ Similarly we find $q_{81} = .04$ and $q_{82} = .06.$ Arbitrarily let $\ell_{80} = 1000,$ so

$$\ell_{81} = (.98)(1000) = 980,$$

$$\ell_{82} = (.96)(980) = 940.80,$$

and

$$\ell_{83} = (.94)(940.80) = 884.352.$$

Then

$$_2q_{80.5} \;=\; 1-\frac{\ell_{82.5}}{\ell_{80.5}} \;=\; 1-\frac{\frac{1}{2}(940.80+884.352)}{\frac{1}{2}(1000+980)}$$

$$=\; 1-\frac{912.576}{990} \;=\; .07821.$$

6. We start with

$$\overset{o}{e}_{x:\overline{3}|} \;=\; \int_0^3 {}_tp_x\,dt$$

$$=\; \int_0^1 {}_tp_x\,dt + p_x\int_0^1 {}_tp_{x+1}\,dt + p_x\cdot p_{x+1}\int_0^1 {}_tp_{x+2}\,dt.$$

Under the constant force assumption,

$$\int_0^1 {}_tp_x\,dt \;=\; \int_0^1 (p_x)^t\,dt \;=\; \left.\frac{(p_x)^t}{\ln p_x}\right|_0^1 \;=\; \frac{1-p_x}{-\ln p_x}.$$

But

$$p_x = e^{-\mu_x} \text{ and } \frac{1-p_x}{-\ln p_x} \;=\; \frac{1-e^{-\mu_x}}{\mu_x},$$

so

$$\overset{o}{e}_{x:\overline{3}|} \;=\; \frac{1-e^{-\mu_x}}{\mu_x} + e^{-\mu_x}\left(\frac{1-e^{-\mu_{x+1}}}{\mu_{x+1}}\right) + e^{-\mu_x}\cdot e^{-\mu_{x+1}}\left(\frac{1-e^{-\mu_{x+2}}}{\mu_{x+2}}\right)$$

$$=\; .9754 + (.9512)(.9744) + (.9512)(.9493)(.9730) \;=\; 2.78084.$$

7. $${}_{.90}q_{[60]+.60} \;=\; {}_{.40}q_{[60]+.60} + (1-{}_{.40}q_{[60]+.60}) \cdot {}_{.50}q_{[60]+1}$$

$$= \frac{.40q_{[60]}}{1-.60q_{[60]}} + \left(1-\frac{.40q_{[60]}}{1-.60q_{[60]}}\right) \cdot .50q_{[60]+1}$$

Here

$$q_{[60]} \;=\; 1-\frac{\ell_{[60]+1}}{\ell_{[60]}} \;=\; 1-\frac{79{,}954}{80{,}625} \;=\; .00832248$$

and

$$q_{[60]+1} \;=\; 1-\frac{\ell_{62}}{\ell_{[60]+1}} \;=\; 1-\frac{78{,}839}{79{,}954} \;=\; .01394552,$$

so

$$\begin{aligned} {}_{.90}q_{[60]+.60} \;&=\; \frac{(.40)(.00832248)}{1-(.60)(.00832248)} \\ &\quad + \left(1-\frac{(.40)(.00832248)}{1-(.60)(.00832248}\right)(.50)(.01394552) \\ &=\; .0033457 + (1-.0033457)(.50)(.01394552) \;=\; .01029. \end{aligned}$$

8. Recall that $A_{50} = v \cdot q_{50} + v \cdot p_{50} \cdot A_{51}$. Then

$$\begin{aligned} A_{51} - A_{50} \;&=\; A_{51} - v \cdot q_{50} - v \cdot p_{50} \cdot A_{51} \\ &=\; A_{51}(1-v \cdot p_{50}) - v \cdot q_{50} \\ &=\; A_{51}\left(1-\frac{.98}{1.02}\right) - \frac{.02}{1.02} \;=\; .004, \end{aligned}$$

which implies $A_{51} = .60199$.

