Supplement to Models for Quantifying Risk, 5th Edition

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We have received input that our text is not always clear about the distinction between a full gross premium and an expense augmented premium, which does not include a provision for profit.

The 14 pages that follow provide an expanded presentation on this issue and can be used to replace the corresponding pages in the 2012 printing of *Models for Quantifying Risk*, 5th Edition.

If the funding scheme is limited to t years, where $t < n,$ then the APV of the m^{thly} annuity in the denominator of the premium formula becomes $\ddot{a}^{(m)}_{x:\overline{t}|}$ and the premium symbol is adjusted to include the pre-subscript t as in the Equation (9.8) set. For example, for t-pay n-year term insurance with immediate payment of claims we would have

$$ {}_tP^{(m)}\left(\bar{A}^{1}_{x:\overline{n}|}\right) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}^{(m)}_{x:\overline{t}|}}. \tag{9.29}$$

The numerical calculation of m^{thly} funding payments follows directly from the approximate calculation of the associated m^{thly} annuity-due in the denominator of the premium formula. The details of such calculations are presented in Section 8.4.4 and its associated exercises.

9.6 FUNDING PLANS INCORPORATING EXPENSES

Recall the observation made earlier in this chapter that the annual funding payments determined by the equivalence principle, which we called net annual premiums in the life insurance context, provide only for the contingent benefit payment. In practice, of course, the price of an insurance (or other contingent payment) product must be set higher than the net premium in order to generate revenue to pay expenses of operation and the contingent benefit payments, as well as providing a profit margin to the insurer. The total annual premium charged for an insurance product is called the *gross annual premium* or the *contract premium*. A premium determined to cover benefits and expenses, but not profit, is called the *expense-augmented premium*. In this text we will use G to denote an expense-augmented premium and G^* to denote a gross (or contract) premium.

It is a simple matter to extend the equivalence principle to incorporate expenses and therefore to calculate expense-augmented premiums. For ease of illustration we assume that the expenses allocated to a particular contingent contract are fixed costs known in advance. Then the *expense-augmented equivalence principle* states that the APV of the expense-augmented funding scheme equals the APV of the benefit payment plus the APV of the expense charges allocated to the contract.

For illustration we assume that expense charges allocated to a contract are of the following four types:

(1) A percentage of the gross premium itself.

(2) A fixed amount per unit of benefit payment.

(3) A fixed (or percentage of benefit) amount incurred when the benefit payment is made.

(4) A fixed amount for the contract itself, regardless of benefit amount.

The analysis of corporate operational expenses leading to the determination of expense charges to be included in the price of each product is a complex issue that will vary according to the type of business under discussion. In any case the mechanics of this expense analysis are beyond the scope of this text.

11.4 INCORPORATION OF EXPENSES

Recall that all reserve expressions developed in Chapter 10 and thus far in this chapter are for *benefit reserves* only, by which we mean that they are based on benefit premiums (also called net premiums). In Section 9.6 we saw how to incorporate expense factors into an expense-augmented equivalence principle to determine expense-augmented premiums.

It is now a simple matter to include the expense factors, along with the expense-augmented premium, to determine *expense-augmented reserves*. (When profit margins are also included in the premium, producing a gross (or contract) premium, the resulting reserves are referred to as *gross premium reserves*.) The general prospective formula for the t^{th} benefit reserve, given by Equation (10.2), is now modified to read

$$\begin{aligned} {}_tV^G &= (APV \text{ of future benefits and expenses}) \\ &\quad -(APV \text{ of future expense-augmented premiums}), \end{aligned} \tag{11.20}$$

where the symbol ${}_tV^G$ denotes the t^{th} expense-augmented reserve. This is illustrated in the following example.

EXAMPLE 11.5

Give an expression for the t^{th} prospective expense-augmented reserve for the whole life contract described in Example 9.8.

