7.0 Introduction

This chapter describes how optimal risky portfolios are constructed.

Asset allocation and security selection are examined first by using two risky mutual funds: a long-term bond fund and a stock fund.

Next, a risk-free asset is added to the portfolio to determine the optimal asset allocation.

Finally, it will be shown that the best attainable capital allocation line emerges when security selection is introduced.

7.1 Diversification and Portfolio Risk

Investors are exposed to two general types of risk:

1. Market Risk (a.k.a. systematic risk or non-diversifiable risk). Market risk arises from uncertainty in the general economy associated with conditions such as the business cycle, interest rates, exchange rates, etc.
2. Firm Specific Risk (a.k.a. non-systematic risk, unique risk, or diversifiable risk). Firm specific risk arises from factors directly attributable to a firm's operations, such as its research and development opportunities or personnel changes.

By holding a well diversified set of stocks, investors can significantly reduce their exposure to firm specific risk, but the risk that remains after extensive diversification is Market Risk.

These types of risks can be shown graphically as follows:

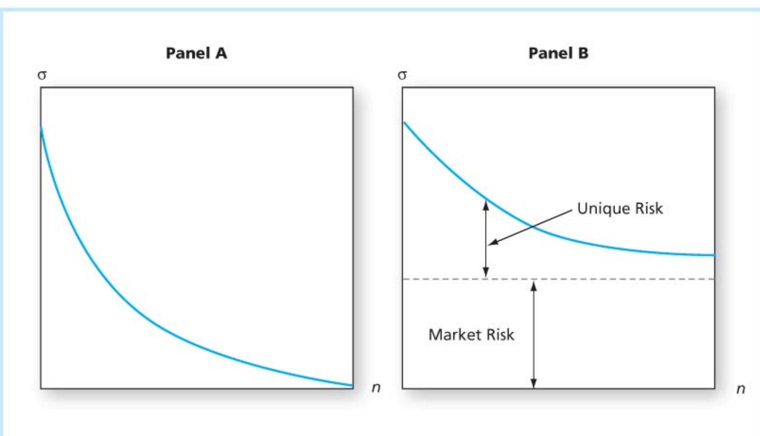


Figure 7.1 Portfolio risk as a function of the number of stocks in the portfolio
Panel A: All risk is firm specific. ***Panel B:*** Some risk is systematic, or marketwide.

7.2 Portfolios of Two Risky Assets

In this section, we demonstrate how efficient diversification is arrived at by constructing risky portfolios which produce the lowest possible risk for any given level of expected return.

Consider a risky portfolio consisting of two risky mutual funds:

a bond fund of long-term debt securities, denoted D, and a stock fund, denoted E.

Statistics for Two Mutual Funds are as follows:

	Debt		Equity
Expected return, E(r)	8%		13%
Standard deviation, σ	12%		20%
Covariance, Cov(r_D, r_E)		72	
Correlation coefficient, ρ_{DE}		.30	

Assume a proportion denoted by w_D is invested in the bond fund, and the remainder, (1 - w_D) denoted w_E, is invested in the stock fund.

As was demonstrated in Chapter 6,

i. the expected return on the portfolio can be computed as $E(r_p) = w_D E(r_D) + w_E E(r_E)$.

ii. the variance of the two-asset portfolio can be computed as $\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$

since the covariance can be computed from the correlation coefficient, ρ_{DE}, as

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E,$$

the variance can be also computed as $\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \rho_{DE}\sigma_D\sigma_E$

Notice that although the expected return is unaffected by correlation between returns, portfolios with assets having low or negative correlations reduce the overall portfolio risk.

Therefore, portfolios of less than perfectly correlated assets always offer better risk-return opportunities and the lower the correlation between the assets, the greater the gain in efficiency.

Degrees of correlation:

When $\rho = 1$, there is no benefit to diversification.

When $\rho < 1$, the portfolio weights which produce a minimum variance portfolio can be solved using

$$w_{Min}(D) \ = \ \frac{\sigma^2{}_E - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_{D,}r_E)}.$$

i. In this case, the portfolio weights that solve this minimization problem turn out to be: $w_{Min}(D) = .82$ and $w_{Min}(E) = 1\text{-}.82 = .18$.

ii. Using the data in the table above, <u>this</u> minimum-variance portfolio has a standard deviation of $\sigma_{Min} = [(.82^2 \times 12^2) + (.18^2 \times 20^2) + (2 \times .82 \times .18 \times 72)]^{1/2} = 11.45\%$.

When $\rho = -1$, indicating perfect negative correlation,

i. the variance of the portfolio simplifies to $\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2$

ii. a perfectly hedged position can be obtained by choosing the portfolio proportions that solve the equation $w_D\sigma_D \text{ - } w_E\sigma_E = 0$.

The solution to the equation is $w_D = \dfrac{\sigma_E}{\sigma_D + \sigma_E}$ and $w_E = \dfrac{\sigma_D}{\sigma_D + \ \sigma_E} = 1 - w_D$.

These weights drive the standard deviation of the portfolio to zero. Also, it is possible to derive the portfolio variance to zero with perfectly positively correlated assess as well, but this would require short sales.

Using the equations above, and varying the weights of the portfolio, the following data can be generated:

			Portfolio Standard Deviation for given correlations			
w_D	w_E	$E(r_p)$	$\rho =$ -1	$\rho = 0$	$\rho = .30$	$\rho = 1$
0.00	1.00	**13.00**	**20.00**	**20.00**	**20.00**	20.00
0.10	0.90	12.50	16.80	18.04	18.40	19.20
0.40	0.60	11.00	7.20	12.92	14.20	16.80
0.50	0.50	10.50	4.00	11.66	13.11	16.00
0.60	0.40	10.00	0.80	10.76	12.26	15.20
0.90	0.10	8.50	8.80	10.98	11.56	12.80
1.00	0.00	**8.00**	**12.00**	**12.00**	**12.00**	12.00

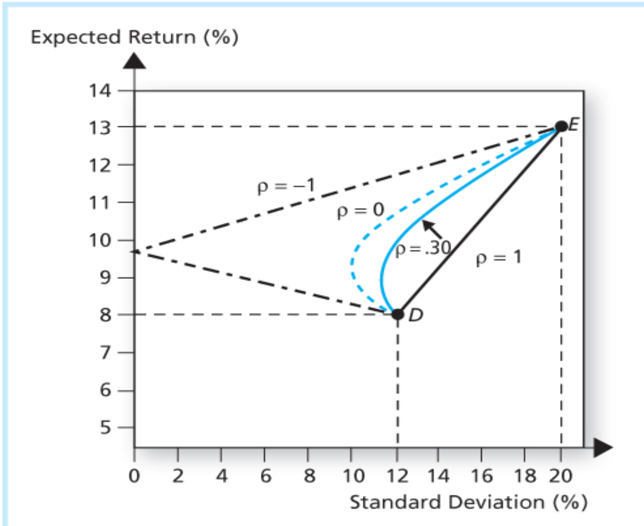
Minimum Variance Portfolio

	$\rho =$ -1	$\rho = 0$	$\rho = .30$
W_D	***0.6250***	0.7353	0.8200
W_E	***0.3750***	0.2647	0.1800
$E(r_p)$	9.8750	9.3235	8.9000
σ_p	**0.0000**	10.2899	11.4473

Notes: When $\rho = -1$, the minimum variance for <u>this</u> portfolio occurs when

$$w_{\min}(D; \rho = -1) = \frac{\sigma_E}{\sigma_D \ + \ \sigma_E} = \frac{20}{12 + 20} = .625, \text{ and } w_{Min}(E; \rho = -1) = 1 - .625 = .375.$$

The relationship between expected return and standard deviation, for a given level of correlation between the two funds, is illustrated below.



Selection of the optimal portfolio set depends upon the degree of risk aversion (risk-return tradeoffs desired).

To summarize, we conclude that:

a. the expected return of any portfolio is the weighted average of the asset expected returns
b. benefits from diversification arise when correlation is less than perfectly positive
   i. the lower the correlation, the greater the potential benefit from diversification.
   ii. when perfect negative correlation exists, a perfect hedging opportunity exists and a zero-variance portfolio can be constructed.

7.3 Asset Allocation with Stocks, Bonds and Bills

In this section, two concepts will be demonstrated:

1. Determining the weights associated with the optimal risky portfolio P (consisting of a stock fund and bond fund).
2. Determining the optimal proportion of the complete portfolio (consisting of an investment in the optimal risky Portfolio P and one in a risk free component (T-Bills)) to invest in the risky component.

For ease of reference, we restate the characteristics of all securities involved in the complete portfolio:

i. Bond Fund: $E(r_D) = .08$ and $\sigma_D = .12$

ii. Stock Fund: $E(r_E) = .13$ and $\sigma_E = .20$

$Cov(r_{D,} r_E) = 72$ and the investor's coefficient of risk aversion, A = 4.

iii. T-Bills: $r_f = .05$

Step 1: It can be shown that the weights associated with the optimal risky portfolio P can be determined using the following equations:

$$w_D = \frac{[E(r_D)-r_f]\sigma_E^2 - [E(r_E)-r_f]Cov(r_D,r_E)}{[E(r_D)-r_f]\sigma_E^2 + [E(r_E)-r_f]\sigma_D^2 - [E(r_D)-r_f+E(r_E)-r_f]Cov(r_D,r_E)}$$, and

$$w_E = 1 - w_D$$

Using the data above, we compute the following:

$$w_D = \frac{(8-5)400 - (13-5)72}{(8-5)400 + (13-5)144 - (8-5+13-5)72} = .40$$ and thus, $w_E = 1 - .40 = .60$

Step 2: Determine the expected return and standard deviation of the optimal risky portfolio:

$$E(r_p) = (.4\times 8)+(.6\times 13) = 11\% \text{ and } \sigma_p = [(.4^2\times 144)+(.6^2\times 400)+(2\times .4\times .6\times 72)]^{1/2} = 14.2\%$$

Step 3: Determine the optimal proportion of the complete portfolio to invest in the risky component. Using the following equation, $y = \frac{E(r_p)-r_f}{A\sigma_P^2} = \frac{.11 - .05}{4\times 14.2^2} = .7439$, the investor would place 74.39% of his/her wealth in Portfolio P and 25.61% in T-Bills.

Step 4: Determine the percentage of wealth placed in bond and in stocks:

$yw_D = .4 * .7439 = 29.76\%,$ while the investment in stocks will be

$yw_E = .6 * .7439 = 44.63\%.$

****A graphical representation of all major concepts considered thus far is shown below****

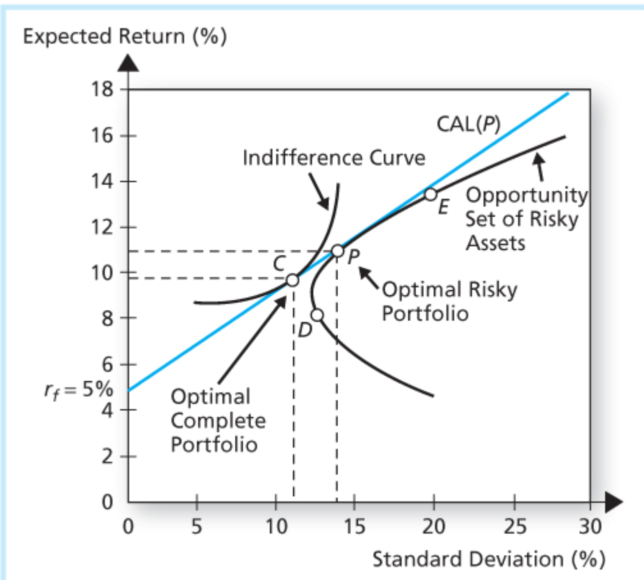


Figure 7.8 Determination of the optimal complete portfolio

Recall that C represents the complete portfolio, D represents the bond portfolio, E represents the Equity portfolio, and CAL (P) represents the capital allocation line of the optimal Risky portfolio P. Two noteworthy points:

1. The CAL of the optimal risky portfolio P has a slope of $S_P = \frac{11-5}{14.2} = .42$. This exceeds the slope of any portfolio consider thus far, and therefore produces the highest reward to variability ratio.

