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INTRODUCTORY COMMENTS

This study guide is designed to help in the preparation for the Society of Actuaries Exam C. The exam covers the topics of modeling (including risk measure), model estimation, construction and selection, credibility and simulation.

The study manual is divided into two volumes. The first volume consists of a summary of notes, illustrative examples and problem sets with detailed solutions on the modeling and model estimation topics. The second volume consists of notes examples and problem sets on the credibility and simulation topics, as well as 14 practice exams.

The practice exams all have 35 questions. The level of difficulty of the practice exams has been designed to be similar to that of the past 4-hour exams. Some of the questions in the problem sets are taken from the relevant topics on SOA exams that have been released prior to 2009 but the practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official reference text for the exam, "Loss Models" by Klugman, Panjer and Willmot. I have been, perhaps, more thorough than necessary on a couple of topics, such as maximum likelihood estimation, Bayesian credibility and applying simulation to hypothesis testing.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study guide is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into sections anywhere from 4 to 14 pages, with suggested time frames for covering the material. There are over 330 examples in the notes and over 800 exercises in the problem sets, all with detailed solutions. The 14 practice exams have 35 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Questions in the problem sets that have come from previous SOA exams are identified as such. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are aver 1600 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study guide.

I suggest that you work through the study guide by studying a section of notes and then attempting the exercises in the problem set that follows that section. My suggested order for covering topics is
(1) modeling (includes risk measures), (2) model estimation , (Volume 1) ,
(3) credibility theory , and (4) simulation , (Volume 2).

It has been my intention to make this study guide self-contained and comprehensive for all Exam C topics, but there are occasional references to the Loss Models reference book (4th edition) listed in the SOA catalog. While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations. In order for the review notes in this study guide to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study guide. The study guide begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance.

Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org .

If you have any questions, comments, criticisms or compliments regarding this study guide, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from www.actexmadriver.com .

It is my sincere hope that you find this study guide helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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MODELING

MODELING SECTION 1 - PROBABILITY REVIEW

Basic Probability, Conditional Probability and Independence

Exam C applies probability and statistical methods to various aspects of loss modeling and model estimation. A good background in probability and statistics is necessary to fully understand models and the modeling that is done. In this section of the study guide, we will review fundamental probability rules. The suggested time frame for this section (not including exercises) is two hours.

LM-1.1 Basic Probability Concepts

Sample point and probability space

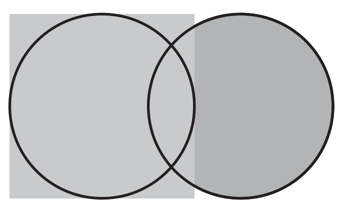
A sample point is the simple outcome of a random experiment. The probability space (also called sample space) is the collection of all possible sample points related to a specified experiment. When the experiment is performed, one of the sample points will be the outcome. An experiment could be observing the loss that occurs on an automobile insurance policy during the course of one year, or observing the number of claims arriving at an insurance office in one week. The probability space is the "full set" of possible outcomes of the experiment. In the case of the automobile insurance policy, it would be the range of possible loss amounts that could occur during the year, and in the case of the insurance office weekly number of claims, the probability space would be the set of integers $\{0, 1, 2, ...\}$.

Event

Any collection of sample points, or any subset of the probability space is referred to as an event. We say "event A has occurred" if the experimental outcome was one of the sample points in A.

Union of events A and B

$A \cup B$ denotes the union of events A and B, and consists of all sample points that are in either A or B.



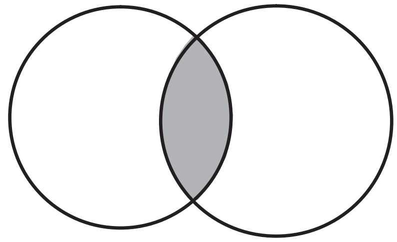
$A \cup B$

Union of events $A_1, A_2, ..., A_n$

$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ denotes the union of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are in at least one of the A_i's. This definition can be extended to the union of infinitely many events.

Intersection of events $A_1, A_2, \ldots, A_n$

$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ denotes the intersection of the events $A_1, A_2, \ldots, A_n$, and consists of all sample points that are simultaneously in all of the A_i's.



$A \cap B$

Mutually exclusive events $A_1, A_2, \ldots, A_n$

Two events are mutually exclusive if they have no sample points in common, or equivalently, if they have **empty intersection**. Events $A_1, A_2, \ldots, A_n$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$, where $\emptyset$ denotes the empty set with no sample points. Mutually exclusive events cannot occur simultaneously.

Exhaustive events $B_1, B_2, \ldots, B_n$

If $B_1 \cup B_2 \cup \cdots \cup B_n = S$, the entire probability space, then the events $B_1, B_2, \ldots, B_n$ are referred to as exhaustive events.

Complement of event A

The complement of event A consists of all sample points in the probability space that are **not in A**. The complement is denoted $\overline{A}$, $\sim A$, A' or A^c and is equal to $\{x : x \notin A\}$. When the underlying random experiment is performed, to say that the complement of A has occurred is the same as saying that A has not occurred.

Subevent (or subset) A of event B

If event B contains all the sample points in event A, then A is a subevent of B, denoted $A \subset B$. The occurrence of event A implies that event B has occurred.

Partition of event A

Events $C_1, C_2, \ldots, C_n$ form a partition of event A if $A = \bigcup_{i=1}^{n} C_i$ and the C_i's are mutually exclusive.