SOLUTION

At duration t, given that the contingent contract is still in effect, the APV of the future expense-augmented premium income is $G \cdot \ddot{a}_{x+t}$, where G is defined in Example 9.8. The APV of the future percent of premium expense charges is $.10G \cdot \ddot{a}_{x+t}$, and the APV of the future fixed per \$1000 of benefit expense charges is $2\ddot{a}_{x+t}$. The APV of the benefit payment itself plus the settlement expense together is $1020A_{x+t}$. Thus we have

$${}_tV^G = 1020A_{x+t} + (.10G+2)\ddot{a}_{x+t} - G \cdot \ddot{a}_{x+t} = 1020A_{x+t} - (.90G-2)\ddot{a}_{x+t}. \quad \square$$

Returning to Section 9.6, we can separate the amount of the level benefit premium P from the expense-augmented premium G. The remainder of G represents the amount of annual premium needed to fund the expenses of administering the contract. In other words, we define

$$EP = G - P \tag{11.21}$$

to be the annual *expense premium* for the contract.

The notion of separating the expense-augmented premium into benefit premium and expense premium components naturally extends to the reserve. As already covered extensively in

Chapter 10, the prospective benefit reserve is the APV of future benefits minus the APV of future benefit premiums. Similarly we define the t^{th} prospective *expense reserve* to be

$$_tV^E = (APV\ of\ future\ expenses) - (APV\ of\ future\ expense\ premiums). \quad (11.22)$$

It should be clear that

$$_tV^G = {_tV^B} + {_tV^E}, \quad (11.23)$$

where $_tV^B$ is the t^{th} benefit reserve previously denoted by simply $_tV$.

EXAMPLE 11.6

Give an expression for the t^{th} prospective expense reserve for the whole life contract described in Example 9.8.

SOLUTION

First we find the level expense premium as

$$EP \cdot \ddot{a}_x = .75G + .10G \cdot a_x + 10 + 2a_x + 20A_x,$$

where G is defined in Example 9.8. Then the expense reserve is

$$_tV^E = 20A_{x+t} + (.10G+2)\ddot{a}_{x+t} - EP \cdot \ddot{a}_{x+t}.$$ ❐

Recall how the concept of the present value of loss (at issue) random variable, L_x, introduced in Section 9.2, was easily extended to the present value of loss (at duration t) random variable, $_tL_x$, defined in Section 10.1.4. In the same way, the expense-augmented present value of loss (at issue) random variable, L_x^G, defined in Section 9.6, is easily extended to the expense-augmented present value of loss (at duration t) random variable, which we denote by $_tL_x^G$. This is pursued in Exercise 11-15.

Recall that we expanded Equation (10.30b), which had presumed level benefit premiums and a level failure benefit, into Equation (11.16b), which generalized Equation (10.30b) for non-level benefits and benefit premium. Now we generalize further to include expenses.

Let G_t denote the expense-augmented premium for the t^{th} contract year, r_t denote the percent-of-premium expense factor for that year, e_t denote the fixed expense for that year, and s_t denote the settlement expense associated with a benefit paid at the end of the t^{th} contract year. Then Equation (11.16b) is generalized to

$$[_tV + G_{t+1}(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) = (b_{t+1}+s_{t+1}) \cdot q_{x+t} + p_{x+t} \cdot {_{t+1}V}. \quad (11.24)$$

Note that Equation (11.24) allows for the reserve interest rate to also vary by contract year, for maximum generality. In many applications, i_{t+1} will be set as a constant.

11.7 GAIN AND LOSS ANALYSIS

We now continue our analysis of financial gain or loss under a contingent contract which we began in Section 10.6, this time in the more realistic environment of gross premiums and *either* gross premium reserves or expense-augmented reserves.

We begin with Equation (11.24), found in Section 11.4, and modify it to read

$$[{}_tV^G + G^*_{t+1}(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - [(b_{t+1}+s_{t+1})\cdot q_{x+t} + p_{x+t}\cdot {}_{t+1}V^G], \qquad (11.30a)$$

which uses the gross premium but expense-augmented (rather than gross premium) reserves. Recall that this expression is written with maximum generality to allow the benefit, gross premium, interest rate, and all expense factors to vary by contract year. In practice, many of these parameters would be constant over contract years.

