A&J

Study Manual for

SOA Exam FM/ CAS Exam 2

Electronic Product/ No Returns

Preface

Understanding that the purpose of purchasing this manual is to pass the exam, I am trying my best to help you. If this is the first time you sit for actuarial exam, let me remind you this: It is not easy! Yet, it is not impossible. Passing or not does not hinge on anything else, but the adequacy of your preparation.

This manual is written in a way that appeals the intuition of the reader in various topics with the attempt to keep the reader interested in the topics. I do this by giving illustrations and examples when explaining new concepts to the readers. Besides, I experienced and do understand the pain of the students who face difficulties and find it time-consuming when they encounter some jargons in the process of studying. Hence, I try to avoid using those words, and introduce them to the readers only when it is necessary.

I hope that while reading the notes you can have pencil and paper with you. I highly suggest you to try the examples before you look at the solutions. This is to keep your understanding of the concept on track, and to give you a deep impression of what the traps related to the topic you are studying might be. Some examples are not exam-liked, yet its purpose is to help you to get a deeper grasp on the concepts.

Besides the study manual, I have also prepared you calculator notes separately. The calculator notes is intended to demonstrate to you how you can make use of BA II Plus calculator to answer in the exam. For more information on other functions of the calculator, you might need to read the manual that comes along with the purchase of it.

For part II of this study manual pertaining Derivatives Market, I believe that the textbook by Robert L McDonald has done a great job in illustrating the concepts. I high recommend that you refer to it in your studies. Attributable to this, my intention of part II is not to expound on the topics, but to summarize and point out the key concepts of the material. I will try to explain the concepts using illustration to provide the reader more perspective of viewing on derivatives.

Personally, when I sat for this exam, I had zero knowledge in finance, and I didn't attend any single class prior to it. I can still remember that one month before the exam FM, I was sitting for my university's Introductory Calculus final exam. Just in a month time, I managed to prepare myself for the exam, and I passed it in the first trial. I was eighteen by that time. Hence I would like to assure you the following:

1. You are smart enough for this exam;
2. You are not too young for this exam;
3. Zero knowledge in finance should not be a discouraging factor to take this exam;
4. It is totally OKAY if you never attend any relevant class before the exam

All you need is just hard work and proper guideline.

Should you have any questions, feel free to raise them to me at anjstudymanual@gmail.com. I will reply you in 1-3 business days. You can also add me in your contact list of messenger for further clarification of your problems.

All the best for the exam, and enjoy the reading.

Alvin Soh

Sample Reading

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Part A - Theory of Interest

Chapter 1
Measurement of Interest

This chapter discusses about:

1. The concept of accumulation function and effective rate of interest
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5. The relationship between nominal rate of interest and effective interest rate
6. Discount rate
7. Force of interest

1.1 Accumulation Function and Effective Rate of Interest

Let me begin by asking you one question: Assume that everything is constant, do you want me to give you $1000 now, or $1000 after 10 years?

Of course you will choose to receive it now. Why? This is because you can make use of the money now, to buy your favorite study manual for exam FM, and to register for the exam, and then you might use the remaining for exam M. Compare this to 10 years later. If you only have $1000 ten years from now, you do not have money to spend for the manuals, exams etc. You will have to wait for 10 years with no extra $1000.

This shows that my offer was definitely not fair. Time is gold; it is the coin of our life. To make it fair, I should offer an additional amount of money after 10 years to compensate for the time that you have to wait. Perhaps, I should offer $1000 now and $1250 after 10 years. This concept is called time value of money.

The additional amount of $250 in this case is called interest.

Now, let say you have just got your salary of $5000, (You may change the figure if you think that it is too low), and after cutting off the income tax, daily expenses, savings, etc, you are left with $1000. You want to invest this $1000 somewhere, so that after t years this amount of money grows.

At this moment, assume that your best friend, Alvin is in an urgent need of money, and he wants to borrow $1000 from you. He promises that he will return $1045 to you after 1 year. While you are negotiating the amount to be returned to you, XXX, a guy who used to bully you during your childhood comes to you to borrow money as well. He needs $1000, and he promises to pay you back $1100 after one year. To whom will you lend your $1000?

Caring ones might lend the money to Alvin. Let's see which decision gives more profit (interest):

<u>Lending $1000 to Alvin</u>
The profit (interest) that you earn is 1045-1000= $45. In percentage, the interest that you earn in one year time is:

$$i = \frac{1045 - 1000}{1000} = 0.045$$

Lending $1000 to XXX

The interest that you earn is 1100-1000= $100. The effective interest that you earn in one year will be:

$$i = \frac{1100 - 1000}{1000} = 0.1$$

This shows that lending money to XXX earns more interest. In terms of accumulation, lending money to Alvin will earn 104.5% of the initial amount whereas XXX will give 110%. In mathematical expression, Alvin and XXX give the accumulation factors of 1.045 and 1.1 respectively.

Now, let us generalize the idea. Let us define A(t) to be the amount function. A(t) is the amount accumulated after t years. In the above illustration, A(0) is simply 1000, the money to be invested, the principal. For Alvin's offer, the A(1) is 1045, and for XXX's, 1100.

In mathematical symbols, it is intuitive that the effective interest is:

$$A(t-1)*(1+i) = A(t)$$

$$\Rightarrow 1+i = \frac{A(t)}{A(t-1)}$$

$$\Rightarrow i = \frac{A(t)}{A(t-1)} - 1 = \frac{A(t) - A(t-1)}{A(t-1)}$$

Now, in order to make the calculation easier, we do not want to include the initial amount in the amount function. Let us define a(t), accumulation function, such that:

$A(t) = ka(t)$, where k is the initial amount invested, which is also called principal

Hence, applying this formula to the above illustration, we have k=$1000, and a(t) offered by Alvin and XXX are a(t)= 1.045 and 1.1 respectively. In the coming section, we shall go into the details for some commonly used accumulation functions.

Obviously, if given a(t), we can also obtain the interest rate by the same reasoning:

$$a(t-1)*(1+i) = a(t)$$

$$\Rightarrow 1+i = \frac{a(t)}{a(t-1)}$$

$$\Rightarrow i = \frac{a(t)}{a(t-1)} - 1 = \frac{a(t) - a(t-1)}{a(t-1)}$$

Well, you might wonder, what is an effective interest rate? Why is it so important?

An effective interest rate is the proportion that you earn when you invest your money into a fund for ONE period of time. The period can be a day, a month, a year, 4 years or anything else. For example, "The bank offers an effective rate of interest of 50% over 5 years", it means, if you invest $1 into the bank, after 5 years, you will get $1*1.5= $1.50.

However, an effective interest rate is usually measured monthly or annually. For example, "The bank offers effective interest rate of 5% annually", this means investing $1 in the bank you will get accumulated value of $1.05 at the end of one year, and $1*1.05^6= $1.34 after 6 years. In other words, for this bank, it offers an annual effective interest rate of 5%, or equivalently, an effective interest rate of 34% for 6 years. Don't worry if you still can't get it. We shall go into the details of it later on.

Example 1.1

You have \$15000 cash and you would like to invest this amount of money for 8 years to earn interest. Bank A offers the accumulation function of $a(t)=1 + 0.07t$ for the first 5 years. After 5 years, the bank will return the interest accumulated and change the accumulation function to $a(t)= 1+0.005t^2$ for the years after year 5. Assume that you keep the interest after 5 years under your pillow.

At the same time, Bank B offers $a(t)=1.07^t$ for all t.

Decide which bank to invest in for the maximum profit.

Solution 1.1

To determine which bank offers the better return, we simply find the final accumulated amount, and the larger of the two is the better choice.

For Bank A after 5 years,

$$A(5) = 15000a(5) = 15000[1+0.07(5)] = 20250$$

Since bank returns the interest earned, you get back \$5250 and you earn no further income or interest for it. (Why?)

After 8 years,

$$A(8) = 15000a(3) = 15000[1+0.005(3)^2] = 15675$$

Hence, the total interest earned is 5250+ 675 = \$5935

For Bank B at the end of 8 years,

$$A(8) = 15000a(8) = 15000[1.07^8] = 25772.79$$

The total interested earned is 25772.79- 15000= 10772.79
No doubt, you should go for **Bank B**.

 #1

Example 1.2

Jessica invests \$3400 into Fund A whose accumulation function is $a(t)= 1.05^t$. At the same time, Ron would like to invest \$4200 into Fund B, which allows the money to be accumulated at $a(t)= 1.04^t$.

(a) Find t at which the accumulated amount of Jessica and Ron is the same.
(b) Manager of Fund B would like to exploit arbitrage (take advantage of different interest rate and make free lunch) by investing Ron's \$4200 into Fund A. Find the profit made by the manager after 10 years, assuming that Ron doesn't withdraw the money throughout the years.

Solution 1.2

(a) We want to find t at which $A_{Jessica}(t)= A_{Ron}(t)$. Hence,

$$3400(1.05^t) = 4200(1.04^t)$$
$$\Rightarrow \frac{3400}{4200} = \left(\frac{1.04}{1.05}\right)^t$$
$$\Rightarrow 0.8095 = 0.99047^t$$
$$\Rightarrow t = 22.07$$

At the end of **year 22**, the accumulated value of Jessica and Ron will be the same.

(b) First, we need to determine the revenue of the manager. Subtracting the liability then will give us the profit.

