a/s/m

Actuarial Study Materials

Study Manual for
Exam P/Exam 1

Probability

16-th Edition

by

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Note:
NO RETURN IF OPENED

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INTRODUCTION

Before you start studying for actuarial examinations you need to familiarize yourself with the following:

Fundamental Rule for Passing Actuarial Examinations
You should greet every problem you see when you are taking the exam with these words: "Been there, done that."

I will refer to this Fundamental Rule as the *BTDT Rule*. If you do not follow the BTDT Rule, neither this manual, nor any book, nor any tutorial, will be of much use to you. And I do want to help you, so I must beg you to follow the BTDT Rule. Allow me now to explain its meaning.

If you are surprised by any problem on the exam, you are likely to miss that problem. Yet the difference between a 5 and a 6 is one problem. This "surprise" problem has great marginal value. There is simply not enough time to think on the exam. *Thinking is always the last resort on an actuarial exam*. You may not have seen this very problem before, but you must have seen a problem like it before. If you have not, you are not prepared.

If you have not thoroughly studied all topics covered on the exam you are taking, you must have subconsciously wished to spend more time studying … a half a year's, or even a year's worth, more. But, clearly, the biggest reward for passing an actuarial examination is *not having to take it again*. By spending extra hours, days, or even weeks, studying and memorizing *all* topics covered on the exam, you are saving yourself possibly as much as a year's worth of your life. Getting a return of one year on an investment of one day is better than anything you will ever make on Wall Street, or even lotteries (unearned wealth is destructive, thus objects called "lottery winnings" are smaller than they appear). Please study thoroughly, without skipping any topic or any kind of a problem. To paraphrase my favorite quote from Ayn Rand: *for zat, you will be very grateful to yourself*.

I had once seen Harrison Ford being asked what he answers to people who tell him: "*May the Force be with you!*"? He said: "*Force Yourself!*" That's what you need to do. You have much more will than you assume, so go make yourself study.

Good luck.

Krzys' Ostaszewski
Bloomington, Illinois, September 2004

P.S. I want to thank my wife, Patricia, for her help and encouragement in writing of this manual. I also want to thank Hal Cherry for his help and encouragement. Any errors in this work are mine and mine only. If you find any, please be kind to let me know about them, because I definitely want to correct them.

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SECTION 1: GENERAL PROBABILITY

Basic probability concepts

Probability concepts are defined for elements and subsets of a certain set, a universe under consideration. That universe is called the *probability space,* or *sample space*. It can be a finite set, a countable set (a set whose elements can be put in a sequence, in fact a finite set is also countable), or an infinite uncountable set (e.g., the set of all real numbers, or any interval on the real line). Subsets of a given probability space for which probability can be calculated are called *events*. An event represents something that can possibly happen. In the most general probability theory, not every set can be an event. But this technical issue does not come up on any lower level actuarial examinations. It should be noted that while not all subsets of a probability space must be events, it is always the case that the empty set is an event, the entire space S is an event, a complement of an event is an event, and a set-theoretic union of any sequence of events is an event. The entire probability space will be usually denoted by S (always in this text) or Ω (in more theoretical probability books). It encompasses everything that can possibly happen.

The simplest possible event is a one-element set. If we perform an experiment, and observe what happens as its outcome, such a single observation is called an *elementary event*. Another name commonly used for such an event is: a *sample point*.

A *union* of two events is an event, which combines all of their elements, regardless of whether they are common to those events, or not. For two events A and B their union is denoted by $A \cup B$. For a more general finite collection of events $A_1, A_2, \ldots, A_n$, their union consists of all elementary events that belong to any one of them, and is denoted by $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$. An infinite union of a sequence of sets $\bigcup_{n=1}^{\infty} A_n$ is also defined as the set that consists of all elementary events that belong to any one of them.

An *intersection* of two events is an event, whose elements belong to both of the events. For two events A and B their intersection is denoted by $A \cap B$. For a more general finite collection of events $A_1, A_2, \ldots, A_n$, their intersection consists of all elementary events that belong to all of them, and is denoted by $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$. An infinite intersection of a sequence of sets $\bigcap_{n=1}^{\infty} A_n$ is also defined as the set that consists of all elementary events that belong to all of them.

Two events are called *mutually exclusive* if they cannot happen at the same time. In the language of set theory this simply means that they are disjoint sets, i.e., sets that do not have any elements in common, or $A \cap B = \varnothing$. A finite collection of events $A_1, A_2, \ldots, A_n$ is said to be *mutually exclusive* if any two events from the collection are mutually exclusive. The concept is defined the same way for infinite collections of events.

We say that a collection of events forms *exhaustive outcomes,* or that this collection forms a *partition* of the probability space, if their union is the entire probability space, and they are mutually exclusive.

An event A is a *subevent* (although most commonly we use the set-theoretic concept of a *subset)* of an event $B,$ denoted by $A \subset B,$ if every elementary event (sample point) in A is also contained in B. This relationship is a mathematical expression of a situation when occurrence of A automatically implies that B also occurs. Note that if $A \subset B$ then $A \cup B = B$ and $A \cap B = A.$

For an event E, its *complement*, denoted by E^C, consists of all elementary events (i. e., elements of the sample space S) that do not belong to E. In other words, $E^C = S - E,$ where S is the entire probability space. Note that $E \cup E^C = S$ and $E \cap E^C = \varnothing.$ Recall that $A - B = A \cap B^C$ is the set difference operation.

An important rule concerning complements of sets is expressed by *DeMorgan's Laws*:

$$(A \cup B)^C = A^C \cap B^C, \qquad (A \cap B)^C = A^C \cup B^C,$$

$$\left(\bigcup_{i=1}^{n} A_i \right)^C = \left(A_1 \cup A_2 \cup \ldots \cup A_n \right)^C = A_1^C \cap A_2^C \cap \ldots \cap A_n^C = \bigcap_{i=1}^{n} A_i^C,$$

$$\left(\bigcap_{i=1}^{n} A_i \right)^C = \left(A_1 \cap A_2 \cap \ldots \cap A_n \right)^C = A_1^C \cup A_2^C \cup \ldots \cup A_n^C = \bigcup_{i=1}^{n} A_i^C,$$

$$\left(\bigcup_{n=1}^{\infty} A_n \right)^C = \bigcap_{n=1}^{\infty} A_n^C, \qquad \left(\bigcap_{n=1}^{\infty} A_n \right)^C = \bigcup_{n=1}^{\infty} A_n^C.$$

The *indicator function* for an event E is a function $I_E : S \to \mathbb{R},$ where S is the entire probability space, and $\mathbb{R}$ is the set of real numbers, defined as $I_E(x) = 1$ if $x \in E,$ and $I_E(x) = 0$ if $x \notin E.$

The simplest, and commonly used, example of a probability space is a set consisting of two elements $S = \{0,1\},$ with 1 corresponding to "success" and 0 representing "failure". An experiment in which only such two outcomes are possible is called a *Bernoulli Trial*. You can view taking an actuarial examination as an example of a Bernoulli Trial. Tossing a coin is also an example of a Bernoulli trial, with two possible outcomes: heads or tails (you get to decide which of these two would be termed success, and which one is a failure).

Another commonly used finite probability space consists of all outcomes of tossing a fair six-faced die. The sample space is $S = \{1,2,3,4,5,6\}$, each number being a sample point representing the number of spots that can turn up when the die is tossed. The outcomes 1 and 6, for example, are mutually exclusive (more formally, these outcomes are events $\{1\}$ and $\{6\}$). The outcomes $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},$ usually written just as 1, 2, 3, 4, 5, 6, are exhaustive for this probability space and form a partition of it. The set $\{1, 3, 5\}$ represents the event of obtaining an odd number when tossing a die. Consider the following events in this die-tossing sample space:

$A = \{1, 2, 3, 4\}$ = "a number less than 5 is tossed",
$B = \{2, 4, 6\}$ = "an even number is tossed",
$C = \{1\}$ = "a 1 is tossed",
$D = \{5\}$ = "a 5 is tossed".

Then we have:

$$A \cup B = \{1,2,3,4,6\} = D^C,$$
$$A \cap B = \{2,4\},$$
$A \cap D = \varnothing$, i.e., events A and D are mutually exclusive,
$$(B \cup C)^C = \{1,2,4,6\}^C = \{3,5\} = \{1,3,5\} \cap \{2,3,4,5,6\} = B^C \cap C^C.$$

Another rule concerning operations on events:

$$A \cap (E_1 \cup E_2 \cup \ldots \cup E_n) = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n),$$
$$A \cup (E_1 \cap E_2 \cap \ldots \cap E_n) = (A \cup E_1) \cap (A \cup E_2) \cap \ldots \cap (A \cup E_n),$$

so that the distributive property holds the same way for unions as for intersections. In particular, if $E_1, E_2, \ldots, E_n$ form a partition of *S,* then for any event *A*

$$A = A \cap S = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n) = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n).$$

As the events $A \cap E_1, A \cap E_2, \ldots, A \cap E_n$ are also mutually exclusive, they form a partition of the event *A*. In the special case when $n = 2$, $E_1 = B$, $E_2 = B^C$, we see that $A \cap B$ and $A \cap B^C$ form a partition of *A*.

Probability (we will denote it by Pr) is a function that assigns a number between 0 and 1 to each event, with the following defining properties:

- $\Pr(\varnothing) = 0$,
- $\Pr(S) = 1$,

and

- If $\{E_n\}_{n=1}^{+\infty}$ is a sequence of mutually exclusive events, then $\Pr\left(\bigcup_{n=1}^{+\infty} E_n\right) = \sum_{n=1}^{+\infty} \Pr(E_n)$.

While the last condition is stated for infinite unions and a sum of a series, it applies equally to finite unions of mutually exclusive events, and the finite sum of their probabilities.

A *discrete probability space,* or *discrete sample space* is a probability space with a countable (finite or infinite) number of sample points. The assignment of probability to each elementary event in a discrete probability space is called the *probability function,* sometimes also called *probability mass function*.

For the simplest such space described as Bernoulli Trial, probability is defined by giving the probability of success, usually denoted by p. The probability of failure, denoted by q is then equal to $q = 1 - p$. Tossing a fair coin is a Bernoulli Trial with $p = 0.5$. Taking an actuarial examination is a Bernoulli Trial and we are trying to get your p to be as close to 1 as possible.

When a Bernoulli Trial is performed until a success occurs, and we count the total number of attempts, the resulting probability space is discrete, but infinite. Suppose, for example, that a fair coin is tossed until the first head appears. The toss number of the first head can be any positive integer and thus the probability space is infinite. Repeatedly taking an actuarial examination creates the same kind of discrete, yet infinite, probability space. Of course, we hope that after reading this manual, your probability space will not only be finite, but a *degenerate* one, consisting of one element only.

Tossing an ordinary die is an experiment with a finite probability space $\{1,2,3,4,5,6\}$. If we assign to each outcome the same probability of $\frac{1}{6}$, we obtain an example of what is called a *uniform probability function*. In general, for a finite discrete probability space, a uniform probability function assigns the same probability to each sample point. Since probabilities have to add up to 1, if there are n points in a finite probability space, uniform probability function assigns to each point equal probability of $\frac{1}{n}$.

In a finite probability space, for an event E, we always have $\Pr(E)=\sum_{e\in E}\Pr(\{e\})$. In other words, you can calculate probability of an event by adding up probabilities of all elementary events contained in it. For example, when rolling a fair die, the probability of getting an even number is

$$\Pr(\{2\})+\Pr(\{4\})+\Pr(\{6\})=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}.$$

Discrete probability space is only one type of a probability space. Other types analyzed involve a continuum of possible outcomes, and are called *continuous probability spaces*. For example we assume usually that an automobile physical damage claim is a number between 0 and the car's value (possibly after deductible is subtracted), i.e., it can be any real number between those two boundary values.

Some rules concerning probability:

- If $A\subset B$ then $\Pr(A)\le\Pr(B)$,
- $\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$,
- $\Pr(A\cup B\cup C)=$
  $=\Pr(A)+\Pr(B)+\Pr(C)-\Pr(A\cap B)-\Pr(A\cap C)-\Pr(B\cap C)+\Pr(A\cap B\cap C)$.
- $\Pr(E^C)=1-\Pr(E)$.
- $\Pr(A)=\Pr(A\cap E)+\Pr(A\cap E^C)$.
- If events $E_1,E_2,\ldots,E_n$ form a partition of the probability space S then:

$$\Pr(A)=\Pr(A\cap E_1)+\Pr(A\cap E_2)+\ldots+\Pr(A\cap E_n).$$

The last statement is commonly referred to as *The Law of Total Probability*.

SECTION 1

In some of the actuarial examinations you will see the words "percentage" and "probability" used interchangeably. If you are told, for example, that 75% of a certain population watch the television show *White Collar,* you should interpret this 75% = 0.75 as the probability that a person randomly chosen from this population watches this television show.

Exercise 1.1. May 2003 Course 1 Examination, Problem No. 1, also P Sample Exam Questions, Problem No. 1, and Dr. Ostaszewski's online exercise posted April 18, 2009

A survey of a group's viewing habits over the last year revealed the following information:
(i) 28% watched gymnastics,
(ii) 29% watched baseball,
(iii) 19% watched soccer,
(iv) 14% watched gymnastics and baseball,
(v) 12% watched baseball and soccer,
(vi) 10% watched gymnastics and soccer,
(vii) 8% watched all three sports.
Calculate the percentage of the group that watched none of the three sports during the last year.

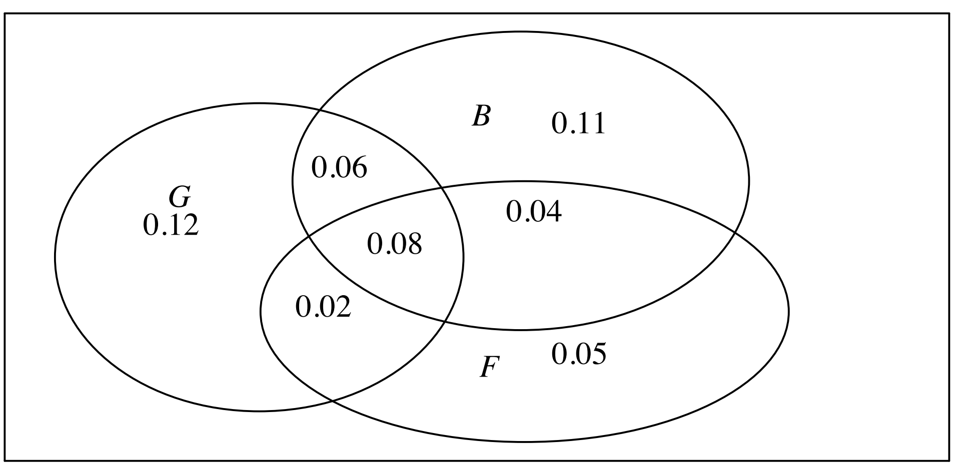
A. 24 B. 36 C. 41 D. 52 E. 60

Solution.

Treat the groups' preferences as events: G – watching gymnastics, B – watching baseball, F – watching soccer (F for "football", as the rest of the world calls it, because we use S for the probability space). We have the following probabilities:

$$\Pr(G)=0.28, \qquad \Pr(B)=0.29, \qquad \Pr(F)=0.19,$$
$$\Pr(G\cap B)=0.14, \qquad \Pr(B\cap F)=0.12, \qquad \Pr(G\cap F)=0.10,$$
$$\Pr(G\cap B\cap F)=0.08.$$

Therefore, the non-overlapping pieces of these sets have the probabilities shown in the figure below.



We are interested in the "area" (probability) outside of the ovals, i.e.,

$$1-0.12-0.02-0.06-0.08-0.05-0.04-0.11=0.52.$$

We can also formally calculate it as:

$$\Pr\left((G\cup B\cup F)^C\right)=1-\Pr(G\cup B\cup F)=1-\Pr(G)-\Pr(B)-\Pr(F)+$$
$$+\Pr(G\cap B)+\Pr(G\cap F)+\Pr(B\cap F)-\Pr(G\cap B\cap F)=$$
$$=1-0.28-0.29-0.19+0.14+0.12+0.10-0.08=0.52.$$

Answer D.

Exercise 1.2. November 2001 Course 1 Examination, Problem No. 9, also P Sample Exam Questions, Problem No. 8, and Dr. Ostaszewski's online exercise posted August 18, 2007

Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

A. 0.26 B. 0.38 C. 0.40 D. 0.48 E. 0.62

Solution.