Similarly,

$$\begin{aligned} {}^{2}A_{51} - {}^{2}A_{50} \;&=\; {}^{2}A_{51} - v' \cdot q_{50} - v' \cdot p_{50} \cdot {}^{2}A_{51} \\ &=\; {}^{2}A_{51}(1-v' \cdot p_{50}) - v' \cdot q_{50} \\ &=\; {}^{2}A_{51}\left(1-\frac{.98}{(1.02)^2}\right) - \frac{.02}{(1.02)^2} \;=\; .005, \end{aligned}$$

which implies ${}^{2}A_{51} = .41725$.

Then

$$Var(Z_{51}) \;=\; {}^{2}A_{51} - A_{51}^{\,2} \;=\; .41725 - (.60199)^2 \;=\; .05486.$$

9. Since $\bar{Z}_{40} = v^{T_{40}}$, we have the transformation $z = v^t$ so $t = \frac{-\ln z}{\delta}$. The transformation is decreasing so we have

$$F_{Z_{40}}(z) \;=\; S_{T_{40}}\left[-\frac{\ln z}{\delta}\right].$$

But $S_{T_{40}}(t) \;=\; {}_tp_{40} \;=\; 1-\frac{t}{\omega-40} \;=\; 1-\frac{t}{70}$ under a uniform distribution with $\omega = 110$. Therefore,

$$F_{Z_{40}}(z) \;=\; 1-\frac{-\frac{\ln z}{\delta}}{70} \;=\; 1+\frac{\ln z}{3.50},$$

since $\delta = .05$. Then

$$f_{Z_{40}}(z) \;=\; \frac{d}{dz}F_{Z_{40}}(z) \;=\; \frac{1}{3.50z},$$

and finally

$$f_{Z_{40}}(.80) \;=\; \frac{1}{(3.50)(.80)} \;=\; .35714.$$

10. (a) From Equation (5.51) we have

$$(IA)_x \;=\; \sum_{k=1}^{\infty} k\cdot v^k \cdot {}_{k-1|}q_x \;=\; \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_x + \sum_{k=2}^{\infty}(k-1)\cdot v^k \cdot {}_{k-1|}q_x.$$

Let $r = k-1$ so $k = r+1$. Then

$$(IA)_x \;=\; A_x + \sum_{r=1}^{\infty} r\cdot v^{r+1}\cdot {}_{r|}q_x$$

$$=\; A_x + v\cdot p_x \sum_{r=1}^{\infty} r\cdot v^r \cdot {}_{r-1|}q_{x+1} \;=\; A_x + {}_1E_x \cdot (IA)_{x+1}.$$

(b) Note that $A_{\overset{\;1}{35}:\overline{1|}} = v = .9434$. Then

$${}_1E_{35} \;=\; v\cdot p_{35} \;=\; (.9434)(.9964) \;=\; .94000.$$

Using the result from part (a), we have

$$(IA)_{35} \;=\; A_{35} + {}_1E_{35}\cdot(IA)_{36}.$$

Then

$$(IA)_{36} \;=\; \frac{(IA)_{35} - A_{35}}{{}_1E_{35}} \;=\; \frac{3.711-.130}{.940} \;=\; 3.80957.$$

11. The relationship given by Equation (5.64b) holds for the second moment functions using an interest rate based on 2δ. We have

$$ {}^2\bar{A}^{\,1}_{x:\overline{2|}} \;=\; \frac{i'}{\delta'}\cdot {}^2A^{\,1}_{x:\overline{2|}} \;=\; \frac{(1.12)^2-1}{\ln(1.12)^2}\left[\frac{.10}{(1.12)^2}+\frac{(.90)(.20)}{(1.12)^4}\right] $$

$$ = 1.12240(.07972+.11439) \;=\; .21787. $$

12. Recall that

$$ Var(\bar{a}_{\overline{T_x|}}) \;=\; Var(\bar{Y}_x) \;=\; \frac{{}^2\bar{A}_x-\bar{A}_x^{\,2}}{\delta^2}. $$