Revenue:
$$4200(1.05)^{10} = 6841.36$$

Liability
$$4200(1.04)^{10} = 6217.03$$

Hence, the profit is 6841.36- 6217.03= **$624.33**

Example 1.3

Fund ABC offers an effective interest of 5.5% reinvested each year from 1 Jan 2000 until 31 Dec 2007. Fund XYZ offers product XYZ whose accumulation function is a(t)= 1+0.06t from 1 Jan 2000 until 31 Dec 2004. From 1 Jan 2005 onwards, the product returns the interest earned and reinvest the principal amount at $a(t)= 1.035^t$. If you are the manager of Fund XYZ, how much the free lunch will you get if Mr. Ignorant purchases your product at $1680 and will not withdraw the money until year 2009?

Solution 1.3

Notice that Fund ABC offers an effective annual interest rate of 5.5% for 8 years. This is actually a compound interest (compound, as the question says "reinvested each year"), which will be covered soon and will be greatly tested in exam FM. This means that the interest earned in the first year will not be returned, but is reinvested into the fund for another year, and so on.

For Fund ABC, the accumulated amount after 8 years is:
$$A(8) = 1680a(8) = 1680(1.055)^8 = 2578.27$$

The interest earned is 2578.27- 1680= 898.27

For Fund XYZ, after 5 years the product gives:
$$A(5) = 1680a(5) = 1680[1+0.06(5)] = 2184$$

Since the product gives back the interest earned (2184-1680= $504) in the first 5 years, the principal goes back to $1680 and is invested at $a(t)= 1.035^t$.
$$A(3) = 1680a(3) = 1680[1.035]^3 = 1862.65$$

The interest earned from 1 Jan 2005 until 31 Dec 2007 is 1862.65-1680= $182.65

After solving these, you can get the free lunch by investing the $1680 into Fund ABC, and gain an arbitrage of 898.27- 504- 182.65= **$211.62**

1.2 Simple Interest

Don't be deceived by its name. It is not simple at all.

The accumulation function of simple interest is:

$$a(t) = 1 + it$$

If the interest offered is 6%, and the initial value invested is \$2400, after 7 years, you have an accumulated value of:

$$A(7) = 2400a(7) = 2400[1 + 0.06(7)] = 3408$$

Usually the t of the accumulation function refers to the number of years. However, mathematics is something living, and it may refer to the number of months, (or even hours?!) according to the context.

From calculus course we know that a(t)=1+it is a linear function, where i is a constant. Nevertheless, despite the fact that the simple interest rate offered is a constant, the effective rate of interest is NOT constant. Let us look at it mathematically:

To find the effective interest rate, i_t from period t-1 to t,

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} = \frac{(1+it) - (1+i(t-1))}{1+i(t-1)} = \frac{i}{1+i(t-1)}$$

This shows that the effective rate of interest declines as t increases. Under the simple interest rate, the interest is given based on the initial value. It doesn't take into consideration of the reinvestment of the interest earned. Thus, under this method of accumulating fund, it is "not fair" to the investors. Let us have a look at the following illustration:

Mr. Sam, FSA, wants to run a business of selling actuarial manuals. He needs \$15000 to run the business and he comes to you for money. He offers a simple interest rate of 4.5% each year, and at the end of 8 years, he promises to give you back the principal (initial value invested) plus interest. What is the accumulated value?

It is quite straightforward. The accumulated value is

$$A(8) = 15000[1 + 0.045(8)] = 20400$$

However, at a closer examination, you find that he is borrowing some part of your money "for free", that is, the part of money which he doesn't pay you the interest. Notice that, after one year, the accumulated value is:

$$A(1) = 15000[1 + 0.045(1)] = 15675$$

This means that, after one year, he should have accredited you an interest of 15675-15000= \$675, but he keeps this amount for another 7 years. At the end of year 2,

$$A(2) = 15000[1 + 0.045(2)] = 16350$$

Meaning, he should have granted you an interest of 16350-15000= \$1350 at the end of 2 years, and so forth. Watch out! This is the place where it gets "unfair". At the beginning of year 2, he no longer owes you \$15000, but \$15675. Hence, at the end of year 2, should he not give the accumulated value of:

$$A(2) = 15675[1 + 0.045(1)] = 16380.38\,?$$

Where does the extra $30.38 come from? It is from the interest earned in the first year, $675. It is just $675*(0.045)= $30.38.

Under the simple interest, Mr. Sam is making use of an additional $675 of yours every year without paying you interest for that. Because of the compounding effect of the interest not paid, the effective rate of interest for each year drops.

In real world situation, a car loan amortization is calculated under the simple interest. We will cover that in Chapter 4.

Example 1.4

A bank offered a loan to Uncle John at a simple interest rate of 5%. At the same time, Uncle John found that a fund in the market was offering a saving interest under a simple interest rate of 7.5%. He thought that it was a good time to get a free lunch, and he decided to borrow $30000 from the bank and invested it into the fund, together with his own additional $10000.

(a) Find the total amount of money that Uncle John earns at the end of year 10.
(b) Find the effective rate of interest of the fund for year 5.
(c) Find the effective rate of interest (return) of the portfolio throughout the 10 years.

Solution 1.4

(a) The accumulated value of loan at the end of year 10:

$$A(10) = 30000[1+0.05(10)] = 45000$$

The accumulated value of the fund at the end of year 10:

$$A(10) = 40000[1+0.075(10)] = 70000$$

Hence, the profit is $70000-$45000= **$25000**

(b)

$$A(4) = 40000[1+0.075(4)] = 52000$$

$$A(5) = 40000[1+0.075(5)] = 55000$$

$$i^* = \frac{A(5)-A(4)}{A(4)} = \frac{55000-52000}{52000} = 0.05769$$

The answer is **5.769%**.

(c) Since the initial value of the portfolio is only $10000, and the profit is $25000, the return is **250%**.

1.3 Compound Interest

As we have seen in the previous section, a simple interest is “unfair”. In this section we shall see another accumulation function named compound interest.

Compound carries the meaning of reinvestment. For example, assume that you own a hotel worth 50 million. After 10 years, this hotel gives you a return of 50 million. If you wish to earn more in the future, you might consider investing this 50 million for another hotel. Doing so, you have 2 hotels contributing 50 million each to you at the end of 20 years. After 20 years, you are still greedy and feel that 100 million is not enough for you. So, you invest this 100 million again for another 2 hotels, so you have 4 hotels. Effectively, at the end of 30 years, you have 400 million assets: 200 million from the worth of the 4 hotels (assume no price appreciation or depreciation) and 200 million generated from the 4 hotels from year 20 to 30.

Compare this to not investing the return of hotel. You will only have assets of \$200 million, \$50 million is from the hotel, and the profit generated of \$150 million for 30 years.

This is the power of compounding, as Albert Einstein said: “The compound interest is the greatest invention of mankind”.

Now, let us look at its mathematics. Let say Bank ABC offers annual effective interest rate of i_t for year t. Calculate the accumulated value (AV) after n years, if k is invested today.

At t=1,

$$AV_1 = k(1+i_1)$$

Remember that the value is reinvested every year. Hence, at the beginning of year 2, the value is already AV_1. Reinvesting this, and at the end of year 2 gives:

$$AV_2 = AV_1(1+i_2) = k(1+i_1)(1+i_2)$$

Compounding in this way, we have:

$$AV_n = k(1+i_1)(1+i_2)...(1+i_n) = k\prod_{t=1}^{n}(1+i_t)$$

For comparison, you might try to prove why the following is true:

Simple Interest	Compound Interest
$a(m+n) = a(m)+a(n)-1$	$a(m+n) = a(m)a(n)$
$a(m-n) = a(m)-a(n)+1$	$a(m-n) = \frac{a(m)}{a(n)}$

In the exam, the question assumes the interest rate is a compound interest rate if it does not specifically mention under which method the fund is accumulated.

Example 1.5

You have just won a lottery of \$100,000 and you would like to invest it for 10 years. You found three funds that offer the following:

- Bank A offers an annual effective interest rate of 5.5 for the first 3 years, 4.7% for the next 3 years, and 7.5% for the last 4 years
- Bank B offers an annual effective interest rate of 6.75% for 10 years

- Bank C offers a simple interest rate of 8% for 10 years
- Bank D, whose manager, Alvin offers 4% for the first 2 years, i for the next 6 years, and i^2+2i for the last 2 years.

(a) Calculate the accumulated value of fund A, B and C if you were to invest your money into them.
(b) Alvin wants to convince you to invest into fund D. He offers the interest rate i, such that you as a rational investor will invest into his fund. Find the minimum i that Alvin possibly offers.

Solution 1.5

(a)

$$AV_A = 100,000(1.055)^3(1.047)^3(1.075)^4 = 179,982.91$$

$$AV_B = 100,000(1.0675)^{10} = 192,167.01$$

$$AV_C = 100,000[1+0.08(10)] = 180,000$$

(b) To convince you, Alvin must offer the total interest that is at least the same as fund B. To find the value of i,

$$AV_B = AV_D$$

$$\Rightarrow 100,000(1.0675)^{10} = 100,000(1.04)^2(1+i)^6(1+i^2+2i)^2$$

$$\Rightarrow 1.0675^{10} = 1.04^2(1+i)^6[(1+i)^2]^2$$

$$\Rightarrow 1.0675^{10} = 1.04^2(1+i)^{10}$$

$$\Rightarrow i = \sqrt[10]{\frac{1.0675^{10}}{1.04^2}} - 1 = 0.05916$$

Hence, Alvin has to offer i which is more than **5.916%**

1.4 Present Value

Imagine that you have a time machine by which you can travel only to the future. You bring along a little girl, whose height is just 1.2m. You bring her together into the time machine, and bring her to the time after 10 years. You realize that she has grown to 1.6m. You find that this is interesting, and you bring her to the time after 5 years from the present. You find that her height is 1.4m. After that, she says that she is hungry, and you bring her back to the present for food.