An important difference between Expression (11.30a) and its net counterpart given by the left side of Equation (10.51) is that the net case expression always equals zero but this gross case profit expression does not necessarily equal zero, since the gross premium has been set to include a profit margin, but the expense-augmented reserves do not consider profit. When the gross case Expression (11.30a) is evaluated using the parameters *anticipated* to apply in the $(t+1)^{st}$ contract year, we refer to the resulting value as the *anticipated profit* for the $(t+1)^{st}$ year, which we denote by $P(0)$. That is,

$$P(0) = [{}_tV^G + G^*_{t+1}(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - [(b_{t+1}+s_{t+1})\cdot q_{x+t} + p_{x+t}\cdot {}_{t+1}V^G], \qquad (11.30b)$$

where we use unprimed symbols for anticipated experience.

Next we consider the notion of gain or loss by source, introduced in Section 10.6 in the net premium case. Here we have three potential sources of gain or loss (interest, mortality, and expenses), and we adopt a different approach to calculating them than was used in Section 10.6. In this case we must specify the *order* in which the gain-by-source calculations are to be made. We illustrate that notion here by choosing the order interest, then mortality, and then expenses.

Now we evaluate Expression (11.30a) using the *actual* interest rate earned in the $(t+1)^{st}$ year, but still using anticipated experience for mortality and expenses, producing

$$P(1) = [{}_tV^G + G^*_{t+1}(1-r_{t+1}) - e_{t+1}](1+i'_{t+1}) - [(b_{t+1}+s_{t+1})\cdot q_{x+t} + p_{x+t}\cdot {}_{t+1}V^G], \qquad (11.31a)$$

where we use i'_{t+1} in place of i_{t+1} to denote the actual interest rate earned in the $(t+1)^{st}$ year.

Next we evaluate Expression (11.30a) using actual interest *and* mortality, but still using anticipated expenses, producing

$$P(2) = [{}_tV^G + G^*_{t+1}(1-r_{t+1}) - e_{t+1}](1+i'_{t+1}) - [(b_{t+1}+s_{t+1})\cdot q'_{x+t} + p'_{x+t}\cdot {}_{t+1}V^G]. \qquad (11.31b)$$

Next we evaluate Expression (11.30a) using actual interest, mortality, *and* expenses, producing

$$P(3) \;=\; [{}_tV^G + G^*_{t+1}(1-r'_{t+1}) - e'_{t+1}](1+i'_{t+1}) - [(b_{t+1}+s'_{t+1})\cdot q'_{x+t} + p'_{x+t}\cdot {}_{t+1}V^G]. \quad (11.31c)$$

(Note that the benefit, gross premium, and expense-augmented reserve parameters are constant throughout these calculations.) Finally, we define the gain from interest to be

$$G^I \;=\; P(1) - P(0), \quad (11.32a)$$

the gain from mortality to be

$$G^M \;=\; P(2) - P(1), \quad (11.32b)$$

and the gain from expenses to be

$$G^E \;=\; P(3) - P(2). \quad (11.32c)$$

Note that

$$G^T \;=\; G^I + G^M + G^E \;=\; P(3) - P(0), \quad (11.33)$$

which shows that the total gain can be calculated by subtracting the anticipated profit from the profit calculated using actual experience throughout.

The theory developed here is illustrated in the following example.

EXAMPLE 11.9

Consider a block of fully discrete whole life policies issued at age 40 with face amount 50,000. On the assumed (or anticipated) mortality, interest, and expense bases, the gross annual premium per policy is 685.00, the tenth-year expense-augmented reserve is 3950.73, and the eleventh-year expense-augmented reserve is 4602.49. The assumed interest rate is 6%, the assumed mortality rate for the eleventh year is $q_{50} = .00592$, and the assumed expenses are 5% of the gross premium and 300 to process a death claim. In the eleventh year, there are 1000 policies in force at the beginning of that year and five deaths occur in the year. Actual expenses in the eleventh year are 6% of the gross premium and 100 to process each death claim, and the actual earned interest rate is 6.5%. Calculate, in order, the gain from mortality, the gain from expenses, and the gain from interest on a single policy.

SOLUTION

Note first that the beginning-of-year expense is percent of premium only, so the term e_{t+1} can be ignored. Using assumed experience throughout, we calculate

Note also that we have taken a numerical approach here to the calculations of each gain by source. We could also calculate each gain by a formula that would bypass the need to do the various profit calculations and then find the various gains by subtracting appropriate profit amounts. This approach is pursued in Exercise 11-20. The same approach presented in this section can also be applied to other types of contingent contracts. This is explored in Exercises 11-21 and 11-22.