We know that

$$ \bar{A}_x \;=\; 1-\delta\cdot\bar{a}_x \;=\; 1-10\delta $$

and

$$ {}^2\bar{A}_x \;=\; 1-2\delta\cdot{}^2\bar{a}_x \;=\; 1-14.75\delta. $$

Then

$$ Var(\bar{Y}_x) \;=\; \frac{(1–14.75\delta)-(1–10\delta)^2}{\delta^2} \;=\; \frac{5.25\delta-100\delta^2}{\delta^2} \;=\; 50, $$

which solves for $\delta=.035$. Finally

$$ \bar{A}_x \;=\; 1-(.035)(10) \;=\; .65. $$

13. The force of mortality is constant, so ${}_tp_x=e^{-\mu t}$ for all t. Then we can calculate

$$ \begin{aligned} \ddot{a}_x \;&=\; 1+v\cdot p_x+v^2\cdot{}_2p_x+\cdots \\ &=\; 1+e^{-\delta}\cdot e^{-\mu}+e^{-2\delta}\cdot e^{-2\mu}+\cdots \\ &=\; 1+e^{-(\mu+\delta)}+(e^{-(\mu+\delta)})^2+\cdots \\ &=\; \frac{1}{1-e^{-(\mu+\delta)}} \;=\; \frac{1}{1-e^{-[.01+\ln(1.04)]}} \;=\; 20.82075, \end{aligned} $$

so

$$ {}_{5|}\ddot{a}_x \;=\; \ddot{a}_x-\ddot{a}_{x:\overline{5|}} \;=\; 16.27875. $$

Then $S > {}_{5|}\ddot{a}_x$ if 17 payments are made, which occurs if (x) survives to age $x+21$. This means that

$$ Pr(S > {}_{5|}\ddot{a}_x) \;=\; {}_{21}p_x \;=\; e^{-(21)(.01)} \;=\; .81058. $$

14. Recall that $\bar{A}_x = \frac{i}{\delta} \cdot A_x$, and $A_x = 1 - d \cdot \ddot{a}_x$, so we have

$$\begin{aligned} \bar{A}_x &= \frac{i}{\delta}(1 - d \cdot \ddot{a}_x) \\ &= \frac{i}{\delta} - \frac{id}{\delta} \cdot \ddot{a}_x \\ &= \frac{i}{\delta} - \frac{i-d}{\delta} \cdot \ddot{a}_x. \end{aligned}$$

15. The APV of the contract is

$$\begin{aligned} APV &= 1 + 2 \cdot v \cdot p_x + 3 \cdot v^2 \cdot {}_2p_x \\ &= 1 + (2)(.90)(.80) + (3)(.90)^2(.80)(.75) \\ &= 3.898. \end{aligned}$$

The present value of payments actually made is $1 + 2v = 2.80$ if only two payments are made, and $2.80 + 3v^2 = 5.23$ if all three payments are made. Then for the present value of payments actually made to exceed the APV, survival to time $t = 2$ is required, the probability of which is $(.80)(.75) = .60$.

16. We have

$$1000P\left(\bar{A}_{x:\overline{n}|}\right) = \frac{1000\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{(1000)(.804)}{\ddot{a}_{x:\overline{n}|}}.$$

We calculate $\ddot{a}_{x:\overline{n}|}$ from

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d},$$

where, in turn, we find $A_{x:\overline{n}|}$ from

$$\begin{aligned} \bar{A}_{x:\overline{n}|} &= \bar{A}^{1}_{x:\overline{n}|} + {}_nE_x \\ &= \frac{i}{\delta} \cdot A^{1}_{x:\overline{n}|} + {}_nE_x = \frac{.05}{\ln(1.05)} \cdot A^{1}_{x:\overline{n}|} + .600 = 804, \end{aligned}$$

which gives us $A^{1}_{x:\overline{n}|} = \frac{(.804 - .600) \cdot \ln(1.05)}{.05} = .200.$ Then

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x = .200 + .600 = .800,$$

so

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - .800}{\frac{.04}{1.04}} = 5.20$$

and finally

$$1000P\left(\bar{A}_{x:\overline{n}|}\right) = \frac{(1000)(.804)}{5.20} = 154.62.$$