Let C be the event that a patient visits a chiropractor, and T be the event that a patient visits a physical therapist. We are given that $\Pr(C\cap T)=0.22$, $\Pr\left(C^C\cap T^C\right)=\Pr\left((C\cup T)^C\right)=0.12$, and $\Pr(C)=\Pr(T)+0.14$. Therefore,

$$1-0.12=0.88=\Pr(C\cup T)=\Pr(C)+\Pr(T)-\Pr(C\cap T)=$$
$$=\Pr(T)+0.14+\Pr(T)-0.22=2\Pr(T)-0.08.$$

This implies that

$$\Pr(T)=\frac{0.88+0.08}{2}=0.48.$$

Answer D.

Conditional probability

The concept of conditional probability is designed to capture the relationship between probabilities of two or more events happening. The simplest version of this relationship is in the question: given that an event B happened, how does this affect the probability that A happens? In order to tackle this question, we define the *conditional probability of A given B* as:

$$\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}.$$

In order for this definition to make sense we must assume that $\Pr(B)>0$. This concept of conditional probability basically makes B into the new probability space, and then takes the probability of A to be that of only the part of A inside of B, scaled by the probability of B in relation to the probability of the entire S (which is, of course, 1). Note that the definition implies that

$$\Pr(A \cap B) = \Pr(A|B) \cdot \Pr(B),$$

of course as long as $\Pr(B) > 0$.

Note also the following properties of conditional probability:

• For any event E with positive probability, the function $A \mapsto \Pr(A|E)$, assigning to any event A its conditional probability $\Pr(A|E)$ meets the conditions of the definition of probability, thus it is itself a probability, and has all properties of probability that we listed in the previous section. For example,

$$\Pr(A^C|E) = 1 - \Pr(A|E),$$

or

$$\Pr(A \cup B|E) = \Pr(A|E) + \Pr(B|E) - \Pr(A \cap B|E).$$

We also have:

• If $A \subset B$ then $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{\Pr(A)}{\Pr(B)}$, and $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)} = \dfrac{\Pr(A)}{\Pr(A)} = 1$.

• If $\Pr(A_1 \cap A_2 \cap \ldots A_n) > 0$ then

$$\Pr(A_1 \cap A_2 \cap \ldots A_n) = \Pr(A_1) \cdot \Pr(A_2|A_1) \cdot \Pr(A_3|A_1 \cap A_2) \cdot \ldots \cdot \Pr(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

Exercise 1.3. May 2003 Course 1 Examination, Problem No. 18, also P Sample Exam Questions, Problem No. 7, and Dr. Ostaszewski's online exercise posted August 11, 2007

An insurance company estimates that 40% of policyholders who have only an auto policy will renew next year and 60% of policyholders who have only a homeowners policy will renew next year. The company estimates that 80% of policyholders who have both an auto and a homeowners policy will renew at least one of those policies next year. Company records show that 65% of policyholders have an auto policy, 50% of policyholders have a homeowners policy, and 15% of policyholders have both an auto and a homeowners policy. Using the company's estimates, calculate the percentage of policyholders that will renew at least one policy next year.

A. 20 B. 29 C. 41 D. 53 E. 70

Solution.

Let A be the event that a policyholder has an auto policy, and H be the event that a policyholder has a homeowners policy, and R be the event that a policyholder renews a policy. We are given: $\Pr(A) = 0.65$, $\Pr(H) = 0.50$, $\Pr(A \cap H) = 0.15$, $\Pr(R|A \cap H^C) = 0.40$, $\Pr(R|H \cap A^C) = 0.60$, and $\Pr(R|A \cap H) = 0.80$. We are looking for $\Pr(R)$. Note that

$$\Pr(A \cap H^C) = \Pr(A - H) = \Pr(A - (A \cap H)) = \Pr(A) - \Pr(A \cap H) = 0.65 - 0.15 = 0.50,$$

and

$$\Pr\left(A^C \cap H\right) = \Pr\left(H - A\right) = \Pr\left(H - \left(H \cap A\right)\right) = \Pr\left(H\right) - \Pr\left(A \cap H\right) = 0.50 - 0.15 = 0.35.$$

Also note that the events $A \cap H^C$, $A^C \cap H$, and $A \cap H$ form a partition of the probability space considered. Therefore,

$$\Pr\left(R\right) = \Pr\left(R \cap \left(A \cap H^C\right)\right) + \Pr\left(R \cap \left(A^C \cap H\right)\right) + \Pr\left(R \cap \left(A \cap H\right)\right) =$$

$$= \Pr\left(R \middle| A \cap H^C\right) \cdot \Pr\left(A \cap H^C\right) + \Pr\left(R \middle| A^C \cap H\right) \cdot \Pr\left(A^C \cap H\right) +$$

$$+ \Pr\left(R \middle| A \cap H\right) \cdot \Pr\left(A \cap H\right) = 0.4 \cdot 0.5 + 0.6 \cdot 0.35 + 0.8 \cdot 0.15 = 0.53.$$

Answer D.

Exercise 1.4. May 2003 Course 1 Examination, Problem No. 5, also P Sample Exam Questions, Problem No. 9, and Dr. Ostaszewski's online exercise posted August 25, 2007

An insurance company examines its pool of auto insurance customers and gathers the following information:

(i) All customers insure at least one car.
(ii) 70% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

A. 0.13 B. 0.21 C. 0.24 D. 0.25 E. 0.30

Solution.

Always start by labeling the events. Let *C* (stands for Corvette) be the event of insuring a sports car (not *S*, because we reserve this for the entire probability space), and *M* be the event of insuring multiple cars. Note that M^C is the event of insuring exactly one car, as all customers insure at least one car. We are given that $\Pr\left(M\right) = 0.70$, $\Pr\left(C\right) = 0.20$, and $\Pr\left(C \middle| M\right) = 0.15$. We need to find $\Pr\left(M^C \cap C^C\right)$. You need to recall De Morgan's Laws and then we see that:

$$\Pr\left(M^C \cap C^C\right) = \Pr\left(\left(M \cup C\right)^C\right) = 1 - \Pr\left(M \cup C\right) = 1 - \Pr\left(M\right) - \Pr\left(C\right) + \Pr\left(M \cap C\right) =$$

$$= 1 - \Pr\left(M\right) - \Pr\left(C\right) + \Pr\left(C \middle| M\right)\Pr\left(M\right) = 1 - 0.70 - 0.20 + 0.15 \cdot 0.70 = 0.205.$$

Answer B.

Independence of events

We say that events *A* and *B* are *independent* if

$$\Pr\left(A \cap B\right) = \Pr\left(A\right) \cdot \Pr\left(B\right).$$

Note that any event of probability 0 is independent of any other events. For events with positive probabilities, independence of *A* and *B* is equivalent to $\Pr\left(A \middle| B\right) = \Pr\left(A\right)$ or $\Pr\left(B \middle| A\right) = \Pr\left(B\right)$. This means that two events are independent if occurrence of one of them has no effect on the probability of the other one happening. You must remember that the concept of independence should never be confused with the idea of two events being mutually exclusive. If two events have positive probabilities and they are mutually exclusive, then they must be dependent, as if one of them happens, the other one cannot happen, and the conditional probability of the second event given the first one must be zero, while the unconditional (regular) probability is not zero. It can be shown, and should be memorized by you that if *A* and *B* are independent, then so are A^C and *B,* as well as *A* and B^C, and so are A^C and B^C.

For three events *A, B,* and *C,* we say that they are *independent*, if $\Pr(A\cap B)=\Pr(A)\cdot\Pr(B)$, $\Pr(A\cap C)=\Pr(A)\cdot\Pr(C)$, $\Pr(B\cap C)=\Pr(B)\cdot\Pr(C)$, and

$$\Pr(A\cap B\cap C)=\Pr(A)\cdot\Pr(B)\cdot\Pr(C).$$

In general, we say that events $A_1,A_2,\ldots,A_n$ are *independent* if, for any finite collection of them $A_{i_1},A_{i_2},\ldots,A_{i_k}$, where $1\le i_1<i_2<\ldots<i_k\le n$, we have

$$\Pr\left(A_{i_1}\cap A_{i_2}\cap\ldots\cap A_{i_k}\right)=\Pr\left(A_{i_1}\right)\cdot\Pr\left(A_{i_2}\right)\cdot\ldots\cdot\Pr\left(A_{i_k}\right).$$

Exercise 1.5. May 2003 Course 1 Examination, Problem No. 37, also P Sample Exam Questions, Problem No. 17, and Dr. Ostaszewski's online exercise posted October 13, 2007
An insurance company pays hospital claims. The number of claims that include emergency room or operating room charges is 85% of the total number of claims. The number of claims that do not include emergency room charges is 25% of the total number of claims. The occurrence of emergency room charges is independent of the occurrence of operating room charges on hospital claims. Calculate the probability that a claim submitted to the insurance company includes operating room charges.

A. 0.10 B. 0.20 C. 0.25 D. 0.40 E. 0.80

Solution.
As always, start by labeling the events. Let *O* be the event of incurring operating room charges, and *E* be the event of emergency room charges. Then, because of independence of these two events,

$$0.85=\Pr(O\cup E)=\Pr(O)+\Pr(E)-\Pr(O\cap E)=\Pr(O)+\Pr(E)-\Pr(O)\cdot\Pr(E).$$

Since $\Pr\left(E^C\right)=0.25=1-\Pr(E)$, it follows that $\Pr(E)=0.75$. Therefore

$$0.85=\Pr(O)+0.75-0.75\cdot\Pr(O),$$

and $\Pr(O)=0.40$.
Answer D.

The Bayes Theorem
Note that as long as $0<\Pr(E)<1$:

$$\Pr(A)=\Pr(A\cap E)+\Pr\left(A\cap E^C\right)=\Pr(A|E)\cdot\Pr(E)+\Pr\left(A|E^C\right)\cdot\Pr\left(E^C\right).$$

Therefore, if also $\Pr(A) > 0$,

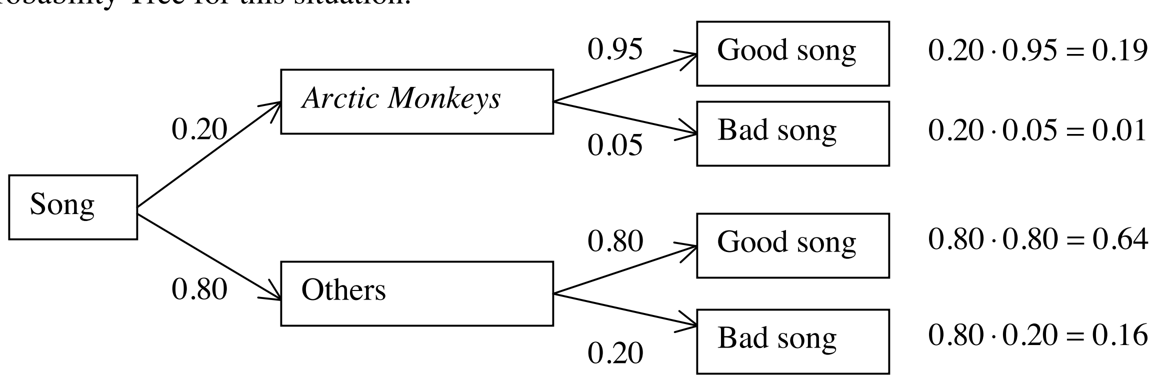
$$\Pr(E|A) = \frac{\Pr(E \cap A)}{\Pr(A)} = \frac{\Pr(A|E)\cdot\Pr(E)}{\Pr(A|E)\cdot\Pr(E) + \Pr(A|E^C)\cdot\Pr(E^C)}.$$

The more general version of the above statement is this very important *Bayes Theorem:*
If events (of positive probability) $E_1, E_2, \ldots, E_n$ *form a partition of the probability space under consideration, and A is an arbitrary event with positive probability, then for any* $i = 1, 2, \ldots, n$

$$\Pr(E_i|A) = \frac{\Pr(A|E_i)\cdot\Pr(E_i)}{\sum_{j=1}^{n}\Pr(A|E_j)\cdot\Pr(E_j)} = \frac{\Pr(A|E_i)\cdot\Pr(E_i)}{\Pr(A|E_1)\cdot\Pr(E_1) + \ldots + \Pr(A|E_n)\cdot\Pr(E_n)}.$$

In solving problems involving the Bayes Theorem (also known as the *Bayes Rule)* the key step is to identify and label events and conditional events in an efficient way. In fact, labeling events named in the problem should always be your starting point in any basic probability problem. The typical pattern you should notice in all Bayes Theorem is the "flip-flop": reversal of the roles of events *A* and E_i in the conditional probability. The values of $\Pr(E_i)$ are called *prior probabilities,* and the values of $\Pr(E_i|A)$ are called *posterior probabilities*. The applications of Bayes Theorem occur in situations in which all $\Pr(E_i)$ and $\Pr(A|E_i)$ probabilities are known, and we are asked to find $\Pr(E_i|A)$ for one of the *i's*.

Problems involving the Bayes Theorem can also be conveniently handled by using a *probability tree* expressing all probabilities involved. Let us illustrate this with an example. Suppose that you have a collection of songs on your iPod and you play them at a party. 20% of your songs are by *Arctic Monkeys,* and 80% by other performers. If you pick a song randomly from *Arctic Monkeys* songs, the probability that it is a good song is 0.95. For other performers, this probability is 0.80. You pick a song randomly from your iPod, play it, and the song turns out to be bad. What is the probability that you picked a song by *Arctic Monkeys*? Consider the figure below, i.e., the Probability Tree for this situation:



The probabilities listed on the right are

$\Pr(\text{A random song is by } \textit{Arctic Monkeys} \text{ and is good}) = 0.19,$

$\Pr(\text{A random song is by } \textit{Arctic Monkeys} \text{ and is bad}) = 0.01,$

$$\Pr(\text{A random song is not by } \textit{Arctic Monkeys} \text{ and is good}) = 0.64,$$

$$\Pr(\text{A random song is not by } \textit{Arctic Monkeys} \text{ and is bad}) = 0.16.$$

Therefore,

$$\Pr(\text{A song is by } \textit{Arctic Monkeys} | \text{That song is bad}) = \frac{0.01}{0.01+0.16} = \frac{1}{17}.$$

Notice that we simply take the probability of both things happening (a song by *Arctic Monkeys* and bad) from the right column and put it in the numerator, and the sum of all ways that the song is bad from the right column and put that in the denominator. You can also get the same result by direct application of the Bayes Theorem:

$$\Pr(\text{A song is by } \textit{Arctic Monkeys} | \text{That song is bad}) =$$

$$= \frac{\Pr(\text{Bad song} | \textit{Arctic Monkeys}) \cdot \Pr(\textit{Arctic Monkeys})}{\Pr(\text{Bad song} | \textit{Arctic Monkeys}) \cdot \Pr(\textit{Arctic Monkeys}) + \Pr(\text{Bad song} | \text{Other}) \cdot \Pr(\text{Other})} =$$

$$= \frac{0.05 \cdot 0.20}{0.05 \cdot 0.20 + 0.20 \cdot 0.80} = \frac{0.01}{0.01+0.16} = \frac{1}{17}.$$

Exercise 1.6. May 2003 Course 1 Examination, Problem No. 31, also P Sample Exam Questions, Problem No. 22, also Dr. Ostaszewski's online exercise posted March 1, 2008

A health study tracked a group of persons for five years. At the beginning of the study, 20% were classified as heavy smokers, 30% as light smokers, and 50% as nonsmokers. Results of the study showed that light smokers were twice as likely as nonsmokers to die during the five-year study, but only half as likely as heavy smokers. A randomly selected participant from the study died over the five-year period. Calculate the probability that the participant was a heavy smoker.

