EXCERPTS FROM ACTEX CALCULUS REVIEW MANUAL

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INTRODUCTORY COMMENTS

This study guide is designed to review mathematical topics, particularly algebra and calculus, that are needed as background and are "prerequisite" topics for the sequence of Society of Actuaries and Casualty Actuarial Society professional examinations. Until 2004 the professional exams tested material on these topics. This study guide has been adapted from the an exam preparation manual for the algebra and calculus topics as it was covered on professional exams in the past.

Most of the examples in the notes and many of the problems in the problem sets are taken from old Society examinations. The first five comprehensive tests are taken from professional exams that were held between 2000 and 2004.

If you have any comments, criticisms or compliments regarding this study guide, please contact the publisher, ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you bring them to my attention. Any errors that are found will be posted in an errata file at the ACTEX website, www.actexmadriver.com .

It is my sincere hope that you find this study guide helpful and useful in reviewing algebra and calculus.

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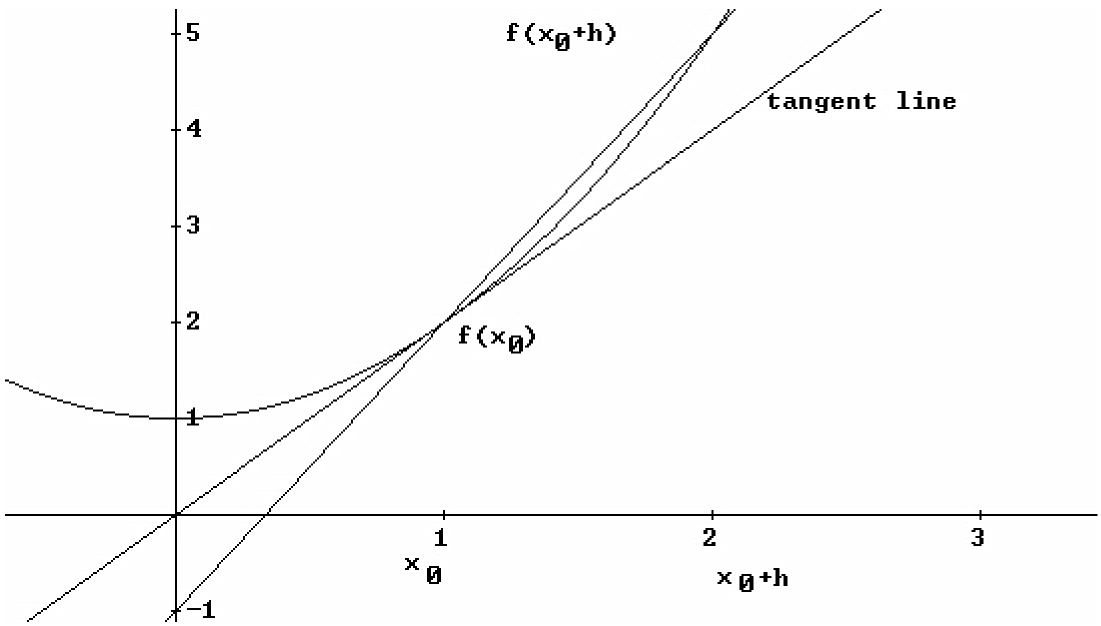
SECTION 6 - DIFFERENTIATION

Geometric interpretation of derivative: The derivative of the function $f(x)$ at the point $x = x_0$ is the slope of the line tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$. The derivative of $f(x)$ at $x = x_0$ is denoted $f'(x_0)$ or $\left.\frac{df}{dx}\right|_{x=x_0}$.

This is also referred to as **the derivative of f with respect to x at the point $x = x_0$.**
The algebraic definition of $f'(x_0)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

As can be seen from the graphs below, $\frac{f(x_0+h)-f(x_0)}{h}$ is the slope of the line joining the two points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$. As h approaches 0, the line approaches the tangent line to the graph.



The derivative as a rate of change: Perhaps the most important interpretation of the derivative $f'(x_0)$ is as the **"instantaneous" rate at which the function is increasing or decreasing as x increases (if $f' > 0$, the graph of $y = f(x)$ is rising, with the tangent line to the graph having positive slope, and if $f' < 0$, the graph of $y = f(x)$ is falling)**, and if $f'(x_0) = 0$ then the tangent line at that point is horizontal (has slope 0). This interpretation is the one most commonly used when analyzing physical, economic or financial processes.

Existence of $f'(x_0)$: We say that $f'(x_0)$ exists (or we say that f is differentiable at x_0) if the limit $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists and is finite. When $f'(x_0)$ exists, the graph will be "smooth" (no sharp corner) at the point $(x_0, f(x_0))$. **Note that if f is differentiable at x_0, then f must be continuous at x_0 .**

It is possible to define **one-sided derivatives** from the right and from the left:
derivative from the right - $f'_+(x_0) = \lim\limits_{h\to 0^+}\frac{f(x_0+h)-f(x_0)}{h} = \lim\limits_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}$,
derivative from the left - $f'_-(x_0) = \lim\limits_{h\to 0^-}\frac{f(x_0+h)-f(x_0)}{h} = \lim\limits_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$.
In order for $f'(x_0)$ to exist, it must be true that f is continuous at x_0 and also that $f'_+(x_0) = f'_-(x_0)$ and are finite (the derivatives from the right and left must be equal).

Higher order derivatives: The second derivative of f at x_0 is the derivative of $f'(x)$ at the point x_0. It is denoted $f''(x_0)$ or $f^{(2)}(x_0)$ or $\left.\frac{d^2f}{dx^2}\right|_{x=x_0}$. The n-th order derivative of f at x_0 (n repeated applications of differentiation) is denoted $f^{(n)}(x_0) = \left.\frac{d^nf}{dx^n}\right|_{x=x_0}$.

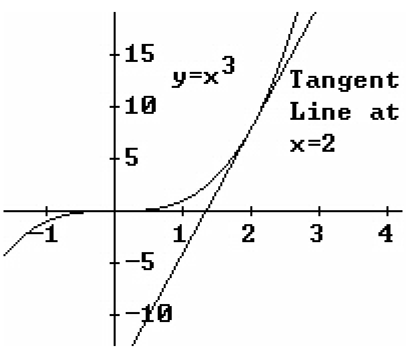
Example 6-1: (i) For the function $f(x) = x^3$, the derivative at $x = 2$ is
$f'(2) = \lim\limits_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}\frac{(2+h)^3-8}{h} = \lim\limits_{h\to 0}\frac{12h+6h^2+h^3}{h} = \lim\limits_{h\to 0}(12 + 6h + h^2) = 12.$
In a similar way, for any x, it can be shown that if $f(x) = x^3$, then $f'(x) = 3x^2$.
Then, the second derivative at $x = 2$ is
$f''(2) = \lim\limits_{h\to 0}\frac{f'(2+h)-f'(2)}{h} = \lim\limits_{h\to 0}\frac{3(2+h)^2-12}{h} = \lim\limits_{h\to 0}\frac{12h+3h^2}{h} = \lim\limits_{h\to 0}(12 + 3h) = 12$.
There are a number of rules for finding the derivatives of commonly used functions - these are summarized a little later in these notes.

Example 6-1 continued

(ii) For the function $f(x) = |x|$, if $x > 0$ then $f'(x) = 1$ and if $x < 0$ then $f'(x) = -1$. At $x = 0$, the derivative from the right is $f'_+(0) = 1$, and from the left it is $f'_-(0) = -1$. It follows that the $f'(0)$ does not exist since the right and left hand derivatives are not equal at $x = 0$. Note that in the graph below, the sharp corner in the graph at the point $x = 0$ corresponds to the derivative not existing at that point.