DeMorgan's Laws

(i) $(A \cup B)' = A' \cap B'$, to say that $A \cup B$ has not occurred is to say that A has not occurred <u>and</u> B has not occurred ; this rule generalizes to any number of events;

$$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$$

(ii) $(A \cap B)' = A' \cup B'$, to say that $A \cap B$ has not occurred is to say that either A has not occurred <u>or</u> B has not occurred (or both have not occurred) ; this rule generalizes to any number of events, $\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$

Indicator function for event A

The function $I_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$ is the indicator function for event A, where x denotes a sample point. $I_A(x)$ is 1 if event A has occurred.

Some important rules concerning probability are given below.

(i) $P[S] = 1$ if S is the entire probability space (when the underlying experiment is performed, some outcome must occur with probability 1).

(ii) $P[\emptyset] = 0$ (the probability of no face turning up when we toss a die is 0).

(iii) If events $A_1, A_2, ..., A_n$ are mutually exclusive (also called disjoint) then

$$P[\bigcup_{i=1}^{n} A_i] = P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = \sum_{i=1}^{n} P[A_i] \,. \quad (1.1)$$

This extends to infinitely many mutually exclusive events.

(iv) For any event A, $0 \leq P[A] \leq 1$.

(v) If $A \subset B$ then $P[A] \leq P[B]$.

(vi) For any events A, B and C, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$. (1.2)

(vii) For any event A, $P[A'] = 1 - P[A]$. (1.3)

(viii) For any events A and B, $P[A] = P[A \cap B] + P[A \cap B']$ (1.4)

(ix) For exhaustive events $B_1, B_2, ..., B_n$, $P[\bigcup_{i=1}^{n} B_i] = 1$. (1.5)

If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive, they form a partition of the entire probability space, and for any event A,

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] = \sum_{i=1}^{n} P[A \cap B_i] \,. \quad (1.6)$$

(x) The words "percentage" and "proportion" are used as alternatives to "probability". As an example, if we are told that the percentage or proportion of a group of people that are of a certain type is 20%, this is generally interpreted to mean that a randomly chosen person from the group has a 20% probability of being of that type. This is the "long-run frequency" interpretation of probability. As another example, suppose that we are tossing a fair die. In the long-run frequency interpretation of probability, to say that the probability of tossing a 1 is $\frac{1}{6}$ is the same as saying that if we repeatedly toss the coin, the proportion of tosses that are 1's will approach $\frac{1}{6}$.

LM-1.2 Conditional Probability and Independence of Events

Conditional probability arises throughout the Exam C material. It is important to be familiar and comfortable with the definitions and rules of conditional probability.

Conditional probability of event A given event B

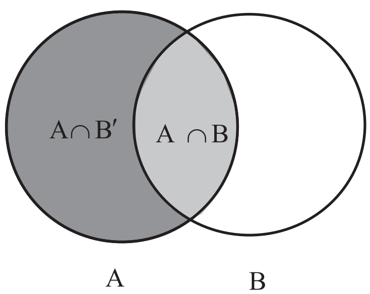
If $P(B) > 0$, then $\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$ is the conditional probability that event A occurs given that event B has occurred. By rewriting the equation we get $P(A \cap B) = P(A|B) \cdot P(B)$.

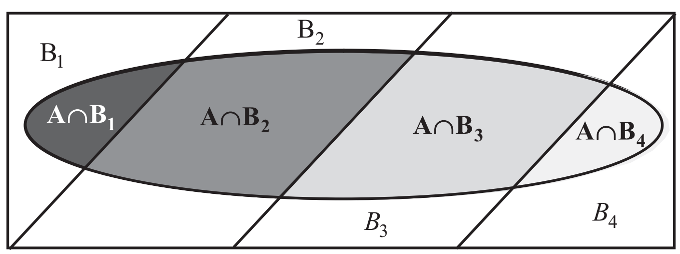
Partition of a Probability Space

Events $B_1, B_2, ..., B_n$ are said to form a partition of a probability space S if
(i) $B_1 \cup B_2 \cup \cdots \cup B_n = S$ and (ii) $B_i \cap B_j = \emptyset$ for any pair with $i \neq j$.

A partition is a disjoint collection of events which combines to be the full probability space.
A simple example of a partition is any event B and its complement B'.

If A is any event in probability space S and $\{B_1, B_2, ..., B_n\}$ is a partition of probability space S, then
$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$.
A special case of this rule is $P(A) = P(A \cap B) + P(A \cap B')$ for any two events A and B.





Bayes rule and Bayes Theorem

For any events A and B with $P(A) > 0$, $\boldsymbol{P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}}$. (1.7)

If $B_1, B_2, ..., B_n$ form a partition of the entire sample space S, then

$$P(B_j|A) = \frac{P(A|B_j) \cdot P(B_j)}{\sum_{i=1}^{n} P(A|B_i) \cdot P(B_i)} \quad \text{for each } j = 1, 2, ..., n. \tag{1.8}$$

The values of $P(B_j)$ are called prior probabilities, and the value of $P(B_j|A)$ is called a posterior probability. Variations on this rule are very important in Bayesian credibility.

Independent events A and B

If events A and B satisfy the relationship $\boldsymbol{P(A \cap B) = P(A) \cdot P(B)}$, then the events are said to be independent or stochastically independent or statistically independent. The independence of (non-empty) events A and B is equivalent to $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

Mutually independent events $A_1, A_2, ..., A_n$

The events are mutually independent if

(i) for any A_i and A_j , $P(A_i \cap A_j) = P(A_i) \times P(A_j)$, and
(ii) for any A_i , A_j and A_k , $P(A_i \cap A_j \cap A_k) = P(A_i) \times P(A_j) \times P(A_k)$,
and so on for any subcollection of the events, including all events:

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n) = \prod_{i=1}^{n} P(A_i) \ . \tag{1.9}$$

Here are some rules concerning conditional probability and independence. These can be verified in a fairly straightforward way from the definitions given above.