A. 0.20 B. 0.25 C. 0.35 D. 0.42 E. 0.57

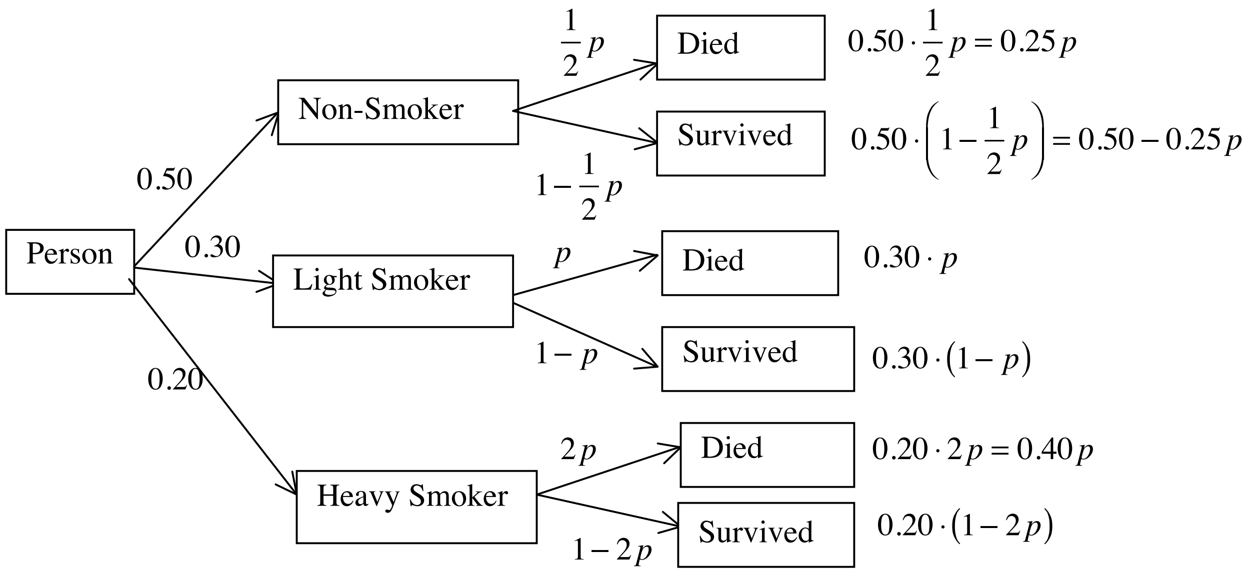
Solution.

Let H be the event of studying a heavy smoker, L be the event of studying a light smoker, and N be the event of studying a non-smoker. We are given that $\Pr(H) = 0.20$, $\Pr(L) = 0.30$, and $\Pr(N) = 0.50$. Additionally, let D be the event of a death within five-year period. We know that $\Pr(D|L) = 2\Pr(D|N)$ and $\Pr(D|L) = \frac{1}{2}\Pr(D|H)$. We are looking for $\Pr(H|D)$. Using the Bayes Theorem, we conclude:

$$\Pr(H|D) = \frac{\Pr(D|H) \cdot \Pr(H)}{\Pr(D|N) \cdot \Pr(N) + \Pr(D|L) \cdot \Pr(L) + \Pr(D|H) \cdot \Pr(H)} =$$

$$= \frac{2\Pr(D|L) \cdot 0.2}{\frac{1}{2}\Pr(D|L) \cdot 0.5 + \Pr(D|L) \cdot 0.3 + 2\Pr(D|L) \cdot 0.2} = \frac{0.4}{0.25+0.3+0.4} \approx 0.4211.$$

Answer D. Alternatively, we can draw a Probability Tree, as shown in the figure below:



Therefore,

$$\Pr(H|D)=\frac{0.40p}{0.25p+0.30p+0.40p}=\frac{0.4}{0.25+0.3+0.4}\approx 0.4211.$$

Answer D, again.

Exercise 1.7. November 2001 Course 1 Examination, Problem No. 4, also P Sample Exam Questions, Problem No. 21, and Dr. Ostaszewski's online exercise posted February 23, 2008

Upon arrival at a hospital's emergency room, patients are categorized according to their condition as critical, serious, or stable. In the past year:

(i) 10% of the emergency room patients were critical;
(ii) 30% of the emergency room patients were serious;
(iii) The rest of the emergency room patients were stable;
(iv) 40% of the critical patients died;
(v) 10% of the serious patients died; and
(vi) 1% of the stable patients died.

Given that a patient survived, what is the probability that the patient was categorized as serious upon arrival?

A. 0.06 B. 0.29 C. 0.30 D. 0.39 E. 0.64

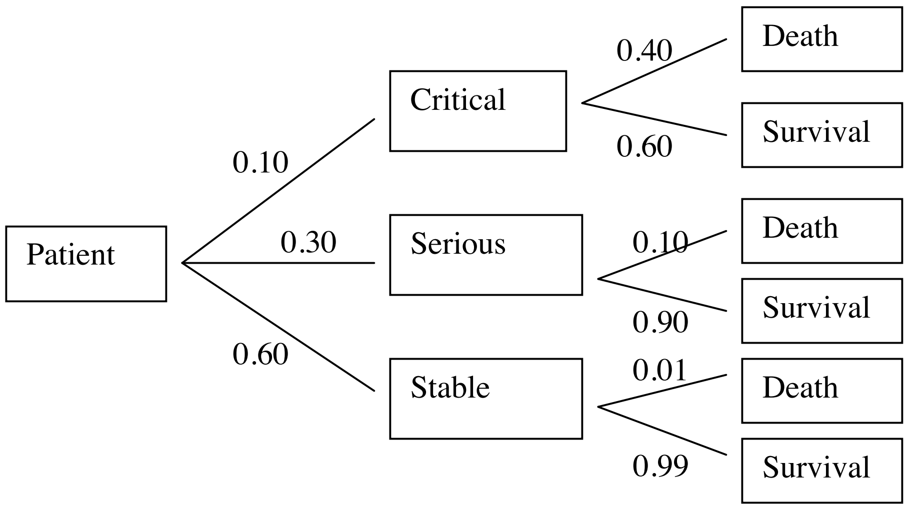
Solution.

Let U be the event that a patient survived, and E_S be the event that a patient was classified as serious upon arrival, E_C -- the event that a patient was critical, and E_T -- the event that the patient was stable. We apply the Bayes Theorem:

$$\Pr(E_S|U) = \frac{\Pr(U|E_S)\cdot\Pr(E_S)}{\Pr(U|E_C)\cdot\Pr(E_C)+\Pr(U|E_S)\cdot\Pr(E_S)+\Pr(U|E_T)\cdot\Pr(E_T)} =$$

$$= \frac{0.9\cdot 0.3}{0.6\cdot 0.1+0.9\cdot 0.3+0.99\cdot 0.6} \approx 0.2922.$$

Answer B. This answer could have been also derived using the probability tree shown below:



Exercise 1.8. May 2003 Course 1 Examination, Problem No. 8, also P Sample Exam Questions, Problem No. 19, and Dr. Ostaszewski's online exercise posted January 12, 2008

An auto insurance company insures drivers of all ages. An actuary compiled the following statistics on the company's insured drivers:

Age of Driver	Probability of Accident	Portion of Company's Insured Drivers
16-20	0.06	0.08
21-30	0.03	0.15
31-65	0.02	0.49
66-99	0.04	0.28

A randomly selected driver that the company insures has an accident. Calculate the probability that the driver was age 16-20.

A. 0.13 B. 0.16 C. 0.19 D. 0.23 E. 0.40

Solution.

This is a standard application of the Bayes Theorem. Let A be the event of an insured driver having an accident, and let

B_1 = Event: driver's age is in the range 16-20,
B_2 = Event: driver's age is in the range 21-30,
B_3 = Event: driver's age is in the range 30-65,
B_4 = Event: driver's age is in the range 66-99.

Then

$$\Pr(B_1|A) = \frac{\Pr(A|B_1)\Pr(B_1)}{\Pr(A|B_1)\Pr(B_1) + \Pr(A|B_2)\Pr(B_2) + \Pr(A|B_3)\Pr(B_3) + \Pr(A|B_4)\Pr(B_4)} =$$

$$= \frac{0.06 \cdot 0.08}{0.06 \cdot 0.08 + 0.03 \cdot 0.15 + 0.02 \cdot 0.49 + 0.04 \cdot 0.28} = 0.1584.$$

Answer B.

Combinatorial probability

If we have a set with *n* elements, there are *n* ways to pick one of them to be the first one, then $n-1$ ways to pick another one of them to be number two, etc., until there is only one left. This implies that there are $n \cdot (n-1) \cdot \ldots \cdot 1$ ways to put all the elements of this set in order. The expression $n \cdot (n-1) \cdot \ldots \cdot 1$ is denoted by $n!$ and termed *n-factorial*. The orderings of a given set of *n* elements are called *permutations*. In general, a *permutation* is an ordered sample from a given set, not necessarily containing all elements from it. If we have a set with *n* elements, and we want to pick an ordered sample of size *k,* then we have $n - k$ elements remaining, and since all of their orderings do not matter, the total number of such ordered *k*-samples (i.e., permutations) possible is $\frac{n!}{(n-k)!}$.

A *combination* is an unordered sample (without replacement) from a given finite set, i.e., its subset. Given a set with *n* elements, there are $\frac{n!}{(n-k)!}$ ordered *k*-samples that can be picked from it. But a combination does not care what the order is, and since *k* elements can be ordered in $k!$ ways, the number of combinations (i.e., subsets) of size *k* that can be picked is reduced by the factor of $k!$ and is therefore equal to: $\frac{n!}{k!(n-k)!}$. This expression $\frac{n!}{k!(n-k)!}$ is written usually as $\binom{n}{k}$ and read as *n choose k*.

In addition to unordered samples without replacement, i.e., combinations, we often consider *samples with replacement,* which are, as the names indicates, samples obtained by taking elements of a finite set, with elements returned to the set after they have been picked. Those problems are best handled by common sense and some practice. Every time an element is picked this way, its chances of being picked are simply the ratio of the number of elements of its type to the total number of elements in the set.

There are also some other types of combinatorial principles that might be useful in probability calculations that we will list now.

Given *n* objects, of which n_1 are of type 1, n_2 are of type 2, …, and n_m are of type *m,* with $n_1 + n_2 + \ldots + n_m = n,$ the number of ways to order all *n* objects, with objects of each type indistinguishable from other objects of the same type, is $\frac{n!}{n_1!n_2!\ldots n_m!}$, sometimes denoted by $\begin{pmatrix} & n & & \\ n_1 & n_2 & \ldots & n_m \end{pmatrix}$.

Given *n* objects, of which n_1 are of type 1, n_2 are of type 2, …, and n_m are of type *m,* with $n_1 + n_2 + \ldots + n_m = n$, the number of ways to choose a subset of size $k \leq n$ (without replacement), with k_1 objects of type 1, k_2 objects of type 2, …, and k_m are of type *m,* with $k_1 + k_2 + \ldots + k_m = k,$ is $\binom{n_1}{k_1} \cdot \binom{n_2}{k_2} \cdot \ldots \cdot \binom{n_m}{k_m}$.

The $\binom{n}{k}$ expression plays a special role in the *Binomial Theorem,* which states that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

We also have this general application of $\binom{n}{k}$: In the power series expansion of $(1+t)^n$ the coefficient of t^k is $\binom{n}{k}$. There is also the following multivariate version: In the power series expansion of $(t_1 + t_2 + \ldots + t_m)^n$ the coefficient of $t_1^{n_1} \cdot t_2^{n_2} \cdot \ldots \cdot t_m^{n_m}$, where $n_1 + n_2 + \ldots + n_m = n,$ is $\begin{pmatrix} & n & & \\ n_1 & n_2 & \ldots & n_m \end{pmatrix}$.

The combinatorial principles are very useful in calculating probabilities. The probability of the outcome desired is calculated as the ratio of the number of favorable outcomes to the total number of outcomes. We will illustrate this in some exercises below.

Exercise 1.9. November 2001 Course 1 Examination, Problem No. 1, also P Sample Exam Questions, Problem No. 4, and Dr. Ostaszewski's online exercise posted June 30, 2007
An urn contains 10 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44. Calculate the number of blue balls in the second urn.

A. 4 B. 20 C. 24 D. 44 E. 64

Solution.

For $i = 1,2$ let R_i be the event that a red ball is drawn from urn i, and B_i be the event that a blue ball is drawn from urn i. Then, if x is the number of blue balls in urn 2, we can assume that drawings from two different urns are independent and obtain

$$0.44 = \Pr\left(\left(R_1 \cap R_2\right) \cup \left(B_1 \cap B_2\right)\right) = \Pr\left(R_1 \cap R_2\right) + \Pr\left(B_1 \cap B_2\right) = \Pr\left(R_1\right) \cdot \Pr\left(R_2\right) +$$

$$+ \Pr\left(B_1\right) \cdot \Pr\left(B_2\right) = \frac{4}{10} \cdot \frac{16}{x+16} + \frac{6}{10} \cdot \frac{x}{x+16} = \frac{1}{5} \cdot \left(\frac{32}{x+16} + \frac{3x}{x+16}\right).$$

Therefore, by multiplying both sides by 5 we get

$$2.2 = \frac{32}{x+16} + \frac{3x}{x+16} = \frac{3x+32}{x+16}.$$

This can be immediately turned into an easy linear equation, whose solution is $x = 4$.

Answer A.

Exercise 1.10. February 1996 Course 110 Examination, Problem No. 1, also Dr. Ostaszewski's online exercise posted June 19, 2010

A box contains 10 balls, of which 3 are red, 2 are yellow, and 5 are blue. Five balls are randomly selected with replacement. Calculate the probability that fewer than 2 of the selected balls are red.

A. 0.3601 B. 0.5000 C. 0.5282 D. 0.8369 E. 0.9167

Solution.

This problem uses sampling with replacement. The general principle of combinatorial probability is that in order to find probability sought, we need to take the ratio of the number of outcomes giving our desired result to the total number of outcomes. The total number of outcomes is always calculated more easily, and in this case, we have 10 ways to choose the first ball, again 10 ways to choose the second one (as the choice is made with replacement), etc. Thus the total number of outcomes is 10^5. We are interested in outcomes that give us only one red ball, or no red balls. The case of no red balls is easy: in such a situation we simply choose from the seven non-red balls five times, and the total number of such outcomes is 7^5. Now let us look at the case of exactly one red ball. Suppose that the only red ball chosen is the very first one. Then we have three choices in the first selection, and 7^4 choices in the remaining selections, for a total of $3 \cdot 7^4$, as the consecutive selections are independent. But the red ball could be in any of the five spots, not just the first one. This raises the total number of outcomes with only one red ball to $5 \cdot 3 \cdot 7^4$. Therefore, the total number of outcomes giving the desired result (no red balls or one red ball) is

$$5 \cdot 3 \cdot 7^4 + 7^5 = 15 \cdot 7^4 + 7 \cdot 7^4 = 22 \cdot 7^4.$$

The desired probability is

$$\frac{22 \cdot 7^4}{10^5} = 2.2 \cdot 0.7^4 \approx 0.52822.$$

Answer C. Note that this problem can be also solved using the methodologies of a later section in this manual, by treating each ball drawing is a Bernoulli Trial with probability of success being 0.3 (3 red balls out of 10), and the number of successes X in five drawings as having the binomial distribution with $n = 5, p = 0.3$. Then

$$\Pr(X<2)=\Pr(X=0)+\Pr(X=1)=\binom{5}{0}\cdot 0.3^0\cdot 0.7^5+\binom{5}{1}\cdot 0.3^1\cdot 0.7^4=$$

$$=\frac{7^5}{10^5}+\frac{5\cdot 3\cdot 7^4}{10^5}=\frac{22\cdot 7^4}{10^5}=2.2\cdot 0.7^4\approx 0.52822.$$

Answer C, again.

Exercise 1.11. February 1996 Course 110 Examination, Problem No. 7, also Dr. Ostaszewski's online exercise posted June 26, 2010

A class contains 8 boys and 7 girls. The teacher selects 3 of the children at random and without replacement. Calculate the probability that the number of boys selected exceeds the number of girls selected.

A. $\frac{512}{3375}$ B. $\frac{28}{65}$ C. $\frac{8}{15}$ D. $\frac{1856}{3375}$ E. $\frac{36}{65}$

Solution.

The number of boys selected exceeds the number of girls selected if there are two or three boys in the group selected. First, the total number of outcomes is the total number of ways to choose 3 children out of 15, without consideration for order, and that is $\binom{15}{3}$. If we choose two boys and one girl, there are $\binom{8}{2}$ ways to choose the boys and $\binom{7}{1}$ ways to choose the girl, for a total of $\binom{8}{2}\cdot\binom{7}{1}$. If we choose three boys, there are $\binom{8}{3}$ ways to pick them, and $\binom{7}{0}$ ways to choose no girls. Thus the desired probability is

$$\frac{\binom{8}{2}\cdot\binom{7}{1}+\binom{8}{3}\cdot\binom{7}{0}}{\binom{15}{3}}=\frac{28\cdot 7+56\cdot 1}{455}=\frac{196+56}{455}=\frac{252}{455}=\frac{36}{65}.$$

Answer E.