(iii) The function $f(x) = \begin{cases} 1-x^2 & \text{for } x \le 0 \\ x^2 & \text{for } x > 0 \end{cases}$ is differentiable at all points except $x = 0$. $f(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} f(x) = 1 \neq 0 = \lim_{x\to 0^+} f(x)$, and therefore, f cannot be differentiable at $x = 0$ (however, for $x > 0$, $f'(x) = 2x$ and for $x < 0$, $f'(x) = -2x$, and as $x \to 0$, $f'(x) \to 0$ from both the right and the left).

(iv) The function $f(x) = \frac{1}{x}$ is differentiable with $f'(x) = -\frac{1}{x^2}$ at all points except $x = 0$. Note that $f'(x) \to -\infty$ as $x \to 0$ from both the left and the right.

(i)


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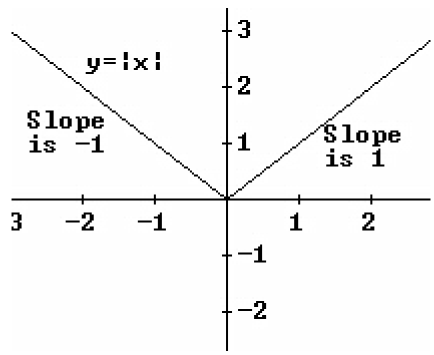

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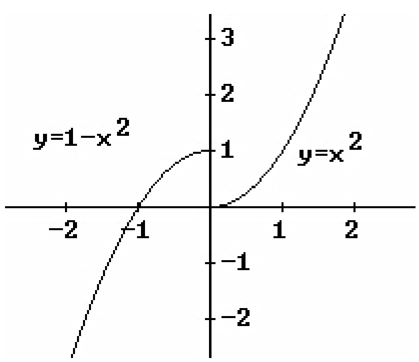

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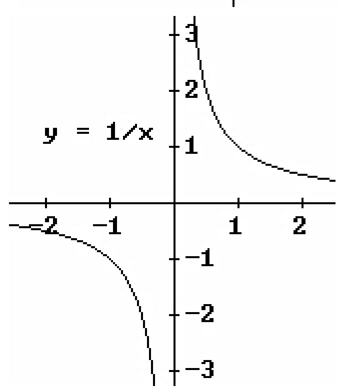


Note that in (ii), for $x > 0$, $f'(x) = 1$ and the graph is a straight line with slope 1, and also, for $x < 0$, $f'(x) = -1$ and the graph is a straight line with slope -1. **In general, if a function has a constant derivative (over some interval), the graph of that function will be a straight line (over that interval).** □

Equation of tangent line to the graph of $y = f(x)$ at the point x_0:

$\frac{y-f(x_0)}{x-x_0} = f'(x_0)$ (point-slope form) .

Example 6-1(i) (continued): At the point $x = 2$, the first derivative of $f(x) = x^3$ was found to be $f'(2) = 12$. This will be the slope of the tangent line at the point $(2, 8)$ on the graph. The equation of the tangent line at that point will be $\frac{y-8}{x-2} = 12$, which can be written in the form $y = 12x - 16$. The graph of this tangent line is plotted in the diagram above. □

Rules of differentiation:	$f(x)$	$f'(x)$
	c (a constant)	0
Power rule -	cx^n ($n \in \mathbb{R}$)	cnx^{n-1}
	$g(x) + h(x)$	$g'(x) + h'(x)$
Product rule -	$g(x) \cdot h(x)$	$g'(x) \cdot h(x) + g(x) \cdot h'(x)$
	$u(x)v(x)w(x)$	$u'vw + uv'w + uvw'$
Quotient rule -	$\frac{g(x)}{h(x)}$	$\frac{h(x)g'(x)-g(x)h'(x)}{[h(x)]^2}$
Chain rule -	$g(h(x))$	$g'(h(x)) \cdot h'(x)$

Note that in applying the chain rule, we can think of h as a variable in the function $g(h)$, and the derivative is $g'(h)$. For instance, if $g(x) = 3x^3$ and $h(x) = sin\, x$, then $g'(x) = 9x^2$ so that $g'(h) = 9h^2$, and $g'(h(x)) = 9(h(x))^2$. Then, according to the chain rule, the derivative of $g(h(x))$ (with respect to x) is $g'(h(x)) \cdot h'(x) = 9(h(x))^2 \cdot h'(x) = 9(sin\, x)^2 \cdot (cos\, x)$.

Another notation that is sometimes used to express the chain rule is
$\frac{dg}{dx} = \frac{d\,g(h(x))}{dx} = \frac{d\,g(h)}{dh} \cdot \frac{d\,h(x)}{dx} = \frac{dg}{dh} \cdot \frac{dh}{dx} = g'(h) \cdot h'(x) = g'(h(x)) \cdot h'(x)$.
Additional examples of the chain rule are given below.

Differentiation of Trigonometric and Exponential Functions

The differentiation rules above and the derivatives in the following list are essential for the Course 1 examination.

$f(x)$	$f'(x)$
$a^x \; (a > 0)$	$a^x \ln a$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$log_b x$	$\frac{1}{x \ln b}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sec x$	$(\sec x)(\tan x)$
$\tan x$	$\sec^2 x$
$Arctan\, x$	$\frac{1}{1+x^2}$
$Arcsin\, x$	$\frac{1}{\sqrt{1-x^2}}$

In addition, from the chain rule we get the following derivatives:

$e^{g(x)}$	$g'(x) \cdot e^{g(x)}$
$\ln(g(x))$	$\frac{g'(x)}{g(x)}$
$\sin(g(x))$	$\cos(g(x)) \cdot g'(x)$
$Arctan(g(x))$	$\frac{g'(x)}{1+[g(x)]^2}$

The chain rule applied to the natural log function $\frac{d}{dx} \ln(g(x)) = \frac{g'(x)}{g(x)}$ has some important applications in a particular situation in which we are told that the rate of change of a function (which means the derivative) is proportional to the function itself. Functions that satisfy this property are of the form $g(x) = Be^{cx}$, where it can seen that $g'(x) = cBe^{cx}$. This will be addressed in detail later in these notes.

Example 6-2: If $f(x) = xe^x$, then the n-th derivative, $f^{(n)}(x) = ?$
Solution: $f(x) = xe^x$. Applying the product rule results in
$f'(x) = x \cdot \frac{d}{dx}\ e^x + (\frac{d}{dx}\ x) \cdot e^x = xe^x + e^x = (x+1)e^x$,
$f^{(2)}(x) = (x+1)e^x + 1 \cdot e^x = (x+2)e^x$. Continuing in this way (or using mathematical induction) results in $f^{(n)}(x) = (x+n)e^x$. □

Example 6-3: What is the derivative of $f(x) = 4x(x^2+1)^3$?
Solution: We apply the product rule and chain rule: $f(x) = g(x) \cdot h(x)$,
where $g(x) = 4x$, $h(x) = (x^2+1)^3$, $g'(x) = 4$, $h'(x) = 3(x^2+1)^2 \cdot 2x$.
$f'(x) = 4x \cdot 3(x^2+1)^2 \cdot 2x + 4(x^2+1)^3 = 4(x^2+1)^2(7x^2+1)$.
Notice that $h(x) = (x^2+1)^3 = [w(x)]^3 = h(w(x))$, where $h(w) = w^3$ and $w(x) = x^2+1$. The chain rule tells us that
$h'(x) = h'(w) \cdot w'(x) = 3w^2 \cdot (2x) = 3(x^2+1)^2 \cdot (2x)$. □

In applying the chain rule to the function $h(w(x))$ the resulting derivative is $\frac{d}{dx}\ h(w(x)) = h'(w(x)) \cdot w'(x)$. This can also be written in the differential notation $\frac{d}{dx}\ h(w(x)) = \frac{dh}{dw} \cdot \frac{dw}{dx}$, where h can be regarded as a function of w with derivative $\frac{dh}{dw}$.
In Example 6-3 we have $h(w) = w^3$ so that $\frac{dh}{dw} = 3w^2$, and $w(x) = x^2+1$, so that $\frac{dw}{dx} = 2x$. Then $\frac{d}{dx}\ (x^2+1)^3 = \frac{d}{dx}$
$h(w(x)) = \frac{dh}{dw} \cdot \frac{dw}{dx} = 3w^2 \cdot (2x) = 3(x^2+1)^2 \cdot (2x)$.