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any events A and B (1.10)

(ii) $P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$ for any events A and B (1.11)

(iii) If $B_1, B_2, ..., B_n$ form a partition of the sample space S, then for any event A,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i) \cdot P(B_i) \ ; \tag{1.12}$$

as a special case, for any events A and B, we have

$$P(A) = P(A \cap B) + P(A \cap B') = P(A|B) \cdot P(B) + P(A|B') \cdot P(B') \tag{1.13}$$

(iv) If $P(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$, then
$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

(v) $P(A') = 1 - P(A)$ and $P(A'|B) = 1 - P(A|B)$ (1.14)

(vi) if $A \subset B$ then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$, and $P(B|A) = 1$

(vii) if A and B are independent events then A' and B are independent events, A and B' are independent events, and A' and B' are independent events

(viii) since $P(\emptyset) = P(\emptyset \cap A) = 0 = P(\emptyset) \cdot P(A)$ for any event A, it follows that $\emptyset$ is independent of any event A

Example LM1-1:
Suppose a fair six-sided die is tossed. We define the following events:

$A =$ "the number tossed is ≤ 3" $= \{1, 2, 3\}$, $B =$ "the number tossed is even" $= \{2, 4, 6\}$,
$C =$ "the number tossed is a 1 or a 2" $= \{1, 2\}$,
$D =$ "the number tossed doesn't start with the letters 'f' or 't'" $= \{1, 6\}$.

The conditional probability of A given B is

$$P(A|B) = \frac{P(\{1,2,3\} \cap \{2,4,6\})}{P(\{2,4,6\})} = \frac{P(\{2\})}{P(\{2,4,6\})} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Events A and B are not independent, since $\frac{1}{6} = P(A \cap B) \neq P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,
or alternatively, events A and B are not independent since $P(A|B) \neq P(A)$.

$P(A|C) = 1 \neq \frac{1}{2} = P(A)$, so that A and C are not independent.
$P(B|C) = \frac{1}{2} = P(B)$, so that B and C are independent

(alternatively, $P(B \cap C) = P(\{2\}) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = P(B) \cdot P(C)$).
It is not difficult to check that both A and B are independent of D. □

IMPORTANT NOTE: The following manipulation of event probabilities arises from time to time:
$\boldsymbol{P(A) = P(A|B) \cdot P(B) + P(A|B') \cdot P(B')}$. If we know the conditional probabilities for event A given some other event B and its complement B', and if we know the (unconditional) probability of event B, then we can find the probability of event A. One of the important aspects of applying this relationship is the determination of the appropriate events A and B.

Example LM1-2:
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. An Urn is chosen at random, and a ball is randomly selected from that Urn. Find the probability that the ball chosen is white.

Solution:
Let B be the event that Urn I is chosen and B' is the event that Urn II is chosen. The implicit assumption is that both Urns are equally likely to be chosen (this is the meaning of "an Urn is chosen at random"). Therefore, $P(B) = \frac{1}{2}$ and $P(B') = \frac{1}{2}$. Let A be the event that the ball chosen in white. If we know that Urn I was chosen, then there is $\frac{1}{2}$ probability of choosing a white ball (2 white out of 4 balls, it is assumed that each ball has the same chance of being chosen); this can be described as $P(A|B) = \frac{1}{2}$.

In a similar way, if Urn II is chosen, then $P(A|B') = \frac{3}{5}$ (3 white out of 5 balls). We can now apply the relationship described prior to this example. $P(A \cap B) = P(A|B) \cdot P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$, and

$P(A \cap B') = P(A|B') \cdot P(B') = \left(\frac{3}{5}\right)\left(\frac{1}{2}\right) = \frac{3}{10}$. Finally,
$P(A) = P(A \cap B) + P(A \cap B' \ = \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$.

The order of calculations can be summarized in the following table

	A
B	**1.** $P(A \cap B) = P(A|B) \cdot P(B)$
B'	**2.** $P(A \cap B') = P(A|B') \cdot P(B')$
	3. $P(A) = P(A \cap B) + P(A \cap B')$

□

Example LM1-3:
Urn I contains 2 white and 2 black balls and Urn II contains 3 white and 2 black balls. One ball is chosen at random from Urn I and transferred to Urn II, and then a ball is chosen at random from Urn II. The ball chosen from Urn II is observed to be white. Find the probability that the ball transferred from Urn I to Urn II was white.

Solution:
Let B denote the event that the ball transferred from Urn I to Urn II was white and let A denote the event that the ball chosen from Urn II is white. We are asked to find $P(B|A)$.

From the simple nature of the situation (and the usual assumption of uniformity in such a situation, meaning all balls are equally likely to be chosen from Urn I in the first step), we have $P(B) = \frac{1}{2}$ (2 of the 4 balls in Urn I are white), and by implication, it follows that $P[B'] = \frac{1}{2}$.

If the ball transferred is white, then Urn II has 4 white and 2 black balls, and the probability of choosing a white ball out of Urn II is $\frac{2}{3}$; this is $P(A|B) = \frac{2}{3}$.

If the ball transferred is black, then Urn II has 3 white and 3 black balls, and the probability of choosing a white ball out of Urn II is $\frac{1}{2}$; this is $P(A|B') = \frac{1}{2}$.