17. The condition $E[L_x]=0$ implies $P=P_x$. Then

$$Var(L_x) = \left(1+\frac{P_x}{d}\right)^2 \cdot \left({}^2A_x - A_x{}^2 \right)$$
$$= \left(\frac{1}{d\cdot \ddot{a}_x}\right)^2 \cdot \left({}^2A_x - A_x{}^2 \right)$$
$$= .30,$$

so

$${}^2A_x - A_x{}^2 = (.30)(d\cdot \ddot{a}_x)^2.$$

Then observe that

$$E\left[L_x^*\right] = A_x\left(1+\frac{P}{d}\right)-\frac{P}{d}$$
$$= (1-d\cdot\ddot{a}_x)\left(1+\frac{P}{d}\right)-\frac{P}{d}$$
$$= 1-d\cdot\ddot{a}_x+\frac{P}{d}-P\cdot\ddot{a}_x-\frac{P}{d} = -.20,$$

so $1-(P_x+d)\ddot{a}_x = -.20$, or $P_x+d=\frac{1.20}{\ddot{a}_x}$.

Now observe that

$$Var(L_x^*) = \left(1+\frac{P}{d}\right)^2 \cdot \left({}^2A_x - A_x{}^2 \right)$$
$$= \left(\frac{P+d}{d}\right)^2 (.30)(d\cdot\ddot{a}_x)^2$$
$$= \left(\frac{1.20}{d\cdot\ddot{a}_x}\right)^2 (.30)(d\cdot\ddot{a}_x)^2$$
$$= (1.20)^2(.30) = .43200.$$

18. From Example 7.5 we know that

$$P = \bar{P}(\bar{A}_x) = \lambda,$$

and from earlier results we know that

$$\bar{A}_x = \frac{\lambda}{\lambda+\delta}$$

and

$${}^2\bar{A}_x = \frac{\lambda}{\lambda+2\delta}.$$

Then from Equation (7.24a) we have

$$Var\left[\bar{L}(\bar{A}_x)\right] = \left(1+\frac{\lambda}{\delta}\right)^2 \cdot \left[\frac{\lambda}{\lambda+2\delta} - \left(\frac{\lambda}{\lambda+\delta}\right)^2\right]$$

$$= \left(\frac{\lambda+\delta}{\delta}\right)^2 \cdot \left[\frac{\lambda}{\lambda+2\delta} - \left(\frac{\lambda}{\lambda+\delta}\right)^2\right]$$

$$= \frac{\lambda(\lambda+\delta)^2}{\delta^2(\lambda+2\delta)} - \frac{\lambda^2}{\delta^2}$$

$$= \frac{\lambda(\lambda^2+2\lambda\delta+\delta^2) - \lambda^2(\lambda+2\delta)}{\delta^2(\lambda+2\delta)}$$

$$= \frac{\lambda^3+2\lambda^2\delta+\lambda\delta^2-\lambda^3-2\lambda^2\delta}{\delta^2(\lambda+2\delta)}$$

$$= \frac{\lambda}{\lambda+2\delta}, \text{ as required.}$$

19. The expense-augmented equation of value is

$$G \cdot \ddot{a}_{x:\overline{20}|} = A_x + .05 + .02\ddot{a}_x + .03G \cdot \ddot{a}_{x:\overline{20}|}.$$

Then

$$G = \frac{A_x + .05 + .02\ddot{a}_x}{(1-.03)\ddot{a}_{x:\overline{20}|}}$$

$$= \frac{1-(.04)(20)+.05+(.02)(20)}{(.97)(10)}$$

$$= .06701.$$

20. The value of $1000\ {}_1V_{x:\overline{3}|}$ is best calculated retrospectively as

$$1000\ {}_1V_{x:\overline{3}|} = 1000\left(P_{x:\overline{3}|} \cdot \frac{1}{{}_1E_x} - v \cdot q_x \cdot \frac{1}{{}_1E_x}\right)$$

$$= 1000\left(\frac{.33251}{(1.06)^{-1}(.90)} - \frac{.10}{.90}\right) = 280.51.$$