There is an alternative model that we need to examine here. Suppose the gross premium, which has been defined to include a profit margin, is calculated without using explicit profit factors, but rather has an implicit profit margin built into it by using more conservative pricing assumptions. In this case there is no distinction between the gross premium and the expense-augmented premium (i.e., $G = G^*$).

When the reserves are then calculated from this premium using the same assumptions, we likewise find no distinction between gross premium reserves and expense-augmented reserves. The important outcome in this case is that $P(0)$ will again be zero. Then when Expression (11.30a) is evaluated using one or more actual factors, the profit (or loss) is revealed. In this case we act as if we do not "anticipate" any profit, although we certainly do because of the conservative pricing assumptions.

11.8 EXERCISES

11.1 Modified Benefit Reserves

11-1 If benefit premiums are modified for the entire premium paying period of n years, show each of the following:

(a) ${}_tV^{NLP}_{x:\overline{n}|} - {}_tV^{M}_{x:\overline{n}|} = (\beta - P) \cdot \ddot{a}_{x+t:\overline{n-t}|}$ (b) ${}_tV^{M}_{x:\overline{n}|} = 1 - (\beta + d) \cdot \ddot{a}_{x+t:\overline{n-t}|}$

11-2 Consider a fully continuous unit whole life insurance issued at age x, under which modified continuous reserves accumulate using modified benefit premium rate $\bar{\alpha}(r)$ at time r, for $0 < r \le 5$, and modified benefit premium rate $\bar{\beta}$ at time $r \ge 5$. The premium rate $\bar{\alpha}(r)$ is defined as $\bar{\alpha}(0) = .25\bar{\beta}$, increasing linearly to $\bar{\alpha}(5) = \bar{\beta}$. Show that

$$\bar{\beta} = \frac{\bar{A}_x}{\bar{a}_x - .75\bar{a}_{x:\overline{5}|} + .15(\bar{I}\bar{a})_{x:\overline{5}|}}.$$

11-3 For an h-pay, n-year unit endowment insurance issued at age x, with reserves calculated by the FPT method, show that

$$ {}^{h}_{t}V^{FPT}_{x:\overline{n}|} = {}^{h-1}_{t-1}V^{NLP}_{x+1:\overline{n-1}|},$$

where $t < h < n$.

11-4 As an extension of Example 11.2, show that, under the two-year FPT reserving method, ${}_1V^F = {}_2V^F = 0$ and $\beta = P_{x+2}$.

11.2 Benefit Reserves at Fractional Durations

11-5 Show that the expression for r given in Equation (11.9) reduces to $r = 1 - s$ under the UDD assumption.

11-6 Show that the t^{th} year mean reserve for a unit insurance can be written as

$$ {}_{t-1/2}V = \frac{1}{2}\left[(1+v\cdot p_{x+t-1})\cdot {}_tV + v\cdot q_{x+t-1}\right]. $$

11.3 Generalization to Non-Level Benefits and Premiums

11-7 Write the general retrospective formula which is the counterpart to the prospective formula given by Equation (11.15).

11-8 A 3-year term insurance issued to (x) has a decreasing failure benefit, paid at the end of the year of failure. The interest rate is $i = .06$. Calculate the initial reserve for the second year, given the following values:

$$ \begin{array}{lll} b_1 = 200 & b_2 = 150 & b_3 = 100 \\ q_x = .03 & q_{x+1} = .06 & q_{x+2} = .09 \end{array} $$

11-9 A 2-year endowment contract issued to (x) has a failure benefit of 1000 plus the reserve at the end of the year of failure and a pure endowment benefit of 1000. Given that $i = .10$, $q_x = .10$, and $q_{x+1} = .11$, calculate the net level benefit premium.

11-10 A whole life contract issued to (40) pays a benefit, at the end of the year of failure, of b_k for failure in the k^{th} year. The net premium P is equal to P_{20}, and the benefit reserves satisfy ${}_tV = {}_tV_{20}$, for $t = 0,1,\ldots,19$. Furthermore, $q_{40+k} = q_{20+k} + .01$, for $k = 0,1,\ldots,19$. Given that ${}_{11}V_{20} = .08154$ and $q_{30} = .008427$, calculate b_{11}.