All of the information needed has been identified. We do calculations in the following order:

1. $P[A \cap B] = P[A|B] \cdot P[B] = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$
2. $P[A \cap B'] = P[A|B'] \cdot P[B'] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$
3. $P[A] = P[A \cap B] + P[A \cap B'] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
4. $P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{1/3}{7/12} = \frac{4}{7}$. □

Example LM1-4:
Three dice have the following probabilities of throwing a "six": p, q, r, respectively. One of the dice is chosen at random and thrown (each is equally likely to be chosen). A "six" appeared. What is the probability that the die chosen was the first one?

Solution:
The event " a 6 is thrown" is denoted by "6"

$$P[\text{die 1}|\text{"6"}] = \frac{P[(\text{die 1}) \cap (\text{"6"})]}{P[\text{"6"}]} = \frac{P[\text{"6"}|\text{die 1}] \cdot P[\text{die 1}]}{P[\text{"6"}]} = \frac{p \cdot \frac{1}{3}}{P[\text{"6"}]} .$$

But

$$\begin{aligned} P[\text{"6"}] &= P[(\text{"6"}) \cap (\text{die 1})] + P[(\text{"6"}) \cap (\text{die 2})] + P[(\text{"6"}) \cap (\text{die 3})] \\ &= P[\text{"6"}|\text{die 1}] \cdot P[\text{die 1}] + P[\text{"6"}|\text{die 2}] \cdot P[\text{die 2}] + P[\text{"6"}|\text{die 3}] \cdot P[\text{die 3}] \\ &= p \cdot \tfrac{1}{3} + q \cdot \tfrac{1}{3} + r \cdot \tfrac{1}{3} = \tfrac{p+q+r}{3} \rightarrow P[\text{die 1}|\text{"6"}] = \frac{p \cdot \frac{1}{3}}{P[\text{"6"}]} = \frac{p \cdot \frac{1}{3}}{(p+q+r) \cdot \frac{1}{3}} = \frac{p}{p+q+r}. \end{aligned}$$ □

MODELING - PROBLEM SET 1

Review of Probability - Section 1

1. A survey of 1000 people determines that 80% like walking and 60% like biking, and all like at least one of the two activities. How many people in the survey like biking but not walking?

   A) 0 B) .1 C) .2 D) .3 E) .4

2. A life insurer classifies insurance applicants according to the following attributes:
   M - the applicant is male
   H - the applicant is a homeowner
   Out of a large number of applicants the insurer has identified the following information:
   40% of applicants are male, 40% of applicants are homeowners and
   20% of applicants are female homeowners.

   Find the percentage of applicants who are male and do not own a home.

   A) .1 B) .2 C) .3 D) .4 E) .5

3. Let A, B, C and D be events such that $B = A'$, $C \cap D = \emptyset$, and
   $P[A] = \frac{1}{4}$, $P[B] = \frac{3}{4}$, $P[C|A] = \frac{1}{2}$, $P[C|B] = \frac{3}{4}$, $P[D|A] = \frac{1}{4}$, $P[D|B] = \frac{1}{8}$

   Calculate $P[C \cup D]$.

   A) $\frac{5}{32}$ B) $\frac{1}{4}$ C) $\frac{27}{32}$ D) $\frac{3}{4}$ E) 1

4. You are given that $P[A] = .5$ and $P[A \cup B] = .7$.
   Actuary 1 assumes that A and B are independent and calculates $P[B]$ based on that assumption.
   Actuary 2 assumes that A and B mutually exclusive and calculates $P[B]$ based on that assumption.
   Find the absolute difference between the two calculations.

   A) 0 B) .05 C) .10 D) .15 E) .20

5. A test for a disease correctly diagnoses a diseased person as having the disease with probability .85. The test incorrectly diagnoses someone without the disease as having the disease with a probability of .10. If 1% of the people in a population have the disease, what is the chance that a person from this population who tests positive for the disease actually has the disease?

   A) .0085 B) .0791 C) .1075 D) .1500 E) .9000

6. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl 1. Find the probability that bowl 1 still has 5 black and 5 white balls.

   A) $\frac{2}{3}$ B) $\frac{3}{5}$ C) $\frac{6}{11}$ D) $\frac{1}{2}$ E) $\frac{6}{13}$

7. People passing by a city intersection are asked for the month in which they were born. It is assumed that the population is uniformly divided by birth month, so that any randomly passing person has an equally likely chance of being born in any particular month. Find the minimum number of people needed so that the probability that no two people have the same birth month is less than .5 .

A) 2 B) 3 C) 4 D) 5 E) 6

8. In a T-maze, a laboratory rat is given the choice of going to the left and getting food or going to the right and receiving a mild electric shock. Assume that before any conditioning (in trial number 1) rats are equally likely to go the left or to the right. After having received food on a particular trial, the probability of going to the left and right become .6 and .4, respectively on the following trial. However, after receiving a shock on a particular trial, the probabilities of going to the left and right on the next trial are .8 and .2, respectively. What is the probability that the animal will turn left on trial number 2?

A) .1 B) .3 C) .5 D) .7 E) .9

9. In the game show "Let's Make a Deal", a contestant is presented with 3 doors. There is a prize behind one of the doors, and the host of the show knows which one. When the contestant makes a choice of door, at least one of the other doors will not have a prize, and the host will open a door (one not chosen by the contestant) with no prize. The contestant is given the option to change his choice after the host shows the door without a prize. If the contestant switches doors, what is the probability that he gets the door with the prize?

A) 0 B) $\frac{1}{6}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

10. A supplier of a testing device for a type of component claims that the device is highly reliable, with $P[A|B] = P[A'|B'] = .95$, where

$A =$ device indicates component is faulty, and
$B =$ component is faulty .