2. The formula for the optimal weights (shown on the prior page) were determined by maximizing the function $\underset{w_i}{Max}\, S_p = \frac{E(r_p) - r_f}{\sigma_p}$ subject to the constraint that the portfolio weights sum to 1.0 (i.e. $w_D + w_E = 1$).

By doing so, we end up with the weights that result in the risky portfolio with the highest reward-to-variability ratio.

7.4 The Markowitz Portfolio Selection Model

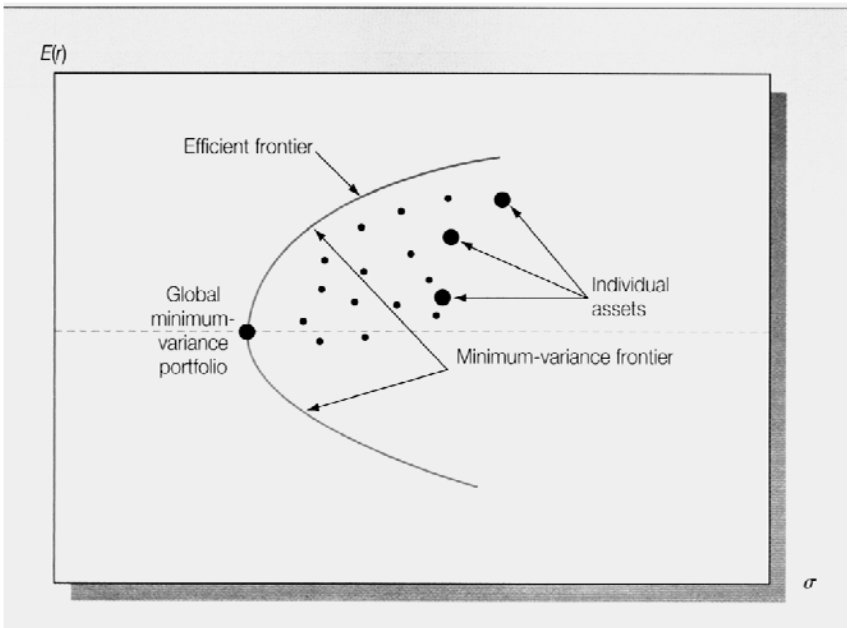
The steps involved in portfolio construction when considering the case of many risky securities and a risk-free asset can be generalized as follows:

Step 1: Identify the risk-return combinations available from the set of risky assets.

Step 2: Identify the optimal portfolio of risky assets by finding the portfolio weights that result in the steepest CAL.

Step 3: Choose an appropriate complete portfolio by mixing the risk free asset with the optimal risky portfolio.

1. The risk-return combinations available to the investor can be summarized by the minimum-variance frontier of risky assets.



   i. The frontier represents a graph of the lowest possible variances that can be attained for a given portfolio expected return.

   ii. Individual assets lie to the right inside of the frontier, when short sales are allowed (since there is a possibility that a single security may lie on the frontier when short sales are not allowed). This also supports the notion that diversified portfolios lead to portfolios with higher expected returns and lower standard deviations that a portfolio consisting of a single risky asset).

   iii. Portfolios lying on the minimum-variance frontier from the global minimum-variance portfolio upward are candidates for the optimal portfolio.

   The upper portion of the frontier is called the efficient frontier of risky assets.

   The lower portion of the frontier is inefficient, and ruled out, since there is a portfolio with the same standard deviation and a greater expected return positioned directly above it.

2. A search for the CAL with the highest reward-to-variability ratio (i.e. the steepest slope) through the optimal portfolio, *P*, and tangent to the efficient frontier is determined next.
3. Finally, the individual investor chooses the appropriate mix between the optimal risky portfolio *P* and T-bills.

   Note: This involves spreadsheet number crunching which is beyond what the CAS examiners can expect candidates to demonstrate on the exam.

The Markowitz Portfolio Selection Model restates step 1 of the process described above.

There are two equivalent approaches to determine the efficient frontier of risky assets:

Approach 1: Draw horizontal lines at different levels of expected returns. Look for the portfolio with the lowest standard deviation that plots on each horizontal line (these are shown by squares in the graph below), and discard those plotting on horizontal lines below the global minimum variance portfolio (since they are inefficient).

Approach 2: Draw vertical lines representing the standard deviation constraint. Look for and plot the portfolio with the highest expected return on a given vertical line. These are represented by circles in the graph below.

A graphical representation of these approaches is shown below:

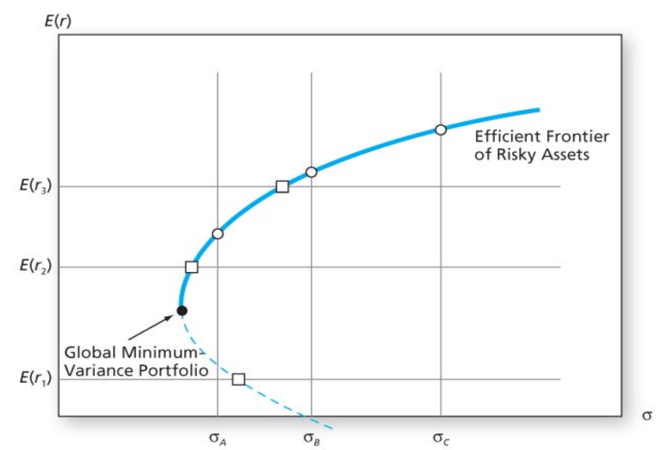


Figure 7.12 The efficient portfolio set

Capital Allocation and the Separation Property

Having established the efficient frontier, the next step is to incorporate the risk-free asset into the complete portfolio.

Once the risk free asset has been selected (and thus, its rate of return established), portfolio managers will choose <u>the</u> portfolio along the efficient frontier that is tangent to the CAL generating the highest reward-to-risk ratio (i.e. the CAL having the steepest slope).

Having done this, a portfolio manager (in theory) will offer the same risky portfolio to all clients, regardless of their degree of risk aversion. The investor's complete portfolio will lie somewhere along the CAL described above based solely on their degree of risk aversion.

The above fact gives rise to a result known as the separation property.

Once the optimal risky portfolio has been chosen (which is a purely technical process), separation among investor choices for their complete portfolio is solely a function of their personal preference for risk. More risk averse investors will invest more in the risk free asset and less in the optimal risky portfolio.

In practice, managers will offer different "optimal" portfolios to their clients, due to client's preference for dividend yields, tax considerations and other preferences.

The Power of Diversification

What can portfolio managers expect from diversification and what are the limits to the benefits of diversification?

Consider a diversification strategy that consists of n-equally weighted securities (i.e. the portfolio ignores optimal weighting among securities within the portfolio).

The average variance can be expressed as $\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$.

The average covariance can be expressed as $\overline{Cov} = \frac{1}{n(n-1)}\sum_{\substack{j=1 \\ j\neq i}}^{n}\sum_{i=1}^{n} Cov(r_i, r_j)$.

Note: There are n variance terms and n(n-1) covariance terms.

Based on the above, the portfolio variance can be expressed as $\sigma_P^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{Cov}$

Scenario 1: When the average covariance is 0 (when securities are uncorrelated), $\sigma_P^2 \to 0$ as n gets large.

Scenario 2: When economy-wide risk factors impart positive correlation, the irreducible risk of a diversified portfolio approaches $\overline{Cov}$ as n gets large. Firm specific risk is diversified away, but for large n,

$$\sigma_P^2 \Rightarrow \frac{1}{Large\ \ n}\bar{\sigma}^2 + \left(1 - \frac{1}{Large\ \ n}\right)\overline{Cov} \Rightarrow \overline{Cov}$$

This concept can also be demonstrated when viewing the relationship between systematic risk and security correlations.

For example, assume all securities have a common standard deviation, σ , and all security pairs have a common correlation coefficient ρ . The covariance between all pairs of securities is $\rho\sigma^2$, and the portfolio variance becomes $\sigma_P^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$

When $\rho = 0$, we obtain the insurance principle: the portfolio variance approaches 0 as n gets large.

When $\rho = 1$, portfolio variance equals σ^2 .

As n gets large, $\sigma_P^2 = \rho\sigma^2$

We conclude that when diversified portfolios are held, the contribution to portfolio risk of a particular security will depend on the covariance of that security's return with those of other securities, and not on the security's variance.

Asset Allocation and Security Selection.

Q. Why Distinguish between Asset allocation and Security selection?

1. Demand for sophisticated security selection has increased tremendously due to society’s need and ability to save for the future (e.g. for retirement, college, health care, etc).
2. Amateur investors are at a disadvantage in their ability select the proper securities to be held in their risky portfolio due to the widening spectrum of financial markets and financial instruments.
3. Strong economies of scale result when sophisticated investment analysis is conducted.

Since large investment companies are likely to invest in both domestic and international markets, management of <u>each</u> asset-class portfolio needs to be decentralized due to the expertise required. This requires that security selection of each asset-class portfolio be optimized independently.

7.5 Risk Pooling, Risk Sharing, And Risk of Long Term Investments

Spreading investments across time, so that the average return reflects returns in several investment periods, offers an benefit known as "time diversification," rendering long-term investing safer than short-term investing.

How risk increases when the horizon of a risky investment lengthens is analogous to risk pooling, whereby an insurer aggregates a large portfolio (or pool) of uncorrelated risks.

The impacts of risk pooling versus risk sharing (where the firm spreads or shares the risk of a fixed portfolio among many owners) are widely misunderstood, and the application of "the insurance principle" to long-term investing also has widespread misconceptions.

Risk Pooling and the Insurance Principle

Risk pooling means merging uncorrelated risky assets to reduce risk.
For insurance, risk pooling entails selling many uncorrelated insurance policies (a.k.a. the insurance principle).

Again, the insurance principle is sometimes misapplied to long-term investments by incorrectly extending what it implies about average returns to predictions about total returns.

Risk pooling and risk sharing are two complementary, but distinct, tools in risk management.
Start with risk pooling. Assume Jon, who holds a $1 billion portfolio, *P*.

- The % of the portfolio invested in a risky asset, *A*, is *y*, leaves the fraction 1 - *y* invested in the risk-free rate.
- Asset *A*'s risk premium is *R*, and its standard deviation is σ .
- From (6.3) and (6.4), the risk premium of the complete portfolio *P* is $R_P = yR$, its standard deviation is $\sigma_P = y\sigma$, and the Sharpe ratio is $S_P = R/\sigma$.

Jon identifies another risky asset, *B*, with the same risk premium and standard deviation as *A*.
Jon estimates that the correlation (and therefore covariance) between the two investments is zero, and considers the potential this offers for risk reduction through diversification.

Given the benefits that Jon anticipates from diversification, he decides to take a position in asset *B* equal in size to his existing position in asset *A.*

- He transfers another fraction, *y*, of wealth from the risk-free asset to asset *B*.
- This leaves his total portfolio allocated as follows:
  * the fraction y is still invested in asset *A*,
  * an additional investment of y is invested in *B*, and
  * 1 - 2*y* is in the risk-free asset.