Now, you think that it is reasonable to deduce that her height follows the function a(t)= 1.2+0.4t. (Imagine what happens if you bring her to time after 60 years?) With this function, you can know the height of the girl at each point of time.

A similar concept can be applied in financial world. In the previous sections, we only mentioned about accumulated value, e.g. what is the future (accumulated) value of $1000 after 5 years if the annual effective interest is 6%? In this case, the accumulation function is $a(t)= 1.06^t$. In other words, $1000 will become $1338.23 at t=5. (How do you get it?)

Now, assume that there is only one bank in the whole universe, and it offers 6% effective annual interest. You go to the time machine, and you go to year 40 from now. You walk around, and you realize that there is a note of $1000. You happily put that money in your pocket and you travel back to the present. How much is in your pocket now?

The concept of calculating the amount is not hard. At year 40, the amount is $1000. Assuming that k is invested in the present in the universal bank for 40 years such that in year 40 it is $1000, we know the equation holds:

$$1000 = k(1.06)^{40}$$

$$\Rightarrow k = \frac{1000}{1.06^{40}} = 97.22$$

The value of k is called present value (PV). Remember that, previously we asked: How much will it be if I invest $k now after n years under an annual effective interest rate of i. Now, in this section, it is the other way round: How much do we need now, so that there is $k after n years under an annual effective interest rate of i? Previously, we wanted to know how much to accumulate; now, we want to know, how much do we need to discount.

Let us look at an illustration. You just have a new baby, and you would like to save a fixed amount of money in the bank, so that after 20 years, you will have enough money for the baby boy's tertiary education. You survey the prices of colleges, and considering inflation and everything, you conclude that you will need $250,000 for him. Assume i= 6.5%, how much do you need to put in your saving account today, so that you will have enough money to meet the need?

$$PV = \frac{AV}{(1+i)^n}$$

$$\Rightarrow PV = \frac{250,000}{1.065^{20}} = 70949.26$$


#2

This shows that you need to save $70949.26 in the bank for your boy's need after 20 years.

The key equation of this section is

$$AV = PVa(t) \Rightarrow PV = \frac{AV}{a(t)}$$

Let us define another commonly used symbol, v. It is defined to be $\frac{1}{1+i}$. v is a discount factor, multiplying it to any amount will discount it back to present value. For instance, what is the present value of 500 after 1 year, 430 after 2 years, 780 after 3 years, if the annual effective interest rate is 3%?

$$v = \frac{1}{1.03} = 0.97087$$

$$PV = 500v + 430v^2 + 780v^3 = 1604.56$$


#3

Try to understand the concept of present value by the following examples.

Example 1.6

You are in need of money, and you go to Mr. Smith for that. He lends you $994.32 today, and you have to repay him $1000 after 2 years. Calculate the annual effective rate of interest.

Solution 1.6

Viewing the situation 2 years from now,

$$1000 = 994.32(1+i)^2 \Rightarrow i = 0.28521\%$$

Viewing the situation at the present by discounting $1000,

$$994.32 = \frac{1000}{(1+i)^2} \Rightarrow i = 0.28521\%$$

You might find that viewing the situation at the present by discounting the FV is not familiar or does not make any sense to you. Try to rationalize it (since, indeed it is rational and is derived from the first view) because you need to develop such capability for later chapters.

Example 1.7

Fund A is invested at an effective rate of interest of 4.5%.
Fund B is invested at an effective rate of interest of 6.7%.
Fund C is invested at an effective rate of interest of i%.

(a) At the end of 7 years, Fund A and B are the same which is $5500. Find the difference of Fund A and B at the end of 13 years
(b) At t=0, Fund A is $1500. After 6 years, the AV of fund A is withdrawn by the investor and invested in Bank B at effective rate of 6.7% for another 6 years. He has a creditor, to whom the investor has to pay $3800 at the end of 12 years. Calculate the additional amount he has to invest after 6 years into fund B.
(c) At the end of 3 years, Fund B is twice the amount in Fund A. Calculate the initial value of Fund A if the present value of Fund B is $4000.

(d) Calculate the interest rate, i offered by Bank C so that the present value of Fund C equals the total of the present values of Fund A and B, if the value of Fund A and B are $4300 and $5700 respectively at t=12. Given that Fund C at t=13 is $11500.

Solution 1.7

This example is written more complicatedly so that the reader might have a stronger grasp on the material. Don't give up if you can't understand them at the first look!

(a) Since Fund A and B are the same at the end of year 7, the present value of the funds is not important. So, to calculate the value at the end of year 13, we just accumulate them respectively:

$$AV_{13}^{A} = 5500(1.045)^6 = 7162.43$$

$$AV_{13}^{B} = 5500(1.067)^6 = 8116.13$$

Hence, the difference is 8116.13- 7162.43= **$953.70**

(b) Assume that the investor travels to t=6. Fund A will grow to:

$$AV_6 = 1500(1.045)^6 = 1953.39$$

Looking to the future, the present value of the debt is:

$$PV_6 = \frac{3800}{(1.067)^6} = 2575.11$$

The reason 6.7% is used instead of 4.5% is because the investor has already decided to move the money into bank B. Hence, $3800 should be discounted at 6.7%.

Since the value of debt at t=6 is $2575.11 and the accumulated value from Fund A is $1953.39, additional 2575.11- 1953.39= **$621.73** is needed.

You might solve it in another way. You accumulate the $1953.39 at 6.7% to the end of year 12 to see the how much is lacked:

$$AV_{12} = 1953.39(1.067)^6 = 2882.54$$

$$3800 - 2882.54 = 917.46$$

Discounting the amount lacked to t=6 at 6.7% gives:

$PV_6 = \frac{917.46}{1.067^6} = 621.73$, which gives the same answer.

(c) First, we find the accumulated value of Fund B at t=3:

$$AV_3 = 4000(1.067)^3 = 4859.07$$

Since Fund B is the twice of Fund A at t=3, AV_3 of Fund A is 4859.07/2= $2429.54. Hence, the initial value can be obtained by discounting it at 4.5%:

$$PV = \frac{2429.54}{1.045^3} = 2128.99$$

The present value is **$2128.99**

(d) We need to discount the value of Fund A and Fund B:

$$PV_A = \frac{4300}{1.045^{12}} = 2535.55$$

$$PV_B = \frac{5700}{1.067^{12}} = 2617.59$$

PV_C is then 2535.55+ 2617.59= \$5153.15. From the relationship of PV and AV,

$$PVa(t) = AV \Rightarrow a(t) = \frac{AV}{PV} \Rightarrow (1+i)^{13} = \frac{11500}{5153.15} \Rightarrow i = 6.36955\%$$

Hence, the annual effective interest rate at which Fund C is invested is **6.36955%**.

Example 1.8

Mr. Rich lends Mr. Poor \$1000 with an effective annual interest of 10%. If Mr. Poor returns the interest at the end of year 5, and clears all the debts at the end of year 10,
(a) How much is the interest that Mr. Poor pays to Mr. Rich?
(b) How much does Mr. Poor pay to Mr. Rich at the end of year 10?
(c) What is the present value of the interest paid to Mr. Rich at the end of year 5?
(d) What is the present value of the amount that Mr. Poor pays to Mr. Rich at the end of year 10?
(e) Sum your answer in c and d.

Solution 1.8

(a) $A(5) = 1000(1.1)^5 = 1610.51$

The interest payment at the end of year 5 is 1610.51- 1000= **\$610.51**

(b) $A(10) = 1000(1.1)^5 = 1610.51$ (1000 because the interest has already been cleared)

\$1610.51 is paid at the end of year 10.

(c) $PV_{Interest} = \frac{610.51}{1.1^5} = 379.08$

The present value of interest paid is **\$379.08**.

(d) $PV_{Debt} = \frac{1610.51}{1.1^{10}} = 620.92$

The present value of the debt plus interest at the end of year 10 is **\$620.92**.

(e) They add up to be **\$1000**, which is the initial value that Mr. Rich lends to Mr. Poor. Do you see something about it?

1.5 Nominal Rate of Interest

Everyone has a name; more than one name sometimes. Let us look at some "different names" for effective rate of interest. In this chapter, remember that "compounded", "payable" and "convertible" carry the same meaning. Let's see what they mean.

A bank offers an effective interest rate of 3% every 6 months. If we open an account and save \$5000, at the end of 6 years we would have $\$5000(1.03)^{(6)(2)}= \$7128.80$. We are clear about this saying now.

To describe the same offer, the bank might state it in a different way: "A nominal interest rate of 6% convertible/ compounded/ payable semiannually". Where does the 6% come from? It is the double of 3%, since it is the semiannual granted interest. The 6% in this case is called nominal rate of interest or more specifically, nominal rate of interest payable semiannually. This is the "different name" of effective interest rate.

The word "nominal" means "existing in name only". Therefore, an annual nominal rate of 6% does not give interest of 6% per year. Let us look at the following illustration.

Bank A offers an effective rate of interest 5%
Bank B offers nominal rate of interest 5% payable semiannually.

If you deposit \$1000 in Bank A, it will grow to \$1050 next year, whereas if you deposit \$1000 into Bank B, you are actually offered an effective rate of interest 2.5% per 6 months. Meaning to say, you are earning an effective interest rate of 2.5% compounded twice (6 months X 2= 1 year). Hence, at the end of the year, your fund of \$1000 will grow to $\$1000(1.025)^2= \$1050.63$.