You plan to use the testing device on a large batch of components of which 5% are faulty. Find the probability that the component is faulty given that the testing device indicates that the component is faulty .

A) 0 B) .05 C) .15 D) .25 E) .50

11. An insurer classifies flood hazard based on geographical areas, with hazard categorized as low, medium and high. The probability of a flood occurring in a year in each of the three areas is

Area Hazard	low	medium	high
Prob. of Flood	.001	.02	.25

The insurer's portfolio of policies consists of a large number of policies with 80% low hazard policies, 18% medium hazard policies and 2% high hazard policies. Suppose that a policy had a flood claim during a year. Find the probability that it is a high hazard policy.

A) .50 B) .53 C) .56 D) .59 E) .62

12. One of the questions asked by an insurer on an application to purchase a life insurance policy is whether or not the applicant is a smoker. The insurer knows that the proportion of smokers in the general population is .30, and assumes that this represents the proportion of applicants who are smokers. The insurer has also obtained information regarding the honesty of applicants:

40% of applicants that are smokers say that they are non-smokers on their applications,
none of the applicants who are non-smokers lie on their applications.

What proportion of applicants who say they are non-smokers are actually non-smokers?

A) 0 B) $\frac{6}{41}$ C) $\frac{12}{41}$ D) $\frac{35}{41}$ E) 1

13. When sent a questionnaire, 50% of the recipients respond immediately. Of those who do not respond immediately, 40% respond when sent a follow-up letter. If the questionnaire is sent to 4 persons and a follow-up letter is sent to any of the 4 who do not respond immediately, what is the probability that at least 3 never respond?

A) $(.3)^4 + 4(.3)^3(.7)$
B) $4(.3)^3(.7)$
C) $(.1)^4 + 4(.1)^3(.9)$
D) $.4(.3)(.7)^3 + (.7)^4$
E) $(.9)^4 + 4(.9)^3(.1)$

14. A fair coin is tossed. If a head occurs, 1 fair die is rolled; if a tail occurs, 2 fair dice are rolled. If Y is the total on the die or dice, then $P[Y = 6] =$

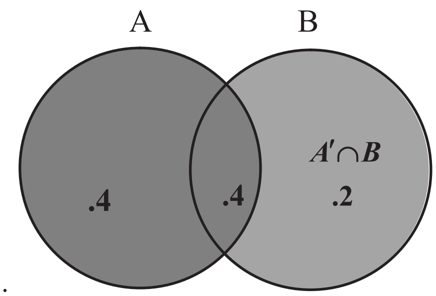
A) $\frac{1}{9}$ B) $\frac{5}{36}$ C) $\frac{11}{72}$ D) $\frac{1}{6}$ E) $\frac{11}{36}$

15. In Canada's national 6-49 lottery, a ticket has 6 numbers each from 1 to 49, with no repeats. Find the probability of matching exactly 4 of the 6 winning numbers if the winning numbers are all randomly chosen.

A) .00095 B) .00097 C) .00099 D) .00101 E) .00103

MODELING - PROBLEM SET 1 SOLUTIONS

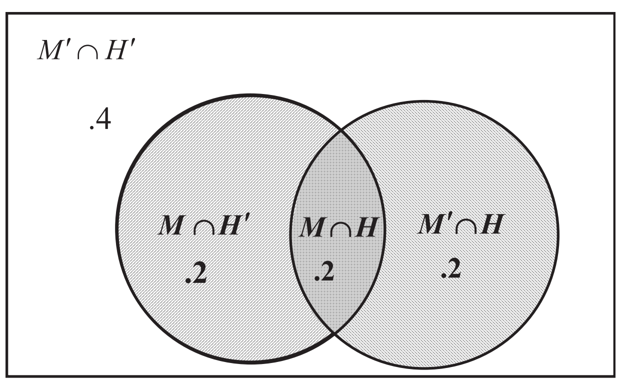
1. Let $A =$ "like walking" and $B =$ "like biking" . We use the interpretation that "percentage" and "proportion" are taken to mean "probability".
We are given $P(A) = .8$, $P(B) = .6$ and $P(A \cup B) = 1$.
From the diagram below we can see that since $A \cup B = A \cup (B \cap A')$ we have
$P(A \cup B) = P(A) + P(A' \cap B) \rightarrow P(A' \cap B) = .2$ is the proportion of people who like biking but (and) not walking . In a similar way we get $P(A \cap B') = .4$



An algebraic approach is the following. Using the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get $1 = .8 + .6 - P(A \cap B) \rightarrow P(A \cap B) = .4$. Then, using the rule
$P(B) = P(B \cap A) + P(B \cap A')$, we get $P(B \cap A') = .6 - .4 = .2$. Answer: C

2. $P[M] = .4$, $P[M'] = .6$, $P[H] = .4$, $P[H'] = .6$, $P[M' \cap H] = .2$,
We wish to find $P[M \cap H']$. From probability rules, we have

$.6 = P[H'] = P[M' \cap H'] + P[M \cap H']$, and
$.6 = P[M'] = P[M' \cap H] + P[M' \cap H'] = .2 + P[M' \cap H']$.
Thus, $P[M' \cap H'] = .4$ and then $P[M \cap H'] = .2$. The following diagram identifies the component probabilities.



The calculations above can also be summarized in the following table. The events across the top of the table categorize individuals as male (M) or female (M'), and the events down the left side of the table categorize individuals as homeowners (H) or non-homeowners (H').