Exercise 1.12. May 1983 Course 110 Examination, Problem No. 39, also Dr. Ostaszewski's online exercise posted July 3, 2010

A box contains 10 white marbles and 15 black marbles. If 10 marbles are selected at random and without replacement, what is the probability that x of the 10 marbles are white for $x = 0, 1, \ldots,$

10?

A. $\dfrac{x}{10}$ B. $\dbinom{10}{x}\left(\dfrac{2}{5}\right)^{x}\left(\dfrac{3}{5}\right)^{10-x}$ C. $\dfrac{\dbinom{10}{x}\dbinom{15}{10-x}}{\dbinom{25}{10}}$ D. $\dfrac{\dbinom{10}{x}}{\dbinom{25}{10}}$ E. $\dfrac{\dbinom{10}{x}}{\dbinom{25}{x}}$

Solution.

The total number of ways to pick 10 marbles out of 25 is $\dbinom{25}{10}$. This is the total number of possible outcomes. How many favorable outcomes are there? There are $\dbinom{10}{x}$ ways to choose x white marbles from 10, and $\dbinom{15}{10-x}$ ways to choose $10-x$ black marbles out of 15. This gives a total number of favorable outcomes as $\dbinom{10}{x}\cdot\dbinom{15}{10-x}$, and the desired probability as

$$\frac{\dbinom{10}{x}\cdot\dbinom{15}{10-x}}{\dbinom{25}{10}}.$$

Answer C.

PRACTICE EXAMINATION NUMBER 6

1. An insurance company examines its pool of auto insurance customers and gathers the following information:
(i) All customers insure at least one car.
(ii) 64% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.
Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

A. 0.16 B. 0.19 C. 0.26 D. 0.29 E. 0.31

2. The lifetime of a machine part has a continuous distribution on the interval (0, 40) with probability density function f_X, where $f_X(x)$ is proportional to $(10+x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 5.

A. 0.03 B. 0.13 C. 0.42 D. 0.58 E. 0.97

3. An insurer's annual weather-related loss, X, is a random variable with density function

$$f_X(x)=\begin{cases}\dfrac{2.5\cdot 200^{2.5}}{x^{3.5}}, & \text{for } x>200,\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the difference between the 25-th and 75-th percentiles of X.

A. 124 B. 148 C. 167 D. 224 E. 298

4. A device runs until either of two components fails, at which point the device stops running. The joint density function of the lifetimes of the two components, both measured in hours, is $f_{X,Y}(x,y)=\dfrac{x+y}{8}$ for $0<x<2$ and $0<y<2$. What is the probability that the device fails during its first hour of operation?

A. 0.125 B. 0.141 C. 0.391 D. 0.625 E. 0.875

5. Let $X_{(1)},X_{(2)},\ldots,X_{(6)}$ be the order statistics from a random sample of size 6 from a distribution with density function

$$f_X(x) = \begin{cases} 2x, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is $E\left(X_{(6)}\right)$?

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{5}{6}$ D. $\frac{6}{7}$ E. $\frac{12}{13}$

6. Let X be a normal random variable with mean 0 and variance $a > 0$. Calculate $\Pr\left(X^2 < a\right)$.

A. 0.34 B. 0.42 C. 0.68 D. 0.84 E. 0.90

7. An urn contains 100 lottery tickets. There is one ticket that wins \$50, three tickets that win \$25, six tickets that win \$10, and fifteen tickets that win \$3. The remaining tickets win nothing. Two tickets are chosen at random, with each ticket having the same probability of being chosen. Let X be the amount won by the one of the two tickets that gives the smaller amount won (if both tickets win the same amount, then X is equal to that amount). Find the expected value of X.

A. 0.1348 B. 0.0414 C. 0.2636 D. 0.7922 E. Does not exist

8. (X_1, X_2, X_3) is a random vector with a multivariate distribution with the expected value $(0,0,0)$ and the variance/covariance matrix:

$$\begin{bmatrix} 4 & 1.5 & 1 \\ 1.5 & 1 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix}.$$

If a random variable W is defined by the equation $X_1 = aX_2 + bX_3 + W$ and it is uncorrelated with the variables X_2 and X_3 then the coefficient a must equal:

A. 1 B. $\frac{4}{3}$ C. $\frac{5}{3}$ D. 2 E. $\frac{7}{3}$

9. A random variable X has the exponential distribution with mean $\frac{1}{\lambda}$. Let $[\![x]\!]$ be the greatest integer function, denoting the greatest integer among those not exceeding x. Which of the following is the correct expression for the expected value of $N = [\![X]\!]$?

A. $\left[\!\left[\dfrac{1}{\lambda}\right]\!\right]$ B. $\left[\!\left[\dfrac{1}{\lambda}\right]\!\right]-\dfrac{1}{2}$ C. $\dfrac{1}{[\![\lambda]\!]}-\dfrac{1}{2}$ D. $\dfrac{e^{\lambda}}{e^{\lambda}-1}$ E. $\dfrac{1}{e^{\lambda}-1}$

10. X and Y are independent and both distributed uniformly from 0 to 20. Find the probability density function of $Z = 25X - 10Y$.

A. $f_Z(z) = 1.5$ where non-zero
B. Stepwise formula:

$$f_Z(z) = \begin{cases} \dfrac{200-z}{100000}, & -200 \le z < 0, \\ \dfrac{1}{300}, & 0 \le z < 300, \\ \dfrac{500+z}{100000}, & 300 \le z \le 500. \end{cases}$$

C. $f_Z(z) = \dfrac{1}{200}$ where non-zero

D. $f_Z(z) = 200e^{-\frac{1}{200}z}$, for $z > 0$, zero otherwise
E. Stepwise formula:

$$f_Z(z) = \begin{cases} \dfrac{200+z}{100000}, & -200 \le z < 0, \\ \dfrac{1}{500}, & 0 \le z < 300, \\ \dfrac{500-z}{100000}, & 300 \le z \le 500. \end{cases}$$

11. Let $X_{(1)}, X_{(2)}, \ldots, X_{(8)}$ be the order statistics from a random sample $X_1, X_2, \ldots, X_8$ of size 8 from a continuous probability distribution. What is the probability that the median of the distribution under consideration lies in the interval $\left[X_{(2)}, X_{(7)}\right]$?

A. $\dfrac{110}{128}$ B. $\dfrac{112}{128}$ C. $\dfrac{119}{128}$ D. $\dfrac{124}{128}$ E. Cannot be determined

12. In a block of car insurance business you are considering, there is a 50% chance that a claim will be made during the upcoming year. Once a claim is submitted, the claim size has the Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$, for which the mean is given by the formula $\dfrac{\theta}{\alpha - 1}$, and the second moment is given by the formula $\dfrac{2\theta^2}{(\alpha-2)(\alpha-1)}$. Only one claim will happen during the year. Determine the variance of the unconditional distribution of the claim size.

A. 62500 B. 437500 C. 500000 D. 750000 E. 1000000

13. A random vector (X,Y) has the bivariate normal distribution with mean $(0,0)$ and the variance-covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find the probability that (X,Y) is in the unit circle centered at the origin.

A. 0.2212 B. 0.3679 C. 0.3935 D. 0.6321 E. 0.7788

14. You are given that *X* and *Y* both have the same uniform distribution on [0, 1], and are independent. $U = X + Y$ and $V = \dfrac{X}{X+Y}$. Find the joint probability density function of (*U, V*) evaluated at the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$.

A. 0 B. $\dfrac{1}{4}$ C. $\dfrac{1}{3}$ D. $\dfrac{1}{2}$ E. 1

15. Three fair dice are rolled and *X* is the smallest number of the three values resulting (if more than one value is the smallest one, we still use that value). Find $\Pr(X = 3)$.

A. $\dfrac{36}{216}$ B. $\dfrac{37}{216}$ C. $\dfrac{38}{216}$ D. $\dfrac{39}{216}$ E. $\dfrac{40}{216}$

16. The moment-generating function of a random variable *X* is $M_X(t) = \dfrac{1}{4} + \dfrac{3}{4} \cdot \dfrac{1}{1-t}$, for $t > 1$. Calculate the excess of $\Pr\left(X > \dfrac{3}{4} + \dfrac{\sqrt{15}}{2}\right)$ over its best upper bound given by the (two-sided) Chebyshev's Inequality.

A. 0.35 B. 0.22 C. 0 D. – 0.15 E. – 0.20

17. The time to failure *X* of an MP3 player follows a Weibull distribution, whose survival function is $s_X(x)=e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$ for $x>0$. It is known that $\Pr(X>3)=\frac{1}{e}$, and that $\Pr(X>6)=\frac{1}{e^4}$. Find the probability that this MP3 player is still functional after 4 years.

A. 0.0498 B. 0.0821 C. 0.1353 D. 0.1690 E. 0.2231

18. You are given that the hazard rate for a random variable *X* is $\lambda_X(x)=\frac{1}{2}x^{-\frac{1}{2}}$ for $x>0$, and zero otherwise. Find the mean of *X*.

A. 1 B. 2 C. 2.5 D. 3 E. 3.5

19. Let *P* be the probability that an MP3 player produced in a certain factory is defective, with *P* assumed *a priori* to have the uniform distribution on [0, 1]. In a sample of one hundred MP3 players, 1 is found to be defective. Based on this experience, determine the posterior expected value of *P*.

A. $\frac{1}{100}$ B. $\frac{2}{101}$ C. $\frac{2}{99}$ D. $\frac{1}{50}$ E. $\frac{1}{51}$

20. You are given that $\Pr(A)=\frac{2}{5}$, $\Pr(A\cup B)=\frac{3}{5}$, $\Pr(B|A)=\frac{1}{4}$, $\Pr(C|B)=\frac{1}{3}$, and $\Pr(C|A\cap B)=\frac{1}{2}$. Find $\Pr(A|B\cap C)$.

A. $\frac{1}{3}$ B. $\frac{2}{5}$ C. $\frac{3}{10}$ D. $\frac{1}{2}$ E. $\frac{1}{4}$

21. Let $X_{(1)},\ldots,X_{(n)}$ be the order statistics from the uniform distribution on [0, 1]. Find the correlation coefficient of $X_{(1)}$ and $X_{(n)}$.

A. $-\frac{1}{n}$ B. $-\frac{1}{n+1}$ C. 0 D. $\frac{1}{n+1}$ E. $\frac{1}{n}$

22. A random variable *X* has the log-normal distribution with density $f_X(x) = \frac{1}{x\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(\ln x - \mu)^2}$ for $x > 0$, and 0 otherwise, where μ is a constant. You are given that $\Pr(X \le 2) = 0.4$. Find $E(X)$.

A. 4.25 B. 4.66 C. 4.75 D. 5.00 E. Cannot be determined

23. You are given a continuous random variable with the density $f_X(x) = \frac{1}{2}x + \frac{1}{2}$ for $-1 \le x \le 1$, and 0 otherwise. Find the density of $Y = X^2$, for all points where that density is nonzero.

A. $\frac{1}{2\sqrt{y}}$ B. $2y$ C. $\frac{3}{2y}$ D. $\frac{4}{3}y^2$ E. $\frac{1}{y\ln 2}$

24. An insurer has 10 independent one-year term life insurance policies. The face amount of each policy is 1000. The probability of a claim occurring in the year under consideration is 0.1. Find the probability that the insurer will pay more than the total expected claim for the year.

A. 0.01 B. 0.10 C. 0.16 D. 0.26 E. 0.31

25. An insurance policy is being issued for a loss with the following discrete distribution:

$$X = \begin{cases} 2, & \text{with probability } 0.4, \\ 20, & \text{with probability } 0.6. \end{cases}$$

Your job as the actuary is to set up a deductible *d* for this policy so that the expected payment by the insurer is 6. Find the deductible.

A. 1 B. 5 C. 7 D. 10 E. 15

26. For a Poisson random variable *N* with mean λ find $\lim_{\lambda \to 0} E(N | N \ge 1)$.

A. ∞ B. 0 C. 1 D. e E. Cannot be determined

27. *X* is a normal random variable with mean zero and variance $\frac{1}{2}$ and *Y* is distributed exponentially with mean 1. X and Y are independent. Find the probability $\Pr\left(Y > X^2\right)$.

A. $\frac{1}{\sqrt{e}}$ B. $\sqrt{\frac{e}{\pi}}$ C. $\frac{1}{\sqrt{2\pi}}$ D. $\frac{1}{2}$ E. $\frac{\sqrt{2}}{2}$

28. $X_{(1)}, X_{(2)}, \ldots, X_{(400)}$ are order statistics from a continuous probability distribution with a finite mean, median m and variance σ^2. Let Φ be the cumulative distribution function of the standard normal distribution. Which of the following is the best approximation of $\Pr\left(X_{(220)} \leq m\right)$ using the Central Limit Theorem?

A. $\Phi(0.05\sigma)$ B. 0.0049 C. 0.0532 D. 0.0256 E. $\Phi\left(\frac{20}{\sigma}\right)$

29. N is a Poisson random variable such that $\Pr(N \leq 1) = 2 \cdot \Pr(N = 2)$. Find the variance of N.

A. 0.512 B. 1.121 C. 1.618 D. 3.250 E. 5.000

30. There are two bowls with play chips. The chips in the first bowl are numbered 1, 2, 3, …, 10, while the chips in the second bowl are numbered 6, 7, 8, …, 25. One chip is chosen randomly from each bowl, and the numbers on the two chips so obtained are compared. What is the probability that the two numbers are equal?

A. $\frac{1}{2}$ B. $\frac{1}{5}$ C. $\frac{1}{10}$ D. $\frac{1}{40}$ E. $\frac{1}{50}$

PRACTICE EXAMINATION NUMBER 6
SOLUTIONS

1. P Sample Exam Questions, Problem No. 10, also Dr. Ostaszewski's online exercise posted September 1, 2007

An insurance company examines its pool of auto insurance customers and gathers the following information:
(i) All customers insure at least one car.
(ii) 64% of the customers insure more than one car.
(iii) 20% of the customers insure a sports car.
(iv) Of those customers who insure more than one car, 15% insure a sports car.
Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

A. 0.16 B. 0.19 C. 0.26 D. 0.29 E. 0.31

Solution.
Always start by labeling the events. Let *C* be the event of insuring a sports car (not *S*, because we reserve this for the entire probability space), and *M* be the event of insuring multiple cars. Note that M^C is the event of insuring exactly one car, as all customers insure at least one car. We are given that: $\Pr(M)=0.64$, $\Pr(C)=0.20$, and $\Pr(C|M)=0.15$. We need to find $\Pr(M^C \cap C^C)$.
We recall De Morgan's Law and obtain

$$\Pr(M^C \cap C^C)=\Pr\left((M \cup C)^C\right)=1-\Pr(M \cup C)=1-\Pr(M)-\Pr(C)+\Pr(M \cap C)=$$
$$=1-\Pr(M)-\Pr(C)+\Pr(C|M)\Pr(M)=1-0.64-0.20+0.15\cdot 0.64=0.256.$$

Answer C.

2. P Sample Exam Questions, Problem No. 35, also Dr. Ostaszewski's online exercise posted May 31, 2008

The lifetime of a machine part has a continuous distribution on the interval (0, 40) with probability density function f_X, where $f_X(x)$ is proportional to $(10+x)^{-2}$. Calculate the probability that the lifetime of the machine part is less than 5.

A. 0.03 B. 0.13 C. 0.42 D. 0.58 E. 0.97

Solution.
We know the density has the form $C(10+x)^{-2}$ for $0 < x < 40$ (and is zero otherwise), where *C* is a certain constant. We determine the constant *C* from the standard condition $\int_0^{40} f_X(x)dx=1$:

$$1=\int_0^{40} C(10+x)^{-2}dx=-C(10+x)^{-1}\Big|_0^{40}=\frac{C}{10}-\frac{C}{50}=\frac{2}{25}C,$$

and therefore $C=\frac{25}{2}=12.5$. The probability that $X<5$ is found as the integral of the density over the interval (0, 5):

$$\int_0^5 12.5(10+x)^{-2}dx=-12.5(10+x)^{-1}\Big|_{x=0}^{x=5}=12.5\cdot\left(\frac{1}{10}-\frac{1}{15}\right)=0.4167.$$

Answer C.