Note: This strategy is analogous to pure risk pooling; Jon has taken on additional risky (albeit uncorrelated) bets, and his risky portfolio is larger than it was previously.
Denote Jon’s new portfolio as *Z*.

Compute the risk premium of portfolio *Z* from (7.2), its variance from (7.3), and thus its Sharpe ratio.

- *R* denotes the excess return of each asset and the excess return of the risk-free asset is zero.
- When calculating portfolio variance, use the fact that covariance is zero. T

For Portfolio *Z*:

$R_z = yR + yR + (1-2y)0 = 2yR$ (double R_P)

$\sigma_Z^2 = y^2\sigma^2 + y^2\sigma^2 + 2y^2\sigma^2$ (double the variance of *P*)

$\sigma_Z = \sqrt{\sigma_Z^2} = y\sigma\sqrt{2}$ ($\sqrt{2}$ = 1.41 times the standard deviation of *P*)

$S_Z = R_Z / \sigma_Z = 2yR / y\sigma\sqrt{2} = \sqrt{2R/\sigma}$ ($\sqrt{2}$ = 1.41 times Sharpe ratio of P)

Results:

- The Sharpe ratio of *Z* is higher than that of *P* by the factor $\sqrt{2}$.
- Its excess rate of return is double that of *P*, yet its standard deviation is only times larger.
- The problem is that by increasing the scale of the risky investment, the standard deviation of the portfolio also increases by $\sqrt{2}$.

Imagine that instead of two uncorrelated assets, Jon has access to many.

- Repeating our analysis, we would find that with *n* assets the Sharpe ratio under strategy *Z* increases (relative to its original value) by a factor of $\sqrt{n}$ to $\sqrt{n} \times R / \sigma$.
- But the total risk of the pooling strategy *Z* will increase by the same multiple to $\sigma\sqrt{n}$.

This analysis shows the opportunities and limitations of pure risk pooling:

- Pooling increases the scale of the risky investment (from *y* to 2*y*) by adding an additional position in another, uncorrelated asset.
- This addition of another risky bet also increases the size of the risky budget.
  So risk pooling by itself does not reduce risk, despite the fact that it benefits from the lack of correlation across policies.

The insurance principle tells us only that risk increases less than proportionally to the number of policies insured when the policies are uncorrelated; hence, the Sharpe ratio—increases.
But this effect is not sufficient to actually reduce risk.

This might limit the potential economies of scale of an ever-growing portfolio of a large insurer.

- Each policy written requires the insurer to set aside additional capital to cover potential losses.
- The insurer invests its capital until it needs to pay out on claims.
- Selling more policies increases the total position in risky investments and therefore the capital that must be allocated to those policies.
- As the insurer invests in more uncorrelated assets (insurance policies), the Sharpe ratio continuously increases (good), but since more funds are invested in risky policies, the overall risk of the portfolio rises (bad).
- As the number of policies grows, the risk of the pool will grow - despite "diversification" across policies. Eventually, that growing risk will overwhelm the company's available capital.

Insurance analysts often think of the insurance principle in terms of the probability of loss declining with risk pooling.

- This interpretation relates to the fact that the Sharpe ratio (profitability) increases with risk pooling.
- But to equate the declining probability of loss to reduction in total risk is erroneous; the latter is measured by overall standard deviation, which increases with risk pooling.

Thus risk pooling allows neither investors nor insurers to shed risk. However, the increase in risk can be overcome when risk pooling is augmented by risk sharing.

Risk Sharing

Consider a variation on the risk pooling portfolio *Z.*

Suppose Jon identified several attractive insurance policies and wishes to invest in all of them.

Consider the case of two policies, so the pool will have the same properties as portfolio *Z*.

- If Jon invested in this two-policy pool, his total risk would be $\sigma_z = y\sigma\sqrt{2}$
- But if this is more risk than he is willing to bear, what might he do?

The answer is risk sharing, the selling of shares in an attractive risky portfolio to limit risk and yet maintain the Sharpe ratio (profitability) of the resultant position.

Suppose every time a new risky asset is added to the portfolio, Jon sells off a portion of his investment in the pool to maintain the total funds invested in risky assets unchanged.

For example, when a second asset is added, he sells half of his position to other investors.

While the total investment budget directed into risky assets is therefore unchanged, it is equally divided between assets *A* and *B*, with weights in each of *y*/2.

The risk-free component of his complete portfolio remains fixed with weight 1 - *y*. C call this strategy *V.*

If you compare the risk-pooling strategy *Z* with the risk-pooling-plus-risk-sharing strategy *V*, you will notice that they both entail an investment in the pool of two assets; the only difference between them is that the risk-sharing strategy sells off half the combined pool to maintain a risky portfolio of fixed size.

While the weight of the total risky pool in strategy *Z* is 2*y*, in the risk-sharing strategy, the risky weight is only one-half that level.

Therefore, we can find the properties of the risk sharing portfolio by substituting y for 2*y* in each formula or, equivalently, substituting *y*/2 for y in the following table.

Risk Pooling: Portfolio *Z*	**Risk Sharing: Portfolio *V***
$R_z = 2yR$	$R_V = 2\left(y/2\right)R = yR$
$\sigma_Z^2 = 2y^2\sigma^2$	$\sigma_V^2 = 2\left(y/2\right)^2\sigma^2 = y^2\sigma^2/2$
$\sigma_z = \sqrt{\sigma_Z^2} = y\sigma\sqrt{2}$	$\sigma_V = \sqrt{\sigma_V^2} = y\sigma\sqrt{2}$
$S_z = R_z/\sigma_z = 2yR/y\sigma\sqrt{2} = \sqrt{2}R/\sigma$	$S_V = R_V/\sigma_V = \sqrt{2}R/\sigma$

- Portfolio V matches the attractive Sharpe ratio of portfolio Z, but with lower volatility.
- Thus risk sharing combined with risk pooling is the key to the insurance industry.
- True diversification means spreading a portfolio of fixed size across many assets, not merely adding more risky bets to an ever-growing risky portfolio.

To control total risk, Jon sold off a fraction of the pool of assets.

- This implies that a portion of those assets must now be held by someone else. For example:

  If the assets are insurance policies, other investors must be sharing the risk, perhaps by buying shares in the insurer.

  Alternatively, insurers often "reinsure" their risk by selling off portions of the policies to other investors or insurance companies, thus explicitly sharing the risk.

Generalize Jon's example to the case of more than two assets. Suppose the risky pool has *n* assets.

- Then the volatility of the risk-sharing portfolio will be $\sigma_V = y\sigma\sqrt{n}$, and its Sharpe ratio will be $\sqrt{n}R / \sigma$.
- Both of these improve as n increases.

With risk sharing, one can set up an insurer of any size, building a large portfolio of policies and limiting total risk by selling shares among many investors.

- As the Sharpe ratio increases with the number of policies written, while the risk to each diversified shareholder falls, the size of ever-more-profitable insurer appears unlimited.
- In reality, two problems put a damper on this process.

  First, burdens related to problems of managing very large firms will sooner or later eat into the increased gross margins.

  More important, the issue of "too big to fail" may emerge. The possibility of error in assessing the risk of each policy or misestimating the correlations across losses on the pooled policies (or worse yet, intentional underestimation of risk) can cause an insurer to fail.

Investment for the Long Run

Turn to the implications of risk pooling and risk sharing for long-term investing.

Think of extending an investment horizon for another period (which adds the uncertainty of that period's risky return) analogous to adding another risky asset or insurance policy to a pool of assets.

Consider an investment in a risky portfolio over the next 2 years, a "long-term investment."

How should you compare this decision to a "short-run investment"?

Compare these two strategies over the same period (2 years).

- The short-term investment therefore must be interpreted as investing in the risky portfolio over 1 year and in the risk-free asset over the other.
- Assuming the risky return on the first year is uncorrelated with that of the second, it is clear that the "long-term" strategy is analogous to portfolio *Z*.

  This is because holding on to the risky investment in the second year (rather than withdrawing to the risk-free rate) piles up more risk, just as selling another insurance policy does.

Also, the long-term investment may be considered analogous to risk pooling.

- While extending a risky investment to the long run improves the Sharpe ratio (as does risk pooling), it also increases risk. Thus "time diversification" is not really diversification.
- An investor can capture the improved Sharpe ratio that accrues from long-term risk taking and still limit overall risk by reducing the fraction of his portfolio invested in the risky asset (analogous to Jon selling off a fraction of his insurance pool).

Thus a smaller fraction of the portfolio in the risky asset invested over a longer horizon is preferable to investing a larger fraction in the same risky investment for a short period and then switching to the risk-free investment for the remainder of the investment horizon.

Questions from the 2003 exam

5. (4 points) The universe of available securities includes two risky stock funds, X and Y, and T-Bills. The expected returns and standard deviations for the universe are as follows:

Asset	Expected Return	Standard Deviation
X	10%	20%
Y	20%	40%
T-Bills	5%	0%

The correlation coefficient between funds X and Y is -0.3.

a. (2.5 points) Calculate the expected return and standard deviation for the optimal risky portfolio, P.

b. (0.50 points) Find the slope of the capital allocation line, CAL, supported by T-Bills and optimal portfolio P, from above.

c. (1 point) If an investor has a coefficient of risk aversion, A, equal to 4, what proportion will he or she invest in fund X, fund Y, and in T-Bills?

Show all work.

7. (1.5 points) Identify which one of the following portfolios cannot lie on the efficient frontier and demonstrate why.

Portfolio	Expected Return %	Standard Deviation
W	15	36
X	12	15
Y	5	7
Z	9	21

Questions from the 2004 exam

4. (3.5 points)

a. (1 point) Explain the significance of the Separation Property in determining optimal complete portfolios for two clients with different degrees of risk aversion.

b. (1.5 points) Assuming an investor cannot borrow, graph, in expected return-standard deviation space, the relationship between:

- Capital Allocation Line
- Efficient Frontier of Risky Assets
- Indifference Curve

Be sure to label the axes, and each line/curve.

c. (1 point) Briefly describe the importance of the following points from the graph in part b. above:

1. The point where the Efficient Frontier and Capital Allocation Line meet
2. The point where the Indifference Curve and Capital Allocation Line meet

Questions from the 2005 exam

5. (3 points) You are given the following information about two risky assets and a risk-free asset.

Asset	Expected Return	Standard Deviation
A	15%	20%
B	20%	15%
Risk-free	5%	0%

Short sales are not allowed.

a. (2 points) Calculate the standard deviation of the optimal risky portfolio if the correlation coefficient is 0.1.

SHOW ALL WORK.

b. (0.5 point) What is the standard deviation of the optimal risky portfolio if the correlation coefficient is -1 ?

c. (0.5 point) What is the standard deviation of the optimal risky portfolio if the correlation coefficient is 1?

6. (1 point)

You are given the following information about stocks and gold.

Asset	Expected Return	Standard Deviation
Stocks	15%	25%
Gold	10%	30%

Demonstrate graphically why an investor would choose to hold gold, in light of the apparent inferiority of gold with respect to expected return and volatility.

Be sure to completely and clearly label all information on the graph.

Questions from the 2006 exam

5. (1.5 points) You are considering three portfolios in which to allocate your investment:

- A risk-free asset that yields 4.0%.
- A stock fund with an expected return of 22.0% and a standard deviation of 30.0%.
- A bond fund with an expected return of 8.0% and a standard deviation of 10.0%.