Example 6-4: Use the power rule, chain rule and product rule to derive the quotient rule.
Solution: Suppose that $f(x) = \frac{g(x)}{h(x)}$. Then $f(x)$ can be written in the form
$f(x) = g(x) \cdot [h(x)]^{-1}$, and applying the product rule results in
$f'(x) = g'(x) \cdot\ [h(x)]^{-1} + g(x) \cdot \frac{d}{dx}\ [h(x)]^{-1}$.
Applying the chain rule and power rule to $[h(x)]^{-1}$ we get
$\frac{d}{dx}\ [h(x)]^{-1} = -\ [h(x)]^{-2} \cdot h'(x)$.
Then $f'(x) = g'(x) \cdot\ [h(x)]^{-1} + g(x) \cdot (-\ [h(x)]^{-2} \cdot h'(x))$
$= g'(x) \cdot h(x) \cdot\ [h(x)]^{-2} + g(x) \cdot (-\ [h(x)]^{-2} \cdot h'(x)) = \frac{h(x)g'(x) - g(x)h'(x)}{[h(x)]^2}$. □

Example 6-5: A cube of ice melts, without changing shape, at the uniform rate of 4 cm^3/min. Find the rate of change of the surface area of the cube, in cm^2/min, when the volume of the cube is 125.

Solution: If the volume of the cube is V then the length of a side is $V^{1/3}$ and the surface area is $6V^{2/3}$ (6 faces on the cube). Given that $\frac{dV}{dt} = -4$ (since the ice cube is melting, volume of the cube is decreasing and the rate of change, or derivative, of the volume, with respect to time, is negative), we have $\frac{d}{dt}6V^{2/3} = 6 \cdot \frac{2}{3} \cdot V^{-1/3} \cdot \frac{dV}{dt}$.

When the volume is $V = 125$, the rate of change of the surface area is

$6 \cdot \frac{2}{3} \cdot (125)^{-1/3} \cdot (-4) = -\frac{16}{5}$. □

Implicit differentiation: A relationship between x and y may be given that is not in functional form, in other words, there may not be an explicit representation of y in terms of x. For example, in the relationship $y^2 + x\,ln\,y + x^3 - 4 = 0$, it is not possible to solve for y directly as a function of x. Making the "implicit" assumption that y is a function of x (say $y(x)$), it is still possible to obtain the derivative of y with respect to x. This is done by using the various rules of differentiation that were mentioned earlier; the notation y' refers to the derivative of y with respect to x. In the example mentioned, differentiating the expression results in $2yy' + ln\,y + x \cdot \frac{y'}{y} + 3x^2 = 0$.

Solving for y' results in $y' = -\frac{3x^2 + ln\,y}{2y + \frac{x}{y}}$.

The point $(0, 2)$ lies on the curve, and the derivative of y with respect to x at that point is $y' = \frac{dy}{dx} = -\frac{0 + ln\,2}{4 + \frac{0}{2}} = -\frac{ln\,2}{4}$.

Given a relationship between y and x, it is possible to assume that an "implicit inverse" function exists, even if it is not possible to solve for x in terms of y explicitly. Assuming that x is a function of y (say $x(y)$) and differentiating the expression with respect to y, we can solve for the derivative of the inverse function; this is done in a mechanical fashion by reversing the roles of x and y symbolically. Applying this to the example just considered we get

$\frac{d}{dy}[y^2 + x\,ln\,y + x^3 - 8] = 0 \rightarrow 2y + \frac{x}{y} + (\frac{dx}{dy}) \cdot ln\,y + 3x^2(\frac{dx}{dy})] = 0$, and solving for $\frac{dx}{dy}$ results in $\frac{dx}{dy} = -\frac{2y + \frac{x}{y}}{3x^2 + ln\,y}$. At the point $(0, 2)$ ($x = 0, y = 2$) the value of $\frac{dx}{dy}$ is $-\frac{4}{ln\,2}$.

Note that $\frac{dx}{dy} = \frac{1}{dy/dx}$. This will always be the case for the derivative of an inverse function.

Example 6-6: Find the slope of the line tangent to the curve $y^3 - x^2y + 6 = 0$ at the point $(1, -2)$.

Solution: Applying implicit differentiation results in
$3y^2y' - 2xy - x^2y' = 0 \rightarrow y' = \frac{2xy}{3y^2 - x^2}$. At the point $(1, -2)$ we have $y' = -\frac{4}{11}$. □

Example 6-7: Given $y = f(x) = x^3 + x^5$, let g be the inverse function of f (so that $x = g(y)$). Find $g'(2)$.

Solution: For inverse functions $y = f(x)$ and $x = g(y)$, we have $g(f(x)) = g(y) = x$ and also $f(g(y)) = f(x) = y$. Once we notice that $f(1) = 2$, it follows that $g(2) = 1$.
Also for the inverse functions $f(x)$ and $g(y)$ we have $g'(y) = \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{f'(x)}$.
Therefore, $g'(2) = \frac{1}{f'(1)}$. Since $f'(x) = 3x^2 + 5x^4$, and $f'(1) = 8$, we have $g'(2) = \frac{1}{8}$ (note that we can find $g'(2)$ even though we do not have an explicit representation for $x = g(y)$). Suppose that we wish to find $g'(3)$. Then $g'(3) = \frac{1}{f'(c)}$, where $3 = f(c) = c^3 + c^5$. Therefore, in order to find $g'(3)$ using the inverse function rule for differentiation, we must know the x-value (called c here) so that $3 = f(c)$. □

Related Rates and Differentiation of Parametrically Defined Relationships:
Situations can arise in which two or more factors are functions of the same variable, say t (often t denotes time), such as when $x(t)$ and $y(t)$ are parametric functions. When x and y are defined in this way, it is possible to express the rate of change of y with respect to x as

$$\boldsymbol{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

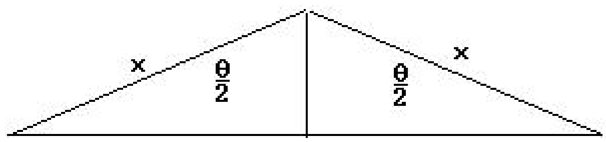
Note that this can be written as $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$, so that there is an implicit assumption that y is a function of t, and t is a function of x (the inverse function of $x(t)$).

If $x(t)$ and $y(t)$ are the horizontal and vertical positions of a point at time t then the velocity vector has horizontal component $\frac{dx}{dt}$ and vertical component $\frac{dy}{dt}$. The length of the vector is $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$.