11-11 A continuously decreasing 25-year term insurance issued to (40) has benefit rate $b_t = 1000\bar{a}_{\overline{25-t}|}$ for failure at time t. The continuous net premium rate is $\bar{P} = 200$. Given also that $i = .05$ and $\bar{A}_{50:\overline{15}|} = .60$, find the benefit reserve at time $t = 10$.

11-12 Solve Thiele's differential equation, given by Equation (11.19), to reach the retrospective reserve expression given by Equation (11.18).

11.4 Incorporation of Expenses

11-13 For the 20-pay whole life insurance described in Exercise 9-26, find the expense-augmented reserve at (a) duration 10 and (b) duration 20, given the additional values $\ddot{a}_{x+10} = 16.5$, $\ddot{a}_{x+20} = 12.5$, and $\ddot{a}_{x+10:\overline{10}|} = 7$.

11-14 Show that ${}_tV = {}_tV^G - {}_tV^E$, where ${}_tV^G$ is given in Example 11.5 and ${}_tV^E$ is given in Example 11.6.

11-15 We now define ${}_tL_x^G$ as the expense-augmented present value of prospective loss measured at time *t*, given that $K_x^* > t$ (i.e., the contract has not yet failed at time *t*). For the whole life contract described in Example 9.8, show that

$$E\left[{}_tL_x^G \mid K_x^* > t\right] = {}_tV^G,$$

the expense-augmented reserve, as determined in Example 11.5.

11-16 Consider the expense-augmented premium recursion relationship given by Equation (11.24). Suppose the premium is paid continuously at annual rate $\overline{G}_t$ at time *t*, and fixed expenses are paid continuously at annual rate $\overline{e}_t$ at time *t*. State Thiele's differential equation in this general case including expenses.

11.5 Introduction to Universal Life Insurance

11-17 The account value roll forward process under a universal life contract is often done on a monthly basis. Suppose the contract in Example 11.7 receives *annual* contributions of 5000, earns interest at $i^{(12)} = .03$, assesses expense charges at 50% of contribution plus 10 per month, and estimates monthly mortality rates at 1/12 the corresponding annual rate. Calculate the account values at the ends of each of the first three months.

11-18 In practice, the net amount at risk under a universal life contract paying a failure benefit fixed at amount *B* is often defined as the excess of *B* over the prior period ending account value plus the current period net contribution before deducting fixed expenses. The cost of insurance is then defined as the mortality rate times the net amount at risk, without the discount factor. Rework Example 11.8 under these definitions of NAR and COI.

11.6 Introduction to Deferred Variable Annuities

11-19 A five-year deferred variable annuity is issued to (60) who makes annual contributions of 5,000 each. The percent-of-contribution expense rate is 60% in the first year and 10% in subsequent years. There is an annual expense charge of 2% of the prior year's account value, assessed at the beginning of each year. For convenience, assume the interest rate credited to the account is constant at 8%, and the failure benefit is the account value. At age 65 the account value is used to purchase an annual annuity-due based on 6% interest and the survival model of Appendix A. Calculate the annual annuity payment.

11.7 Gain and Loss Analysis

11-20 Referring to Example 11.9, write each of $P(0), P(1), P(2),$ and $P(3)$ symbolically and do the appropriate subtractions to reach each of the following results:

(a) $G^M = (q_{x+t} - q'_{x+t})(b+s-{}_{t+1}V)$

(b) $G^E = G^*(r-r')(1+i) + (s-s') \cdot q'_{x+t}$

(c) $G^I = [{}_tV + G^*(1-r')](i'-i)$

11-21 A block of 1000 fully discrete 20-year term insurance policies of face amount 10,000 were issued to independent lives all age 40, of which 990 remain in force after three policy years. The gross premium and expense-augmented reserves are $G^* = 90$, ${}_3V = 100$, and ${}_4V = 125$. For the fourth policy year, the anticipated interest rate is $i = .05$, the anticipated mortality rate is $q_{43} = .003$, and the anticipated percent-of-premium expense rate is $r = .03$. In the fourth policy year, the actual interest rate, mortality rate, and percent-of-premium expense rate were .04, .002, and .025, respectively. Calculate, in order, each of the following gains by source for the 990 policies together:

(a) Gain from interest
(b) Gain from mortality
(c) Gain from expenses

11-22 An annual premium deferred annuity is now in its payout phase, paying 10,000 at the end of each year. The contract holder is currently age 70. The only expense is 5% of the benefit payment, payable at the end of the year for surviving contract holders only. The contract reserves are calculated from the life table in Appendix A at $i = .06$. For the year of age from 70 to 71, the anticipated interest and mortality rates are .06 and .02, respectively, and the actual interest and mortality rates are .055 and .025, respectively. Calculate, in order, (a) the gain from mortality and (b) the gain from interest.

under a single contingent contract (such as an insurance policy) or for a block of such contracts. The results are known as *projected asset shares*.

Suppose we have a contingent payment contract funded by a level annual contract premium G^*. The contract pays in the event of the failure (such as death) of a specified entity of interest or in the event of withdrawal from the contingent contract. The payment due in the event of failure in Year k is denoted $b_k^{(1)}$ and the payment due in the event of withdrawal is denoted $b_k^{(2)}$; in either case the benefit is paid at the end of the year. As in Example 14.3, expenses are paid at the beginning of each year and are of both the percent-of-premium and contract constant types.

The projected asset share at duration k, which is the actuarial accumulated value of premiums minus expected benefits and expenses, is denoted by ${}_kAS$. All notation used in this section is summarized in the following table.

Symbol	Concept
G^*	Annual contract premium
$b_k^{(1)}$	Benefit paid at end of Year k for failure during Year k
$b_k^{(2)}$	Benefit paid at end of Year k for withdrawal during Year k
r_k	Percent-of-premium expense factor paid at beginning of Year k
e_k	Fixed contract expense paid at beginning of Year k
i	Effective annual rate of interest (presumed constant)
$q_{x+k-1}^{(1)}$	Conditional probability of failure during Year k, given that the contract is still in force at time $k-1$
$q_{x+k-1}^{(2)}$	Conditional probability of withdrawal during Year k, given that the contract is still in force at time $k-1$
$p_{x+k-1}^{(\tau)}$	Conditional probability of the contract staying in force through Year k, given that it is still in force at time $k-1$
${}_kAS$	The projected asset share associated with the contract at the end of Year k

We denote the initial asset share at time 0 by ${}_0AS$, and note that ${}_0AS$ may or may not equal zero. Successive values of ${}_kAS$ are then found recursively by expanding the discussion in Section 11.4 to include multiple decrements. For $k=1$ we have

$$\left[{}_0AS + G^*(1-r_1) - e_1\right](1+i) \;=\; b_1^{(1)} \cdot q_x^{(1)} + b_1^{(2)} \cdot q_x^{(2)} + {}_1AS \cdot p_x^{(\tau)}, \tag{14.5a}$$

so

$$ {}_1AS \;=\; \frac{\left[{}_0AS + G^*(1-r_1) - e_1\right](1+i) - b_1^{(1)} \cdot q_x^{(1)} - b_1^{(2)} \cdot q_x^{(2)}}{p_x^{(\tau)}}. \tag{14.5b}$$

For k in general we have

$$\left[{}_{k-1}AS + G^*(1-r_k) - e_k\right](1+i) \;=\; b_k^{(1)} \cdot q_{x+k-1}^{(1)} + b_k^{(2)} \cdot q_{x+k-1}^{(2)} + {}_kAS \cdot p_{x+k-1}^{(\tau)} \tag{14.6a}$$

so

$$ {}_kAS = \frac{\left[{}_{k-1}AS + G^*(1-r_k) - e_k\right](1+i) - b_k^{(1)} \cdot q_{x+k-1}^{(1)} - b_k^{(2)} \cdot q_{x+k-1}^{(2)}}{p_{x+k-1}^{(\tau)}}. \tag{14.6b} $$

EXAMPLE 14.4

Consider the five-year endowment insurance described in Example 14.3. If the contract premium is $G^* = 200.00$[2] and the initial asset share is ${}_0AS = 50$, find ${}_kAS$ for $k = 1,2,3,4,5$.