The correlation between the returns of this stock fund and this bond fund is -0.10.

a. (1 point) Calculate the percentages of the optimal portfolio that would be allocated to the stock fund and to the bond fund.

b. (0.5 point) Given the answer from part a. above, calculate the expected return of the optimal portfolio.

Questions from the 2007 exam

2. (2.5 points) Available securities include two risky stock funds, X and Y, and T-bills. Data for the securities follows:

	Expected Return	Standard Deviation
Fund X	8%	22%
Fund Y	27%	70%
T-bills	6%	0%

The correlation coefficient between funds X and Y is -0.15.

a. (1.5 points) Determine the expected return and the standard deviation of the optimal risky portfolio.

b. (1 point) "A" is the index of an investor's risk aversion. Calculate how much an investor with a risk aversion index of 4 will invest in each of funds X and Y and in T-bills.

SHOW ALL WORK.

Questions from the 2008 exam

1. (3 points) A fund manager recently attended a presentation on a new weather catastrophe bond offering and is considering adding the catastrophe bond to his current optimal risky portfolio. Assume the following:

	Expected Return	Standard Deviation
Current optimal risky portfolio	8%	12%
New catastrophe bond	95%	300%
Treasury bills (T-bills)	5%	0%

The fund manager has estimated the correlation coefficient between his current optimal risky portfolio and the new catastrophe bond to be 0.1.

a. (1.75 points) Determine the expected return and the standard deviation of the new optimal risky portfolio if the new catastrophe bond is added.

b. (0.5 point) Assume the utility function $U = E(r) - 0.005A\sigma^2$ and risk aversion parameter A = 3.

Calculate the proportion of the new optimal complete portfolio that should be invested in T-bills.

c. (0.75 point) Assume that the fund manager invested $100,000 in the new optimal risky portfolio and T-bills in the proportions determined in part b. above at the beginning of the year.

One year later, the investments have grown to $78,000 in the risky portfolio fund and $37,800 in T-bills. The investor's risk aversion and the expected returns and standard deviation of the new optimal risky portfolio and T-bills remain unchanged.

The investor wants to rebalance the portfolio by either buying or selling T-bills.

Determine the amount of T-bills required to rebalance the portfolio and state whether the investor must buy or sell these.

Questions from the 2009 exam

2. (2.25 points) Given the following information:

	Expected Return	Standard Deviation
T-Bills	4%	0%
Bond Portfolio	8%	15%
Stock Portfolio	20%	40%

The correlation coefficient of the stock and bond portfolios is 10%.

a. (1.25 points) Calculate the weight to invest in each asset to construct the optimal risky portfolio.

b. (1 point) Calculate the Sharpe Ratio for the optimal risky portfolio.

SHOW ALL WORK.

3. (1.75 points) Given the following information about the universe of available risky securities:

- Stocks are identically distributed.
- The expected return for each stock is 10%.
- The standard deviation of the return for each stock is 50%.
- The correlation coefficient between any pair of stocks is 0.4.

a. (0.5 point) Calculate the variance of an equally weighted risky portfolio of 20 stocks.

b. (0.5 point) Calculate both the firm-specific risk and systematic risk in this portfolio.

c. (0.75 point) Calculate the number of stocks necessary for the portfolio's standard deviation to be less than 32%.

SHOW ALL WORK.

Questions from the 2011 exam

1. (2.25 points) A portfolio is being constructed for an investor using the following assets and assumptions:

	Asset		
	D	E	F
Expected Return	0.12	0.14	0.04
Standard Deviation	0.15	0.10	0.00

- The utility function is $U = E\left(r_C\right) - 0.5A\sigma_C^2$

  The subscript c refers to the optimal portfolio.
- The coefficient of risk aversion, A, is 7.
- The weight given to asset D in the optimal risky portfolio is 0.16.
- The reward-to-volatility ratio is 1.03.
- The investor is allowed to borrow at the risk-free rate.

a. (1.75 points) Calculate the share of the optimal complete portfolio invested in the risk-free asset that would maximize the investor's utility.

b. (0.5 point) Describe the result in part a above in terms of the optimal risky portfolio.

Questions from the 2012 exam

1. (3 points) Given the following information:

Asset	Expected Return	Standard Deviation
A	12%	> 0%
B	10%	> 0%
C	6%	> 0%
D	3%	= 0%

- The coefficient of risk aversion is 3.
- The weight given to Asset A in the optimal risky portfolio is 40%.
- The standard deviation of the optimal risky portfolio is 16%.
- The slope of the capital allocation line is 0.35.

a. (2.5 points) Construct an optimal investment plan using some of each asset (A, B, C, and D) and justify your proposed plan. Include brief descriptions of the proportion to be invested in each asset and the overall expected risk and return results.

b. (0.5 point) Explain why the proportion invested in Asset D may be the only difference between plans for investors with the same assets.

Solutions to questions from the 2003 Exam:

5. (4 points)

a. (2.5 points) Calculate the expected return and standard deviation for the optimal risky portfolio, P.

The expected rate of return on a portfolio is a weighted average of the expected rate of return on each component asset, with portfolio proportions as weights. Given two assets with expected returns of E(R_1) and E(R_2) and weights of w_1 and w_2, the expected return on the portfolio is computed as

$$E(r_P) = \sum_{i=1}^{2} E(r_i) * w_i$$

When two risky assets with variances σ_D^2 and σ_E^2 are combined into a portfolio with portfolio weights W_D and W_E the portfolio variance σ_P^2 is given by $\sigma_P^2 = W_D^2\sigma_D^2 + W_E^2\sigma_E^2 + 2W_D W_E \rho_{DE}\sigma_D\sigma_E$ where $Cov(r_D,\ r_E) = \rho_{DE}\sigma_D\sigma_E$, and

$$W_D = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]\mathrm{Cov}(r_D, r_E)}{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) - r_f + E(r_E) - r_f]\mathrm{Cov}(r_D, r_E)}$$

In light of the above, first determine the weights for the optimal risky portfolio:

$$W_X = \frac{[.10 - .05](.40)^2 - [.20 - .05]*(-.30)*(.20)*(.40)}{[.10 - .05].4^2 + [.20 - .05].2^2 - [.10 - .05 + .20 - .05](-.30)*(.20)*(.40)} = \frac{0.0116}{0.0188} = 0.6170$$

Thus, $W_Y = 1 - W_X = 1 - .6170 = .3830$

Therefore, $E(r_P) = .617(.10) + .383(.2) = .1383$, and

$$\sigma_P^2 = (.617)^2(.20)^2 + (.383)^2(.40)^2 + 2*0.617*0.383*(-.3)(.2)(.4) = .02735$$

b. (0.50 points) Find the slope of the capital allocation line, CAL, supported by T-Bills and optimal portfolio P, from above.

The formula for the optimal weights (shown on the prior page) were determined by maximizing the function $\underset{w_i}{Max}\, S_P = \frac{E(r_P) - r_f}{\sigma_P}$ subject to the constraint that the portfolio weights sum to 1.0 (i.e. w_D+ w_E = 1).

As shown in chapter 6, the formula for the optimal weights is $y = \frac{E(r_P) - r_f}{.01 \times A\sigma_P^2} = \frac{13.83\ -\ 5}{.01 \times 4 \times 16.54^2} = 0.8069$, which is the portion to be invested in the risky portfolio.

c. (1 point) If an investor has a coefficient of risk aversion, A, equal to 4, what proportion will he or she invest in fund X, fund Y, and in T-Bills?

The proportions invested are as follows:

.8069 (.617) = .4979 should be invested in fund X.

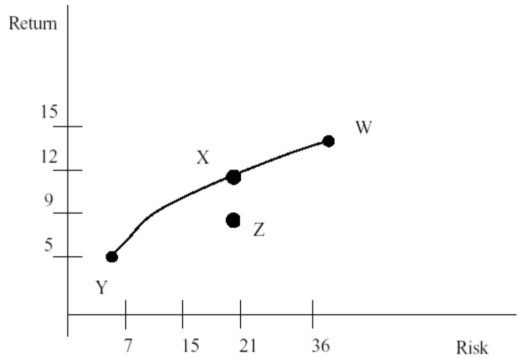
.8069 (1 - .617) = .8069 (.383) =.3090 should be invested in fund Y, and

1.0 - .8069 = .1931 should be invested in T-Bills.

Note: These proportions sum to 1.00

Solutions to questions from the 2003 Exam (continued):

7. (1.5 points) Identify which one of the following portfolios cannot lie on the efficient frontier and demonstrate why.



Portfolio Z cannot lie on the efficient frontier.

The efficient frontier is the set of potential portfolios that have the minimum standard deviation for any given return, and the maximum return for a given standard deviation.

If there exists a portfolio that has a greater return for the same or lower standard deviation than a second portfolio, then the second portfolio cannot be on the efficient frontier. Portfolio X has both a greater return and a lower standard deviation than Portfolio Z. Therefore Portfolio Z cannot be on the efficient frontier.

Solutions to questions from the 2004 Exam:

4. (3.5 points)

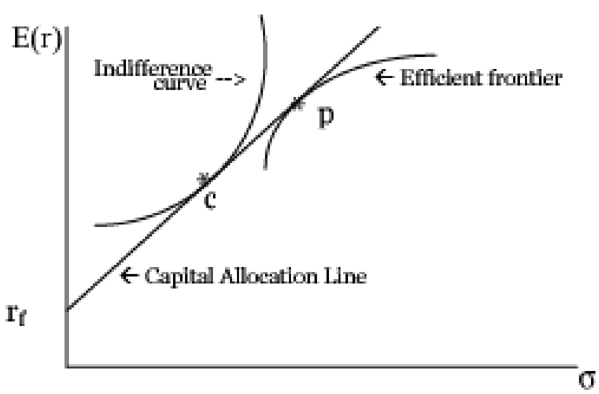
a. (1 point) Explain the significance of the Separation Property in determining optimal complete portfolios for two clients with different degrees of risk aversion.

Determining the complete portfolio under the Separation Property can be divided into 2 steps:

Step 1: Determine the optimal risky portfolio. This will be the same for both investors despite their different degrees of risk aversion.

Step 2: Determine the proportion to invest in the risky portfolio and the proportion to invest in the risk-free asset. Different proportions will be used for the two clients depending on their respective degrees of risk aversion.

b. (1.5 points) Assuming an investor cannot borrow, graph, in expected return-standard deviation space, the relationship between Capital Allocation Line, the Efficient Frontier of Risky Assets and Indifference Curve



Solutions to questions from the 2004 Exam - continued:

4c. (1 point) Briefly describe the importance of the following points from the graph in part b. above:

1. The point where the Efficient Frontier and Capital Allocation Line meet
2. The point where the Indifference Curve and Capital Allocation Line meet

1. Point p represents the optimal risky portfolio.
2. Point c represents the investor's complete portfolio and depends on the amount invested at the risk-free rate and the amount invested in the risky portfolio.

Solutions to questions from the 2005 Exam:

5. (3 points)

a. (2 points) Calculate the standard deviation of the optimal risky portfolio if the correlation coefficient is 0.1.