3. P Sample Exam Questions, Problem No. 61, and Dr. Ostaszewski's online exercise posted December 15, 2007

An insurer's annual weather-related loss, X, is a random variable with density function

$$f_X(x)=\begin{cases}\dfrac{2.5\cdot 200^{2.5}}{x^{3.5}}, & \text{for } x>200,\\ 0, & \text{otherwise.}\end{cases}$$

Calculate the difference between the 25-th and 75-th percentiles of X.

A. 124 B. 148 C. 167 D. 224 E. 298

Solution.

The cumulative distribution function of X is given by

$$F_X(x)=\int_{200}^{x}\frac{2.5\cdot 200^{2.5}}{t^{3.5}}dt=-\frac{200^{2.5}}{t^{2.5}}\Bigg|_{200}^{x}=1-\frac{200^{2.5}}{x^{2.5}}$$

for $x>200$. Therefore, the p-th percentile x_p of X is given by

$$0.01p=F\left(x_p\right)=1-\frac{200^{2.5}}{x_p^{\;2.5}}=1-\left(\frac{200}{x_p}\right)^{\frac{5}{2}},$$

or $(1-0.01p)^{\frac{2}{5}}=\frac{200}{x_p}$, resulting in $x_p=\frac{200}{(1-0.01p)^{\frac{2}{5}}}$. It follows that

$$x_{75}-x_{25}=\frac{200}{0.25^{\frac{2}{5}}}-\frac{200}{0.75^{\frac{2}{5}}}=123.8292.$$

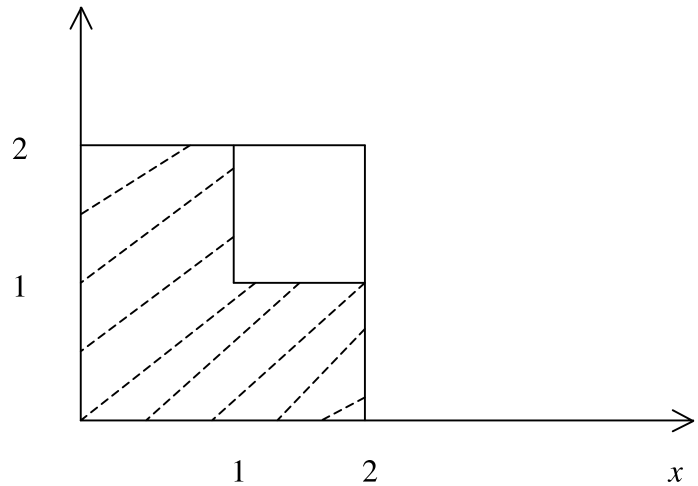
Answer A.

4. P Sample Exam Questions, Problem No. 77, also Dr. Ostaszewski's online exercise posted March 14, 2009

A device runs until either of two components fails, at which point the devise stops running. The joint density function of the lifetimes of the two components, both measured in hours, is $f_{X,Y}(x,y) = \frac{x+y}{8}$ for $0 < x < 2$ and $0 < y < 2$. What is the probability that the device fails during its first hour of operation?

A. 0.125 B. 0.141 C. 0.391 D. 0.625 E. 0.875

Solution.
Probability that the device fails within the first hour is calculated by integrating the joint density function over the shaded region shown below.



This is best done by integrating over the un-shaded region and then subtracting the result from 1:

$$\Pr(\{X<1\}\cup\{Y<1\}) = 1 - \int_1^2\int_1^2 \frac{x+y}{8}dx\,dy = 1 - \int_1^2 \left.\frac{x^2+2xy}{16}\right|_{x=1}^{x=2} dy =$$

$$= 1 - \frac{1}{16}\int_1^2 (4+4y-1-2y)\,dy = 1 - \frac{1}{16}\int_1^2 (3+2y)\,dy = 1 - \frac{1}{16}\left(3y+y^2\right)\Big|_1^2 = 1 - \frac{6}{16} = 0.625.$$

Answer D.

5. May 1983 Course 110 Examination, Problem No. 50, and Dr. Ostaszewski's online exercise posted October 2, 2010

Let $X_{(1)}, X_{(2)}, \ldots, X_{(6)}$ be the order statistics from a random sample of size 6 from a distribution with density function

$$f_X(x) = \begin{cases} 2x, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is $E\left(X_{(6)}\right)$?

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{5}{6}$ D. $\frac{6}{7}$ E. $\frac{12}{13}$

Solution.

The CDF of the original distribution is $F_X(x)=\int_0^x 2t\,dt = t^2\Big|_{t=0}^{t=x} = x^2$ for $0 < x < 1$. We also have $F_X(x)=0$ for $x \le 0$, and $F_X(x)=1$ for $x \ge 1$. Hence, $F_{X_{(6)}}(x)=\Pr\left(X_{(6)} \le x\right)=\left(F_X(x)\right)^6 = x^{12}$ for $0<x<1$. Also, $F_{X_{(6)}}(x)=0$ for $x \le 0$ and $F_{X_{(6)}}(x)=1$ for $x>1$. Therefore, for $0<x<1$, $s_{X_{(6)}}(x)=1-x^{12}$, and $s_{X_{(6)}}(x)=0$ for $x \ge 1$. We conclude that

$$E\left(X_{(6)}\right)=\int_0^1\left(1-x^{12}\right)dx=\left(x-\frac{x^{13}}{13}\right)\Bigg|_0^1=\frac{12}{13}.$$

Answer E.

6. February 1996 Course 110 Examination, Problem No. 2, and Dr. Ostaszewski's online exercise posted October 9, 2010

Let X be a normal random variable with mean 0 and variance $a>0$. Calculate $\Pr\left(X^2<a\right)$.

A. 0.34 B. 0.42 C. 0.68 D. 0.84 E. 0.90

Solution.

Since $a>0$, and a is the variance, $\sqrt{a}$ is the standard deviation of X. Therefore, if we denote a standard normal random variable by Z, and the standard normal CDF by Φ, we get

$$\Pr\left(X^2<a\right)=\Pr\left(-\sqrt{a}<X<\sqrt{a}\right)=\Pr\left(\frac{-\sqrt{a}-0}{\sqrt{a}}<\frac{X-0}{\sqrt{a}}<\frac{\sqrt{a}-0}{\sqrt{a}}\right)=\Pr(-1<Z<1)=$$

$$=\Phi(1)-\Phi(-1)=\Phi(1)-\left(1-\Phi(1)\right)=2\Phi(1)-1=2\cdot 0.8413-1=0.6826.$$

Answer C.

7. Dr. Ostaszewski's online exercise posted March 12, 2005

An urn contains 100 lottery tickets. There is one ticket that wins \$50, three tickets that win \$25, six tickets that win \$10, and fifteen tickets that win \$3. The remaining tickets win nothing. Two tickets are chosen at random, with each ticket having the same probability of being chosen. Let X be the amount won by the one of the two tickets that gives the smaller amount won (if both tickets win the same amount, then X is equal to that amount). Find the expected value of X.

A. 0.1348 B. 0.0414 C. 0.2636 D. 0.7922 E. Does not exist

Solution.

Note that you cannot have X = 50, because there is only one \$50 ticket. Thus, the possible values of X are: 25, 10, 3, and 0. Furthermore, X = 25 when we choose one \$50 ticket and one \$25 ticket, are and there $\binom{1}{1}\cdot\binom{3}{1}$ ways to do that, or when we choose two \$25 tickets, and there are $\binom{3}{2}$ ways to do that. Since there are $\binom{100}{2}$ ways to choose 2 tickets out of 100,

$$\Pr(X=25)=\frac{\binom{1}{1}\cdot\binom{3}{1}+\binom{3}{2}}{\binom{100}{2}}=\frac{1\cdot 3+3}{4950}=\frac{6}{4950}=\frac{1}{825}.$$

Furthermore, X = 10 for one \$10 ticket and one higher amount ticket, or two \$10 tickets, so that

$$\Pr(X=10)=\frac{\binom{4}{1}\cdot\binom{6}{1}+\binom{6}{2}}{\binom{100}{2}}=\frac{4\cdot 6+15}{4950}=\frac{39}{4950}=\frac{13}{1650},$$

while X = 3 for one \$3 ticket and one higher amount ticket, or two \$3 tickets, so that,

$$\Pr(X=3)=\frac{\binom{10}{1}\cdot\binom{15}{1}+\binom{15}{2}}{\binom{100}{2}}=\frac{10\cdot 15+105}{4950}=\frac{255}{4950}=\frac{17}{330},$$

and finally, X = 0, one \$0 ticket and one other ticket, or two \$0 tickets, so that

$$\Pr(X=0)=\frac{\binom{25}{1}\cdot\binom{75}{1}+\binom{75}{2}}{\binom{100}{2}}=\frac{25\cdot 75+2775}{4950}=\frac{4650}{4950}=\frac{31}{33}.$$

Thus

$$E(X)=25\cdot\frac{1}{825}+10\cdot\frac{13}{1650}+3\cdot\frac{17}{330}+0\cdot\frac{31}{33}=\frac{87}{330}\approx 0.2636.$$

Answer C.

8. Dr. Ostaszewski's online exercise posted March 19, 2005

(X_1,X_2,X_3) is a random vector with a multivariate distribution with the expected value $(0,0,0)$ and the variance/covariance matrix:

$$\begin{bmatrix} 4 & 1.5 & 1 \\ 1.5 & 1 & 0.5 \\ 1 & 0.5 & 1 \end{bmatrix}.$$

If a random variable *W* is defined by the equation $X_1 = aX_2 + bX_3 + W$ and it is uncorrelated with the variables X_2 and X_3 then the coefficient *a* must equal:

A. 1 B. $\frac{4}{3}$ C. $\frac{5}{3}$ D. 2 E. $\frac{7}{3}$

Solution.
We have $W = X_1 - aX_2 - bX_3$, and therefore,

$$\mathrm{Cov}(W, X_2) = \mathrm{Cov}(X_1 - aX_2 - bX_3, X_2) = \mathrm{Cov}(X_1, X_2) - a\mathrm{Var}(X_2) - b\mathrm{Cov}(X_3, X_2) =$$
$$= 1.5 - a - 0.5b = 0,$$
$$\mathrm{Cov}(W, X_3) = \mathrm{Cov}(X_1 - aX_2 - bX_3, X_3) = \mathrm{Cov}(X_1, X_3) - a\mathrm{Cov}(X_2, X_3) - b\mathrm{Var}(X_3) =$$
$$= 1 - 0.5a - b = 0,$$

Hence, $3 - 2a = 1 - 0.5a$, so that $2 = 1.5a$, and $a = \frac{4}{3}$.
Answer B.

9. Dr. Ostaszewski's online exercise posted October 16, 2010
A random variable *X* has the exponential distribution with mean $\frac{1}{\lambda}$. Let $[\![x]\!]$ be the greatest integer function, denoting the greatest integer among those not exceeding *x*. Which of the following is the correct expression for the expected value of $N = [\![X]\!]$?

A. $\left[\!\left[\frac{1}{\lambda}\right]\!\right]$ B. $\left[\!\left[\frac{1}{\lambda}\right]\!\right] - \frac{1}{2}$ C. $\frac{1}{[\![\lambda]\!]} - \frac{1}{2}$ D. $\frac{e^{\lambda}}{e^{\lambda} - 1}$ E. $\frac{1}{e^{\lambda} - 1}$

Solution.
Note that *N* is a discrete non-negative random variable, so that its expected value can be calculated as (note that $e^{-\lambda} < 1$, because $\lambda > 0$):

$$E(N) = \sum_{n=1}^{+\infty} \Pr(N \geq n) = \sum_{n=1}^{+\infty} \Pr([\![X]\!] \geq n) = \sum_{n=1}^{+\infty} \Pr(X \geq n) = \sum_{n=1}^{+\infty} e^{-\lambda n} = \frac{e^{-\lambda}}{1 - e^{-\lambda}} = \frac{1}{e^{\lambda} - 1}.$$

Answer E.

10. Dr. Ostaszewski's online exercise posted May 14, 2005
X and *Y* are independent and both distributed uniformly from 0 to 20. Find the probability

density function of $Z = 25X - 10Y$.

A. $f_Z(z) = 1.5$ where non-zero
B. Stepwise formula:

$$f_Z(z) = \begin{cases} \dfrac{200 - z}{100000}, & -200 \le z < 0, \\ \dfrac{1}{300}, & 0 \le z < 300, \\ \dfrac{500 + z}{100000}, & 300 \le z \le 500. \end{cases}$$

C. $f_Z(z) = \dfrac{1}{200}$ where non-zero
D. $f_Z(z) = 200e^{-\frac{1}{200}z}$, for $z > 0$, zero otherwise
E. Stepwise formula:

$$f_Z(z) = \begin{cases} \dfrac{200 + z}{100000}, & -200 \le z < 0, \\ \dfrac{1}{500}, & 0 \le z < 300, \\ \dfrac{500 - z}{100000}, & 300 \le z \le 500. \end{cases}$$

Solution.
This problem can be solved in a simplified way, but we will discuss three possible solutions, in a drawn-out fashion, in order to fully explain possible approaches such problems involving sums or differences of random variables. To begin with, note the following: $f_X(x) = \dfrac{1}{20}$ for $0 < x < 20$, and zero otherwise, as well as $f_Y(y) = \dfrac{1}{20}$ for $0 < y < 20$, and zero otherwise, and finally $f_{X,Y}(x,y) = \dfrac{1}{20} \cdot \dfrac{1}{20} = \dfrac{1}{400}$ for $0 < x < 20$ and $0 < y < 20$, and zero otherwise. There are actually three possible approaches to solving this: the multivariate transformation approach, the convolution approach, and the CDF approach. You should know all of them for Exam P/1, so all three will be presented here.

• *Multivariate transformation*

Consider a transformation $W = 10Y$, $Z = 25X - 10Y$, i.e., $(W,Z) = \Phi(X,Y) = (10Y, 25X - 10Y)$, whose inverse is $\Phi^{-1}(W,Z) = \left(\dfrac{W+Z}{25}, \dfrac{W}{10}\right)$. You might wonder: how do I know that I am supposed to pick $W = 10Y$? You are not supposed to pick anything. Your objective is to find a second function of X and Y such that you will be able to find the inverse of the transformation so obtained. There is no unique answer. In this case, $W = Y$ would do the job, so would $W = 25X + 10Y$, and so would inifinitely many other choices. The key point is that you must be able to find Φ^{-1} and then find the determinant of its derivative (the *Jacobian)*. Let us find that derivative now (we switch to lower case variables because this is what we will use in the density):

$$\left(\Phi^{-1}\right)'(w,z)=\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{bmatrix}=\begin{bmatrix} \frac{1}{25} & \frac{1}{25} \\ \frac{1}{10} & 0 \end{bmatrix}.$$

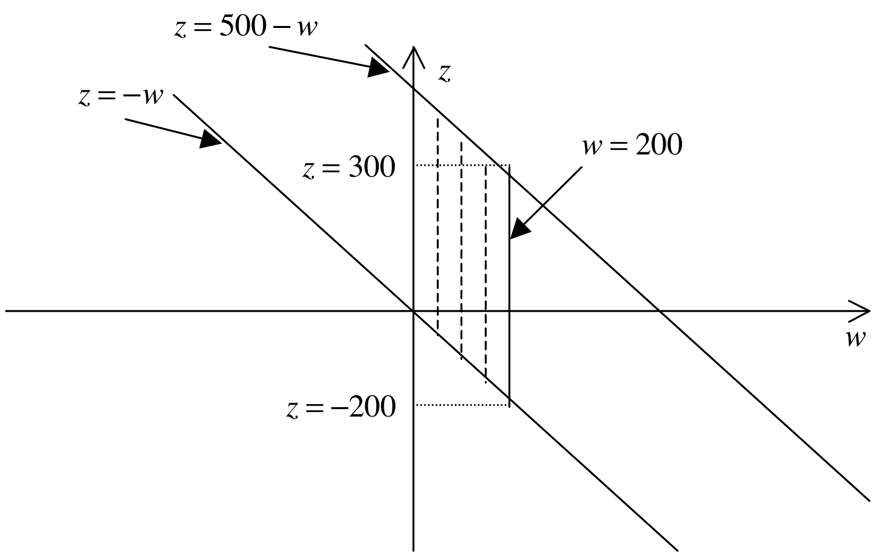
Therefore,

$$\frac{\partial(x,y)}{\partial(w,z)}=\det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{bmatrix}=-\frac{1}{250}.$$

This gives

$$f_{W,Z}(w,z)=f_{X,Y}\left(x(w,z),y(w,z)\right)\cdot\left|\frac{\partial(x,y)}{\partial(w,z)}\right|=\frac{1}{400}\cdot\frac{1}{250}=\frac{1}{100000}.$$

We also have to figure out the ranges for w and z. As $w = 10y$ and $0 < y < 20,$ we have $0 < w < 200.$ Also, as $0 < x < 20$ and $x = \frac{z+w}{25},$ so that $0 < \frac{z+w}{25} < 20,$ and this means that $-w < z < 500 - w$ or equivalently $-z < w < 500 - z.$ Graphically,



Given that, we can now figure out the marginal density of Z:

$$f_Z(z) = \int\limits_{\text{all values of } w} f_{W,Z}(w,z)\,dw =$$

$$= \left\{ \begin{array}{ll} \displaystyle\int_{-z}^{200} \frac{1}{100000}\,dw, & -200 \le z < 0, \\ \displaystyle\int_{0}^{200} \frac{1}{100000}\,dw, & 0 \le z < 300, \\ \displaystyle\int_{0}^{500-z} \frac{1}{100000}\,dw, & 300 \le z \le 500. \end{array} \right\} = \left\{ \begin{array}{ll} \dfrac{200+z}{100000}, & -200 \le z < 0, \\ \dfrac{1}{500}, & 0 \le z < 300, \\ \dfrac{500-z}{100000}, & 300 \le z \le 500. \end{array} \right.$$

Answer E.