In general, the following equation gives the relationship between nominal interest rate and effective interest rate:

$$(1+i)^n = \left(1+\frac{i^{(m)}}{m}\right)^{mn} \Leftrightarrow i^{(m)} = m\left[(1+i)^{\frac{1}{m}}-1\right]$$

where i is the annual effective rate of interest, $i^{(m)}$ is the nominal interest rate payable, n is the number of years, (m) is the number of times that the number of time that the nominal interest rate is compounded per year. Be cautious! The (m) on the top right of $i^{(m)}$ doesn't mean an index!

Remember, in dealing with nominal interest rate, we always want to find the effective rate of interest at each m^{th} period first, and then only we evaluate the accumulation factor for n years.

Take note! $(1+0.12) \neq \left(1+\frac{0.12}{12}\right)^{12}$! You may try it and see. In fact, an effective rate of interest of 12% is less than a nominal rate of interest of 12% convertible monthly.

With this knowledge, try to calculate the accumulated value of the followings:
(a) 5000 invested at a nominal interest rate of 24% compounded monthly for 3 years;
(b) 6500 invested at a nominal interest rate of 12% payable quarter annually for 7 years;
(c) 3700 invested at a nominal interest rate of 10% convertible semiannually for 2 years;
(d) 10000 invested at a nominal rate of interest 5% compounded once every 2 years (?!) for 8 years;
(e) 5000 invested at nominal rate of interest 14% compounded monthly for 4 months.

Answers:

(a) $5000\left(1+\frac{0.24}{12}\right)^{(12)(3)}=10199.44$

(b) $6500\left(1+\frac{0.12}{4}\right)^{(4)(7)}=148715.53$

(c) $3700\left(1+\frac{0.1}{2}\right)^{(2)(2)}=4497.37$

(d) $10000\left[1+(0.05)(2)\right]^{(8)(\frac{1}{2})}=14641$

(e) $5000\left(1+\frac{0.14}{12}\right)^{4}=5237.45$

#4

Well, perhaps (d) might sound a bit weird. The nominal interest rate is compounded once every 2 years. First, we convert the nominal interest rate into effective interest rate for 2 years by dividing the 0.05 by 0.5. Then, we find the annual effective interest rate by equating:

$$(1+i)^2=\left(1+\frac{0.05}{1/2}\right)\Rightarrow i=0.04881$$

After getting the annual effective rate of interest, we just power it by 8 since \$10000 is compounded annually for 8 years.

Consider the following offers provided by a bank:

Package 1: An effective interest rate of 6% per annum
Package 2: A nominal interest rate of 5.9126% convertible semiannually
Package 3: A nominal interest rate of 5.84106% compounded monthly
Package 4: A nominal interest rate of 5.58274% payable daily

You are a fund manager and you put \$1000 into each "package".
Applying what we have learned before,

$$AV_1=1000(1.06)=1060$$

$$AV_2=1000\left(1+\frac{0.059126}{2}\right)^{2}=1060$$

$$AV_3=1000\left(1+\frac{0.0584106}{12}\right)^{12}=1060$$

$$AV_4=1000\left(1+\frac{0.0558274}{365}\right)^{365}=1060$$

Do you see why they are the same? Can you calculate the nominal interest rates if you are only given the 6%? It is calculated using the following formula, changing the m as you wish:

$$i^{(m)}=m\left[(1+i)^{\frac{1}{m}}-1\right]$$

Give yourself a try!

Example 1.9

Jack deposits his annual bonus of $400,000 as the CFO of Lucky Company into Bank A which offers 7.5% nominal rate of interest compounded quarterly. After 6 years, Bank B offers 7.2% nominal rate of interest convertible monthly. Jack has much knowledge in financial mathematics and he chooses bank that offers a higher effective interest rate. As such, he deposits his bonus for 15 years in total.

Predict which bank will Jack choose, and the accumulated value after 15 years.

Solution 1.9

First of all, we need to compare which bank gives higher effective interest rate:

Bank A:

$$1+i=\left(1+\frac{0.075}{4}\right)^{4}=1.077136$$

Bank B:

$$1+i=\left(1+\frac{0.073}{12}\right)^{12}=1.075493$$

This means that Bank A gives higher effective rate of interest. Hence, Jack will continue to deposit his bonus:

$$AV=400,000(1.077136)^{15}=1,219,321.15$$

Jack will choose **Bank A** and his accumulated value after 15 years will be **$1,219,321.15**.

Example 1.10

Mary deposits X into Bank A, which offers 7.5% nominal interest rate payable semiannually. Her husband, Julian deposits X-c into Bank B, which gives 8% nominal interest rate convertible quarter annually. At the end of 8 years, their accumulated values are the same. Find X if c is $400.

Solution 1.10

Since the AV for both funds is the same at the end of year 8, we equate them:

$$X\left(1+\frac{0.075}{2}\right)^{(2)(8)}=(X-400)\left(1+\frac{0.08}{4}\right)^{(4)(8)}$$

$$\Rightarrow 1.8022X=1.88454X-753.82$$

$$\Rightarrow X=9154.97$$

X is **$9154.97**.

Example 1.11

Bank A offers 7% of nominal rate of interest convertible monthly for all depositors. The manager of Bank B, in order to attract more depositors, promises to give an additional of 0.6% to the effective rate of interest offered by Bank A. With such a plan in head, he offers a nominal interest rate of k% compounded semiannually. Find k.

Solution 1.11

Since the adjustment of Bank B manager is based on the effective rate of interest by Bank A, we need to calculate the effective rate of interest offered by Bank A:

$$1+i=\left(1+\frac{0.07}{12}\right)^{12}=1.07229\Rightarrow i=0.07229$$

After the additional of 0.6%, the effective interest rate to be offered by Bank B is 0.07229+ 0.006= 0.07829. Recall that:

$$i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$$

So,

$$i^{(2)} = 2[(1+0.07829)^{\frac{1}{2}} - 1] = 0.076815$$

Hence, k% is **7.6815%**.

Chapter 2
Valuation of Annuities

This chapter discusses about:

1. Annuity immediate, annuity due, deferred annuity
2. Future value of annuity
3. Relationships between present value and future value of annuity
4. Perpetuity
5. Annuities payable at a different frequency than interest is convertible
6. Miscellaneous topics and techniques in annuity valuation
7. Arithmetic increasing and decreasing annuity
8. Geometric increasing and decreasing annuity
9. Continuous annuity

This chapter is the heart of the exam pumping blood to the following chapters. If you can do well in this chapter, you are done with 40% of the material, I would say. The chapters after this build on the understanding of this chapter. With a solid grasp of this chapter, you will find it easy later on.

Please don't give up if you find that it is hard! As your understanding of the material gets mature practice after practice, this chapter will be a piece of cake for you then.

2.1 Annuity Immediate, Annuity Due & Deferred Annuity

An annuity is defined as a series of payments. Let say, if you deposit $100 annually into a bank for 10 years, that is a kind of annuity. If an insurance company pays you $3500 per year for 25 years, that series of payments is an annuity. If you save $100, $200, $300, …, $2000 at the end of year 1, 2, …, 20, that series is also an annuity, called an increasing annuity.

What we are trying to do here is to find the present value of these payments. Recall that we defined discount factor, v, in chapter 1, which is the inverse of accumulation function:

$v = \frac{1}{1+i}$, where i is the effective rate of interest

As you are already familiar: If the annual effective interest rate is 5%, and a bank is going to pay you $200 after 5 years, the present value is $200/1.05^5$= $156.71.

Consider a more complicated case: If the bank is going to pay you $200 after 4 years, $850 after 5.5 years, $780 after 6 years, what is the present value? It is nothing more than summing up the present values of payments:

$$PV = \frac{200}{1.05^4} + \frac{850}{1.05^{5.5}} + \frac{780}{1.05^6} = 1396.54$$

Now, we would like to know, if the bank consistently pays you $1000 per year starting from one year from now for 50 years, what is the present value of this annuity? Applying what we have learned, we have:

$$PV = \frac{1000}{1.05} + \frac{1000}{1.05^2} + \frac{1000}{1.05^3} + ... + \frac{1000}{1.05^{50}}$$
$$= 1000\left(\frac{1}{1.05} + \frac{1}{1.05^2} + \frac{1}{1.05^3} + ... + \frac{1}{1.05^{50}}\right)$$
$$= 1000\left(v + v^2 + v^3 + ... + v^{50}\right)$$
$$= 1000\sum_{t=1}^{50} v^t$$

Since v is the ratio of the series, we get

$$\sum_{t=1}^{50} v^t = \frac{v\left(1-v^{50}\right)}{1-v} = \frac{v\left(1-v^{50}\right)}{\left(1-\frac{1}{1+i}\right)} = \frac{\left(\frac{1}{1+i}\right)\left(1-v^{50}\right)}{\left(\frac{i}{1+i}\right)} = \frac{\left(1-v^{50}\right)}{i}$$

Solving this gives us 1000(18.25593)= $18,255.93.

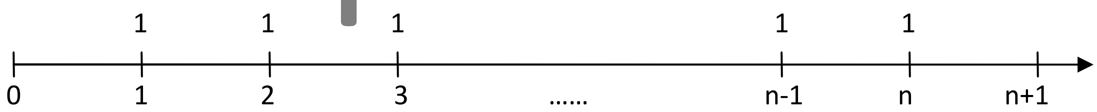
Annuity Immediate

Let us generalize that series of things, and I will introduce you a fancy mathematical symbol:

"If a series of payment is $1 at the end of each year for n years, the present value can be computed as:

$a_{\overline{n}|i} = \frac{\left(1-v^n\right)}{i}$, where the fancy symbol $a_{\overline{n}|i}$ is the present value of the series of payments"

Using time diagram, the payments of the annuity are:



The numbers at the bottom of the horizontal line represent times, whereas the numbers on top of the horizontal line represent payments.