Initial comments: When two risky assets with variances σ_D^2 and σ_E^2 are combined into a portfolio with portfolio weights w_D and w_E, the portfolio variance σ_P^2 is given by $\sigma_P^2 = W_D^2\sigma_D^2 + W_E^2\sigma_E^2 + 2W_D W_E \rho_{DE}\sigma_D\sigma_E$ where

$$Cov\ r_D, r_E = \rho_{DE}\sigma_D\sigma_E \text{ and } W_D = \frac{[E(r_D) - r_f]\sigma_E^2 - [E(r_E) - r_f]Cov(r_D, r_E)}{[E(r_D) - r_f]\sigma_E^2 + [E(r_E) - r_f]\sigma_D^2 - [E(r_D) - r_f + E(r_E) - r_f]Cov(r_D, r_E)}$$

In light of the above, first determine the weights for the optimal risky portfolio:

$$W_A = \frac{[0.15 - 0.05].15^2 - [0.20 - 0.05]*(.1)*(.20)*(.15)}{[.15 - .05].15^2 + [.20 - .05].2^2 - [.15 - .05 + .20 - .05](.10)*(.20)*(.15)} = \frac{0.0018}{0.0075} = 0.24$$

Thus, $W_B = 1 - W_A \quad 1 - 0.24 = 0.76$, and

$$\sigma_P^2 = (0.24)^2 \ (0.20)^2 + (0.76)^2 \ (0.15)^2 + 2*0.24*0.76*0.10*0.20*0.15 = 0.01639$$

$$\sigma_P = \sqrt{0.01639} = 0.1280$$

b. (0.5 point) What is the standard deviation of the optimal risky portfolio if the correlation coefficient is -1 ? Note the following:

Assume short selling is not allowed. Since the correlation coefficient is -1, A and B are perfectly negatively correlated. There is arbitrage opportunity of creating a zero standard deviation portfolio with a return higher than the risk free rate. Finally, for ρ = -1, the portfolio opportunity set is linear, but now it offers a perfect hedging opportunity and the maximum advantage from diversification. Investors would take unlimited position in this portfolio by borrowing at the risk free rate. That is the optimal portfolio. Therefore, the standard deviation of optimal risky portfolio is 0. Thus, in the extreme case of perfect negative correlation, we have a perfect hedging opportunity and can construct a zero-variance portfolio.

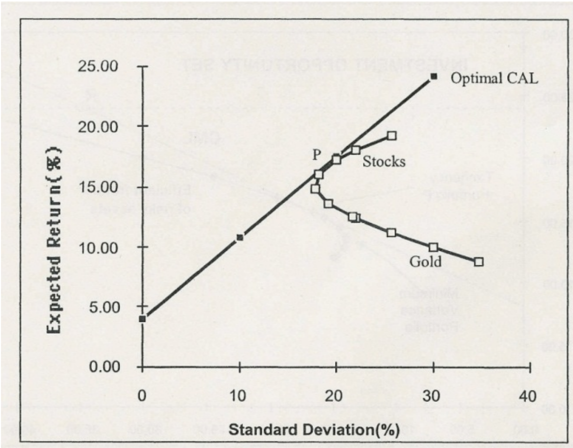
c. (0.5 point) What is the standard deviation of the optimal risky portfolio if the correlation coefficient is 1?

Since the two assets are perfectly correlated, and since asset b provides a greater expected return and smaller standard deviation relative to asset a, and since short sales are not allowed, then all will be invested in asset b. Therefore, the standard deviation will be 15, which is the STD of asset b

Solutions to questions from the 2005 Exam:

6. (1 point) Demonstrate graphically why an investor would choose to hold gold, in light of the apparent inferiority of gold with respect to expected return and volatility.

   Be sure to completely and clearly label all information on the graph.



   As shown above, gold is inferior to stocks in terms of expected return and volatility. However, holding gold may provide diversification benefits if it is independent of (or negatively correlated with) the performance of stocks.

   A risky portfolio, including gold, maybe preferable when one considers a complete portfolio, which depends on the amount invested at the risk-free rate and the amount invested in the risky portfolio.

   Finally, if the correlation between gold and stocks is sufficiently low, gold will be held as a component in a portfolio, specifically the optimal tangency portfolio.

Solutions to questions from the 2006 Exam:

5. (1.5 points)

   a. (1 point) Calculate the percentages of the optimal portfolio that would be allocated to the stock fund and to the bond fund.

   b. (0.5 point) Given the answer from part a. above, calculate the expected return of the optimal portfolio.

Initial comments: The expected rate of return on a portfolio is a weighted average of the expected rate of return on each component asset, with portfolio proportions as weights.

Given two assets with expected returns of $E\left(R_1\right)$ and $E\left(R_2\right)$ and weights of W_1 and W_2, the expected return on the portfolio is computed as $E(r_p)=\sum_{i=1}^{2}E(r_i)*W_i$ When two risky assets with variances σ_D^2 and σ_E^2 are combined into a portfolio with portfolio weights w_D and w_E, the portfolio variance σ_P^2 is given by $\sigma_P^2=W_D^2\sigma_D^2+W_E^2\sigma_E^2+2W_DW_E\rho_{DE}\sigma_D\sigma_E$ where

$Cov\left(r_D,\ r_E\right)=\rho_{DE}\sigma_D\sigma_E$, and

$$W_D=\frac{[E(r_D)-r_f]\sigma_E^2-[E(r_E)-r_f]Cov(r_D,r_E)}{[E(r_D)-r_f]\sigma_E^2+[E(r_E)-r_f]\sigma_D^2-[E(r_D)-r_f+E(r_E)-r_f]Cov(r_D,r_E)}$$

Solutions to questions from the 2006 Exam:

Question 5 - Model Solution 1:

a. Step 1: Write notation for the information given in the problem:

r_f = 4.0%

$E(r_1)$ = 22.0% and σ_1 = 30.0% (stock), where subscript 1 indicates investment in stock fund

$E(r_2)$ = 8.0% and σ_2 = 10% (bond) and ρ_{12} = 0.10, where subscript 2 indicates investment in bond fund

a. Step 2: Write an equation to calculate the percentages of the optimal portfolio that would be allocated to the stock fund and to the bond fund.

The equation to determine the weights for the optimal risky portfolio is

$$W_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]Cov(r_1, r_2)}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]Cov(r_1, r_2)}$$

$$W_1 = \frac{[.22 - .04](.10)^2 - [.08 - .04]*(.10)*(.30)*(.10)}{[.22 - .04].1^2 + [.08 - .04].3^2 - [.22 - .04 + .08 - .04](.10)*(.30)*(.10)}$$

$$= \frac{0.0018 - 0.00012}{0.0018 + .0036 - .00066} = .3544$$

Thus, $W_2 = 1 - W_1 = 1 - 0.3544 = 0.6456$ = portion that would be allocated to the bond fund.

b. Step 1: Write an equation to determine the expected return of the optimal portfolio.

$$E(r_P) = \sum_{i=1}^{2} E(r_i) * W_i$$

b. Step 2: Using the equation in Step 1, the data given in the problem and the results from Step 2, compute the expected return of the optimal portfolio.

Therefore, $E(r_P) = .3544(.22) + .6456(.08) = .078 + .0516 = .1296$

Question 5 - Model Solution 2:

Note: The only difference between the values in model solution 1 and model solution 2 stems from interpretation of the given value for the correlation between the returns of this stock fund and this bond fund. Due to a stray marking in the question, it was interpreted as either positive (+0.10) and negative (-0.10). In this model solution, it was interpreted as negative (-0.10).

a. Step 1: Write notation for the information given in the problem:

r_f =4.0%

$E(r_1)$ = 22.0% and σ_1 = 30.0% (stock), where subscript 1 indicates investment in stock fund

$E(r_2)$ = 8.0% and σ_2 = 10% (bond) and ρ_{12} = - 0.10, where subscript 2 indicates investment in bond fund

Solutions to questions from the 2006 Exam (continued):

a. Step 2: Write an equation to calculate the percentages of the optimal portfolio that would be allocated to the stock fund and to the bond fund.

The equation to determine the weights for the optimal risky portfolio is

$$W_1 = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]Cov(r_1, r_2)}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]Cov(r_1, r_2)}$$

$$W_1 = \frac{[.22 - .04](.10)^2 - [.08 - .04]*(-.10)*(.30)*(.10)}{[.22 - .04].1^2 + [.08 - .04].3^2 - [.22 - .04 + .08 - .04](-.10)*(.30)*(.10)} =$$

$$\frac{0.0018 + 0.00012}{0.0018 + .0036 + .00066} = .3168$$

Thus, $W_2 = 1 - W_1 = 1 - 0.3168 = 0.6832$ = portion that would be allocated to the bond fund.

5b. Step 1: Write an equation to determine the expected return of the optimal portfolio.

$$E(r_P) = \sum_{i=1}^{2} E(r_i) * W_i$$

5b. Step 2: Using the equation in Step 1, the data given in the problem and the results from Step 2, compute the expected return of the optimal portfolio.

Therefore, $E(r_2) = .3168(.22) + .6832(.08) = .124352$

Solutions to questions from the 2007 Exam:

2. (2.5 points)

a. (1.5 points) Determine the expected return and the standard deviation of the optimal risky portfolio.

b. (1 point) "A" is the index of an investor's risk aversion. Calculate how much an investor with a risk aversion index of 4 will invest in each of funds X and Y and in T-bills. SHOW ALL WORK.

Question 2 - Model Solution

The expected rate of return on a portfolio is a weighted average of the expected rate of return on each component asset, with portfolio proportions as weights. Given two assets with expected returns of $E(R_1)$ and $E(R_2)$ and weights of w_1 and w_2, the expected return on the portfolio is computed as $E(r_P) = \sum_{i=1}^{2} E(r_i) * W_i$

When two risky assets with variances σ_X^2 and σ_Y^2 are combined into a portfolio with portfolio weights W_x and W_y the portfolio variance σ_P^2 is given by $\sigma_P^2 = W_x^2\sigma_x^2 + W_y^2\sigma_y^2 + 2W_yW_x\rho_{xy}\sigma_x\sigma_y$ where

$Cov(r_x, r_y) = \rho_{xy}\sigma_x\sigma_y$, and $W_x = \dfrac{[E(r_x) - r_f]\sigma_y^2 - [E(r_y) - r_f]Cov(r_x, r_y)}{[E(r_x) - r_f]\sigma_y^2 + [E(r_y) - r_f]\sigma_x^2 - [E(r_x) - r_f + E(r_y) - r_f]Cov(r_x, r_y)}$

In light of the above, first determine the weights for the optimal risky portfolio:

$$W_X = \frac{[8 - 6](70)^2 - [27 - 6]*(-.15)*(22)*(70)}{[8 - 6]70^2 + [27 - 6]22^2 - [8 - 6 + 27 - 6](-.15)*(22)*(70)} = \frac{14,651}{25,277} = 0.58$$

Thus, $W_Y = 1 - W_X = 1 - .58 = .42$ Therefore, $E(r_P) = .58(8) + .42(27) = 15.98$, and

$$\sigma_P^2 = (.58)^2\ (22)^2 + (.42)^2(70)^2 + 2*0.58*0.42*(-.15)(22)(70) = 162.82 + 864.36 - 112.54 = 914.64$$

$$\sigma_P = \sqrt{914.64} = 30.24$$

Solutions to questions from the 2007 Exam:

Question 2

b. Suppose the utility function is $E\left(r\right)-0.005A\sigma^2$

Given a degree of risk aversion, A, equal to 4, the proper allocation to the risky portfolio is

$y=\dfrac{E(r_P)-r_f}{.01*A\sigma_P^2}=\dfrac{15.98-6}{.01*4*914.64}=\dfrac{9.98}{36.585}=.27$ (i.e. a 27% investment in the optimal risky portfolio).

Stock Fund X	0.27(0.58) = 15.66
Stock Fund Y	0.27(0.42) = 11.34
T-bills	1 – 0.27 = 73.00

Solutions to questions from the 2008 Exam:

Question 1 - Model solution

a. Step 1: Write an equation to determine the expected return and the standard deviation of the new optimal risky portfolio if the new catastrophe bond is added.