Example 6-8: Suppose that x is the length of one of the equal sides of an isosceles triangle, and suppose that θ is the angle between the equal sides. If x is increasing at the rate of $\frac{1}{12}$ m/hr., and θ is increasing at the rate of $\frac{\pi}{180}$ radians/hr., then at what rate, in m^2/hr., is the area of the triangle increasing when $x = 12$ m and $\theta = \frac{\pi}{4}$?

Solution: Area $= A = (x\,cos\,\frac{\theta}{2})(x\,sin\,\frac{\theta}{2}) = \frac{x^2}{2} \cdot sin\,\theta$ (this follows from the identity $sin(a+b) = (sin\,a)(cos\,b) + (cos\,a)(sin\,b)$, so that $sin(c) = 2(sin\,\frac{c}{2})(cos\,\frac{c}{2})$).

From the context of the situation, we have that x and θ are both functions of t, with $\frac{dx}{dt} = \frac{1}{12}$ and $\frac{d\theta}{dt} = \frac{\pi}{180}$. Applying the product rule and the chain rule to differentiate A results in $\frac{dA}{dt} = (x\,sin\,\theta) \cdot \frac{dx}{dt} + (\frac{x^2}{2} \cdot cos\,\theta) \cdot \frac{d\theta}{dt}$. When $x = 12$ and $\theta = \frac{\pi}{4}$, we have $\frac{dA}{dt} = (12\,sin\,\frac{\pi}{4}) \cdot \frac{1}{12} + (\frac{12^2}{2} \cdot cos\,\frac{\pi}{4}) \cdot \frac{\pi}{180} = \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{5}$.



□

L'Hospital's rules for calculating limits: A limit of the form $\lim_{x\to c} \frac{f(x)}{g(x)}$ is said to be in indeterminate form if both the numerator and denominator go to 0, or if both the numerator and denominator go to $\pm\infty$. L'Hospital's rules are:

1. **IF** $\begin{cases} \text{(i) } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f'(c) \text{ exists, } \textbf{and} \\ \text{(iii) } g'(c) \text{ exists and is } \neq 0 \end{cases}$ **THEN** $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$

2. **IF** $\begin{cases} \text{(i) } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0, \textbf{ and} \\ \text{(ii) } f \text{ and } g \text{ are differentiable near } c, \textbf{ and} \\ \text{(iii) } \lim_{x\to c} \frac{f'(x)}{g'(x)} \text{ exists} \end{cases}$ **THEN** $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

In 1 or 2, the conditions $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ can be replaced by the conditions $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, and the point c can be replaced by $\pm\infty$ with the conclusions remaining valid.

In calculating limits, it is sometimes useful to use the "**natural log transformation**"; instead of finding $\lim_{x \to c} f(x) = L$ directly, we find $\lim_{x \to c} ln\, f(x) = b$, and then $L = e^b$. This technique is useful if the limit is of the form "1^∞" or "∞^0" or "0^0" ; this method will often involve L'Hospital's rule to find $\lim_{x \to c} ln\, f(x)$.

Example 6-9: Find $\lim_{x \to 2} \frac{3^{x/2}-3}{3^x-9}$.

Solution: The limits in both the numerator and denominator are 0, so we can apply l'Hospital's rule. $\frac{d}{dx} 3^x = 3^x\, ln\, 3$, and $\frac{d}{dx} 3^{x/2} = 3^{x/2} \cdot \frac{1}{2}\, ln\, 3$, so that $\lim_{x \to 2} \frac{3^{x/2}-3}{3^x-9} = \lim_{x \to 2} \frac{3^{x/2} \cdot \frac{1}{2}\, ln\, 3}{3^x\, ln\, 3} = \frac{1}{6}$. Note that this limit could also have been found by factoring the denominator into $3^x - 9 = (3^{x/2} - 3)(3^{x/2} + 3)$, and then canceling out the factor $3^{x/2} - 3$ in the numerator and denominator. □

Example 6-10: Find $\lim_{n \to \infty} \sqrt{(1 + \frac{1}{2n})^n}$.

Solution: Using the substitution $y = \frac{1}{2n}$, the limit becomes
$\lim_{n \to \infty} \sqrt{(1 + \frac{1}{2n})^n} = \lim_{n \to \infty} (1 + y)^{1/4y} = \lim_{y \to 0} (1 + y)^{1/4y} = L.$
Note that this limit is of the form "1^∞" .
We use the natural log transformation, $\lim_{y \to 0} ln[(1 + y)^{1/4y}] = ln\, L$.
Now, $\lim_{y \to 0} ln[(1 + y)^{1/4y}] = \lim_{y \to 0} \frac{1}{4y}\, ln(1 + y) = \lim_{y \to 0} \frac{ln(1+y)}{4y}$, and applying l'Hospital's rule results in $\lim_{y \to 0} \frac{ln(1+y)}{4y} = \lim_{y \to 0} \frac{1/(1+y)}{4} = \frac{1}{4} = ln\, L$. Then, $L = e^{1/4}$. □

Example 6-11: If $\alpha > 1$, find $\lim_{x \to \infty} \frac{ln\, x}{x^\alpha}$.

Solution: As $x \to \infty$ both numerator and denominator approach ∞. Applying l'Hospital's rule results in $\lim_{x \to \infty} \frac{ln\, x}{x^\alpha} = \lim_{x \to \infty} \frac{1/x}{\alpha x^{\alpha - 1}} = \frac{0}{\infty} = 0$ (since $\alpha > 1$, $\alpha - 1 > 0$).
□

Differentiation of functions of several variables - partial differentiation:

Given the function $f(x,y)$, a function of two variables, the **partial derivative of f with respect to x** at the point (x_0,y_0) is found by differentiating f with respect to x and regarding the variable y as constant - then substitute in the values $x=x_0$ and $y=y_0$. There are various ways in which this partial derivative may be denoted:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \ , \ f_x(x_0,y_0) \ , \ f_1(x_0,y_0) \ , \ \left.\frac{\partial f}{\partial x}\right|_{x=x_0,y=y_0} = \lim_{x\to x_0}\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}.$$

The partial derivative with respect to y is defined in a similar way:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = f_2(x_0,y_0) = \left.\frac{\partial f}{\partial y}\right|_{x=x_0,y=y_0} = \lim_{y\to y_0}\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0} \ .$$

Partial differentiation can be extended to a function of more than two variables.

"Higher order" partial derivatives can be defined - $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x})$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})$; and "mixed partial" derivatives can be defined (the order of partial differentiation does not usually matter) - $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y\,\partial x}$.

The chain rule can be applied to a function of two or more variables. Suppose that $u(x,y)$ and $v(x,y)$ are both functions of the variables x and y, and suppose that $F(u,v)$ is a function of u and v. Then $F(u(x,y),v(x,y))$ is a function of x and y, and

$$\boldsymbol{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}} \quad \text{and} \quad \boldsymbol{\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}}$$

Example 6-12: If $f(x,y)=x^y$ for $x,y>0$ then find $\left.\frac{\partial f}{\partial x}\right|_{(4,\frac{1}{2})}$ and $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(4,\frac{1}{2})}$.