One thing you should always remember. The symbol $a_{\overline{n}|i}$ means the present value of payments of **$1** at the end of each period for n periods. Let say an annuity pays $50 at the end of each period for n periods, the present value is $50a_{\overline{n}|i}$.

This type of annuity is named "annuity immediate" Well, you have to accept the fact that the name does not match its nature. Though it is an annuity immediate, the payment does not start "immediately". It starts after one period, i.e. the end of the first period.

Annuity Due

The annuity that pays "immediately" is called "annuity due". The definition is as followed:

"If a series of payments is $1 at the beginning of each year for n years, the present value can be computed as:

$\ddot{a}_{\overline{n}|i} = \frac{(1-v^n)}{d}$, where the fancy symbol $\ddot{a}_{\overline{n}|i}$ is the present value of the series of payments"
The difference is just that there are double dots on top of the "a".

Using time diagram, the payments of the annuity are:

Similar to annuity immediate: If we were to find the present value of a series of payments of $50 at the beginning of each period for n periods, the present value is $50\ddot{a}_{\overline{n}|i}$

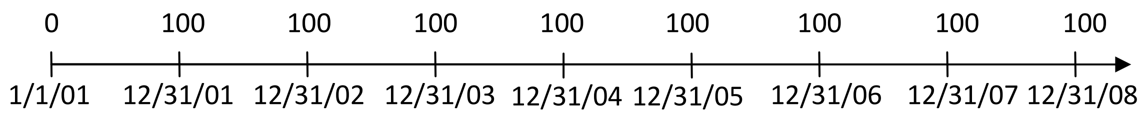
Watch out for the last payment! It is at the end of period n-1, not n! Don't get confused about the little "n" in $\ddot{a}_{\overline{n}|i}$. The little "n" means there are n payments, instead of it is paid until period n.

Every time when we use the phrase- present value, we are always standing at t=0. From there, we look to the right to find the present value.

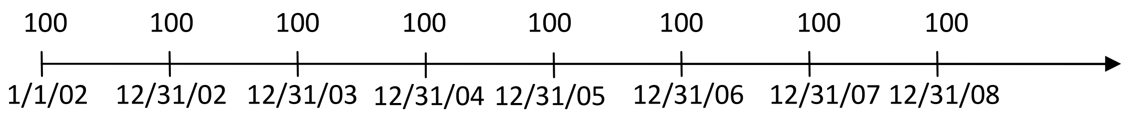
Let us see how to arrive at the formula for annuity due:

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + ... + v^{n-1}$$
$$= \frac{1}{v}\left(v + v^2 + ... + v^{n-1} + v^n\right)$$
$$= \frac{1}{v}\left(\frac{1-v^n}{i}\right)$$
$$= \frac{1-v^n}{\left(\frac{i}{1+i}\right)}$$
$$= \frac{1-v^n}{d}$$

As it is mentioned in chapter 1, you should always be clear on WHEN you are at. Let's say you buy an annuity which pays $100 at the end of each year for 8 years today, on 1 January 2001. You draw the time diagram, and you get this:



The present value of this annuity is $100a_{\overline{8}|i}$. It is an annuity immediate. However, after one year, notice that it has become an annuity due:



Do you see it? The point of time at which you stand to view the annuity is very critical. On 1/1/01 the present value is $100a_{\overline{8}|i}$ whereas on 1/1/02 the present value is $100\ddot{a}_{\overline{n}|i}$.

This leads us to another type of annuity called *deferred annuity*. This annuity is nothing more than an annuity that starts its payment after more than a period of time. For example, an annuity pays 10 payments of $1 at the end of each year, starting at the end of 5 years from now. The present value of the annuity is simply:

$$PV = v^5 a_{\overline{10}|i}$$

The idea is to discount the 10 payments from t=5 to the present.

Another way to solve the present value of deferred annuity is to subtract the present value of $1 at the end of first 5 years from the present value of $1 at the end of first 15 years. We will look into the details in example 2.2.

Example 2.1

Betty purchases an annuity, which pays $40 at the end of each year for 5 years. The nominal rate of interest compounded semiannually is 5.4%.

(a) Find the price of the annuity.
(b) After one year, right before the annuity pays $40, Betty sells the annuity at X. Find X.
(c) After one year, right after the annuity pays $40, Betty sells the annuity at Y. Find Y.

Solution 2.1

SOA likes to trouble the candidates with this type of questions. Although the topic tested is valuation of annuity, they usually won't give you information that can be used directly.

Since we are given the nominal rate of interest payable semiannually instead of annual effective rate, we need to convert it before attempting the questions:

$$i^{(2)} = 0.054$$
$$\Rightarrow 1 + i = 1.054^2$$
$$\Rightarrow i = 11.092\%$$

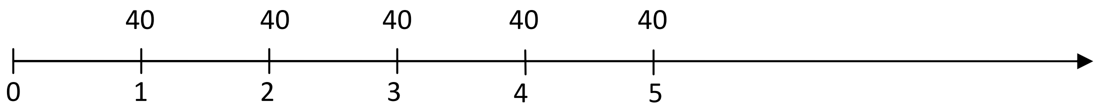
(SOA really likes to play with interest conversion. Make sure you master it.)

(a) This is an annuity immediate, since it pays $40 only after 1 period of time. So,

$$PV = 40a_{\overline{5}|11.092\%} = 40\left(\frac{1-v^5}{0.11092}\right) = 40(3.68736) = 147.49$$

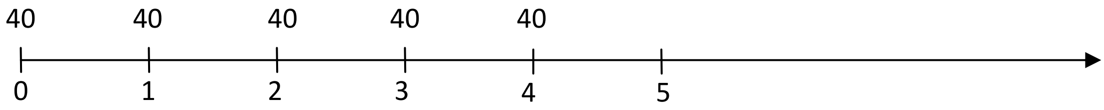
#1
The price of the annuity is **$147.49**.

If you find it hard to determine if it is annuity immediate or annuity due, let the time diagram help you.



The payment is at t=1. Hence it is annuity immediate with 5 payments.

(b) Since the first payment has not yet been made, and there are still 5 payments with the first one coming almost immediately, it is an annuity due with 5 payments.

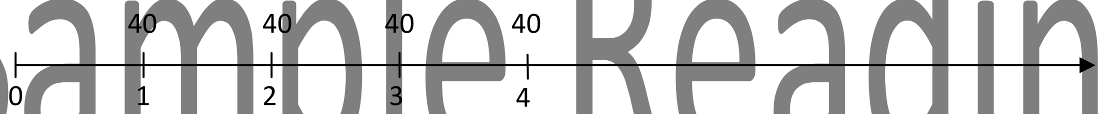


$$PV = 40\ddot{a}_{\overline{5}|11.092\%} = 40\left(\frac{1-v^5}{\left(\frac{0.11092}{1.11092}\right)}\right) = 40(4.09636) = 163.85$$

The price of annuity right before the first payment is **$163.85**.


#2

(c) The first payment has been made. This means that there are 4 more payments to go, with the first payment starts after 1 year. Hence, it is an annuity immediate with 4 payments:



$$PV = 40a_{\overline{4}|11.092\%} = 40\left(\frac{1-v^4}{0.11092}\right) = 40(3.09636) = 123.85$$

The price right after the first payment is **$123.85**.

Example 2.2

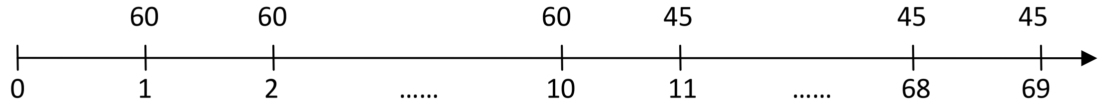
Ken purchases an annuity which pays $60 at the end of each month, starting from 30 September 2009. Starting from the 11th payment, the annuity payment decreases to $45 per month. You are given that the annual effective rate of interest is 6%. Find the present value of the annuity if the last payment of the annuity is on 31 May 2015.

Solution 2.2

Usually, the first step in solving annuity problems is always to convert the interest rate. Since the payment is at the end of each month, the interest rate that we use has to be the effective rate per month. Hence, we have:

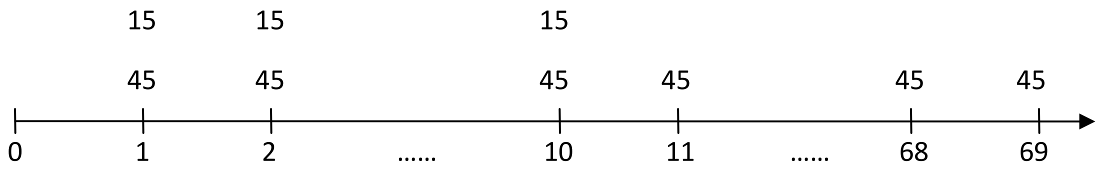
$$1+i = 1.06^{\frac{1}{12}} \Rightarrow i = 0.4868\%$$

Since the annuity pays at the end of each month, we can say that this is an annuity immediate. Starting from 31 September until 31 December, there are 4 payments, from year 2010 until 2014 there are 5(12)= 60 payments, plus the 5 payments in year 2015. Hence, this annuity pays 4+ 60+ 5= 69 payments. If we draw out the diagram, we have:



This is an unusual annuity! Do we have a close form solution for such present value? The answer is, we indeed do not have a wonderful and all inclusive formula for this. However, we can view the annuity payment in other ways. There are 3 ways we can solve this question. Learn all of them, as you will need them in the exam!