Let E(R_1) = the expected return of the new catastrophe bond and E(R_2) the expected return of the current optimal risky portfolio. Once the weights w_1 and w_2 are determined, the expected return on the portfolio is computed as $E(r_P)=\sum_{i=1}^{2}E(r_i)*W_i$.

Also, when two risky assets with variances σ_1^2 and σ_2^2 are combined into a portfolio with portfolio weights W_1 and W_2 the portfolio variance σ_P is given by $\sigma_P^2=W_1^2\sigma_1^2+W_2^2\sigma_2^2+2W_1W_2\rho_{12}\sigma_1\sigma_2$ where

$\text{Cov}(r_1,\ r_2)=\rho_{12}\sigma_1\sigma_2$, and $W_1=\dfrac{[E(r_1)-r_f]\sigma_2^2-[E(r_2)-r_f]Cov(r_1,r_2)}{[E(r_1)-r_f]\sigma_2^2+[E(r_2)-r_f]\sigma_1^2-[E(r_1)-r_f+E(r_2)-r_f]Cov(r_1,r_2)}$

Step 2: Using the equations in Step 1, solve for the W_1, W_2, and the expected return and the standard deviation.

Weight in cat bond = $W_1=\dfrac{(95-5)12^2-(8-5)(0.1*12*300)}{(95-5)12^2+(8-5)300^2-(95-5+8-5)(0.1*12*300)}=.048$, $W_2=.952$

Expected return of new optimal risky portfolio = $E(r_P)=0.048*95\%+0.952*8\%=12.1\%$

Standard deviation of new optimal risky portfolio

$=\sqrt{.048^2*300^2+0.952^2*12^2+2*0.048*0.952*0.1*12*300)}=.192$

b. Given a degree of risk aversion, A = 3, and the form of the utility function, the proper allocation to the risky portfolio is $y*=\dfrac{E(r)-r_f}{.01A\sigma_P^2}=\dfrac{12.15\ -\ 5}{.01*3*19.18^2}=0.6478$ (i.e. a 64.78% investment in the optimal risky portfolio).

Therefore, the proportion of the new optimal complete portfolio to be invested in T-bills = 1.0 - .6478 = .3522.

c. Note: CAS Examiners accepted the following solutions to this part of the problem.

Solutions to questions from the 2008 Exam:

Question 1 - model solution 1- part c

Determine the % invested in the risky portfolio at the end of the year and the % that should be invested to rebalance to the optimal investment percentages.

At the end of year, the % invested in risky portfolio = [78,000/(78,000+37,800)] = 67.36%

However, $78,000 needs to represent 64.78% of the portfolio in order to rebalance the portfolio to being optimally invested in risky portfolio and risk free assets. This suggests the following:

- The total portfolio value must equal $78,000/0.6478 = $120,408. This implies that:
- The T-bill portion should be $120,408 - $78,000 = $42,408

Therefore, rebalancing requires purchase of $42,408 - $37,800 = $4,608 in T-bills

Question 1 - model solution 2- part c

The Total portfolio at t = 1 is $78,000 + $37,800 = $115,800

At the end of year, the % invested in T-bills = 35.22%

Further, the amount of T-bills needed to rebalance must equal .3552 * 115,800 = $40,785

Therefore, there will be a need to buy [$40,785 - $37,800] = $2,985 in T-bills.

Solutions to questions from the 2009 Exam:

Question 2 - model solution

a. Write an equation and compute the weight to invest in each asset to construct the optimal risky portfolio.

$$W_s = \frac{(E(r_s)-r_f)\sigma_b^{\,2} - (E(r_b - r_f)\sigma_{sb}}{(E(r_s)-r_f)\sigma_b^{\,2} + (E(r_b)-r_f)\sigma_s^{\,2} - (E(r_s)-r_f + E(r_b)-r_f)\sigma_{sb}}$$

$$\text{Where } \sigma_{sb} = \rho\sigma_s\sigma_b = .10(.40)(.15) = .006$$

$$W_s = \frac{(.20-.04)(.15)^2 - (.08-.04)(.006)}{(.20-.04)(.15)^2 + (.08-.04)(.40)^2 - (.20-.04+.08-.04)(.006)}$$

$$W_s = \frac{.0036}{.0088} = .382; \quad W_b = 1.0 - W_s = 1.0 - .382 = .618$$

b. Write an equation and calculate the Sharpe Ratio for the optimal risky portfolio.

$$Sharpe\ Ratio = \frac{E(r_P)-r_f}{\sigma_P} = \frac{.1258-.04}{.1865} = \boxed{.46}$$

$$Note:\ E(r_P) = W_s\ E(r_s) + W_b\ E(r_b) = .382(.20) + .618(.08) = .1258$$

$$\sigma_P^{\,2} = w_s^{\,2}\sigma_s^{\,2} + w_b^{\,2}\sigma_b^{\,2} + 2\rho W_s W_b \sigma_s \sigma_b$$

$$= .382^2(.40)^2 + .618^2(.15)^2 + 2(.10)(.40)(.15)(.382)(.618) = .0348$$

$$\sigma_P = .1865$$

Solutions to questions from the 2009 Exam:
Question 3 - model solutions 1 and 3– part a

a. The portfolio variance can be expressed as: $Var_P = \frac{1}{n}\overline{Var} + \frac{n-1}{n}\overline{Cov};\ where\ \overline{Cov} = \sigma_i \sigma_j \rho_{ij}$

Using the givens in the problem, compute

$$Cov = (.5)(.5)(.4) = .10\ and\ Var_p = \frac{1}{20}(.5^2) + \frac{19}{20}(.1) = \boxed{.1075}$$

As n becomes large, firm specific risk is diversified away, and

$$\sigma_P^2 \Rightarrow \frac{1}{Large\ n}\bar{\sigma}^2 + \left(1 - \frac{1}{Large\ n}\right)\overline{Cov} \Rightarrow \overline{Cov}$$

Question 3 - model solution 2– part a

a. $\sigma_P^2 = \sum_{i=1}^{20} W_i^2 \sigma_i^2 + \sum_{i=1}^{20}\sum_{\substack{j=1 \\ i=j}}^{20} 2W_i W_j \sigma_i \sigma_j \rho_{ij}$

when $W_i = \frac{1}{20} = 0.05$, then

$$\sigma_P^2 = (0.05)^2(0.5)^2(20) + 2(.05)^2\left[\frac{20^2 - 20}{2}\right](0.5)^2(0.4) = 0.0125 + 0.095 = 0.1075$$

Solutions to questions from the 2009 Exam:
Question 3 - model solutions 1, 2, and 3– parts b and c

b. If the total risk is 0.1075, and the minimum systematic risk is $\overline{Cov} = .10$, then
firm specific = .1075 - .1 = .075 since .1 is minimum possible and
systematic risk = .10 (the minimum variance the portfolio could ever reach with an infinite number of stocks)

c. A std dev > .32 implies that the variance < .1024. Therefore, using the formula from part a., solve for n.

$$.1024 = \frac{1}{n}(.5^2) + \frac{n-1}{n}(.1)$$

$$.1024 = \frac{.25 + .1n - .1}{n}$$

$$.1024n = .15 + .1n$$

$$.0024n = .15$$

$$n = 62.5, \text{so } \boxed{63 \text{ stocks}}$$

Solutions to questions from the 2011 exam

a. (1.75 points) Calculate the share of the optimal complete portfolio invested in the risk-free asset that would maximize the investor's utility.

b. (0.5 point) Describe the result in part a above in terms of the optimal risky portfolio.

Question 1 - Model Solution 1

a. Calculate $1.0-y$, the share of the optimal complete portfolio invested in the risk-free asset that would maximize the investor's utility. $y=[E(r_P)-r_f]/A\sigma_P^2$

$$r_f=.04(\textit{based on Asset } F);\ W_D=.16\,(\textit{given});\ \textit{thus},\ W_E=1-W_D=.84;\ E(r_D)\ \textit{and}\ E(r_E)\ \textit{given}$$

$$E(r_p)=\sum_{i=1}^{2}E(r_i)*w_i=E(r_p)=W_D*E(r_D)+W_E*E(r_E)=.16(.12)+.84(.14)=13.68\%$$

$$\textit{The reward to volatility ratio}=S=[E(r_p)-r_f]/\sigma_p=(.1368-.04)/\sigma_p=1.03$$

$$1.03\sigma_p=.0968;\ \textit{Thus}\ \sigma_\rho=.094$$

The proportion y, *to be allocated to the risky portfolio*, P, *is*

$$y=[E(r_p)-r_f]/A\sigma_p^2=[.1368-.04]/7(.094)^2=156.5\%$$

$$\textit{The position to be taken in the risk free asset},1.0-y=\ 1.0-1.565=\boxed{-56.5\%}$$

b. The optimal complete portfolio requires more invested in the optimal risky portfolio than the investor has, so the investor must borrow funds at the risk-free rate which will then be invested in the optimal risky portfolio.

Question 1 - Model Solution 2

a.

$$\textit{Given}:\ W_D=.16;\ \textit{Thus},\ W_E=1-W_D=.84$$

$$R_D=0.12-0.04=0.08 \qquad \sigma_D=0.15$$

$$R_E=.14-0.04=0.10 \qquad \sigma_E=.1$$

$$W_D=0.16=[0.08*.10^2-.10*\text{cov}(r_D,r_E)]/[0.08*0.1^2+.10(0.15)^2-(0.08+0.1)*\text{cov}(r_D,r_E)]$$

$$\text{Solve for cov}(r_D,r_E):\ \ 0.000488-0.0288812\,\text{cov}(r_D,r_E)=0.0008-.1*\text{cov}(r_D,r_E)$$

$$\boxed{\text{cov}(r_D,r_E)=0.004382}\ \ \textit{Used below}$$

$$\sigma_P^2=0.15^2*(0.16)^2+(0.1)^2*(0.84)^2+2*0.16*0.84*0.004382=0.0088;\ \ \sigma_P=0.094$$

$$y_P=R_P/A\sigma_P^2=(R_P/\sigma_P)/A\sigma_P=\ S/A\sigma_P=1.03/(7*0.094)=1.565$$

$$1-1.565=\boxed{-0.565}\leftarrow\ \textit{weight to the risk free asset in the complete portfolio}$$

b. It means we should borrow risk free asset and invest in risky portfolio.

Solutions to questions from the 2011 exam

Question 1 - Model Solution 3

a.