	$P(M) = .4$, given		$P(M') = 1 - .4 = .6$
$P(H) = .4$ given $\Downarrow$	$P(M \cap H)$ $= P(H) - P(M' \cap H) = .4 - .2 = .2$	$\Leftarrow$	$P(M' \cap H) = .2$, given
$P(H') = 1 - .4 = .6$	$P(M \cap H') = P(M) - P(M \cap H) = .4 - .2 = .2$		Answer: B

3. Since C and D have empty intersection, $P[C \cup D] = P[C] + P[D]$.

Also, since A and B are "exhaustive" events (since they are complementary events, their union is the entire sample space, with a combined probability of

$$P[A \cup B] = P[A] + P[B] = 1 \text{).}$$

We use the rule $P[C] = P[C \cap A] + P[C \cap A']$, and the rule $P[C|A] = \frac{P[A \cap C]}{P[A]}$ to get

$P[C] = P[C|A] \cdot P[A] + P[C|A'] \cdot P[A'] = \frac{1}{2} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{11}{16}$ and
$P[D] = P[D|A] \cdot P[A] + P[D|A'] \cdot P[A'] = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{3}{4} = \frac{5}{32}$.

Then, $P[C \cup D] = P[C] + P[D] = \frac{27}{32}$. Answer: C.

4. Actuary 1: Since A and B are independent, so are A' and B' .
$P[A' \cap B'] = 1 - P[A \cup B] = .3$.

But $.3 = P[A' \cap B'] = P[A'] \cdot P[B'] = (.5)P[B'] \rightarrow P[B'] = .6 \rightarrow P[B] = .4$.
Actuary 2: $.7 = P[A \cup B] = P[A] + P[B] = .5 + P[B] \rightarrow P[B] = .2$.
Absolute difference is $|.4 - .2| = .2$. Answer: E

5. We define the following events: D - a person has the disease ,
TP - a person tests positive for the disease. We are given $P[TP|D] = .85$ and
$P[TP|D'] = .10$ and $P[D] = .01$. We wish to find $P[D|TP]$.
Using the formulation for conditional probability we have $P[D|TP] = \frac{P[D \cap TP]}{P[TP]}$.
But $P[D \cap TP] = P[TP|D] \cdot P[D] = (.85)(.01) = .0085$, and
$P[D' \cap TP] = P[TP|D'] \cdot P[D'] = (.10)(.99) = .099$. Then,
$P[TP] = P[D \cap TP] + P[D' \cap TP] = .1075 \rightarrow P[D|TP] = \frac{.0085}{.1075} = .0791$.

The following table summarizes the calculations.

$$\begin{array}{lcl}
P[D] = .01 \text{ , given} & \Rightarrow & P[D'] = 1 - P[D] = .99 \\
\quad\Downarrow & & \quad\Downarrow \\
P[D \cap TP] & & P[D' \cap TP] \\
\ = P[TP|D] \cdot P[D] = .0085 & & \ = P[TP|D'] \cdot P[D'] = .099
\end{array}$$

$$\Downarrow$$

$$P[TP] = P[D \cap TP] + P[D' \cap TP] = .1075$$

$$\Downarrow$$

$$P[D|TP] = \frac{P[D \cap TP]}{P[TP]} = \frac{.0085}{.1075} = .0791 \text{ .}$$

Answer: B

6. Let C be the event that bowl 1 has 5 black balls after the exchange.
Let B_1 be the event that the ball chosen from bowl 1 is black, and let B_2 be the event that the ball chosen from bowl 2 is black.

Event C is the disjoint union of $B_1 \cap B_2$ and $B_1' \cap B_2'$ (black-black or white-white picks), so that $P[C] = P[B_1 \cap B_2] + P[B_1' \cap B_2']$.
The black-black combination has probability $(\frac{6}{11})(\frac{1}{2})$, since there is a $\frac{5}{10}$ chance of picking black from bowl 1, and then (with 6 black in bowl 2, which now has 11 balls) $\frac{6}{11}$ is the probability of picking black from bowl 2. This is

$$P[B_1 \cap B_2] = P[B_2|B_1] \cdot P[B_1] = (\tfrac{6}{11})(\tfrac{1}{2}) \ .$$

In a similar way, the white-white combination has probability $(\frac{6}{11})(\frac{1}{2})$.

Then $P[C] = (\frac{6}{11})(\frac{1}{2}) + (\frac{6}{11})(\frac{1}{2}) = \frac{6}{11}$. Answer: C

7. $A_2 =$ event that second person has different birth month from the first.
$P(A_2) = \frac{11}{12} = .9167$.
$A_3 =$ event that third person has different birth month from first and second.

Then, the probability that all three have different birthdays is
$P[A_3 \cap A_2] = P[A_3|A_2] \cdot P(A_2) = (\frac{10}{12})(\frac{11}{12}) = .7639$.
$A_4 =$ event that fourth person has different birth month from first three.

Then, the probability that all four have different birthdays is
$$\begin{aligned} P[A_4 \cap A_3 \cap A_2] &= P[A_4|A_3 \cap A_2] \cdot P[A_3 \cap A_2] \\ &= P[A_4|A_3 \cap A_2] \cdot P[A_3|A_2] \cdot P(A_2) = (\tfrac{9}{12})(\tfrac{10}{12})(\tfrac{11}{12}) = .5729 \ . \end{aligned}$$
$A_5 =$ event that fifth person has different birth month from first four.