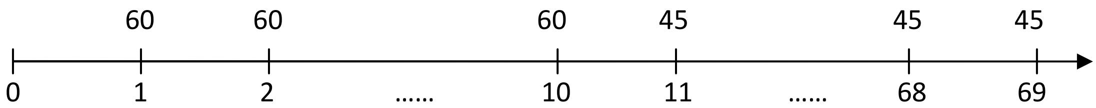
<u>Method 1: Addition of 2 annuities immediate with different "n"</u>



So you see, this is actually a combination of 2 annuities!? The first annuity pays 10 payments of \$15, whereas another one pays 69 payments of \$45. Adding the present values of these annuities will give us the answer that we want.

$$
\begin{aligned}
PV &= 15a_{\overline{10}|0.4868\%} + 45a_{\overline{69}|0.4868\%} \\
&= 15\left(\frac{1-v^{10}}{0.004868}\right) + 45\left(\frac{1-v^{69}}{0.004868}\right) \\
&= 146.06 + 2631.94 \\
&= 2778
\end{aligned}
$$

<u>Method 2: Addition of annuity immediate and deferred annuity</u>



This method is a bit troublesome, yet intuitive. Notice that the first 10 years, there are payments of \$60, and for the remaining 59 payments, they are \$45. We can calculate the present value of annuity by adding these 2 blocks of payments.

The first part is easy. The present value of the first 10 payments is:

$$60a_{\overline{10}|0.4868\%} = 60\left(\frac{1-v^{10}}{0.004868}\right) = 584.24$$

Now, assume that we are now at t=10. From there, if we look forward to those 59 payments, they are just another annuity immediate. So, we can obtain:

$$45a_{\overline{59}|0.4868\%} = 45\left(\frac{1-v^{59}}{0.004868}\right) = 2302.92$$

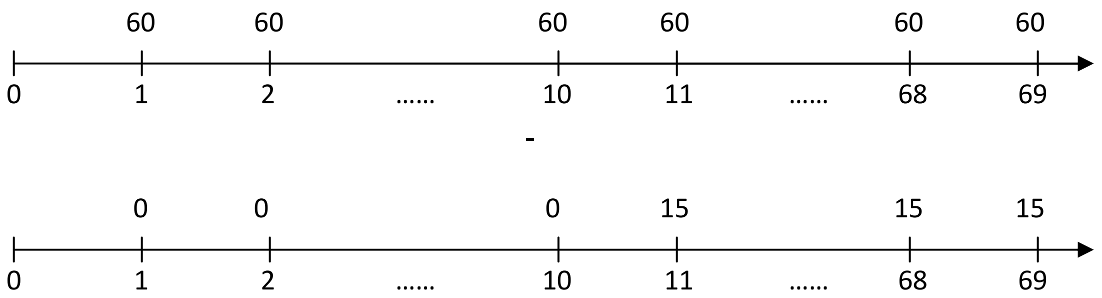
However, we need to find the value of the annuity immediate at t=0. Hence, we need to discount this amount:

$$2302.92v^{10} = 2193.75$$

Hence, the present value of the annuity is:

$$60a_{\overline{10}|0.4868} + 45v^{10}a_{\overline{59}|0.4868\%} = 584.24 + 2193.75 = 2778$$

<u>Method 3: Subtraction of deferred annuity from annuity immediate</u>
This method is troublesome, and no one would use it in the exam. I am here just to show you how methods may vary and yet we can get the same answer. Try to understand it.



Look at the first and second time diagram. Notice that if we subtract the second one from the first, we get exactly the payoff of the annuity. From t=11 until 59, \$60- \$15 is \$45 each payment. So, we can build the equation easily:

$$\begin{aligned} &60a_{\overline{69}|0.4868} - 15v^{10}a_{\overline{59}|0.4868\%} \\ &= 60\left(\frac{1-v^{69}}{0.004868}\right) - 15v^{10}\left(\frac{1-v^{59}}{0.004868}\right) \\ &= 3509.25 - 767.64v^{10} \\ &= 3509.25 - 731.25 \\ &= 2778 \end{aligned}$$

Don't forget to discount the second deferred annuity of \$15 regular payments! So, using the 3 methods, we still have the present value to be **\$2778**.

(Don't create for yourself catastrophe in the exam. Try not to attempt those solutions with traps. But you can always do so during preparation time to challenge yourself or to spice up your studying time!)

<u>**Example 2.3**</u>

An annuity pays \$750 at the end of each month. You are given v^n=0.525982, which represents the discount factor for \$1 at the end of n years. Assume that $i^{(2)}$ is 0.054264. Find present value of the annuity if the annuity pays 60n payments.

<u>**Solution 2.3**</u>

Take note that n represents the number of year, instead of month; the annuity pays regularly every month. So, we need to find the effective monthly interest rate. Let j be the monthly effective rate and i be the annual effective rate. We know that $1+i= (1+j)^{12}$. Hence,

$$v^n = \left(\frac{1}{1+i}\right)^n = \left(\frac{1}{(1+j)^{12}}\right)^n = \left(\frac{1}{1+j}\right)^{12n} = v_j^{12n}$$

So, we have

$$750a_{\overline{60n}|j} = 750\left(\frac{1-v_j^{60n}}{j}\right) = 750\left(\frac{1-\left(v_j^{12n}\right)^5}{\left(1+\frac{i^{(2)}}{2}\right)^{\frac{1}{6}}-1}\right) = 750\left(\frac{1-\left(v^n\right)^5}{1.027132^{\frac{1}{6}}-1}\right) = 750\left(\frac{1-0.525982^5}{0.004472}\right) \approx \boxed{160960}$$

Chapter 7
Advanced Financial Analysis

This chapter discusses about:

1. Yield Curves, spot rates, forward rates
2. Inflation rate
3. Portfolio method and Investment year method
4. Duration
5. Convexity
6. Immunization
7. Cash flow matching

7.1 Yield Curves, Spot Rates, Forward Rates

Yield Rate and Yield Curve

If you open a fixed deposit account, you will notice that you are offered a list of yield rates, which are based on the years of maturity. Most of the time, you will find that for the longer time you deposit your money, the more yield rate you will enjoy. The following table is something you usually see:

Duration	Effective rate per annum
3 months	2.0%
6 months	2.1%
1 year	2.5%
2 years	2.8%
5 years	3.0%

The interest per annum can be viewed as the yield rate. This means that, if \$1000 is deposited for 3 months, the accumulated value after 3 months should be $\$1000(1.02)^{0.25}$ whereas if \$1000 is deposited for 2 years, the accumulated value after 2 years is $\$1000(1.028)^2$. This is the term structure of the interest rates, telling you that for different term of investment, the yield rate varies. If the yield rates against the term of investment are plotted, the graph is called yield curve.

From the above example the effective rate increases with the increasing duration of investment. This is what we usually observe, and there are financial theories to explain it. The most obvious reason is because the risk of not getting back the principal is higher for longer duration of time. Hence, to "compensate for" long term investors, usually a higher yield rate is offered. At the same time, there are also financial theories which argue that yield curve should be negatively sloped, i.e. the yield rate should decrease for longer investment duration. Whatever they say, for the sake of the exam we would just ignore them and focus on the calculation instead of the theories behind.

Spot Rate

Spot rate is another rate that investors might need to consider when making an investment decision. It is the yield rate that the investors is offered now (on the spot) on the investment yield. The yield rates offered in the above table are all spot rates.

You might wonder, what is the difference between yield rate and spot rate?

If an investment has only one future cash flow, then they are the same. For example, using the table above, if an investment gives a cash inflow of $500 after 5 years, then the price (present value) of that investment is $\frac{500}{1.03^5} = \$431.30$. However, if an investment has more than one future cash flow, then the spot rate and yield rate are not the same. Suppose a special bond pays $40 after 1 year, and $50 after 2 years. According to the above table, the price of the bond should be:

$$P = \frac{40}{1.025} + \frac{50}{1.028^2} = 86.34$$

If the yield rate is j, then we have:

$$86.34 = \frac{40}{1+j} + \frac{50}{(1+j)^2}$$

$$\Rightarrow j = 0.027107$$

Do you see the difference? Spot rate is the rate used for single cash flow, whereas yield rate is applied on multiple cash flows, or simply the internal rate of return of an investment. If someone says, "I have an investment that has a yield rate of 7%", it means that his investment is giving a return of 7% per year and it is compounding every year. You may also say that yield rate is the "average" of the spot rates for 1 and 2 years. As you might guess, if we assume that the yield curve is flat, i.e. the spot rate is the same for all durations, then the spot rate equals the yield rate. If you are still unclear about it, do proceed first.

Forward Rate

Forward rate is a 1-year yield rate earned on an investment made in the future. 3-year forward rate of 5% means, your fund of $1000 will earn a yield of $50 for from t=3 until t=4.

How do we find the forward rate, given a set of spot rates? As we will see immediately, there exists a relationship between yield rate, spot rate and forward rate. The following illustration is based on the table below:

Term	**Yield to Maturity**
1 year	5%
2 years	5.5%
3 years	6%
4 years	6.5%
5 years	7%

What is the 3-year forward rate?

As we have already stated, 3-year forward rate refers to the one-year yield rate earned from t=3 to t=4. To ensure that there is no arbitrage opportunity, the investment of $1 now until t=4 at 4-year spot rate must be the same as the investment of $1 now until t=3 at 3-year spot rate, and then from t=3 to t=4 at 3-year forward rate. So, let s_t be the spot rate at time t, f_t be the t-year forward rate,

$$(1+s_4)^4 = (1+s_3)^3(1+f_3)$$

Solving this we get:

$$(1.065)^4 = (1.06)^3(1+f_3)$$
$$\Rightarrow (1+f_3) = \frac{1.065^4}{1.06^3} = 1.080142$$

Hence, the 3-year forward rate is **8.0142%**.