$U = E\left(r_c\right) - 0.5A\sigma_c^2 = r_f + y\left[E\left(r_P\right) - r_f\right] - .5Ay^2\sigma_P^2;\ \ r_f = .04$

$y = \sigma_c / \sigma_P$ or the allocation of the risky portfolio

$dU / dy = E\left(r_P\right) - r_f - A\sigma_P^2 y = 0;$

To maximize y: Set $y = E\left(r_P\right) - r_f / A\sigma_P^2 = [E\left(r_P\right) - r_f] / 7\sigma_P^2 = [E\left(r_P\right) - .04] / 7\sigma_P^2$

Solve for $E\left(r_P\right)$: $E\left(r_P\right) = .16\left(.12\right) + 84\left(.14\right) = .1368$

Solve for $\sigma^2\left(P\right) : \sigma^2\left(P\right) = .16^2\left(.15\right)^2 + .84^2\left(.1\right)^2 + 2\left(.16\right)\left(.84\right)\left(.00438\right) = .00881;$

Solve for $Cov\left(D,E\right) = .00438$ (*see below*)

$.16 = W_{p_2} = [RP_D\left(\sigma_E^2\right) - RP_E Cov\left(D,E\right)] / [RP_D\left(\sigma_E^2\right) + RP_E\left(\sigma_D^2\right) - \left(RP_D + RP_E\right)Cov\left(D,E\right)]$

$\quad = [.08\left(.1\right)^2 - .1Cov\left(D,E\right)] / [.08\left(.1\right)^2 + .1 * .15^2 - .18Cov\left(D,E\right)]$

$"RP" = \textit{Risk premium over risk free rate. Solve for } Cov\left(D,E\right)$

$.16 = [.0008 - .1Cov\left(D,E\right)] / [.0008 + .00225 - .18Cov\left(D,E\right)]$

$\quad = [.0008 - .1Cov\left(D,E\right)] / [.00305 - .18Cov\left(D,E\right)]$

$.000488 - .0288Cov\left(D,E\right) = .0008 - .1Cov\left(D,E\right)$

$.712Cov\left(D,E\right) = .000312 \qquad Cov\left(D,E\right) = .00438$

$y = [.1368 - .04] / (7 * .00881) = 1.5696$

The investor should borrow @ the risk free rate (57%) and invest 100% in the risky portfolio.

b. Borrow at the risk-free rate and invest in the optimal risky portfolio. The final weights on the complete portfolio are: D = 25% (.16 * 1.5657), E = 132%, F = (-57%)

Solutions to questions from the 2011 exam

Question 1 - Model Solution 4

$r_f = .04 \ (asset\ F).\ Must\ find\ \rho_{DE}$

$W_D = \left(E\left(r_D\right)-r_f\right)\sigma_E^2 - \left(E\left(r_E\right)-r_f\right)\sigma_D\sigma_E\rho\ /$

$\quad \left(E\left(r_D + r_f\right)\sigma_E^2 + \left(E\left(r_E\right)-r_f\right)\right)\sigma_D^2 - \left(E\left(r_D\right)-r_f + E\left(r_E\right)-r_f\right)\sigma_D\sigma_E\rho$

$.16 = \left(.12-.04\right).10^2 - \left(.14-.04\right)\left(.15\right)\left(.10\right)\rho_{DE}\ \ /$

$\quad \left(0.12-.04\right).10^2 + \left(.14-.04\right).15^2 - \left(.14-.04+.12-.04\right)\left(.15\right)\left(.10\right)\rho_{DE}$

$.16 = [.0008 - .0015\rho_{DE}]/[.00305 - .0027\rho_{DE}]$

$.000488 - .00432\rho_{DE} = .0008 - .0015\rho_{DE};\ \rho_{DE} = .292$

$E\left(r_p\right)W_D E\left(r_D\right) + W_E E\left(r_E\right) = .16\left(.12\right) + 84\left(.14\right) = .1368$

$\sigma_p^2 = W_D^2 + W_e{}^2\sigma_e^2 + 2W_D W_E \sigma_D \sigma_E \rho_{DE}$

$\quad = .16^2\left(.15\right)^2 + .84^2.10^2 + 2\left(.16\right)\left(.84\right)\left(.15\right)\left(.10\right)\left(.292\right) = .00881$

$y* = [E\left(r_p\right)-r_f]/A\sigma_p^2 = [.1368-.04]/7\left(.00881\right) = \boxed{1.570}\left(risky\right) \rightarrow \boxed{1-y*=-.57}\left(risk-free\right)$

b. Borrow $.57 for every dollar to invest $1.57 in risky assets.

Question 1 - Model Solution 5

W_D in optional risky portfolio = .16; therefore $W_E = 1-1.6 = .84$

Let P be the optional risky portfolio

$E\left(r_P\right) = \left(0.16\right)\left(0.12\right) + \left(.84\right)\left(.14\right) = .1368$

Asset F is the risk-free asset so $r_f = .04$

$[E\left(r_P\right)-r_f\ /\ \sigma_P] =$ reward to volatility ratio = 1.03 = sharpe ratio

$E\left(r_C\right) = r_f + \left[E\left(r_P\right)-r_f\right]\sigma_C / \sigma_P$

$E\left(r_C\right) = .04 + 1.03\sigma_C \rightarrow$ substitute into $\rightarrow U = E\left(r_C\right) - 0.5A\sigma_C^2$

$U = .04 + 1.03\sigma_C - 3.5\sigma_C^2$

To maximize U take 1st derivative in relation to σ_C and set to 0 and solve for σ_C

$0 = 1.03 - 7\sigma_C;\ \ .147143 = \sigma_C \rightarrow$ plug into equation from above

$E\left(r_C\right) = .04 + 1.03\left(.147143\right) = .191557$

The investor's utility is not maximized by investing in risk-free asset at all instead the optimal complete portfolio consists of borrowing an additional 56.57% (of the amount being invested) at the risk-free rate and using the borrowed funds to purchase more of the optional risky portfolio.

b. The investor should invest 156.57% of their funds in the optimal risky portfolio, borrowing the additional 56.57% at the risk-free rate.

Solutions to questions from the 2011 exam

Question 1 - Model Solution 6

$E(r_C) = yE(r_P) + (1-y)r_f \quad E(r_E) = 0.14 \quad E(r_D) = 0.12 \quad W_P = 0.16 \quad \sigma_C = y*\sigma_P$

$\mathrm{E}(\mathrm{r}_P) = \mathrm{W_D E}(\mathrm{r_D}) + (1-\mathrm{W_D})\mathrm{E}(\mathrm{r_E}) = 0.1368$

$\sigma_P^2 = W^2\sigma_D^2 + (1-W)^2\sigma_E^2 + 2\rho W(1-W)\sigma_D\sigma_E$ (Where ρ is unknown)

$y = [E(r_P) - r_f]/A\sigma_P^2 \;; \quad [E(r_C) - r_f]/\sigma_C = 1.03; \quad E(r_C) = yE(r_P) + (1-y)r_f$

$\{[yE(r_P) + (1-y)r_f] - r_f\}/y\sigma_p = 1.03$

$y[E(r_p) - r_f]/y\sigma_P = 1.03$

$[E(r_P) - r_f]^2/\sigma_p^2 = 1.03^2$

$y = 1.03^2/A(E(r_p) - r_f) = 1.03^2/7(0.1368 - 0.04) = 1.56$

$1 - y = \text{weight to riskfree} = \boxed{-56\%}(\text{i.e. borrowing this }\%)$

b. The investor would borrow 56% of their invested amount, then invest their own assets, plus this borrowing, the optimal risky portfolio.

Question 1 - Model Solution 7

$E(r_P) = 0.16(0.12) + 0.84(0.14) = 0.1368$

$S = [0.1368 - 0.04]/7(0.094)^2 = 1.5657$

The portion invested in risk free asset is -0.5657 (borrow -.5657 at risk-free rate)

b. 25.05% (.16 * 1.5657) will be invested in D; 131.52% will be invested in E.
These two items form the optimal risky portfolio. (-.5657 will be borrowed at risk free rate)

Questions from the 2012 exam

1a. (2.5 points) Construct an optimal investment plan using some of each asset (A, B, C, and D) and justify your proposed plan. Include brief descriptions of the proportion to be invested in each asset and the overall expected risk and return results.

Question 1 – Model Solution 1 – part a

Asset D represents the risk free asset, since $\sigma = 0$

$$Given\ S = \frac{E(r_p) - r_f}{\sigma_P} = .35 \text{ where } E(r_p) = \text{Exp return on optimal risky portfolio}$$

$$\text{and } \sigma_P = \text{std of optimal risky portfolio}$$

$$\text{Solve for } \mathrm{E}(r_P) = (.35)\sigma_P + r_f = (.35).16 + .03 = .086$$

$$\text{Solve for} W_B\text{: } \mathrm{E}(r_P) = (.40)(.12) + W_B(.60)(.10) + (1 - W_B).60(.06)$$

$$.086 = .048 + W_B.06 + (1 - W_B).036$$

$$.038 = W_B.024 + .036$$

$$=> W_B = .0833$$

$$\boxed{\begin{array}{l} => W_B^* = .0833 \times .60 = .05 \rightarrow 5\% \text{ weight to asset B} \\ => W_C^* = 1 - .40 - .05 = .55 \rightarrow 55\% \text{ weight to asset C} \end{array}}$$

The std deviation of portfolio including the risk free asset is $y\sigma_P = \sigma_C$, where y is the percentage invested in the optimal risky portfolio

Let Utility function $= U = E(r_C) - .5A\sigma_C^2$

where c represents portfolio including risk free asset . So y that maximizes U is

$$y^* = \frac{(E_P) - r_f}{3\sigma_P^2} = \frac{.086 - .03}{3(.16)^2} = .729$$ and 1.0 - .729 = .271 is invested in the risk free asset,

Explanation:

[I took derivative of U w.r.t. to y and set it equal to zero to solve for y]

$$E(r_C) = yE(R_p) + (1 - y)E(r_D)$$

so

	Proportion invested	E(r)
Asset A	(.729)(.40)=.292	.12
Asset B	(.729)(.05)=.036.	.10
Asset C	(.729)(.55)=0.401	.06
Asset D	.271	.03

$$\text{Overall risk } = \sigma_C = y\sigma_P = (.729)(.16) = .1166$$

$$\text{Overall return} = \sum \text{proportion invested (from above)} \times \text{E(r)} = .07083$$

Solutions to questions from the 2012 exam:

Question 1 – Model Solution 2 – part a

For the complete portfolio

$$\text{Slope} = \frac{E(r_0)(w_R)+(1-w_R)(0.03)-0.03}{w_R(0.16)} = 0.35 \sim\sim\sim \{1\}$$

$$U = E(r_0)(w_R)+(1-w_R)(0.03)-\frac{3}{2}(w_R)^2(0.16)^2$$

$$\frac{dU}{dw_R} = E(r_0)-0.03-3w_R(0.16)^2 = 0 \qquad <-- \text{to maximize utility}$$

$$w_R = \frac{E(r_0)-0.03}{3(0.16)^2}$$

$$E(r_0) = 0.03+3(0.16)^2 w_R \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \{2\}$$

Substitute {2} into {1} to solve for w_R

$$(0.03+0.0768w_R)(w_R)-0.03w_R = 0.35(0.16)w_R$$

$$w_R = 0.7292 = \text{portion to invest in optimal risky portfolio}$$

$$\text{Given Optimal risky portfolio } w_A = 0.4; \quad \text{Given } \sigma_P = 0.16$$

$$0.4^2(\sigma_A^2)+w_B^2(\sigma_B^2)+(0.6-w_B)^2(\sigma_C^2) = 0.16^2$$

$$E(r_P) = 0.4(0.12)+w_B(0.1)+(0.6-w_B)(0.06) \sim\sim\sim\sim \{3\}$$

$E(r_0) = 0.03+3(0.16)^2(0.7292) = 0.086$ = expected return of optimal risky portfolio

Substitute $E(r_0) = 0.086$ into $\{3\}$ and solve for w_B and w_C

$$0.086 = 0.048+w_B(0.1)+(0.6-w_B)(0.06)$$

$$w_B = 0.05$$

$$w_C = 0.6-0.05 = 0.55$$

(weights given to assets b and c in optimal risky portfolio)

For complete portfolio

Portion for risk free asset D = 1 - 0.7292 = 0.2708

Portion for asset A = 0.7292(0.4) = 0.2917

Portion for asset B = 0.7292(0.05) = 0.03646

Portion for asset C = 0.7292(0.55) = 0.4010

$$E(r) = 0.7292(0.086)+0.2708(0.03) = 0.07083$$

$$\sigma = 0.72922(0.16) = 0.1167$$

The proportion shown above is to minimize risk with highest return in optimal risky portfolio, then distributed among optimal risky and risk free asset D based on investor's risk aversion of 3.