The value of $1000\ {}_2V_{x:\overline{3}|}$ is best calculated prospectively as

$$1000\ {}_2V_{x:\overline{3}|} = 1000\left(A_{x+2:\overline{1}|} - P_{x:\overline{3}|}\right)$$

$$= 1000\left((1.06)^{-1} - .33251\right) = 610.89.$$

Then the difference is

$$610.89 - 280.51 = 330.38.$$

21. The loss at issue, denoted L, will be

$$L = \frac{400}{1.10} - (400)(.185825) = 289.31$$

if failure occurs in the first year, which happens with probability q_x. The loss will be

$$L = \frac{400}{(1.10)^2} - 74.33 - \frac{74.33}{1.10} = 188.68$$

if failure occurs in the second year, which happens with probability $p_x \cdot q_{x+1}$. Certainly the loss will be less than 190 if (x) survives to age $x+2$, so we can conclude that the loss is less than 190 with probability p_x.

To find p_x we first use

$$_1V^1_{x:\overline{2}|} = v \cdot q_{x+1} - .185825 = .04145,$$

which solves for $q_{x+1} = .25$. Then we use

$$P^1_{x:\overline{2}|} = \frac{\frac{q_x}{1.10} + \frac{p_x \cdot q_{x+1}}{(1.10)^2}}{1 + \frac{p_x}{1.10}} = \frac{\frac{1-p_x}{1.10} + \frac{.25\,p_x}{(1.10)^2}}{1 + \frac{p_x}{1.10}}$$

$$= \frac{(1.10)(1-p_x) + .25\,p_x}{(1.10)^2 + 1.10\,p_x} = .185825,$$

which solves for $p_x = .83$.

22. Using the recursive relationship

$$({}_9V + P)(1+i) = 1000q_{x+9} + {}_{10}V \cdot p_{x+9},$$

we have

$$(322.87 + 32.88)(1.06) = 12.62 + {}_{10}V(.98738),$$

which solves for ${}_{10}V = 369.13$. At duration 10 the contract is paid up, so the reserve prospectively is $1000A_{x+10} = 369.13$. Then

$$P_{x+10} = \frac{A_{x+10}}{\ddot{a}_{x+10}} = \frac{.36913}{\frac{1-.36913}{.06/1.06}} = .03312.$$

23. From Equation (8.39) we have

$$Var(\Lambda_{11} \mid K_{40} > 10) = v^2(1 - {}_{11}V)^2 \cdot q_{50} \cdot p_{50}.$$

Here we have

$${}_{11}V = 1000 \cdot {}_{11}V_{40} = 1000\left(1 - \frac{\ddot{a}_{51}}{\ddot{a}_{40}}\right) = 117.19.$$

To find p_{50} we use $\ddot{a}_{50} = 1 + v \cdot p_{50} \cdot \ddot{a}_{51}$, so

$$p_{50} = \frac{(\ddot{a}_{50} - 1)(1+i)}{\ddot{a}_{51}} = \frac{(12.2669)(1.06)}{13.0803} = .99408.$$

Then

$$Var(\Lambda_{11} \mid K_{40} > 10) = (1.06)^{-2}(1000 - 117.19)^2(.99408)(.00592) = 4081.93.$$

24. Let L denote ${}_0\bar{L}(\bar{A}_x)$. Recall that

$$Var(L) = \left(\frac{1}{\delta \cdot \bar{a}_x}\right)^2 \cdot ({}^2\bar{A}_x - \bar{A}_x^{\,2}) = \left(\frac{1}{1 - \bar{A}_x}\right)^2 \cdot (.30 - \bar{A}_x^{\,2}) = .20,$$

so

$$(.30 - \bar{A}_x^{\,2}) = .20(1 - 2\bar{A}_x + \bar{A}_x^{\,2})$$

or

$$1.20\bar{A}_x^{\,2} - .40\bar{A}_x - .10 = 0,$$

which solves for $\bar{A}_x = .50$. Then

$${}_{20}\bar{V}(\bar{A}_x) = 1 - \frac{\bar{a}_{x+20}}{\bar{a}_x} = 1 - \frac{1 - \bar{A}_{x+20}}{1 - \bar{A}_x} = 1 - \frac{.30}{.50} = .40.$$