What about, 4-year forward rate? This rate can be obtained in 2 ways at least. The first method is the same as above:

$$(1+s_5)^5 = (1+s_4)^4(1+f_4)$$
$$\Rightarrow 1+f_4 = \frac{(1+s_5)^5}{(1+s_4)^4} = \frac{1.07^5}{1.065^4} = 1.09024$$

Since we have already got the 3-year forward rate, we can also use it to find 4-year forward rate:

$$(1+s_5)^5 = (1+s_3)^3(1+f_3)(1+f_4)$$
$$\Rightarrow 1+f_4 = \frac{(1+s_5)^5}{(1+s_3)^3(1+f_3)} = \frac{1.07^5}{(1.06^3)(1.080142)} = 1.09024$$

Do you see some light here? From the first and second method, we can see that $(1+s_4)^4 = (1+s_3)^3(1+f_3)$. Generalizing it we have:

$$(1+s_t)^t = (1+s_{t-1})^{t-1}(1+f_{t-1})$$

Also, we can further modify the generalization to get:

$$(1+s_t)^t = (1+s_{t-1})^{t-1}(1+f_{t-1}) = (1+s_{t-2})^{t-2}(1+f_{t-2})(1+f_{t-1}) = \ldots = (1+s_1)(1+f_1)\ldots(1+f_{t-2})(1+f_{t-1})$$

We have therefore built the relationship between spot and forward rate. Let me ask you one question: What is the yield rate of a 5-year investment using the table above? The answer is 7%, which is the 5-year spot rate. Remember? If there is only one cash flow, the yield rate equals the spot rate.

Carefully read through this example and try to get the gist of these 3 rates.

Example 7.1

You are given the following information:

Term	Yield to Maturity
1 year	3%
2 years	3.25%
3 years	3.5%
4 years	4%
5 years	4.5%

(a) Find the price of a $1000 bond maturing in 4 years, which pays an annual coupon of $40.
(b) Find the yield rate of the bond.
(c) Find the 1-year, 2-year, 3-year, 4-year, and 5-year forward rates.

(d) If you are told that the 3-year forward rate should be 105% of the one you calculated. Find the new 2-year spot rate, assuming that other information given is correct.
(e) If you are told that the 4-year forward rate should be 108% of the one you calculated, find the new yield rate of the bond assuming that the 1-year spot rate given is incorrect.

Solution 7.1

(a) Remember! This is not the same as how you priced the bond in chapter 6. In chapter 6, we were given the yield rate and asked to find the price. Now we don't have the yield rate. So, we should use the spot rates:

$$P=\frac{40}{1.03}+\frac{40}{1.0325^2}+\frac{40}{1.035^3}+\frac{1040}{1.04^4}=1001.43$$

The bond price is **$1001.43**.

(b) In chapter 6, we are given yield rate to find the price. Having calculated the price, we are going to find the yield rate: $1001.43=40a_{\overline{4}|i}+1000v^4$

With N =4, PV = -1001.43, PMT = 40, FV= 1000, CPT I/Y, the BA II Plus tells us that the yield rate is **3.988157%**.
To revise, what do you expect if you use this yield rate to price the bond? You should quickly say: "Come on! Of course I will get $1001.43!" If you don't see why, apply the basic formula we learned in chapter 6 to give a try.

(c) We have learned that

$$(1+f_{t-1})=\frac{(1+s_t)^t}{(1+s_{t-1})^{t-1}}$$

So, substitute the value one by one and we can get the answers:

$$1+f_1=\frac{(1+s_2)^2}{(1+s_1)}=\frac{1.0325^2}{1.03}=1.035\Rightarrow f_1=0.035$$

$$1+f_2=\frac{(1+s_3)^3}{(1+s_2)^2}=\frac{1.035^3}{1.0325^2}=1.04002\Rightarrow f_2=0.04002$$

$$1+f_3=\frac{(1+s_4)^4}{(1+s_3)^3}=\frac{1.04^4}{1.035^3}=1.055145\Rightarrow f_3=0.055145$$

$$1+f_4=\frac{(1+s_5)^5}{(1+s_4)^4}=\frac{1.045^5}{1.04^4}=1.06524\Rightarrow f_4=0.06524$$

Can you get f_5? If you do, then you must have made some mistakes. We have insufficient information to find it.

(d) This is a subtle question. We found that the 3-year forward rate is 5.5145%. The correct forward rate should be 105% of it, which is 5.790225%. Then,

$$1.05790225=\frac{(1+s_4)^4}{(1+s_3)^3}=\frac{(1+s_4)^4}{(1+s_2^N)^2(1+f_2)}=\frac{1.04^4}{(1+s_2^N)^2(1.04002)}\Rightarrow s_2^N=0.031153$$

Hence, the new 2-year spot rate is **3.1153%**.

(e) This is an even more subtle question. The correct 4-year forward rate is (1.08)(0.06524)= 0.0704592. Since we are told that the wrong spot rate is 1-year spot rate, we, by recursive method, work out the corrected 1-year spot rate:

$$1.0704592 = \frac{(1+s_5)^5}{(1+s_4)^4}$$

$$= \frac{(1+s_5)^5}{(1+s_1^N)(1+f_1)(1+f_2)(1+f_3)}$$

$$= \frac{1.045^5}{(1+s_1^N)(1.035)(1.04002)(1.055145)}$$

$$\Rightarrow s_1^N = 0.024984$$

So, we should find the price of the bond using the corrected 1-year spot rate:

$$P = \frac{40}{1.024984} + \frac{40}{1.0325^2} + \frac{40}{1.035^3} + \frac{1040}{1.04^4} = 1001.62$$

Now, we can solve for the yield rate by asking BA II Plus to solve $1001.62 = 40a_{\overline{4}|i} + Cv^4$, and we get the yield rate is **3.95542%**.

Part B - Derivatives Market

Chapter 3
Options

This chapter discusses about:

1. Call option
2. Mathematics of call option
3. Put option
4. Mathematics of put option

3.1 Call option

Options mean choices. To have a choice is better than having no choice. When you have no choice, it is an obligation. The more choices you have, the better it is. If you are offered a choice, it means that you are offered an opportunity, and, in the financial world, you have to pay for it.

Illustration

Suppose Alvin thinks that the price of gold will rise from \$100 to \$120 in a year. He needs 1 gram of gold to make a ring for his wife one year from today. Of course, the lower the price of gold is, the better it is for him. Understanding his need, the gold seller offers him an option:

"If the gold price is more than \$100 after one year, I will still sell you at \$100. If it is less than \$100, you may purchase 1 gram of gold with the spot price that time"

Is this not advantageous to Alvin? At the end of the year, the maximum amount that Alvin needs to pay is \$100, and he will get 1 gram of gold. For example, if the gold price rises to \$120 as he expects, he only pays \$100 and saves \$20 to get 1 gram of gold. If the gold price is \$80, of course, cheers will be with him.

However, think of the side of gold seller. Is this not a "sure lose" condition? If the price is \$120, the seller is supposed to receive \$120 for 1 gram of gold, but to Alvin, he has to sell at \$100 only. This renders a loss of \$20. If the price goes below \$100, he will not earn any profit.

No doubt, Alvin is on the advantageous side of the game. He has the right to choose, and definitely he will choose the option that is beneficial to him. He has the option; he has the choice, and the opportunity. As we have mentioned, he has to pay for it. The price that Alvin has to pay to the gold seller is called premium. This premium is to compensate the gold seller for offering such a "sure lose" condition. How much should the premium be? You don't have to worry about this for exam FM. The premium is calculated by Black-Scholes Formula which is covered in exam MFE.

Definition

A call option is a contract, which gives the buyer the right to purchase an underlying asset at a strike price, but not an obligation to purchase the asset at the date of expiration.

In this case, the buyer is Alvin. The underlying asset is gold. The strike price is \$100. The date of expiration is 1 year from now. Since a call option is a contract, there must be a party who offers, and other party who accepts. The party who offers the contract is called an option writer (the gold seller), whereas the party who accepts the contract is a called option buyer. When the buyer decides to purchase the underlying asset at a strike price, the option is said to be exercised by the option buyer.

Cash Settlement

One question: Who is the option writer? Must that party be the seller of the underlying asset? The answer is no. We can edit the scenario a little bit:

Suppose Alvin thinks that the price of gold will rise from $100 to $120 in a year. He needs 1 gram of gold to make a ring for his wife one year from today. Of course, the lower the price of gold is, the better it is for him. Understanding his need, **his friend** offers him an option:

"If the gold price is more than $100 after one year, I will **pay you the excess of the spot price over $100**. If it is less than $100, I will pay you nothing."

If you compare this to the above scenario, everything is the same except for the option writer. If the gold price after one year is $120, Alvin will have to pay $120 to the gold seller for 1 gram of gold, BUT his friend will have to compensate him $120- $100= $20. Hence, Alvin simply needs to pay the net of $100, whereas, if the gold price is $80 at that time, Alvin still pays $80 for 1 gram of gold.

Since the settlement of payoff of the call option (the contract) is by cash, we call it *cash settlement*. (If the gold seller is the option writer, can Alvin transact with him on the cash settlement basis?)

Styles of Options

There are many types of call options in the complex financial world. I would like to introduce you three basic ones:

1. European call option: It is a call option that can only be exercised (executed) at the date of expiration
2. American call option: It is the call option that can be exercised any time until the date of expiration
3. Bermuda call option: It is a call option that can be exercised during a certain interval of time before the expiration date

Remember, the styles of option have nothing to do with the trading location of the option! You can definitely trade European option in US, and American option in England!