SOLUTION

Using the recursive relationship given by Equation (14.6b) the following values are obtained. (The details of the calculations are left to the reader as an exercise.[3])

k	${}_kAS$
1	275.90
2	535.10
3	837.90
4	1115.03
5	373.90

❐

The excess (if any) of the assets associated with a contingent contract over the contract liability may be interpreted as an amount of *surplus* generated by that contract. The liability is given by the expense-augmented reserve, and the associated asset value is given by the projected asset share. Then the projected surplus at the end of Year k is given by

$$ U_k = {}_kAS - {}_kV^G. \tag{14.7} $$

EXAMPLE 14.5

Find the surplus U_k, for $k = 1,2,3,4,5$, for the five-year endowment contract described in Examples 14.3 and 14.4.

SOLUTION

Note that although the projected asset shares are determined using the actual contract premium of 200.00, the reserves are determined using the expense-augmented premium of 184.90, which is the premium necessary to cover the benefits and expenses that constitute the contract liability. The following results are obtained directly from the results of Example 14.3(b) and Example 14.4. Note that $U_0 = 50$ since ${}_0AS = 50$ and ${}_0V^G = 0$.

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[2] The premium might exceed the value calculated in Example 14.3 to reflect considerations of competition and profit.

[3] The complete solution can be found on the ACTEX Publications website.

14.6 GAIN AND LOSS ANALYSIS

We consider, for the third and final time, the notion of gain or loss by source, this time in a multiple-decrement and gross premium environment. For the $(t+1)^{st}$ contract year, the general profit expression is an expansion of Expression (11.30a) to include, say, two decrements, producing

$$[{}_tV^G + G*(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - [(b_{t+1}^{(1)} + s_{t+1}^{(1)}) \cdot q_{x+t}^{(1)} + (b_{t+1}^{(2)} + s_{t+1}^{(2)}) \cdot q_{x+t}^{(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1}V^G], \tag{14.39a}$$

where we assume an expense of $s_{t+1}^{(1)}$ to settle a benefit claim due to Cause 1 and an expense of $s_{t+1}^{(2)}$ to settle a benefit claim due to Cause 2 in the $(t+1)^{st}$ year.

The reader should by now understand the ensuing calculations. We first calculate

$$P(0) = [{}_tV^G + G*(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) - [(b_{t+1}^{(1)} + s_{t+1}^{(1)}) \cdot q_{x+t}^{(1)} + (b_{t+1}^{(2)} + s_{t+1}^{(2)}) \cdot q_{x+t}^{(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1}V^G] \tag{14.39b}$$

using the fixed gross premium, expense-augmented reserves, and Cause 1 and Cause 2 benefit amounts, and anticipated experience for all four factors of interest, expenses, Cause 1 failure rate, and Cause 2 failure rate. Then the order of calculating each gain by source is established, and we calculate $P(1)$ by substituting actual for anticipated experience for the factor whose gain is to be calculated first. Then we calculate $P(2)$ using actual experience for the two factors whose gains are to be calculated first and second, but anticipated experience for the other two factors. Then we calculate $P(3)$ using actual experience for the three factors whose gains are to be calculated first, second, and third, but anticipated experience for the fourth factor. Finally we calculate $P(4)$ using actual experience throughout. Then the gain from the factor whose gain is calculated first is

$$G^{F_1} = P(1) - P(0), \tag{14.40a}$$

the gain from the factor whose gain is calculated second is

$$G^{F_2} = P(2) - P(1), \tag{14.40b}$$

the gain from the factor whose gain is calculated third is

$$G^{F_3} = P(3) - P(2), \tag{14.40c}$$

and the gain from the factor whose gain is calculated fourth is

$$G^{F_4} = P(4) - P(3). \tag{14.40d}$$

As before, the total gain is

$$G^T = G^{F_1} + G^{F_2} + G^{F_3} + G^{F_4} = P(4) - P(0). \tag{14.41}$$

Under a life insurance policy, it is often the case that Cause 1 is death and Cause 2 is surrender of (or withdrawal from) the contract. The profit expression given by Expression (14.39a) presumes that both death and withdrawal can occur at any time throughout the contract year. Alternatively, we might assume that death can occur throughout the year, but withdrawal can occur only at year end. In this case the anticipated profit expression is

$$\begin{aligned} P(0) \;=\; & [{}_tV^G + G^*(1-r_{t+1}) - e_{t+1}](1+i_{t+1}) \\ & - [(b_{t+1}^{(1)} + s_{t+1}^{(1)}) \cdot q_{x+t}'^{(1)} + (b_{t+1}^{(2)} + s_{t+1}^{(2)})(1-q_{x+t}'^{(1)}) \cdot q_{x+t}'^{(2)} + p_{x+t}^{(\tau)} \cdot {}_{t+1}V^G], \end{aligned} \tag{14.42}$$

since, with withdrawal not possible within the contract year, mortality is operating in a single-decrement environment and the policyholder must survive death throughout the year in order for the Cause 2 (i.e., withdrawal) benefit to be paid.)