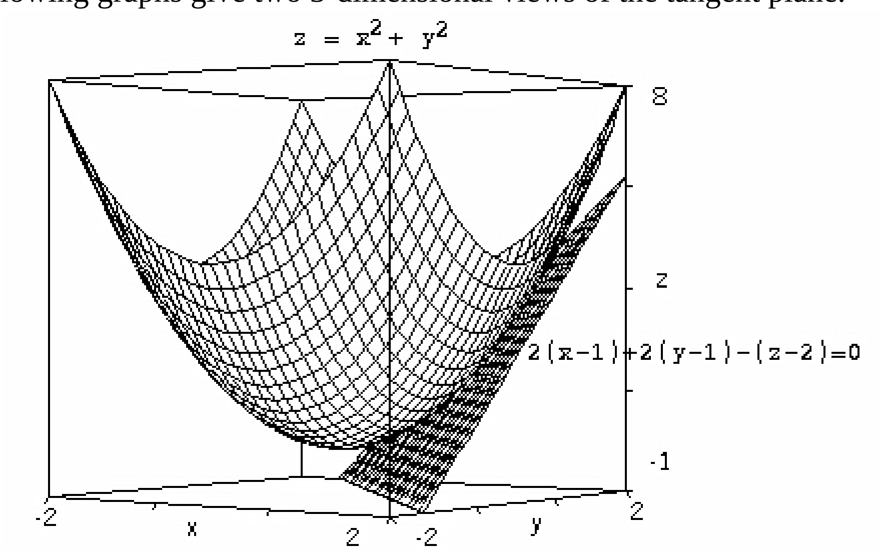
Solution: $\frac{\partial f}{\partial x} = \left. yx^{y-1}\right|_{(4,\frac{1}{2})} = (\frac{1}{2})(4)^{-1/2} = \frac{1}{4}$, and

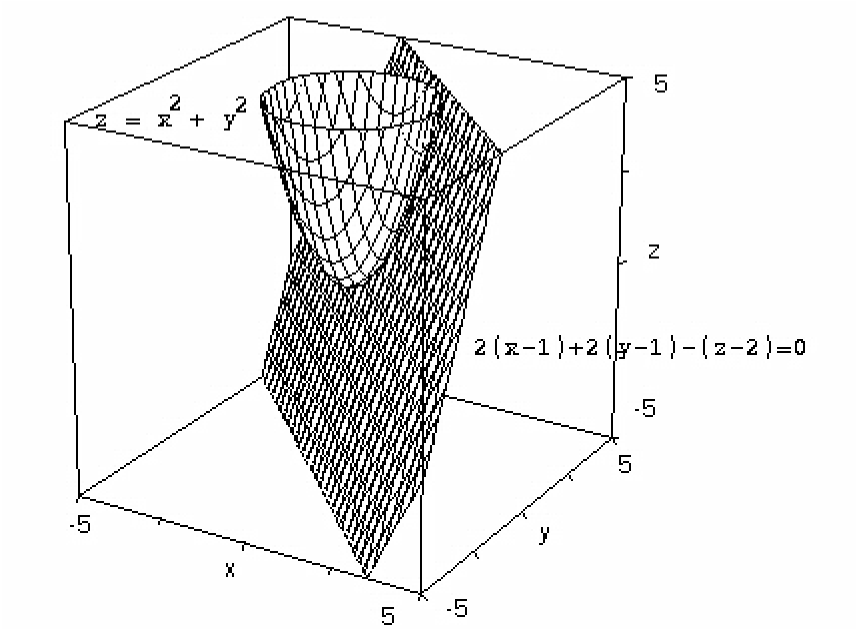
$\frac{\partial f}{\partial y} = x^y\,(ln\,x)$ and $\frac{\partial^2 f}{\partial y^2} = \left. x^y\,(ln\,x)^2\right|_{(4,\frac{1}{2})} = 4^{1/2}\,(ln\,4)^2 = 2\,(ln\,4)^2$. □

Example 6-13: $F(u,v) = u + v^2$, $u(x,y) = xy$, $v(x,y) = x - y^3$. Find $\frac{\partial F}{\partial x}$.
Solution: $\frac{\partial F}{\partial u} = 1,\ \frac{\partial F}{\partial v} = 2v,\ \frac{\partial u}{\partial x} = y,\ \frac{\partial v}{\partial x} = 1.$
$\frac{\partial F}{\partial x} = 1 \cdot y + 2v \cdot 1 = y + 2v = y + 2(x - y^3).$
Alternatively, $F(u,v) = F(xy, x - y^3) = xy + (x - y^3)^2 \rightarrow \frac{\partial F}{\partial x} = y + 2(x - y^3).$ □

If $F(x,y,z)$ is a function of three variables then the relationship $F(x,y,z) = 0$ defines a 3-dimensional surface. If (x_0, y_0, z_0) is a point in the surface, then the equation of the tangent plane to the surface at that point is
$$(x - x_0)\frac{\partial F}{\partial x} + (y - y_0)\frac{\partial F}{\partial y} + (z - z_0)\frac{\partial F}{\partial z} = 0 \ ,$$
where the partial derivatives are evaluated at the point (x_0, y_0, z_0) . It is possible that a 3-dimensional surface is presented in the form $z = f(x,y)$. This can be written in the form $f(x,y) - z = 0$, so that $F(x,y,z) = f(x,y) - z$, and then the tangent plane method just described can be applied.

Example 6-14: Find the equation of the tangent line to the surface $z = x^2 + y^2$ at the point $(x_0, y_0, z_0) = (1,1,2)$.
Solution: The surface can be written in the form $F(x,y,z) = x^2 + y^2 - z = 0$.
Then $\frac{\partial}{\partial x}F = 2x$, $\frac{\partial}{\partial y}F = 2y$, $\frac{\partial}{\partial z}F = -1$. The equation of the tangent plane is
$(x-1)(2) + (y-1)(2) + (z-2)(-1) = 0$, or equivalently, $z = 2x + 2y - 2$.
The following graphs give two 3-dimensional views of the tangent plane.





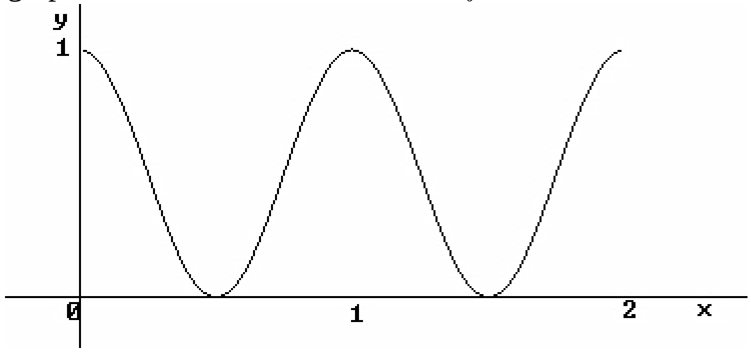
z = x^2 + y^2
5
z
2(x-1)+2(y-1)-(z-2)=0
-5
5
y
-5
x
5
-5

□

PROBLEM SET 6

Integration (Notes Section 8)

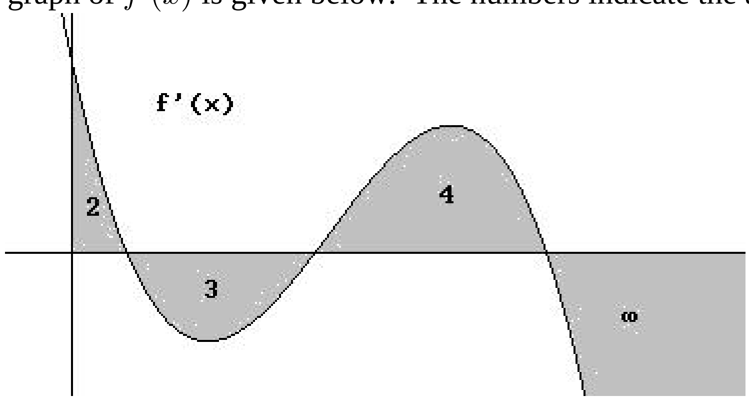
1. The graph of the differentiable function f is shown below:



Which of the following has the largest value?