• *The convolution method*

Recall that if X and Y have a continuous joint distribution and are continuous, then the density of $X+Y$ is $f_{X+Y}(s) = \int_{-\infty}^{+\infty} f_{X,Y}(x, s-x)\,dx$. If X and Y are independent, then

$$f_{X+Y}(s) = \int_{-\infty}^{+\infty} f_X(x) f_Y(s-x)\,dx.$$

In this case, we are adding

• $U = 25X$, which has the uniform distribution on $(0, 500)$, and is independent of

• $V = -10Y$, which has the uniform distribution on $(-200, 0)$.

The density of U is $\frac{1}{500}$, where non-zero, and the density of V is $\frac{1}{200}$, where non-zero. Thus

$$f_{U+V}(s) = \int_{-\infty}^{+\infty} f_U(u)\cdot f_V(s-u)\,du = \int\limits_{\substack{0\le u\le 500 \text{ and} \\ -200\le s-u\le 0}} \frac{1}{500}\cdot\frac{1}{200}\,du = \int\limits_{\substack{0\le u\le 500 \text{ and} \\ s\le u\le s+200}} \frac{1}{500}\cdot\frac{1}{200}\,du =$$

$$= \left\{ \begin{array}{ll} \displaystyle\int_{0}^{s+200} \frac{1}{100000}\,du, & -200 \le s < 0, \\ \displaystyle\int_{s}^{s+200} \frac{1}{100000}\,du, & 0 \le s < 300, \\ \displaystyle\int_{s}^{500} \frac{1}{100000}\,du, & 300 \le s \le 500. \end{array} \right\} = \left\{ \begin{array}{ll} \dfrac{200+s}{100000}, & -200 \le s < 0, \\ \dfrac{1}{500}, & 0 \le s < 300, \\ \dfrac{500-s}{100000}, & 300 \le s \le 500. \end{array} \right.$$

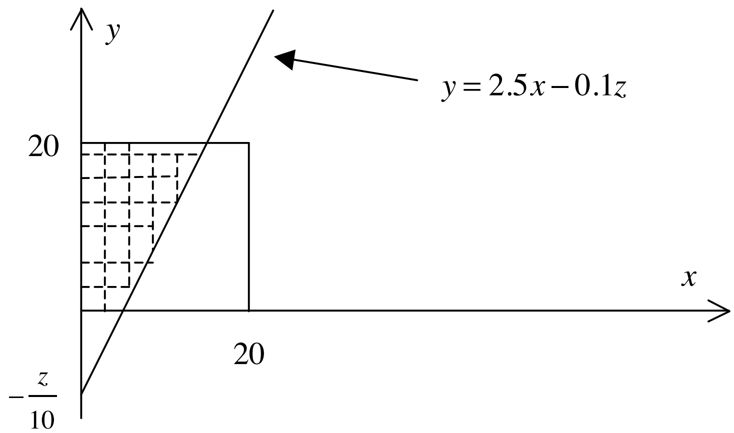
Answer E, again.

• *The CDF method*

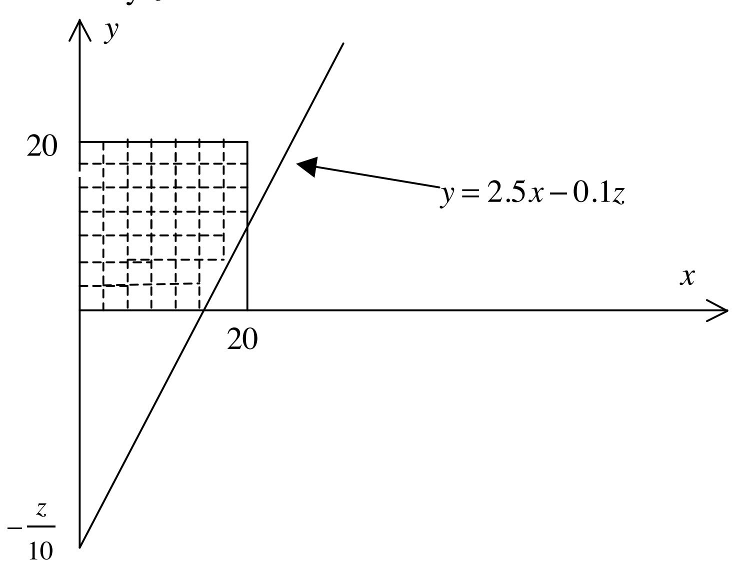
We have $Z = 25X - 10Y$. Let us find the CDF of it directly. We have:

$$F_Z(z) = \Pr(Z \le z) = \Pr(25X - 10Y \le z).$$

This probability can be obtained by simply taking the integral of the joint density of X and Y over the region where $25X - 10Y \le z$. The figure below shows the region where the joint density is allowed to play, and the shaded area is where $25X - 10Y \le z$.



The problem is that the position of the line $y = 2.5x - 0.1z$ can vary as z varies, and we get different results in different cases. The line crosses the point $(20,0)$ when $z = 500$. If $z \ge 500$ then the line does not go through the twenty by twenty square at all and the probability of being above the line is 1. Since the slope of the line is more than 1, the next point where a change occurs is when the line crosses the point (20,20), which occurs for $z = 300$. The value of the CDF of *Z* for any z between 300 and 500 is the area shown in this figure:



The point where the line crosses the *x* axis is at $(x,y) = \left(\frac{z}{25},0\right)$ and the point where it crosses the line *x* = 20 is at $(x,y) = \left(20, 50 - \frac{z}{10}\right)$. The area of the bottom-right triangle left out of the calculation of probability is therefore $\frac{1}{2}\cdot\left(20 - \frac{z}{25}\right)\cdot\left(50 - \frac{z}{10}\right)$, and the corresponding probability is $\frac{1}{400}$ times that, so that the probability we are looking for is:

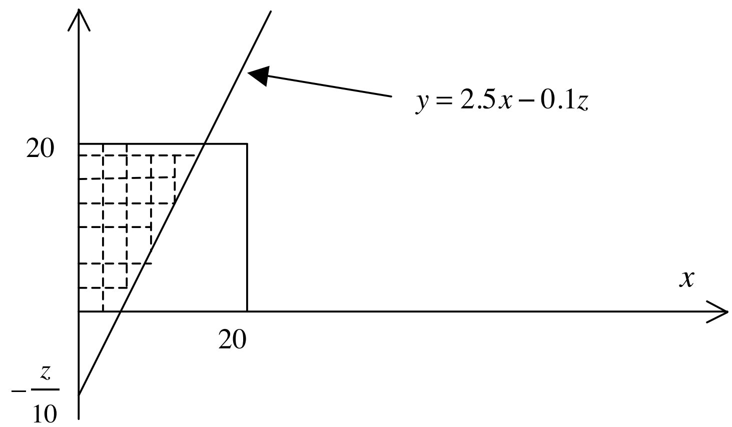
$$F_Z(z) = 1 - \frac{1}{800} \cdot \left(20 - \frac{z}{25}\right) \cdot \left(50 - \frac{z}{10}\right).$$

The corresponding density is:

$$f_Z(z) = F_Z'(z) = -\frac{1}{800} \cdot \left(-\frac{1}{25}\right) \cdot \left(50 - \frac{z}{10}\right) - \frac{1}{800} \cdot \left(20 - \frac{z}{25}\right) \cdot \left(-\frac{1}{10}\right) =$$

$$= \frac{1}{400} - \frac{z}{800 \cdot 25 \cdot 10} + \frac{1}{400} - \frac{z}{800 \cdot 25 \cdot 10} = \frac{500 - z}{100000},$$

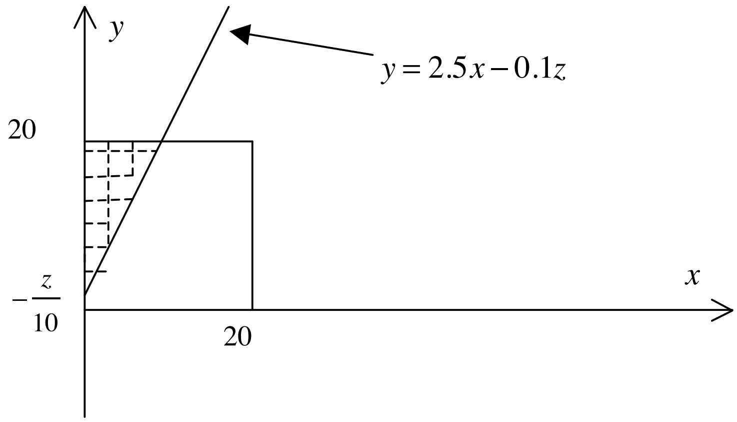
for $300 \le z \le 500$. This is the same answer in this range of values of z that we obtained before. The second case starts with z for which the point (20, 20) is crossed by the line, i.e., $z = 300$. This case is generally described by this figure:



This case ends when the line crosses the origin, i.e., when $z = 0$. Between $z = 0$ and $z = 300$, the probability we want to find is just the area of the shaded region as a fraction of the area of the 20 by 20 square. The bottom side of the shaded region has length $\frac{z}{25}$ (from $y = 0$ and the equation of the line) and the top side has length $8 + \frac{z}{25}$ (from $y = 20$ and the equation of the line), so that its area is $20 \cdot \left(4 + \frac{z}{25}\right) = 80 + \frac{4}{5}z$. As a fraction of the entire area, this is

$$F_Z(z) = \frac{80 + \frac{4}{5}z}{400} = \frac{1}{5} + \frac{z}{500}.$$

Therefore, $f_Z(z) = F_Z'(z) = \frac{1}{500}$ for $0 \le z \le 300$. Again, this is the same answer we obtained before. The final case is when the line crosses the y-axis above the origin, but below 20. When the line crosses the origin, $z = 0$. When the line crosses the y-axis at the point (0, 20), we have $z = -200$. In this case, the figure looks as follows:



When $y = 20$, we have $x = 8 + \frac{z}{25}$, so that the area of the marked triangle is

$$\frac{1}{2}\cdot\left(8+\frac{z}{25}\right)\cdot\left(20+\frac{z}{10}\right).$$

The 20 by 20 square has the area of 400, so that

$$F_Z(z) = \frac{\frac{1}{2}\cdot\left(8+\frac{z}{25}\right)\cdot\left(20+\frac{z}{10}\right)}{400} = \frac{\left(8+\frac{z}{25}\right)\cdot\left(20+\frac{z}{10}\right)}{800}.$$

Therefore,

$$f_Z(z) = F_Z'(z) = \frac{1}{800}\left(\frac{1}{25}\left(20+\frac{z}{10}\right)+\frac{1}{10}\left(8+\frac{z}{25}\right)\right) = \frac{1.6+\frac{z}{125}}{800} = \frac{200+z}{100000},$$

for $-200 \le z \le 0$. This is again the same formula we obtained before.
Answer E.

11. Dr. Ostaszewski's online exercise posted March 26, 2005

Let $X_{(1)}, X_{(2)}, \ldots, X_{(8)}$ be the order statistics from a random sample $X_1, X_2, \ldots, X_8$ of size 8 from a continuous probability distribution. What is the probability that the median of the distribution under consideration lies in the interval $\left[X_{(2)}, X_{(7)}\right]$?

A. $\frac{110}{128}$ B. $\frac{112}{128}$ C. $\frac{119}{128}$ D. $\frac{124}{128}$ E. Cannot be determined

Solution.

Let m be the median of the distribution. We are looking for $\Pr\left(X_{(2)} \le m \le X_{(7)}\right)$. We note the following:

$$\Pr\left(X_{(2)} \le m \le X_{(7)}\right) = 1 - \Pr\left(\left\{m < X_{(2)}\right\} \cup \left\{X_{(7)} < m\right\}\right) = 1 - \Pr\left(m < X_{(2)}\right) - \Pr\left(X_{(7)} < m\right).$$

Now we note that $\Pr\left(m < X_{(2)}\right)$ is the same as the probability that in a random sample $X_1, X_1, \ldots, X_8$ of size 8 there are 0 or 1 elements less than or equal to *m*. As the probability of being less than or equal to *m* is exactly $\frac{1}{2}$, this amounts to performing eight Bernoulli trials with the probability of success of $\frac{1}{2}$ (with "success" being having a piece of the random sample less than the median) and getting only 0 or 1 successes, and that probability equals

$$\binom{8}{0} \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^8 + \binom{8}{1} \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^7 = 1 \cdot \frac{1}{2^8} + 8 \cdot \frac{1}{2^8} = \frac{9}{256}.$$

Similarly, $\Pr\left(X_{(7)} < m\right)$ is the probability that either none or one of eight elements of the random sample $X_1, X_1, \ldots, X_8$ are greater than or equal to *m*. As the probability of being greater than or equal to *m* is exactly $\frac{1}{2}$, this again amounts to performing eight Bernoulli trials with the probability of success (this time "success" means getting a number more than the median) of $\frac{1}{2}$ and getting only 0 or 1 successes, i.e., the probability we are looking for, $\Pr\left(X_{(7)} < m\right)$, is again $\frac{9}{256}$. Hence:

$$\Pr\left(X_{(2)} \le m \le X_{(7)}\right) = 1 - \Pr\left(m < X_{(2)}\right) - \Pr\left(X_{(7)} < m\right) =$$

$$= 1 - \frac{9}{256} - \frac{9}{256} = 1 - \frac{18}{256} = 1 - \frac{9}{128} = \frac{119}{128}.$$

Answer C.

12. Dr. Ostaszewski's online exercise posted April 2, 2005

In a block of car insurance business you are considering, there is a 50% chance that a claim will be made during the upcoming year. Once a claim is submitted, the claim size has the Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$, for which the mean is given by the formula $\frac{\theta}{\alpha - 1}$, and the second moment is given by the formula $\frac{2\theta^2}{(\alpha-2)(\alpha-1)}$. Only one claim will happen during the year. Determine the variance of the unconditional distribution of the claim size.

A. 62500 B. 437500 C. 500000 D. 750000 E. 1000000

Solution.