Which option do you think is more valuable? Meaning, which option do you think gives more opportunities or advantages to the option buyer?

Intuitively, it must be American call option. This is due to the flexibility. The call option buyer can exercise the option anytime that is beneficial to him. Compared to European call option, even if the price of underlying asset far surpasses the strike price, the option buyer still has to wait until the date of expiration. So, the value of option, assuming that all are constant, in decreasing order is:

American Call Option > Bermuda Call Option > European Call option.

This applies to all options; be it a call option, put option or any exotic options.

3.2 Mathematics of Call Options

We have already seen the definition. Let us revise it:

A call option is a contract that gives the buyer the right but not the obligation to buy the underlying asset at a predetermined strike price.

Payoff of Call option

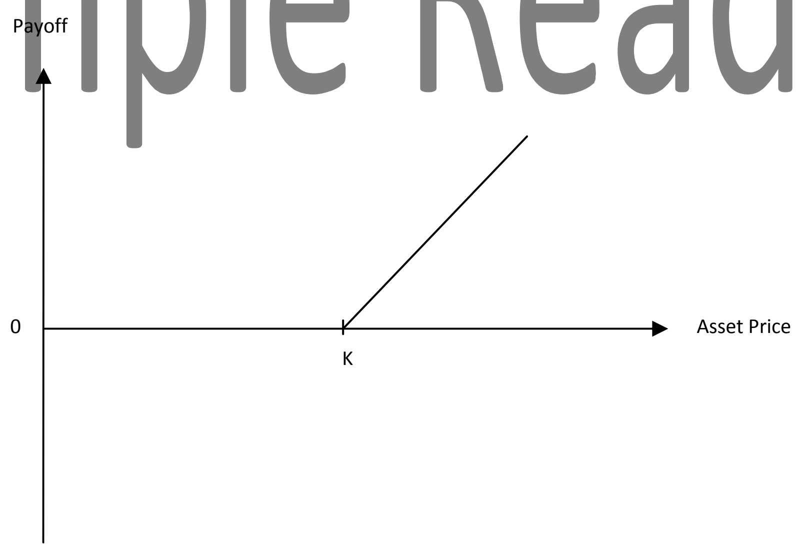
From the previous section I have already illustrated how a call option works. The payoff of the option at the expiration date, T, is equivalent to the excess of spot price at T over the strike price.

In mathematical symbol, if S_T is the spot price at T, K is the strike price, we have the payoff of a call option to be:

$$Payoff = \max\left[0, S_T - K\right]$$

The reason that we choose the maximum of 0 and the difference of spot price and strike price is because the option buyer always has the right to exercise the option ONLY IF the option has positive payoff. If the option has negative payoff, the option buyer will not exercise the option, assuming that the option buyer is mentally healthy (If the spot price is $80, are you willing to purchase it at strike price of $100?)

Hence, if we plot the graph of option payoff vs. spot price, we have:



Call option Payoff vs. Asset Price

As you can see, the payoff of the option is always 0, when the underlying asset price is less than K. As the asset price is more than K, the payoff of the option increases proportionally.

Profit of Call option

Remember that a premium has to be paid to purchase options. The premium is the cost of the option. Regardless of how much payoff the option buyer gets, the premium has to be paid. Hence,

$$Profit = \max\left[0, S_T - K\right] - FV\left(C\right)$$

We need to accumulate the call premium, because the payoff is evaluated at time T, while the premium is paid at the time of purchase of the option. Of course, the premium can be paid at the date of expiration as well, but this is not the practice in the real world situation.

Example 3.1

You purchase a $47.50-strike call from a dealer. The underlying asset is the stock price of ABC. The option is priced at $4.50 while the stock price of ABC is priced at $50. Assuming a risk free rate of 5%, find the profit you make if the stock price is $53.20 at time 2, the expiration date.

Solution 3.1

$47.50-strike call means that the call option has a strike price of $47.50. (Hence K= $47.50). The option is priced at $4.50 means that the premium is $4.50. Hence, we have the profit of the option:

$$Profit = \max\left[0, 53.20 - 47.50\right] - FV\left(4.50\right)$$
$$= 5.7 - 4.5e^{0.05(2)}$$
$$= 0.72673$$

Hence, the profit that you make after 2 years is approximately **$0.73**.

Example 3.2

You are given that a call option has a strike price of $30, and the current underlying stock price is $31.23. The option premium is $3.45, and the lifespan of the option is 3 years. Find the stock price at t=3 if such that:

(a) The profit is $2.30.
(b) The profit is -$1.78.

assuming that risk free rate is 3%.

Solution 3.2

We are already given K= $30, C= $3.45, T=3, r= 0.03. What we need to do is just to substitute these values into the formula.

(a)

$$Profit = \max\left[0, S_T - K\right] - FV\left(C\right)$$
$$\Rightarrow 2.30 = \max\left[0, S_T - 30\right] - 3.45e^{0.03(3)}$$
$$\Rightarrow 6.0749 = \max\left[0, S_T - 30\right]$$
$$\Rightarrow S_T = 36.0749$$

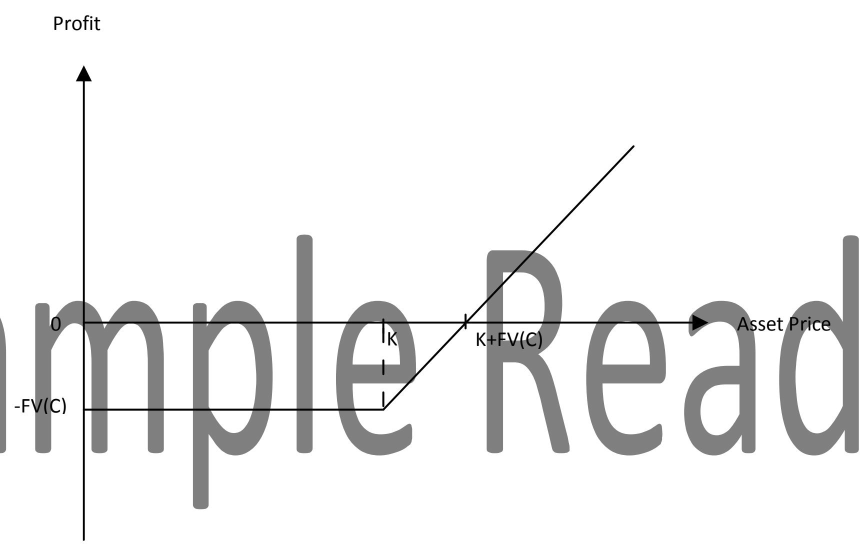
Answer is **$36.07**.

(b)

$$Profit = \max\left[0, S_T - K\right] - FV\left(C\right)$$
$$\Rightarrow -1.78 = \max\left[0, S_T - 30\right] - 3.45e^{0.03(3)}$$
$$\Rightarrow 1.995 = \max\left[0, S_T - 30\right]$$
$$\Rightarrow S_T = 31.995$$

Answer is **$32.00**.

Hence, if we graph the profit of call option vs. underlying asset price, we have:



Call option Profit vs. Asset Price

As you can see, the hope of the call option buyer is that the price of the underlying asset will go not only beyond strike price, but also beyond the strike price plus the future value of premium. Else, the option purchaser is going to make a loss.

Notice that the maximum loss of a purchased call is the future value of the premium. He will not lose more than that. Why? This is because if the underlying asset price is less than the strike price, he simply can walk away and ignore the contract. However, he has already paid the premium. Hence, the maximum loss that he might get is the premium (which at time T is valued as FV(C))

The way that I remember about this option is: a call option buyer always anticipates an increase in underlying asset price. When the underlying asset increases and is more than the strike price, the option pays off, and thus potentially gives profit.

Moneyness

An option is said to be in the money, if the payoff of the option is positive when it is marked to market. For example, let say on 1 January, we purchase a 50-strike call and a 50-strike put with 3 months to expiration. On 1 February, the stock price rises to $55. This means that if we were to exercise the option now, the call option pays $5. The call option is in the money.

An option is said to be at the money, if the strike price equals the spot price of the underlying asset. In the above example, if on 15 February the price of the underlying stock falls back to $50, then both the call and put option are said to be at the money at that time.

An option is said to be out of the money if the option has zero payoff, and the strike price is not the same as underlying spot price. If on 1 March, the price of the underlying stock falls to $45. If we were to exercise the options now, the call option pays nothing. Hence, the call option is out of the money. Oppositely, the put option is in the money.

Author's Biography

Alvin Soh was born in Penang, Malaysia. He is now a final year campus student studying in Petaling Jaya, Malaysia. He is keen in sharing and helping his peers in actuarial exams. His strong passion in mathematics and actuarial career is reflected in his fast progress in actuarial exams. He has passed all the preliminary exams in the mere 1.5 years and he is more than willing to share his study methods and experience to every actuarial student striving to progress in exams. He has passed an advanced level exam (Exam AFE) recently, and is pursuing his CERA and FSA in Finance and Enterprise Risk Management.

The following is the progress of his exams:

Exam Passed	**Sitting**
P/1- Probability	Spring 2007
FM/2- Financial Mathematics	Fall 2007
MLC- Models for Life Contingencies	Spring 2008
MFE/3F- Models for Financial Economics	Fall 2008
C/4- Construction and Evaluation of Actuarial Models	Fall 2008
AFE- Advanced Finance and Enterprise Risk Management	Spring 2009

He is now preparing for the last fellowship exam by the Society of Actuaries in finance and enterprise risk management track- exam FETE (Financial Economics Theory and Engineering).