A) $\int_0^1 f(x)\,dx$ B) $\int_0^2 f'(x)\,dx$ C) $\int_0^2 (f(x))^{1/2} dx$ D) $\int_0^2 f(x)\,dx$ E) $\int_0^2 (f(x))^2 dx$

2. The graph of $f'(x)$ is given below. The numbers indicate the area of the region.



For how many distinct and strictly positive x-values is it true that $f(x) = f(0)$?

A) 0 B) 1 C) 2 D) 3 E) 4

3. An advertiser claims that based on its proposed advertising campaign, the rate at which sales will occur at time t (days) into the campaign is $10,000te^{-t}$ (t is regarded as a continuous variable). According to the advertiser's claim, determine the amount of sales that will occur during the second day of the campaign (nearest 100).

A) 3100 B) 3300 C) 3500 D) 3700 E) 3900

4. A model for world population assumes a population of 6 billion at reference time 0, with population increasing to a limiting population of 30 billion. The model assumes that the rate of population growth at time $t \geq 0$ is $\frac{Ae^t}{(.02A+e^t)^2}$, where t is regarded as a continuous variable. According to this model, at what time will the population reach 10 billion (nearest .1)?

A) .3 B) .4 C) .5 D) .6 E) .6

5. A paint machine is set to spray randomly within a square area 1 unit by 1 unit in dimension. The rate at which the painted area increases is proportional to the area not yet painted. When the sprayer starts, 50% of the area has already been painted, and after 1 unit of time, 75% of the area has been painted. In how many more units of time (after time 1) will 99% of the area be painted (nearest .01 units of time)?

A) 5.00 B) 5.32 C) 5.64 D) 5.96 E) 6.28

6. Two particles start from the origin at $t = 0$ and move along the x-axis. One moves with velocity $v(t) = 2t + 8$, and the other moves with velocity $w(t) = 6t + 2$. Calculate how far from the origin the particles are when they meet again.

A) 3 B) 11 C) 14 D) 20 E) 33

7. $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} \frac{1}{u+\sqrt{u^2+1}} \, du =$

A) $\sqrt{1+x^2} - x$ B) $\frac{1}{x-\sqrt{x^2+1}}$ C) 0 D) ∞ E) $cos\, 0$

8. Let $R(t)$ be the area of the region bounded by the y-axis, a positive continuous function $f(x)$, a negative continuous function $g(x)$, and the line $x = t$, where $t > 0$. Which of the following must be equal to $\frac{dR}{dt}$?

A) $\int_0^t [f(x) - g(x)] \, dx$ B) $\int_0^t [f(x) + g(x)] \, dx$ C) $f'(t) - g'(t)$

D) $f(t) - g(t)$ E) $f(t) + g(t)$

9. If $\int_a^c f(x)\,dx = 6$, $\int_a^b f(x)\,dx = 3$, $\int_b^d g(x)\,dx = -3$, $\int_c^d g(x)\,dx = 7$,
then $\int_b^c [3f(x) - 4g(x)]\,dx =$
A) -31 B) -19 C) 11 D) 30 E) 49

10. Find $\frac{d^2}{dx^2}\int_{x^2}^{x} \sqrt{u}\,du$ for $x > 0$.

11. Let $F(x) = \int_0^{x^{1/3}} \sqrt{1+t^4}\,dt$. $F'(0) =$
A) 0 B) $\frac{1}{3}$ C) $\frac{2}{3}$ D) 1 E) Does not exist

12. Let $f(x) = x^2$. For what value of x does $f(x)$ equal the average of f on $[2,5]$?
A) $\sqrt{\frac{107}{9}}$ B) $\sqrt{13}$ C) $\sqrt{\frac{125}{9}}$ D) $\sqrt{\frac{133}{9}}$ E) 7

13. Find the slope of the tangent line to the graph of $y = \int_0^{x^2} u\,(sin\,u)^{1/3} du$ at $x = \sqrt{\frac{\pi}{2}}$
A) $\frac{\pi}{2}$ B) $\frac{\pi^{3/2}}{2^{1/2}}$ C) π D) $\pi^{3/2}$ E) $2\pi^{3/2}$

14. $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx =$
A) $\frac{3}{8}$ B) $\frac{2}{3}$ C) $\frac{3}{2}$ D) $\frac{9}{4}$ E) $\frac{8}{3}$

15. Which is an antiderivative of $x\,cos\,x$?
A) $x\,sin\,x - cos\,x$ B) $x\,sin\,x + cos\,x$ C) $x\,sin\,x$ D) $\frac{x^2 cos\,x}{2}$ E) $\frac{x^2 sin\,x}{2}$

16. Let f, g, h and k be differentiable functions. For all real x, $f(x) = \int_{g(x)}^{h(x)} k(t)\,dt$.
Then $f'(x) =$
A) $k'(x)$ B) $k(x)$ C) $k(h(x))$
D) $k(h(x)) - k(g(x))$ E) $k(h(x))h'(x) - k(g(x))g'(x)$

17. Let $[x]$ be the greatest integer less than or equal to x. What is $\int_0^2 [t^2]\, dt$?

A) $5 - \sqrt{3} - \sqrt{2}$ B) 2 C) $\frac{8}{3}$ D) $1 + \sqrt{2} + \sqrt{3}$

E) The integral is not defined

18. Find the area enclosed by the graphs of $y = \frac{1}{1+x^2}$ and $|x| = 1$.

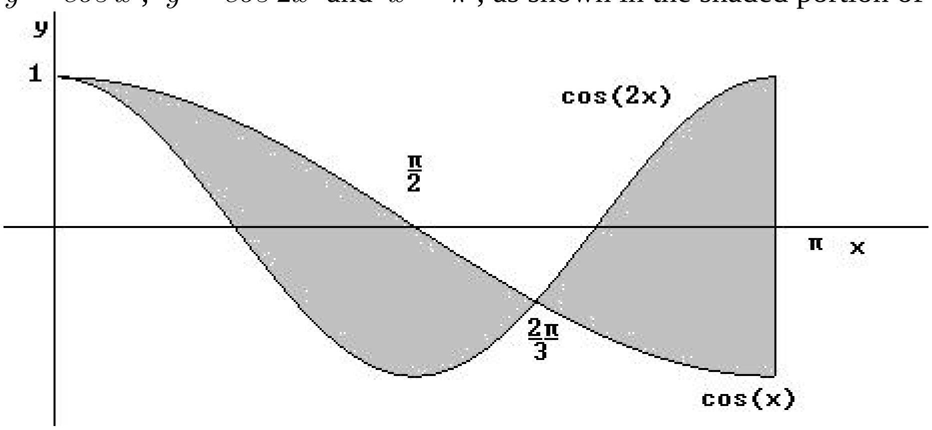
A) π B) $\frac{\pi}{2}$ C) 1 D) 0 E) $\frac{\pi}{4}$

19. If $F(x) = \int_0^x x f(t)\, dt$, then what is $xF'(x)$?

A) $x^2 f(x) + F(x)$ B) $f(x) + F(x)$ C) $xf(x) + F(x)$

D) $xF(x) + f(x)$ E) $xF(x) + xf(x)$

20. Calculate the area of the closed region in the xy-plane bounded by the graphs of $y = cos\, x$, $y = cos\, 2x$ and $x = \pi$, as shown in the shaded portion of the diagram.