Then, the probability that all five have different birthdays is
$$\begin{aligned} P[A_5 \cap A_4 \cap A_3 \cap A_2] &= P[A_5|A_4 \cap A_3 \cap A_2] \cdot P[A_4 \cap A_3 \cap A_2] \\ &= P[A_5|A_4 \cap A_3 \cap A_2] \cdot P[A_4|A_3 \cap A_2] \cdot P[A_3|A_2] \cdot P(A_2) \\ &= (\tfrac{8}{12})(\tfrac{9}{12})(\tfrac{10}{12})(\tfrac{11}{12}) = .3819 \ . \end{aligned}$$
Answer: D

8. $L1 =$ turn left on trial 1, $R1 =$ turn right on trial 1, $L2 =$ turn left on trial 2 .

We are given that $P[L1] = P[R1] = .5$.
$P[L2] = P[L2 \cap L1] + P[L2 \cap R1]$ since $L1, R1$ form a partition .
$P[L2|L1] = .6$ (if the rat turns left on trial 1 then it gets food and has a .6 chance of turning left on trial 2). Then $P[L2 \cap L1] = P[L2|L1] \cdot P[L1] = (.6)(.5) = .3$.

In a similar way, $P[L2 \cap R1] = P[L2|R1] \cdot P[R1] = (.8)(.5) = .4$.

Then, $P[L2] = .3 + .4 = .7$. Answer: D

9. We define the events $A =$ prize door is chosen after contestant switches doors , $B =$ prize door is initial one chosen by contestant . Then $P[B] = \frac{1}{3}$, since each door is equally likely to hold the prize initially. To find $P[A]$ we use the Law of Total Probability.

$$P[A] = P[A|B] \cdot P[B] + P[A|B'] \cdot P[B'] = (0)(\tfrac{1}{3}) + (1)(\tfrac{2}{3}) = \tfrac{2}{3} \ .$$

If the prize door is initially chosen, then after switching, the door chosen is not the prize door, so that $P[A|B] = 0$. If the prize door is not initially chosen, then since the host shows the other non-prize door, after switching the contestant definitely has the prize door, so that $P[A|B'] = 1$.
Answer: E

10. We are given $P[B] = .05$. We can calculate entries in the following table in the order indicated.

	A	A'
B $P[B] = .05$ (given)	$P[A\|B] = .95$ (given) **1.** $P[A \cap B] = P[A\|B] \cdot P[B] = .0475$	$P[A'\|B'] = .95$ (given)
B' $P[B']$ $= 1 - P[B]$ $= .95$	**3.** $P[A \cap B']$ $= P[B'] - P[A' \cap B']$ $= .95 - .9025 = .0475$	**2.** $P[A' \cap B']$ $= P[A'\|B'] \cdot P[B']$ $= .95^2 = .9025$

4. $P[A] = P[A \cap B] + P[A \cap B'] = .095$

5. $P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{.0475}{.095} = .5$. Answer: E

11. This is a classical Bayesian probability situation. Let C denote the event that a flood claim occurred. We wish to find $P(H|C)$.

We can summarize the information in the following table, with the order of calculations indicated.

	L , $P(L) = .8$ (given)	M , $P(M) = .18$ (given)	H , $P(H) = .02$ (given)
C	$P(C\|L) = .001$ (given)	$P(C\|M) = .02$ (given)	$P(C\|H) = .25$ (given)
	1. $P(C \cap L)$ $= P(C\|L) \cdot P(L)$ $= .0008$	**2.** $P(C \cap M)$ $= P(C\|M) \cdot P(M)$ $= .0036$	**3.** $P(C \cap H)$ $= P(C\|H) \cdot P(H)$ $= .005$

4. $P(C) = P(C \cap L) + P(C \cap M) + P(C \cap H) = .0094$.

5. $P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{.005}{.0094} = .532$. Answer: B

12. We identify the following events:

S - the applicant is a smoker, NS - the applicant is a non-smoker $= S'$
DS - the applicant declares to be a smoker on the application
DN - the applicant declares to be non-smoker on the application $= DS'$.

The information we are given is $P[S] = .3$, $P[NS] = .7$, $P[DN|S] = .4$, $P[DS|NS] = 0$. We wish to find $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]}$.

We calculate $.4 = P[DN|S] = \frac{P[DN \cap S]}{P[S]} = \frac{P[DN \cap S]}{.3} \rightarrow P[DN \cap S] = .12$,
and $0 = P[DS|NS] = \frac{P[DS \cap NS]}{P[NS]} = \frac{P[DS \cap NS]}{.7} \rightarrow P[DS \cap NS] = 0$.

Using the rule $P[A] = P[A \cap B] + P[A \cap B']$, and noting that $DS = DN'$ and $S = NS'$ we have

$P[DS \cap S] = P[S] - P[DN \cap S] = .3 - .12 = .18$, and
$P[DN \cap NS] = P[NS] - P[DS \cap NS] = .7 - 0 = .7$, and
$P[DN] = P[DN \cap NS] + P[DN \cap S] = .7 + .12 = .82$.

Then, $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$.
These calculations can be summarized in the order indicated in the following table.