The concepts presented in this section are reviewed in Exercises 14-27 and 14-28.

14.7 EXERCISES

14.1 Actuarial Present Value

14-1 A company hires all new employees at age 25. An employee can leave the company via death while employed (Decrement 1), resignation prior to age 65 (Decrement 2), or retirement at age 65. The company provides the following benefits for its employees:

(a) Employees who retire at age 65 receive continuous retirement income at an annual rate of 500 for each year of employment with the company.

(b) Employees who die while employed receive a one-time death benefit of 200,000 at the precise time of death.

(c) Employees who resign prior to age 65, but survive on to age 65, receive continuous retirement income at an annual rate of 400 for each year of employment with the company (including partial years).

Write expressions involving continuous annuities and/or integrals for the APV at time of hire for each of the three benefits.

14.2 Asset Shares

14-2 Show that

$${}_kAS \;=\; \left[{}_{k-1}AS + G(1-r_k) - e_k\right](1+i) - q_{x+k-1}^{(1)} \cdot (b_k^{(1)} - {}_kAS) - q_{x+k-1}^{(2)} \cdot (b_k^{(2)} - {}_kAS).$$

(This relationship shows that the difference between the withdrawal value and the asset share is important to the progression of the asset share values. If the asset share were paid as a withdrawal value, then the asset share values would progress independently of the withdrawal risk.)

benefit payable immediately, without reduction, if the employee has at least five years of service. The death benefit requires ten years of service, and is set at 50% of the then accrued benefit, reduced as for early retirement. Assume the surviving beneficiary is three years younger than the employee.

Write the APV formulas for each of (a) normal retirement, (b) early retirement, (c) withdrawal, (d) disability, and (e) death. Assume early retirement, withdrawal, disability, or death occur half way through the year of age, on average.

14-26 Assume the employee of Exercise 14-25 is now exact age 56, with a salary of 150,000 in the year from age 55 to age 56 and a salary of 156,000 in the year from age 56 to age 57. Determine each of the following:

(a) The benefit accrual for the year from age 56 to age 57.
(b) The unit credit normal cost.
(c) The accrued liability under the unit credit cost method.

14.6 Gain and Loss Analysis

14-27 Consider the general double-decrement profit expression given by Expression (14.39a). Let the actual earned interest rate in the $(t+1)^{st}$ year be denoted by i^*_{t+1}, and the actual Cause 2 decrement probability be denoted by $q^{*(2)}_{x+t}$. If the gain from interest is calculated first and the gain from the Cause 2 decrement is calculated second, show that the gain from the Cause 2 decrement is

$$G^{(2)} = \left(b^{(2)}_{t+1} + s^{(2)}_{t+1} - {}_{t+1}V^G\right)\left(q^{(2)}_{x+t} - q^{*(2)}_{x+t}\right).$$

14-28 A block of 1000 fully discrete insurances, issued at age 70, are in force at age 79. The gross premium is $G^* = 16$, the ninth gross premium reserve is 115.00, the tenth gross premium reserve is 128.83, the tenth year death benefit is 1000, the tenth year withdrawal benefit is 110, and the assumed interest rate is .06. Expenses are 3 per policy, incurred at the beginning of the year, and there are no claim settlement expenses. Withdrawals can occur only at the end of the contract year. The assumed decrement rates are $q'^{(d)}_{79} = .01$ and $q'^{(w)}_{79} = .10$. During the tenth contract year there are 15 deaths and 100 withdrawals. Calculate, in order, (a) the gain from mortality and (b) the gain from withdrawal on this block of policies. [Note that, except for rounding, in this problem $P(0) = 0$.]