SECTION 10

Let Y be the random claim size. Its probability distribution is mixed: 50% in the Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$ and 50% in a point-mass at 0. Its mean is simply half of the Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$, i.e., half of $\frac{\theta}{\alpha - 1}$, or $\frac{1}{2} \cdot \frac{1000}{2} = 250$. The second moment is also half of the second moment of that Pareto distribution, i.e., half of $\frac{2\theta^2}{(\alpha - 2)(\alpha - 1)}$, or $\frac{1}{2} \cdot \frac{2 \cdot 1000^2}{1 \cdot 2} = 500000$. The variance is therefore $500000 - 250^2 = 437500$. That's Answer B. You could also do this by defining $X = 0$ when there is no claim, and $X = 1$ when there is a claim, with $\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2}$, so that X is a Bernoulli Trial with $p = \frac{1}{2}$. Then we see that $(Y|X = 0)$ is degenerate distribution equal to 0 with probability 1, while $(Y|X = 1)$ is Pareto with $\alpha = 3$ and $\theta = 1000$. Based on this

$$E(Y|X = x) = \begin{cases} 0 & x = 0, \\ \frac{\theta}{\alpha - 1} = \frac{1000}{3 - 1} = 500 & x = 1, \end{cases}$$

so that $E(Y|X = x) = 500x$, and

$$\mathrm{Var}(Y|X = x) = \begin{cases} 0 & x = 0, \\ \frac{2\theta^2}{(\alpha - 2)(\alpha - 1)} - \left(\frac{\theta}{\alpha - 1}\right)^2 = \frac{2 \cdot 1000^2}{(3 - 2)(3 - 1)} - 500^2 = 750000 & x = 1, \end{cases}$$

so that $\mathrm{Var}(Y|X = x) = 750000x$. Therefore

$$E(\mathrm{Var}(Y|X)) = E(750{,}000X) = 750{,}000 \cdot E(X) = 750{,}000 \cdot \frac{1}{2} = 375{,}000,$$

$$\mathrm{Var}(E(Y|X)) = \mathrm{Var}(500X) = 500^2 \cdot \mathrm{Var}(X) = 500^2 \cdot 0.5 \cdot 0.5 = 62{,}500,$$

$$\mathrm{Var}(Y) = E(\mathrm{Var}(Y|X)) + \mathrm{Var}(E(Y|X)) = 375{,}000 + 62{,}500 = 437{,}500.$$

Answer B, again.

13. Dr. Ostaszewski's online exercise posted April 9, 2005

A random vector (X, Y) has the bivariate normal distribution with mean $(0,0)$ and the variance-covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find the probability that (X, Y) is in the unit circle centered at the origin.

A. 0.2212 B. 0.3679 C. 0.3935 D. 0.6321 E. 0.7788

Solution.
First observe that X and Y, being uncorrelated and having the bivariate normal distribution, are independent. We are trying to find $\Pr\left(X^2+Y^2\leq 1\right)$. But X^2+Y^2 has the chi-square distribution with two degrees of freedom, or, equivalently, the gamma distribution with parameters $\alpha=\frac{2}{2}=1$, and $\beta=\frac{1}{2}$. But any gamma distribution with $\alpha=1$ is an exponential distribution whose parameter is $\lambda=\beta$ and mean is $\frac{1}{\beta}$. So the problem simply asks what the probability is that an exponential distribution with mean 2 is less than 1, and that's $1-e^{-\frac{1}{2}}\approx 0.3935$.
Answer C.

14. Dr. Ostaszewski's online exercise posted April 16, 2005
You are given that X and Y both have the same uniform distribution on [0, 1], and are independent. $U=X+Y$ and $V=\frac{X}{X+Y}$. Find the joint probability density function of (U, V) evaluated at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

A. 0 B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$ E. 1

Solution.
We have $X=UV$, and $Y=U-UV$. Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}=\det\begin{bmatrix} v & u\\ 1-v & -u\end{bmatrix}=-uv-u+uv=-u.$$

This implies that

$$f_{U,V}(u,v)=f_{X,Y}\left(x(u,v),y(u,v)\right)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|-u\right|=u.$$

This density at $\left(\frac{1}{2},\frac{1}{2}\right)$ equals $\frac{1}{2}$.
Answer D.

SECTION 10

15. Dr. Ostaszewski's online exercise posted April 23, 2005

Three fair dice are rolled and *X* is the smallest number of the three values resulting (if more than one value is the smallest one, we still use that value). Find $\Pr(X = 3)$.

A. $\frac{36}{216}$ B. $\frac{37}{216}$ C. $\frac{38}{216}$ D. $\frac{39}{216}$ E. $\frac{40}{216}$

Solution.

$\Pr(X = 3)$ is the same as the probability that all three dice show a 3 or more, and at least one shows a 3. This means that the event of all three showing at least 3 happened, but the event of all three of them showing at least a 4 did not happen. This probability is:

$$\underbrace{\left(\frac{2}{3}\right)^3}_{\text{All three show 3,4,5 or 6}} \underbrace{-}_{\text{but not}} \underbrace{\left(\frac{1}{2}\right)^3}_{\text{All three show 4,5, or 6}} = \frac{8}{27} - \frac{1}{8} = \frac{64-27}{216} = \frac{37}{216}.$$

Answer B.

16. Dr. Ostaszewski's online exercise posted April 30, 2005

The moment-generating function of a random variable *X* is $M_X(t) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{1-t}$, for $t > 1$.

Calculate the excess of $\Pr\left(X > \frac{3}{4} + \frac{\sqrt{15}}{2}\right)$ over its best upper bound given by the (two-sided) Chebyshev's Inequality.

A. 0.35 B. 0.22 C. 0 D. – 0.15 E. – 0.20

Solution.

Recall that the moment-generating function (MGF) of a mixed distribution is a weighted average of MGFs of the pieces of the mixture. Also recall that the MGF of the exponential distribution with parameter λ is $M_T(u) = \frac{\lambda}{\lambda - u}$, for $u < \lambda$, so with $\lambda = 1$, for $u < 1$, $M_T(u) = \frac{1}{1-u}$. This looks familiar, but not what we want. Consider a mixture of a degenerate random variable *W*, where $\Pr(W = 0) = 1$, with weight $\frac{1}{4} = 0.25$, and *T*, exponential with $\lambda = 1$, with weight $\frac{3}{4} = 0.75$. Note that $M_W(u) = E\left(e^{uW}\right) = E\left(e^0\right) = 1$. We see then that

$$X = \begin{cases} W & \text{with probability } \frac{1}{4}, \\ T & \text{with probability } \frac{3}{4}. \end{cases}$$

Therefore

$$E(X)=\frac{1}{4}\cdot 0+\frac{3}{4}\cdot 1=\frac{3}{4},$$

Also, the second moment is:

$$E\left(X^2\right)=\frac{1}{4}\cdot 0^2+\frac{3}{4}\cdot\underbrace{(1+1)}_{\substack{\text{Second moment of } T\\ =E(T)+\operatorname{Var}(T)}}=\frac{3}{2}.$$

The variance of X is therefore $\frac{3}{2}-\frac{9}{16}=\frac{24}{16}-\frac{9}{16}=\frac{15}{16}$, and the standard deviation is $\frac{\sqrt{15}}{4}$.

Because this random variable is non-negative almost surely, the (two-sided) Chebyshev's Inequality implies that

$$\frac{1}{4}=\frac{1}{2^2}\geq \Pr\left(\left|X-\frac{3}{4}\right|>2\frac{\sqrt{15}}{4}\right)=\Pr\left(\left|X-\frac{3}{4}\right|>\frac{\sqrt{15}}{2}\right)=$$

$$=\Pr\left(X<\frac{3}{4}-\frac{\sqrt{15}}{2}\right)+\Pr\left(X>\frac{3}{4}+\frac{\sqrt{15}}{2}\right)=0+\Pr\left(X>\frac{3}{4}+\frac{\sqrt{15}}{2}\right).$$

Thus $\Pr\left(X>\frac{3}{4}+\frac{\sqrt{15}}{2}\right)\leq\frac{1}{4}$ is the best upper bound estimate for this probability provided by the (two-sided) Chebyshev's Inequality. The exact probability is

$$\Pr\left(X>\frac{3}{4}+\frac{\sqrt{15}}{2}\right)=\frac{3}{4}\Pr\left(T>\frac{3}{4}+\frac{\sqrt{15}}{2}\right)=\frac{3}{4}e^{-\left(\frac{3}{4}+\frac{\sqrt{15}}{2}\right)}\approx 0.0511.$$

The excess equals approximately 0.0511 – 0.25 = – 0.1989. It is interesting to note that if we use the one-sided Chebyshev's Inequality

$$\Pr(X-\mu\geq k\sigma)=\Pr(X\geq\mu+k\sigma)\leq\frac{1}{1+k^2},$$

with $\mu=\frac{3}{4}$, $\sigma=\frac{\sqrt{15}}{4}$, and $k = 2$, we get $\Pr\left(X>\frac{3}{4}+\frac{\sqrt{15}}{2}\right)\leq\frac{1}{1+2^2}=\frac{1}{5}$, and the excess equals approximately 0.0511 – 0.20 = –0.1489, approximately answer D. But in my opinion, the one-sided version of the Chebyshev's Inequality is not covered on exam P/1.

Answer E.

17. Dr. Ostaszewski's online exercise posted May 7, 2005

The time to failure X of an MP3 player follows a Weibull distribution, whose survival function is $s_X(x)=e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$ for $x>0$. It is known that $\Pr(X>3)=\frac{1}{e}$, and that $\Pr(X>6)=\frac{1}{e^4}$. Find the probability that this MP3 player is still functional after 4 years.

A. 0.0498 B. 0.0821 C. 0.1353 D. 0.1690 E. 0.2231
Solution.
From the information given in the problem we have $s_X(3)=e^{-1}$ and $s_X(6)=e^{-4}$. Therefore, $e^{-\left(\frac{3}{\alpha}\right)^{\beta}}=e^{-1}$, as well as $e^{-\left(\frac{6}{\alpha}\right)^{\beta}}=e^{-4}$. We are looking for $s_X(4)=e^{-\left(\frac{4}{\alpha}\right)^{\beta}}$. But $e^{-\left(\frac{3}{\alpha}\right)^{\beta}}=e^{-1}$ implies that $\left(\frac{3}{\alpha}\right)^{\beta}=1$, or $\alpha=3$. Furthermore, $e^{-\left(\frac{6}{\alpha}\right)^{\beta}}=e^{-2^{\beta}}=e^{-4}$ implies that $\beta=2$. Therefore, we conclude that $s_X(4)=e^{-\left(\frac{4}{\alpha}\right)^{\beta}}=e^{-\left(\frac{4}{3}\right)^{2}}=e^{-\frac{16}{9}}\approx 0.1690.$
Answer D.

18. Dr. Ostaszewski's online exercise posted October 23, 2010

You are given that the hazard rate for a random variable X is $\lambda_X(x)=\frac{1}{2}x^{-\frac{1}{2}}$ for $x>0$, and zero otherwise. Find the mean of X.

A. 1 B. 2 C. 2.5 D. 3 E. 3.5

Solution.

We calculate the survival function of X as $e^{-\int_0^x \frac{1}{2}t^{-\frac{1}{2}}dt}=e^{-\left(\sqrt{t}\right)\Big|_0^x}=e^{\left(-\sqrt{x}+\sqrt{0}\right)}=e^{-\sqrt{x}}$ for $x>0$, and then using the Darth Vader Rule we calculate the mean:

$$E(X)=\underbrace{\int_0^{+\infty} e^{-\sqrt{x}}\,dx}_{\substack{\text{Substitute } z=\sqrt{x},\\ \text{then } 2zdz=dx}}=\int_0^{+\infty}2ze^{-z}dz=2\cdot\underbrace{\int_0^{+\infty}ze^{-z}dx}_{\text{Mean of EXP(1)}}=2.$$

You could also notice that this is the hazard rate $\lambda_X(x)=\frac{\beta}{\alpha}\cdot\left(\frac{x}{\alpha}\right)^{\beta-1}=\frac{\beta}{\alpha^{\beta}}\cdot x^{\beta-1}$ for a Weibull distribution with $\beta=\frac{1}{2}$ and $\alpha=1$, so that $E(X)=\alpha\Gamma\left(1+\frac{1}{\beta}\right)=1\cdot\Gamma(3)=2$, but of course you are not expected to memorize the Weibull distribution any more.
Answer B.

19. Dr. Ostaszewski's online exercise posted October 30, 2010
Let P be the probability that an MP3 player produced in a certain factory is defective, with P assumed *a priori* to have the uniform distribution on [0, 1]. In a sample of one hundred MP3 players, 1 is found to be defective. Based on this experience, determine the posterior expected value of P.

A. $\frac{1}{100}$ B. $\frac{2}{101}$ C. $\frac{2}{99}$ D. $\frac{1}{50}$ E. $\frac{1}{51}$

Solution.
Let us write N for the number of defective items in a sample of 100. Then $N|P=p$ is binomial with $n = 100$ and p being the probability of success in a single Bernoulli Trial. We have

$$f_P(p|N=1)=\frac{f_{P,N}(p,1)}{f_N(1)}=\frac{f_{P,N}(p,1)}{\int_0^1 f_{P,N}(p,1)\,dp}=\frac{f_P(p)f_N(1|P=p)}{\int_0^1 f_P(p)f_N(1|P=p)\,dp}=$$

$$=\frac{1\cdot\begin{pmatrix}100\\1\end{pmatrix}\cdot p^1\cdot(1-p)^{99}}{\int_0^1 1\cdot\begin{pmatrix}100\\1\end{pmatrix}\cdot p^1\cdot(1-p)^{99}\,dp}=\frac{p(1-p)^{99}}{\int_0^1 p(1-p)^{99}\,dp}.$$

Now note that

$$\int_0^1 p(1-p)^{99}\,dp=\underbrace{\left|\begin{array}{ll} u=p & v=-\frac{(1-p)^{100}}{100} \\ du=dp & dv=(1-p)^{99}\,dp \end{array}\right|}_{\text{Integration by parts}}=-\frac{p(1-p)^{100}}{100}\Bigg|_{p=0}^{p=1}+\frac{\int_0^1 (1-p)^{100}\,dp}{100}=$$

$$=0+\frac{1}{100}\cdot\left(-\frac{(1-p)^{101}}{101}\Bigg|_{p=0}^{p=1}\right)=\frac{1}{10100}.$$

This means that $f_P(p|N=1)=10100p(1-p)^{99}$. We then calculate

$$E(P|N=1)=\int_0^1 p\cdot 10100p(1-p)^{99}\,dp=101\int_0^1 p^2\cdot 100(1-p)^{99}\,dp=$$

$$=\underbrace{\left|\begin{array}{ll} u=p^2 & v=-(1-p)^{100} \\ du=2p\,dp & dv=100(1-p)^{99}\,dp \end{array}\right|}_{\text{Integration by parts}}=-101p^2(1-p)^{100}\Big|_{p=0}^{p=1}+\int_0^1 202p(1-p)^{100}\,dp=$$

$$=202\int_0^1 p(1-p)^{100}\,dp=\underbrace{\left|\begin{array}{ll} u=p & v=-\frac{(1-p)^{101}}{101} \\ du=dp & dv=(1-p)^{100}\,dp \end{array}\right|}_{\text{Integration by parts}}=2\int_0^1 (1-p)^{101}\,dp=\frac{1}{51}.$$

If you remember the beta distribution, then the posterior probability distribution of P is beta with

parameters 1 + 1 = 2, and 99 + 1 = 100. Thus, its expected value is $\frac{2}{100+2}=\frac{1}{51}$, same answer.
Answer E.

20. Dr. Ostaszewski's online exercise posted November 6, 2010

You are given that $\Pr(A)=\frac{2}{5}$, $\Pr(A\cup B)=\frac{3}{5}$, $\Pr(B|A)=\frac{1}{4}$, $\Pr(C|B)=\frac{1}{3}$, and $\Pr(C|A\cap B)=\frac{1}{2}$. Find $\Pr(A|B\cap C)$.

A. $\frac{1}{3}$ B. $\frac{2}{5}$ C. $\frac{3}{10}$ D. $\frac{1}{2}$ E. $\frac{1}{4}$

Solution.

First note that $\frac{1}{4}=\Pr(B|A)=\frac{\Pr(A\cap B)}{\Pr(A)}=\frac{5}{2}\cdot\Pr(A\cap B)$, so that $\Pr(A\cap B)=\frac{1}{10}$. Hence,

$$\Pr(A|B\cap C)=\frac{\Pr(A\cap B\cap C)}{\Pr(B\cap C)}=\frac{\Pr(C|A\cap B)\cdot\Pr(A\cap B)}{\Pr(C|B)\cdot\Pr(B)}=$$

$$=\frac{\Pr(C|A\cap B)\cdot\Pr(A\cap B)}{\Pr(C|B)\cdot\left(\Pr(A\cup B)-\Pr(A)+\Pr(A\cap B)\right)}=\frac{\frac{1}{2}\cdot\frac{1}{10}}{\frac{1}{3}\cdot\left(\frac{3}{5}-\frac{2}{5}+\frac{1}{10}\right)}=\frac{1}{2}.$$

Answer D.

21. Dr. Ostaszewski's online exercise posted November 13, 2010

Let $X_{(1)},\ldots,X_{(n)}$ be the order statistics from the uniform distribution on [0, 1]. Find the correlation coefficient of $X_{(1)}$ and $X_{(n)}$.

A. $-\frac{1}{n}$ B. $-\frac{1}{n+1}$ C. 0 D. $\frac{1}{n+1}$ E. $\frac{1}{n}$

Solution.