Solutions to questions from the 2012 exam:

Question 1 – Model Solution 3 – part a

Given A=3; $w_A = 0.4; \sigma_P = 0.16;$ and $\frac{E(r_P)-r_f}{\sigma_P} = 0.35$

From asset D we can conclude that $r_f = 0.03$

$$Given\ \frac{E(r_P)-r_f}{\sigma_P} = 0.35 \Rightarrow \frac{E(r_P)-0.03}{0.16} = 0.35 \Rightarrow E(r_P) = 0.086$$

$$E(r_P) = w_A E(r_A) + w_B E(r_B) + w_C E(r_C) = (0.4)(0.12) + w_B(0.1) + w_C(0.06)$$

$$\Rightarrow 0.086 = 0.048 + w_B(0.1) + (0.6 - w_B)(0.06)$$

$$\Rightarrow 0.038 - 0.036 = w_B(.1 - 0.06)$$

$$\Rightarrow w_B = 5\%$$

$$\Rightarrow w_C = 0.6 - w_B = 55\%$$

=> optimal investment in risky portfolio, assuming $U = E(r) - \frac{1}{2}A\sigma^2$

$$y^* = \frac{E(r_P)-r_f}{A\sigma_P^2} = \frac{0.086-0.03}{(3)(0.16)^2} = 72.9167\%$$

=> Investment in risk-free asset =1 - y* = 27.0833%

Optimal investment plan:

-Invest in A =(0.4)x(0.729167) = 29.167% of funds

-Invest in B = (0.05)(0.729167) = 3.646% of funds

-Invest in C = (0.55)(0.729167) = 40.104% of funds

-Invest in D = 27.0833% of funds

Expected return of complete portfolio:

$$E(r_C) = r_f + y^*\left[E(r_P) - r_f\right] = 0.03 + 0.729167[0.086 - 0.03] = 7.083\%$$

Std deviation of complete portfolio (Note that the portfolio will have same sharp ratio as the risky portfolio)

Solutions to questions from the 2012 exam:

Question 1 – Model Solution 4 – part a

(1) $S_P = \dfrac{E(r_P) - r_f}{\sigma_P} = 0.16 = 0.35 => E(r_P) = 0.086$

(2) P_B – proportion in B in the optimal risky portfolio

P_C – proportion in C in the optimal risky portfolio

(3) $P_B \times 0.1 + P_C \times 0.06 + 0.4 \times 0.12 = 0.086 \}$

$P_B + P_C = 0.6 \qquad \} \rightarrow P_B = 0.05;\ P_C = 0.55$

(4) y = Proportion invested in the optimal risky portfolio

(5) $y^* = \dfrac{E(r_P) - r_f}{A\sigma_P^2} = \dfrac{0.086 - 0.03}{3 \times 0.16^2} = 72.917\%$

(6) The proportion invested in A: 72.917% x 0.4 = 29.167%

The proportion invested in B: 72.91% x 0.05 = 3.646%

The proportion invested in C: 72.917% x 0.55 = 40.104%

The proportion invested in D: 100% - 72.917% = 27.083%

The overall expected return: 0.72917 x 0.086 + 0.27083 x 0.03 = 7.083%

The SD of the overall portfolio: 0.72917 x 0.16 = 0.11667

Question 1 – Model Solution 5 – part a

Given the Slope of CAL $= 0.35 = \dfrac{E(r_P) - r_f}{\sigma_P}$; $r_f = 3\%$; $\sigma_P = 16\%$

$E(r_P) = w_A E(r_A) + w_B E(r_B) + w_C E(r_C), \sum w = 1$

$= 0.4(0.12) + w_B(0.1) + w_C(0.06) = 0.4(0.12) + w_B(0.1) + (0.6 - w_B)(0.06)$

$Key\ step:\ w_C = (0.6 - w_B)$

$0.35 = \dfrac{0.048 + 0.1w_B + 0.036 - 0.06w_B - 0.03}{0.16} = \dfrac{0.054 + 0.04w_B}{0.16}$

Solve for w_B, w_C and $E(r_P)$; $w_B = 0.05 \Rightarrow w_C = 0.55 \Rightarrow E(r_P) = 0.086$

Complete portfolio

$E(r_C) = r_f + y(E(r_P) - r_f), y = \%$ in risky asset

= 0.03+0.056y

Utility: $U = E(r_C) - 1/2A\sigma_C^2 = 0.03 + 0.056y - 1/2(3)\sigma_C^2;\quad \sigma_C = y\sigma_P - 0.16y$

$--> U = 0.03 + 0.056y - 3/2 * 0.16^2 y^2;\quad U' = 0.056 - 3/2(0.16^2)(2y) = 0$

$y^* = 0.7292$

Invest (0.7292)(0.4) = 29.17% in Asset A
(0.7292)(0.05) = 3.65% in Asset B
(0.7292)(0.55) = 40.10% in Asset C
1- 0.7292 = 27.08% in Asset D

The expected return of the complete portfolio is $E(r_C)$ = 0.03 + 0.056y = 7.08%

The expected risk of the complete portfolio (in term of σ) = $\sigma_C = y\sigma_p$ = 0.1167

2012 Question 1

Examiner's Comments:

This part was generally responded to well, with a many candidates receiving full credit.

The most common mistake was ignoring the fact that the item instructed candidates to give the overall expected risk and return results of the "optimal investment plan", which the item specified included some of each asset (A, B, C, & D). Many candidates omitted these calculations for $E\left(r_C\right)$ and σ_C.

Some other responses that lost points related to this mistake include using the expected return on the Optimal Risky Portfolio of $E\left(r_P\right)$=8.6% as the expected return on the Optimal Complete Portfolio, or stating that there wasn't enough information to calculate the standard deviation of the Optimal Complete Portfolio.

The next most common mistake was in calculating the weights given to assets B and C in the Optimal Risky Portfolio. When setting up the equation to solve for the weight in asset B, candidates had to restate the weight given to asset C in terms of the weight given to asset B as in the following equation:

$0.086 = 0.40 * 0.12 + w_B * 0.10 + (1 – 0.40 – w_B) * 0.06$.

However, many candidates set up the equation as follows:

$0.086 = 0.40 * 0.12 + w_B *0.10 + (1 – w_B) *0.06$

Another common mistake was made in calculating y*, the percentage of the Optimal Complete Portfolio that is invested in the Optimal Risky Portfolio. For example, some candidates wrote the y* formula with an extra factor of 2 in the denominator, whether they differentiated to find the formula themselves or simply stated the formula from memory. This is most likely a result of erroneously eliminating the ½ factor during differentiation of $U = E\left(r_C\right) - 1/2A\sigma_C^2$. Other candidates wrote the correct formula but simply made calculation errors in solving for y*

Other common mistakes included candidates failing to realize they had all the information necessary to fully respond to the item. Many candidates stated they needed to know the standard deviations of the individual assets and made assumptions, while others erroneously calculated the standard deviations of the individual assets assuming each one lies on the CAL.

Solutions to questions from the 2012 exam continued

1b. (0.5 point) Explain why the proportion invested in Asset D may be the only difference between plans for investors with the same assets.

Question 1 – Model Solution 1 – part b

Optimal risky portfolio with asset A, B and C are mean variance efficient, so it is optimal. Other combination of A, B and C would have lower expected return or higher risk which are not efficient and thus not the choice for rational investors. When combining optimal risky portfolio with risk free asset D, it would depend on each investor's risk aversion. Risk taking investor would invest more in optimal risky portfolio and less in asset D.

Question 1 – Model Solution 2 – part b

Portfolio selection can be separated into two broad categories (i.e. "separation property")

1. Selection of the optimal risky portfolio. This is the same for all investors

2. Allocation of funds between the optimal risky portfolio & the risk-free asset. This will depend on the investor's risk preferences

=> Asset D is basically the risk-free asset & therefore the proportion invested in it will vary based on risk preferences. The proportions in the other assets (forming the optimal risky portfolio) will be the same for all investors

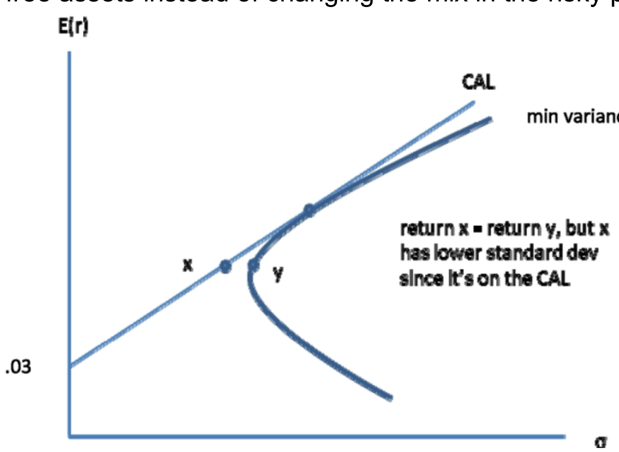
Question 1 – Model Solution 3 – part b

(1) The first step for determining complete portfolio for an investor is the selection of optimal risky portfolio. This will be same for all investors.

(2) The separation among investor choices for their complete portfolio is solely a function of their personal preference for risk.

Question 1 – Model Solution 4 – part b

The best risk return portfolio is determined by where the CAL is tangent to the minimum variance frontier. This is the max slope available. In order to keep this same slope, we must move along the CAL using risk free assets instead of changing the mix in the risky portfolio.



Question 1 – Model Solution 5 – part b

The weights of A, B & C in risky portfolio represent the combination of these three assets that will maximize the sharpe ratio (slope of the CAL). So all investors (with these 3 assets) should hold a risky portfolio with these weights. What will vary is the amount assigned to the risky versus risk free to form the complete portfolio. This will be based on the individuals risk aversion.

Solutions to questions from the 2012 exam continued

Question 1 – Model Solution 6 – part b

In efficient markets, all investors should hold the same risky portfolio. The only reason for any change in the risky portfolio is due to taxes or some other special risk factor such as age or profile of the person. Since all people should hold the same risky portfolio, the only difference in the overall portfolio is the proportion held between the risky portfolio & the risk free asset. That varies due to risk averse-ness. So the more risk averse a person is, the more they will hold in D.

Question 1 – Model Solution 7 – part b

The portion invested in Asset D will be the only difference between plans for investor with the same assets because of the separation property. The determination of the optimal complete portfolio is comprised of 2 independent steps

1. Determine the optimal risky portfolio. This is a technical exercise and will be the same for each investor.
2. Determine proportion invested in the optimal risky portfolio based on the individuals risk aversion level.

This explains why the amount invested in Asset D will be vary based on the individuals risk aversion. The optimal risky portfolio is the same for everyone.

Examiner's Comments:

This part was generally responded to well, with most candidates receiving full credit. Any reasonable answer that fully responded to the item received credit. The most common reason that candidates lost points was due to a response that lacked sufficient detail for full credit. Given the keyword “Explain” we were looking for a more complete explanation of the Separation Property (using the key term Separation Property was not necessary for credit). Many candidates who lost points explained why the Optimal Risky Portfolio would be the same for everyone, or they explained how the amount invested in asset D would vary based on the individual’s risk aversion, but they didn’t put both concepts together to form a full explanation. The key here was completeness of the explanation, not necessarily length.