A) $1 - \frac{\sqrt{3}}{2}$ B) $\sqrt{3}$ C) $\sqrt{3} + \frac{1}{2}$ D) $\frac{3\sqrt{3}}{2}$ E) $3\sqrt{3}$

21. Calculate the area of the closed region in the xy-plane bounded by $y = x - 5$ and $y^2 = 2x + 5$.

A) 8 B) $\frac{74}{3}$ C) $\frac{98}{3}$ D) $\frac{122}{3}$ E) $\frac{128}{3}$

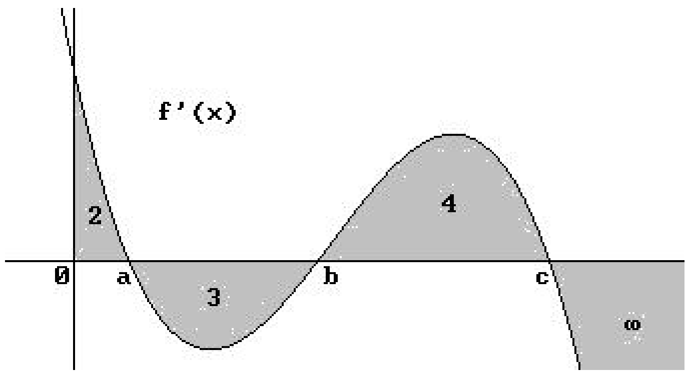
22. What is the area of the closed region bounded by $y = x^2 - |x|$ and the x-axis, between $x = -1$ and $x = 1$?

A) $\frac{1}{12}$ B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) $\frac{2}{3}$ E) $\frac{5}{6}$

PROBLEM SET 6 SOLUTIONS

1. Note that $0 \leq f(x) \leq 1$ for all x between 0 and 1. Thus,
$0 \leq [f(x)]^2 \leq f(x) \leq f(x)^{1/2}$ for x in $[0,2]$.
Then, $\int_0^1 f(x)\,dx \leq \int_0^2 f(x)\,dx$ and $\int_0^2 [f(x)]^2 dx \leq \int_0^2 f(x)\,dx \leq \int_0^2 f(x)^{1/2} dx$.
Also, $\int_0^2 f'(x)\,dx = f(2) - f(0) = 1 - 1 = 0 \leq \int_0^2 f(x)^{1/2} dx$. Answer: C

2.



$2 = \int_0^a f'(x)\,dx = f(a) - f(0)$, so that $f(a) = f(0) + 2$.
Since $f'(x) > 0$ for $0 < x < a$, $f(x)$ is strictly increasing from 0 to a, and we see that $f(x)$ increases by 2 as x goes from 0 to a.
Then from a to b, $f'(x) < 0$, and the area factor of 3 tells us that $f(x)$ decreases by 3 as x goes from a to b (this is true since $-3 = \int_a^b f'(x)\,dx = f(b) - f(a) \rightarrow$
$f(b) = f(a) - 3$).
Therefore, $f(b) = f(a) - 3 = f(0) + 2 - 3 = f(0) - 1$. This means
that $f(x)$ decreases from $f(0) + 2$ to $f(0) - 1$, and must be equal to $f(0)$ for some x between a and b (this is true because $f(x)$ is a differentiable function, which implies that $f(x)$ is a continuous function).

The area factor of 4 between b and c indicated that $f(x)$ increases by 4 as x goes from b to c. Therefore, $f(c) = f(b) + 4 = f(0) + 3$. Since $f(x)$ increases from $f(0) - 1$ to $f(0) + 3$ as x goes from b to c, there must be some x between b and c for which $f(x) = f(0)$.

The negative area factor of $-\infty$ to the right of c, indicates that $f(x)$ decreases without bound as x increases from c. Therefore, since $f(x) = f(0) + 3$, as x increases above c, $f(x)$ decreases to $-\infty$ (as $x \rightarrow \infty$), and therefore there must be an $x > c$ for which $f(x) = f(0)$. There are 3 other values of $x > 0$ for which $f(0) = f(x)$. Answer: D

3. If $F(t)$ denotes the total sales to time t, then $F'(t)$ is the rate at which sales are occurring at time t. Therefore, $F'(t) = 10,000te^{-t}$. The second day runs from $t = 1$ to $t = 2$, so that

$F(2) - F(1) = \int_1^2 F'(t)\,dt = \int_1^2 10,000te^{-t}\,dt = 10,000(-te^{-t} - e^{-t})\Big|_{t=1}^{t=2}$

$= 10,000(-2e^{-2} - e^{-2} + e^{-1} + e^{-1}) = 3,297.5$. Answer: B

4. We define $F(t)$ to be the population at time t. Then $F(0) = 6$, $\lim_{t\to\infty} F(t) = 30$, and $F'(t) = \frac{Ae^t}{(.02A+e^t)^2}$. Then

$F(s) - F(0) = \int_0^s F'(t)\,dt = \int_0^s \frac{Ae^t}{(.02A+e^t)^2}\,dt$

$= -\frac{A}{.02A+e^t}\Big|_{t=0}^{t=s} = \frac{A}{.02A+1} - \frac{A}{.02A+e^s}$,

so that $F(s) = 6 + \frac{A}{.02A+1} - \frac{A}{.02A+e^s}$.

Then $\lim_{s\to\infty} F(s) = 6 + \frac{A}{.02A+1} = 30 \rightarrow \frac{A}{.02A+1} = 24 \rightarrow A = 46.15$.

Therefore, $F(s) = 30 - \frac{46.15}{.923+e^s}$. In order to have $F(t) = 10$, we have

$30 - \frac{46.15}{.923+e^s} = 10 \rightarrow s = .325$. Answer: A

5. Let $A(t)$ be the proportion of the area painted at time t. We are given than $A(0) = .5$ and $A(1) = .75$. We are also given that $A'(t) = K[1 - A(t)]$, since the rate at which the area is being painted is $A'(t)$ and the area not yet painted as of time t is $1 - A(t)$. Therefore, $\frac{d}{dt}[1 - A(t)] = -A'(t) = -K[1 - A(t)]$, so that the rate of change of $1 - A(t)$ is proportional to $1 - A(t)$. Such a function must be of the form Bc^t , where $B > 0$ and $c > 0$. Therefore, $1 - A(t) = Bc^t$

$\rightarrow .5 = 1 - A(0) = Bc^0 = B \rightarrow B = .5$, and

$.25 = 1 - A(1) = Bc = .5c \rightarrow c = .5 \rightarrow A(t) = 1 - (.5)(.5)^t = 1 - (.5)^{t+1}$.

The t for which $A(t) = .99$ must satisfy the equation

$.99 = 1 - (.5)^{t+1} \rightarrow (.5)^{t+1} = .01 \rightarrow t = \frac{ln\,.01}{ln\,.5} - 1 = 5.64$. Answer: C

6. The position of the first particle at time t is

$d_1(t) = \int_0^t v(s)\,ds = \int_0^t (2s + 8)\,ds = t^2 + 8t$, and the position of the second particle is

$d_2(t) = \int_0^t w(s)\,ds = \int_0^t (6s + 2)\,ds = 3t^2 + 2t$. The particles meet when

$d_1(t) = d_2(t) \rightarrow t^2 + 8t = 3t^2 + 2t \rightarrow t = 0, 3 \rightarrow d_1(3) = d_2(3) = 33$. Answer: B

7. Let $f(x) = \int_0^x \frac{1}{u+\sqrt{u^2+1}}\, dx$. Then

$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \frac{1}{u+\sqrt{u^2+1}}\, du = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = \frac{1}{x+\sqrt{x^2+1}}$

But $\frac{1}{x+\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x} = \sqrt{x^2+1} - x$. Answer: A