25. When the failure benefit is a fixed amount plus the benefit reserve, we use the recursive relationship approach. For the first year, where ${}_0V = 0$, we have

$$P(1+i) = q_x(1000 + {}_1V) + p_x \cdot {}_1V = 1000q_x + {}_1V,$$

so

$${}_1V = P(1+i) - 1000q_x = P(1.10) - 100.$$

For the second year we have

$$({}_1V + P)(1+i) = q_{x+1}(1000 + {}_2V) + p_{x+1} \cdot {}_2V,$$

so

$$\begin{aligned} {}_2V &= ({}_1V + P)(1+i) - 1000q_{x+1} \\ &= \left[P(1.10) - 100 + P\right](1.10) - 110 = P \cdot \ddot{s}_{\overline{2}|.10} - 220. \end{aligned}$$

But, prospectively, ${}_2V = 1000$, as we have $P = \dfrac{1000 + 220}{\ddot{a}_{\overline{2}|.10}} = 528.14.$

26. The SDF of T_{xy} is

$$
\begin{aligned}
{}_t p_{xy} = {}_t p_x \cdot {}_t p_y &= (1-.080t^2)(1-.004t^2) \\
&= 1-.084t^2+.00032t^4,
\end{aligned}
$$

for $0 \le t \le 1.$ Then the PDF is given by

$$-\frac{d}{dt}\,{}_t p_{xy} = 2(.084)t - .00128t^3,$$

and the PDF at $t=.50$ is $(2)(.084)(.50)-(.00128)(.50)^3 = .08384.$

27. We seek the value of

$$
\begin{aligned}
{}_5 q_{\overline{80:85}} &= 1 - {}_5 p_{\overline{80:85}} \\
&= 1 - {}_5 p_{80} - {}_5 p_{85} + {}_5 p_{80:85} \\
&= 1-\left(1-\frac{5}{20}\right)-\left(1-\frac{5}{15}\right)+\left(1-\frac{5}{20}\right)\left(1-\frac{5}{15}\right) \\
&= 1-\frac{15}{20}-\frac{10}{15}+\left(\frac{15}{20}\right)\left(\frac{10}{15}\right) = \frac{1}{12}.
\end{aligned}
$$

Alternatively,

$${}_5 q_{\overline{80:85}} = {}_5 q_{80} \cdot {}_5 q_{85} = \frac{5}{20}\cdot\frac{5}{15} = \frac{1}{12}.$$

28. The APV of the benefit is $A_{\overline{xx}} = A_x + A_x - A_{xx}.$ The APV of the premium stream is

$$
\begin{aligned}
P\cdot \ddot{a}_{xx} + .75P(\ddot{a}_{\overline{xx}} - \ddot{a}_{xx}) &= P\cdot \ddot{a}_{xx} + .75P(2\ddot{a}_x - 2\ddot{a}_{xx}) \\
&= P(1.50\ddot{a}_x - .50\ddot{a}_{xx}).
\end{aligned}
$$

We use the given values of A_x and $\ddot{a}_x$ to find d, from

$$A_x = .40 = 1 - d\cdot \ddot{a}_x = 1-10d,$$

so $d=.06,$ and then to find $\ddot{a}_{xx}$ from

$$\ddot{a}_{xx} = \frac{1-A_{xx}}{d} = \frac{.45}{.06} = 7.50.$$

Then

$$
\begin{aligned}
P &= \frac{2A_x - A_{xx}}{1.50\ddot{a}_x - .50\ddot{a}_{xx}} \\
&= \frac{(2)(.40)-.55}{(1.50)(10.00)-(.50)(7.50)} \\
&= .02222.
\end{aligned}
$$

29. Under the assumption of independence, the APV is

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = \frac{.06}{.11} + \frac{.06}{.11} - \frac{.12}{.17} = .38503.$$