If X is uniform on [0, 1], so is $1-X$. The order statistics for a random sample for X are the same as the order statistics for a random sample of $1-X$, but in reverse order. This means that we have $E\left(X_{(1)}\right)=1-E\left(X_{(n)}\right)$ and $\operatorname{Var}\left(X_{(1)}\right)=\operatorname{Var}\left(1-X_{(n)}\right)=\operatorname{Var}\left(X_{(n)}\right)$. Note that $F_{X_{(n)}}(x)=x^n$, i.e., the n-th power of the CDF of uniform distribution on [0, 1], so that $s_{X_{(n)}}(x)=1-x^n$, and

$$E\left(X_{(n)}\right)=\int_0^1\left(1-x^n\right)dx=\frac{n}{n+1}.$$

This implies that

$$E\left(X_{(1)}\right)=1-E\left(X_{(n)}\right)=1-\frac{n}{n+1}=\frac{1}{n+1}.$$

We also have $f_{X_{(n)}}(x)=nx^{n-1}$, and

$$E\left(X_{(n)}^2\right)=\int_0^1 x^2\cdot nx^{n-1}dx=n\int_0^1 x^{n+1}dx=\frac{n}{n+2}.$$

This implies that:

$$\operatorname{Var}\left(X_{(n)}\right)=\frac{n}{n+2}-\frac{n^2}{(n+1)^2}=\frac{n(n+1)^2-n^2(n+2)}{(n+1)^2(n+2)}=\frac{n}{(n+1)^2(n+2)}=\operatorname{Var}\left(X_{(1)}\right).$$

The joint density of $X_{(1)}$ and $X_{(n)}$ is determined from observing that if $X_{(1)}$ is in $(x, x+dx)$ and $X_{(n)}$ is in $(y, y+dy)$, with $x<y$, then $n-2$ pieces of the random sample must be in the interval (x,y), and the probability of this is

$$f_{X_{(1)},X_{(n)}}(x,y)dxdy=\underbrace{\frac{n!}{(n-2)!}}_{\substack{\text{number of ways to pick ordered samples of size 2 from}\\ \text{population of size } n\text{, or just note that there are } n \text{ ways to}\\ \text{pick the first number and } n-1 \text{ ways to pick the second one}}}\cdot\underbrace{(y-x)}_{\substack{\text{probability of}\\ \text{being in the}\\ \text{interval } (x,y)}}{}^{n-2}\cdot\underbrace{dx}_{\substack{\text{probability}\\ \text{of being in}\\ \text{the interval}\\ (x,x+dx)}}\cdot\underbrace{dy}_{\substack{\text{probability}\\ \text{of being in}\\ \text{the interval}\\ (y,y+dy)}}.$$

Therefore,

$$E\left(X_{(1)}X_{(n)}\right)=\int_0^1\int_0^y\frac{xy\cdot n!}{(n-2)!}(y-x)^{n-2}\,dx\,dy=\int_0^1 ny\left(\int_0^y x(n-1)(y-x)^{n-2}\,dx\right)dy=$$

$$=\underbrace{\left|\begin{array}{ll} u=x & v=-(y-x)^{n-1}\\ du=dx & dv=(n-1)(y-x)^{n-2}\,dx\end{array}\right|}_{\text{INTEGRATION BY PARTS}}=\int_0^1 ny\left(\int_0^y (y-x)^{n-1}\,dx\right)dy=$$

$$=\int_0^1 ny\left(\left(-\frac{(y-x)^n}{n}\right)\Bigg|_{x=0}^{x=y}\right)dy=\int_0^1 y^{n+1}dy=\frac{1}{n+2}.$$

This implies that the covariance is $\frac{1}{n+2}-\frac{n}{(n+1)^2}$, and the correlation coefficient is:

$$\frac{\frac{1}{n+2}-\frac{n}{(n+1)^2}}{\frac{n}{(n+1)^2(n+2)}}=\frac{\frac{(n+1)^2-n(n+2)}{(n+1)^2(n+2)}}{\frac{n}{(n+1)^2(n+2)}}=\frac{1}{n}.$$

Answer E.

22. Dr. Ostaszewski's online exercise posted November 20, 2010

A random variable X has the log-normal distribution with density $f_X(x) = \frac{1}{x\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(\ln x - \mu)^2}$ for $x > 0$, and 0 otherwise, where μ is a constant. You are given that $\Pr(X \le 2) = 0.4$. Find $E(X)$.

A. 4.25 B. 4.66 C. 4.75 D. 5.00 E. Cannot be determined

Solution.
X is log-normal with parameters μ and σ^2 if $W = \ln X \sim N(\mu, \sigma^2)$. From the form of the density function in this problem we see that $\sigma = 1$. Therefore

$$\Pr(\ln X \le \ln 2) = \Pr\left(\frac{\ln X - \mu}{1} \le \frac{\ln 2 - \mu}{1}\right) = 0.4.$$

Let $z_{0.6}$ be the 60-th percentile of the standard normal distribution. Let Z be a standard normal random variable. Then $\Pr(Z \le -z_{0.6}) = 0.40$. But $\frac{\ln X - \mu}{1} = \ln X - \mu$ is standard normal, thus $\ln 2 - \mu = -z_{0.6}$, and $\mu = \ln 2 + z_{0.6}$. From the table, $\Phi(0.25) = 0.5987$ and $\Phi(0.26) = 0.6026$.
By linear interpolation,

$$z_{0.6} \approx 0.25 + \frac{0.6 - 0.5987}{0.6026 - 0.5987} \cdot (0.26 - 0.25) = 0.25 + \frac{1}{3} \cdot 0.01 \approx 0.2533.$$

This gives $\mu = \ln 2 + z_{0.6} \approx 0.9465$. The mean of the log-normal distribution is $E(X) = e^{\mu + \frac{1}{2}\sigma^2}$, so that in this case $E(X) \approx e^{0.9465 + \frac{1}{2}} \approx 4.2481$.
Answer A.

23. Dr. Ostaszewski's online exercise posted November 27, 2010

You are given a continuous random variable with the density $f_X(x) = \frac{1}{2}x + \frac{1}{2}$ for $-1 \le x \le 1$, and 0 otherwise. Find the density of $Y = X^2$, for all points where that density is nonzero.

A. $\frac{1}{2\sqrt{y}}$ B. $2y$ C. $\frac{3}{2y}$ D. $\frac{4}{3}y^2$ E. $\frac{1}{y \ln 2}$

Solution.
This is an unusual problem because you cannot use the direct formula for the density, as the transformation $Y = X^2$ is not one-to-one for $-1 \le x \le 1$. So the approach using the cumulative distribution function will work better. We have

$$F_X(x)=\int_{-1}^{x}\left(\frac{1}{2}t+\frac{1}{2}\right)dt=\left(\frac{t^2}{4}+\frac{t}{2}\right)\Bigg|_{t=-1}^{t=x}=\left(\frac{x^2}{4}+\frac{x}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)=\frac{x^2+2x+1}{4}=\frac{(x+1)^2}{4},$$

for $-1\le x\le 1$, 0 for $x<-1$, and 1 for $x>1$. Note that Y takes on only non-negative values. Therefore,

$$F_Y(y)=\Pr(Y\le y)=\Pr\left(X^2\le y\right)=\Pr\left(-\sqrt{y}\le X\le\sqrt{y}\right)=$$

$$=F_X\left(\sqrt{y}\right)-F_X\left(-\sqrt{y}\right)=\frac{\left(\sqrt{y}+1\right)^2}{4}-\frac{\left(-\sqrt{y}+1\right)^2}{4}=\frac{y+2\sqrt{y}+1-y+2\sqrt{y}-1}{4}=\sqrt{y}.$$

Hence, $f_Y(y)=F_Y'(y)=\dfrac{1}{2\sqrt{y}}$.

Answer A.

24. Dr. Ostaszewski's online exercise posted December 4, 2010

An insurer has 10 independent one-year term life insurance policies. The face amount of each policy is 1000. The probability of a claim occurring in the year under consideration is 0.1. Find the probability that the insurer will pay more than the total expected claim for the year.

A. 0.01 B. 0.10 C. 0.16 D. 0.26 E. 0.31

Solution.

The expected claim from each individual policy is $0.10\cdot 1000=100$. The overall expected claim is 1000. The total claim for the year will be more than 1000 if there is more than 1 death. The number of claims N has the binomial distribution with $n = 10$, $p = 0.10$. The probability sought is:

$$\Pr(N>1)=1-\left(\Pr(N=0)+\Pr(N=1)\right)=$$

$$=1-\left(\binom{10}{0}\cdot 0.10^0\cdot 0.90^{10}+\binom{10}{1}\cdot 0.10^1\cdot 0.90^9\right)\approx 0.2639.$$

Answer D.

25. Dr. Ostaszewski's online exercise posted December 11, 2010

An insurance policy is being issued for a loss with the following discrete distribution:

$$X=\begin{cases}2, & \text{with probability } 0.4,\\ 20, & \text{with probability } 0.6.\end{cases}$$

Your job as the actuary is to set up a deductible d for this policy so that the expected payment by the insurer is 6. Find the deductible.

A. 1 B. 5 C. 7 D. 10 E. 15

Solution.

The large loss of 20 will be paid after deductible as $20 - d$, and this can happen with probability 0.60, so that the expected payment of that loss is $0.6 \cdot (20 - d) = 12 - 0.6d$. This amount is equal to 6 if $d = 10$. With that large of a deductible, the smaller loss of 2 must always be absorbed by the insured, so it will not affect the expected payment, and therefore with the deductible of 10, the expected payment is 6. Any smaller deductible will increase the expected payment, and a larger deductible will decrease it. Thus the deductible must equal to 10.

Answer D.

26. Dr. Ostaszewski's online exercise posted December 18, 2010

For a Poisson random variable N with mean λ find $\lim_{\lambda \to 0} E\left(N \middle| N \geq 1\right)$.

A. ∞ B. 0 C. 1 D. e E. Cannot be determined

Solution.

Let $f_N(n) = \frac{\lambda^n}{n!} \cdot e^{-\lambda}$ for $n = 0,1,2,\ldots$ be the probability function of N. We have

$$\lim_{\lambda \to 0} E\left(N \middle| N \geq 1\right) = \lim_{\lambda \to 0} \sum_{n=1}^{+\infty} n \cdot f_{N|N\geq 1}\left(n \middle| N \geq 1\right) = \lim_{\lambda \to 0} \sum_{n=1}^{+\infty} n \cdot \frac{f_N(n)}{\Pr(N \geq 1)} = \lim_{\lambda \to 0} \sum_{n=1}^{+\infty} n \cdot \frac{f_N(n)}{1 - f_N(0)} =$$

$$= \lim_{\lambda \to 0} \frac{1}{1 - f_N(0)} \sum_{n=0}^{+\infty} n \cdot f_N(n) = \lim_{\lambda \to 0} \frac{E(N)}{1 - f_N(0)} = \lim_{\lambda \to 0} \frac{\lambda}{1 - e^{-\lambda}} \underset{\text{de l'Hospital}}{=} \lim_{\lambda \to 0} \frac{1}{e^{-\lambda}} = 1.$$

Answer C.

27. Dr. Ostaszewski's online exercise posted December 25, 2010

X is a normal random variable with mean zero and variance $\frac{1}{2}$ and Y is distributed exponentially with mean 1. X and Y are independent. Find the probability $\Pr\left(Y > X^2\right)$.

A. $\frac{1}{\sqrt{e}}$ B. $\sqrt{\frac{e}{\pi}}$ C. $\frac{1}{\sqrt{2\pi}}$ D. $\frac{1}{2}$ E. $\frac{\sqrt{2}}{2}$

Solution.

We know that $f_X(x) = \frac{1}{\sqrt{\frac{1}{2}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{\frac{1}{2}}}\right)^2} = \frac{1}{\sqrt{\pi}} e^{-x^2}$ for $-\infty < x < +\infty$, and $f_Y(y) = e^{-y}$ for $y > 0$, so that $f_{X,Y}(x,y)=\frac{1}{\sqrt{\pi}}e^{-x^2}e^{-y}$ for $-\infty<x<+\infty$ and $y>0$. Therefore,

$$\Pr\left(Y>X^2\right)=\int_{-\infty}^{+\infty}\int_{x^2}^{+\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}e^{-y}\,dy\,dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}\cdot s_Y\left(x^2\right)dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-x^2}e^{-x^2}dx=$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-2x^2}dx=\sqrt{2}\cdot\frac{1}{2}\cdot\underbrace{\int_{-\infty}^{+\infty}\frac{1}{\frac{1}{2}\cdot\sqrt{2\pi}}e^{-\frac{1}{2}\cdot\left(\frac{x}{\frac{1}{2}}\right)^2}dx}_{\text{Integral of PDF of }N\left(0,\left(\frac{1}{2}\right)^2\right)}=\sqrt{2}\cdot\frac{1}{2}\cdot 1=\frac{\sqrt{2}}{2}.$$

Answer E.

28. Dr. Ostaszewski's online exercise posted January 1, 2011

$X_{(1)}, X_{(2)}, \ldots, X_{(400)}$ are order statistics from a continuous probability distribution with a finite mean, median m and variance σ^2. Let Φ be the cumulative distribution function of the standard normal distribution. Which of the following is the best approximation of $\Pr\left(X_{(220)}\leq m\right)$ using the Central Limit Theorem?

A. $\Phi(0.05\sigma)$ B. 0.0049 C. 0.0532 D. 0.0256 E. $\Phi\left(\frac{20}{\sigma}\right)$

Solution.

Let F_X be the cumulative distribution function of the distribution under consideration. We have:

$$\Pr\left(X_{(220)}\leq m\right)=\sum_{j=220}^{400}\Pr\left(\text{Exactly } j \text{ random sample elements are } \leq m\right)=$$

$$=\sum_{j=220}^{400}\binom{400}{j}\left(F_X(m)\right)^j\left(1-F_X(m)\right)^{400-j}=\sum_{j=220}^{400}\binom{400}{j}\left(\frac{1}{2}\right)^{400}.$$

The last expression is the probability that a binomial random variable with parameters $n = 400$, $p=0.5$, has at least 220 successes. The mean of that binomial distribution is 200 and the variance of it is 100. Denote that binomial variable by N. We have:

$$\Pr\left(N\geq 220\right)\underset{\substack{\text{continuity}\\ \text{correction}}}{=}\Pr\left(N>219.5\right)=\Pr\left(\frac{N-200}{10}>\underbrace{\frac{219.5-200}{10}}_{=1.95}\right)\approx$$

$$\approx 1-\Phi(1.95)=1-0.9744\approx 0.0256.$$

Answer D.

29. Dr. Ostaszewski's online exercise posted January 8, 2011

N is a Poisson random variable such that $\Pr(N \leq 1) = 2 \cdot \Pr(N = 2)$. Find the variance of N.

A. 0.512 B. 1.121 C. 1.618 D. 3.250 E. 5.000

Solution.

Let λ be the mean and the variance of N. We have

$$\Pr(N \leq 1) = \frac{\lambda^0 \cdot e^{-\lambda}}{0!} + \frac{\lambda^1 \cdot e^{-\lambda}}{1!} = 2 \cdot \Pr(N = 2) = 2 \cdot \frac{\lambda^2 \cdot e^{-\lambda}}{2!}.$$

Therefore, $1 + \lambda = \lambda^2$, so that $\lambda^2 - \lambda - 1 = 0$, and $\lambda = \frac{1 \pm \sqrt{1+4}}{2}$. Because the parameter λ must be positive, $\lambda = \frac{1+\sqrt{5}}{2} \approx 1.618$. This is both the mean and the variance of N.

Answer C.

30. Dr. Ostaszewski's online exercise posted January 15, 2011

There are two bowls with play chips. The chips in the first bowl are numbered $1, 2, 3, \ldots, 10$, while the chips in the second bowl are numbered $6, 7, 8, \ldots, 25$. One chip is chosen randomly from each bowl, and the numbers on the two chips so obtained are compared. What is the probability that the two numbers are equal?

A. $\frac{1}{2}$ B. $\frac{1}{5}$ C. $\frac{1}{10}$ D. $\frac{1}{40}$ E. $\frac{1}{50}$

Solution.

There are $10 \cdot 20 = 200$ pairs of chips that can be picked, and of these there are 5 pairs of identical numbers, so that the probability desired is $\frac{5}{200} = \frac{1}{40}$.

Answer D.