8. The area is $R(t) = \int_0^t [f(x) - g(x)]\, dx$. Then $\frac{dR}{dt} = f(t) - g(t)$. Answer: D

9. $\int_a^c f = \int_a^b f + \int_b^c f$, and $\int_b^d g = \int_b^c g + \int_c^d g$, so that $\int_b^c f = 3$ and $\int_b^c g = -10$.
Then, $\int_b^c [3f - 4g] = 49$. Answer: E

10. According to integration rules, if $G(x) = \int_{g(x)}^{h(x)} f(u)\, du$, then
$G'(x) = f[h(x)] \cdot h'(x) - f[g(x)] \cdot g'(x)$. Therefore,
$\frac{d}{dx}\int_{x^2}^{x} \sqrt{u}\, du = \sqrt{x} \cdot 1 - \sqrt{x^2} \cdot 2x = \sqrt{x} - 2x^2$.
Then $\frac{d^2}{dx^2}\int_{x^2}^{x} \sqrt{u}\, du = \frac{d}{dx}(\sqrt{x} - 2x^2) = \frac{1}{2\sqrt{x}} - 4x$. □

11. $F(x) = f(g(x))$, where $g(x) = x^{1/3}$ and $f(z) = \int_0^z \sqrt{1+t^4}\, dt$. Applying the Chain Rule results in
$F'(x) = f'(g(x)) \cdot g'(x) = f'(x^{1/3}) \cdot \frac{1}{3}x^{-2/3} = \sqrt{1+(x^{1/3})^4} \cdot \frac{1}{3}x^{-2/3}$.
At $x = 0$, this becomes $\frac{1}{0^+}$. Answer: E

12. The average of f on the interval $[2,5]$ is $\frac{1}{3} \cdot \int_2^5 x^2 dx = \frac{1}{3} \cdot \frac{117}{3} = 13$.
The value of x for which $f(x)$ is equal to the average is the solution of $x^2 = 13$,
or $x = \sqrt{13}$. Answer: B

13. The tangent line has slope $y' = x^2(sin\, x^2)^{1/3} \cdot 2x$. At $x = \sqrt{\frac{\pi}{2}}$ this slope is
$\frac{\pi}{2} \cdot (sin\, \frac{\pi}{2})^{1/3} \cdot 2\sqrt{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right)^{3/2} = \frac{\pi^{3/2}}{2^{1/2}}$. Answer: B

14. $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_0^3 \frac{x+1-1}{\sqrt{x+1}}\,dx = \int_0^3 [\sqrt{x+1} - \frac{1}{\sqrt{x+1}}]dx$

$= \frac{2}{3}(x+1)^{3/2} - 2(x+1)^{1/2}\Big|_0^3 = \frac{8}{3}$. Answer: E

15. Differentiating each possible answer shows that B is the antiderivative. Answer: B

16. Answer: E

17. $[t^2] = \begin{cases} 0 \text{ if } 0 \le t < 1 \\ 1 \text{ if } 1 \le t < \sqrt{2} \\ 2 \text{ if } \sqrt{2} \le t < \sqrt{3} \\ 3 \text{ if } \sqrt{3} \le t < 2 \end{cases}$.

Then, $\int_0^2 [t^2]\,dt = \int_0^1 0\,dt + \int_1^{\sqrt{2}} 1\,dt + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dt + \int_{\sqrt{3}}^2 3\,dt$

$= (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5 - \sqrt{3} - \sqrt{2}$. Answer: A

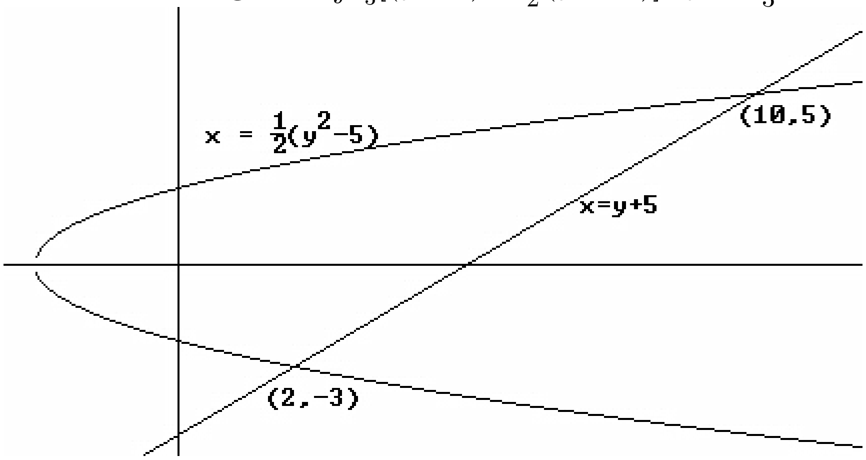
18. $\int_{-1}^1 \frac{1}{1+x^2}\,dx = arctan(x)\Big|_{-1}^1 = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}$. Answer: B

19. $F'(x) = xf(x) + \int_0^x f(t)dt \rightarrow xF'(x) = x^2 f(x) + x\int_0^x f(t)dt = x^2 f(x) + F(x)$.
Answer: A

20. For $0 \le x \le \frac{2\pi}{3}$, $cos(x) \ge cos(2x)$ and for $\frac{2\pi}{3} \le$ x $\le \pi$, $cos(2x) \ge cos(x)$.
The area is $\int_0^{2\pi/3} [cos(x) - cos(2x)]\,dx + \int_{2\pi/3}^{\pi} [cos(2x) - cos(x)]\,dx$

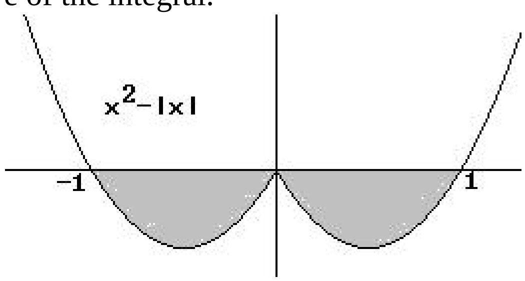
$= [sin(x) - \frac{1}{2}sin(2x)]\Big|_0^{2\pi/3} + [\frac{1}{2}sin(2x) - sin(x)]\Big|_{2\pi/3}^{\pi} = \frac{3\sqrt{3}}{2}$. Answer: D

21. The line and the parabola intersect at y-values that are the solutions of $y+5=\frac{1}{2}(y^2-5)$, so that $y=-3\ (x=2)\ ,\ 5\ (x=10)$.
The graph below indicates the closed region bounded by the line and the parabola.
The area of the region is $\int_{-3}^{5}[(y+5)-\frac{1}{2}(y^2-5)]\,dy=\frac{128}{3}$.

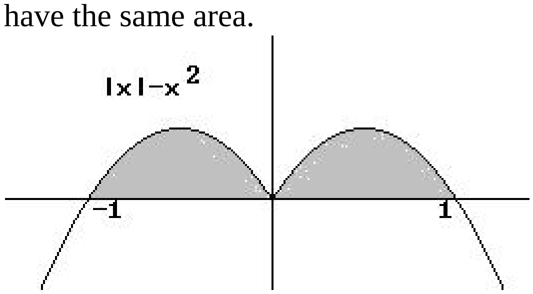


Answer: E

22. Since $|x|\geq x^2$ for $-1\leq x\leq 1$, the area is $\int_{-1}^{1}(|x|-x^2)\,dx$ which, by the symmetry of the graph is equal to $2\int_{0}^{1}(|x|-x^2)\,dx=\frac{1}{3}$.
The region is described in the following graph. The integral is negative, so the area is the negative of the integral.



Therefore, we integrate $-(x^2-|x|)=|x|-x^2$. The two shaded regions in the graph below have the same area.



Answer: C