Recognition of the common shock hazard means that $\mu_x = \mu_y = .06$ as before, but the joint hazard rate is now

$$\mu_{xy} = \mu_x^* + \mu_y^* + \lambda = .04 + .04 + .02 = .10.$$

Now the APV is

$$\bar{A}_{\overline{xy}} = \frac{.06}{.11} + \frac{.06}{.11} - \frac{.10}{.15} = .42424,$$

so the difference is

30. We seek the value of ${}_5p_{50}^{(\tau)} - {}_{10}p_{50}^{(\tau)}$. We have

$${}_5p_{50}^{(\tau)} = {}_5p_{50}'^{(1)} \cdot {}_5p_{50}'^{(2)} = \left(1 - \frac{5}{50}\right)\left(e^{-(.05)(5)}\right) = .70092,$$

and

$${}_{10}p_{50}^{(\tau)} = \left(1 - \frac{10}{50}\right)\left(e^{-(.05)(10)}\right) = .48522,$$

so

$${}_5p_{50}^{(\tau)} - {}_{10}p_{50}^{(\tau)} = .70092 - .48522 = .21570.$$

31. Recall that

$$q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = .96,$$

so $p_x^{(\tau)} = .04.$ Then from Equation (10.26),

$$p_x'^{(1)} = (p_x^{(\tau)})^{q_x^{(1)}/q_x^{(\tau)}} = (.04)^{.48/.96} = .20.$$

32. In the sub-interval $(x, x+.70)$, Decrement 2 cannot occur, so Decrement 1 is operating in a single-decrement environment and is uniformly distributed. Therefore ${}_{.70}q_x'^{(1)} = (.70)(.100) = .070.$ If we assume an arbitrary radix of $\ell_x^{(\tau)} = 1000,$ then we have 70 decrements in the interval so we have 930 survivors at age $x+.70.$ Then there are $(930)(.125) = 116.25$ occurrences of Decrement 2 at age $x+.70,$ and therefore in $(x, x+1],$ so the probability is

$$q_x^{(2)} = \frac{116.25}{1000} = .11625.$$

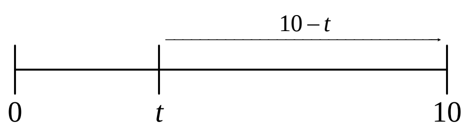
33. (a) The waiting time for the tenth event, denoted S_{10}, has a gamma distribution with parameters $\alpha = 10$ and $\beta = \lambda = 2$. Then

$$E[S_{10}] = \frac{\alpha}{\beta} = \frac{10}{2} = 5.$$

(b) The interarrival time for the eleventh event, denoted T_{11}, has an exponential distribution with parameter $\beta = \lambda = 2$. Then

$$Pr(T_{11} > 2) = e^{-2\beta} = e^{-4} = .01832.$$

34. Note that the process counting the number of infections occurring is a standard (homogeneous) Poisson process with $\lambda = 20$ (per day), so the expected number of infections would be 200 in a 10-day period. But we wish to count the number of *identifications* in the 10-day period, not the number of actual infections. Let $t = 0$ denote the start of the 10-day period.



For an infection occurring at time *t*, the probability of being identified by time 10 is $1 - e^{-(10-t)/7}$, since the time-to-identification has an exponential distribution with mean 7. Thus the rate function for the process counting all occurring infections is the constant $\lambda(t) = 20$, but the rate function for the process counting only infections that get identified by time 10 is the non-constant

$$\lambda'(t) = 20(1 - e^{-(10-t)/7}),$$

which identifies this process as a nonstationary (nonhomogeneous) Poisson process. Then the expected number of identifications in the interval from $t = 0$ to $t = 10$ is given by

$$\begin{aligned} m(10) &= \int_0^{10} \lambda'(t)\, dt \\ &= \int_0^{10} 20(1 - e^{-(10-t)/7})\, dt \\ &= \int_0^{10} 20\, dt - 20\int_0^{10} e^{-10/7} \cdot e^{t/7}\, dt \\ &= 20t\Big|_0^{10} - 20 \cdot e^{-10/7}(7e^{t/7})\Big|_0^{10} \\ &= 200 - 20(.23965)(7)(4.17273 - 1) = 93.55161. \end{aligned}$$