$P(S)$, .3 given	$\Rightarrow$	**1.** $P(NS) = 1 - P(S) = .7$ $\Downarrow$
6. $DS \Leftarrow$ $P(DS)$ $= P(DS \cap S)$ $+ P(DS \cap NS)$ $= .18 + 0 = .18$ $\Downarrow$	**5.** $P(DS \cap S)$ $= P(S) - P(DN \cap S)$ $= .3 - .12 = .18$ $\Uparrow$	**2.** $P(DS|NS) = 0$, given $P(DS \cap NS)$ $= P(DS|NS) \cdot P(NS)$ $= (0)(.7) = 0$
7. DN $P(DN)$ $= 1 - P(DS)$ $= 1 - .18$ $= .82$	**4.** $P(DN|S) = .4$ given $P(DN \cap S)$ $= P(DN|S) \cdot P(S)$ $= (.4)(.3) = .12$	**3.** $P(DN \cap NS) =$ $= P(NS) - P(DS \cap NS)$ $= .7 - 0 = .7$

Then,
8. $P[NS|DN] = \frac{P[NS \cap DN]}{P[DN]} = \frac{.7}{.82} = \frac{35}{41}$. Answer: D

13. The probability that an individual will not respond to either the questionnaire or the follow-up letter is $(.5)(.6) = .3$. The probability that all 4 will not respond to either the questionnaire or the follow-up letter is $(.3)^4$.

$$\begin{aligned} P[\text{3 don't respond}] &= P[\text{1 response on 1st round, no additional responses on 2nd round}] \\ &\quad + P[\text{no responses on 1st round, 1 response on 2nd round}] \\ &= 4[(.5)^4(.6)^3] + 4[(.5)^4(.6)^3(.4)] = 4(.3)^3(.7)\ . \text{ Then,} \end{aligned}$$

$P[\text{at least 3 don't respond}] = (.3)^4 + 4(.3)^3(.7)$. Answer: A

14. If 1 fair die is rolled, the probability of rolling a 6 is $\frac{1}{6}$, and if 2 fair dice are rolled, the probability of rolling a 6 is $\frac{5}{36}$ (of the 36 possible rolls from a pair of dice, the rolls 1-5, 2-4, 3-3, 4-2 and 5-1 result in a total of 6), Since the coin is fair, the probability of rolling a head or tail is .5. Thus, the probability that $Y = 6$ is $(.5)(\frac{1}{6}) + (.5)(\frac{5}{36}) = \frac{11}{72}$.

Answer: C

15. Suppose you have bought a lottery ticket. There are $\binom{6}{4} = 15$ ways of picking 4 numbers from the 6 numbers on your ticket. Suppose we look at one of those subsets of 4 numbers from your ticket. In order for the winning ticket number to match exactly those 4 of your 6 numbers, the other 2 winning ticket numbers must come from the 43 numbers between 1 and 49 that are not numbers on your ticket. There are $\binom{43}{2} = \frac{43\times 42}{2\times 1} = 903$ ways of doing that, and since there are 15 subsets of 4 numbers on your ticket, there are $15 \times 903 = 13,545$ ways in which the winning ticket numbers match exactly 3 of your ticket numbers. Since there are a total of 13,983,816 ways of picking 6 out of 49 numbers, your chance of matching exactly 4 of the winning numbers is $\frac{13{,}545}{13{,}983{,}816} = .00096862$. Answer: B

SIMULATION

SIMULATION - SECTION 1, THE INVERSION METHOD

The material in this section relates to Sections 20.1 and 20.2 of "Loss Models" . The suggested time frame for this section is 2-3 hours.

The objective in performing a simulation is to reproduce the behavior of a random variable by generating observations from another random variable which has the same distribution as the random variable being simulated. For instance, to simulate the flip of a fair coin, which has $P(H) = P(T) = .5$, we can toss a fair die. The simulation can be defined as follows:
the event H is simulated by tossing a 1, 2 or 3 on the die, and the event T is simulated by tossing a 4, 5 or 6 on the die. To see if this is a valid simulation we must check of the simulated events replicate the original probability distribution. In this example, there is .5 chance of tossing a 1, 2 or 3, so the simulation replicates the correct .5 probability of getting a H when a fair coin is tossed. Same for T.

Example SI1-1:
You are given a random number generator that produces sample observations r from the following probability density function: $f(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$.

You use this random number generator to simulate the color of a traffic light facing a randomly-arriving car. The light is green 36% of the time, yellow 13% of the time, and red 51% of the time.

The following table gives the correspondence between the values of r and the color of the traffic light:

Value of r	Color of light
$0 \le r < m$	Green
$m < r \le n$	Yellow
$n < r \le 1$	Red

Determine $n - m$.

Solution:
In order for the simulation to be valid it must reproduce the probabilities for the color of the traffic light. In the simulating distribution X, the probability of Green is

$$P[0 \le X < m] = \int_0^m f(x)\,dx = \int_0^m 2x\,dx = m^2 \ .$$

This must be $P(G) = .36$ in the traffic light color distribution. Thus, $m^2 = .36 \to m = .6$. In a similar way,

$$P[m < X \le n] = \int_m^n 2x\,dx = n^2 - m^2 = P(Y) = .13 \to n^2 = .13 + (.6)^2 = .49 \to n = .7. \quad \square$$

One of the general requirements for simulation is to have a way of obtaining independent uniform random numbers from the interval $(0,1)$. There are various ways of generating such random numbers, but we will simply assume that they will be available when needed. The uniform $(0,1)$ numbers that are used as sometimes referred to as **pseudorandom numbers or pseudouniform random numbers**.

SI -1.1 Simulation of A Discrete Random Variable

Example SI1-2:
We wish to simulate the number of heads in 3 tosses of a fair coin.
The actual distribution and distribution function are

X :	0	1	2	3
$p(x)$:	.125	.375	.375	.125
$F(x)$	.125	.5	.875	1.00

We consider the following partition of the unit interval

$$\underset{|}{0}\quad \underset{|}{.125}\qquad\qquad \underset{|}{.5}\qquad\qquad \underset{|}{.875}\quad \underset{|}{1.00}$$

Using the uniform distribution on $(0,1)$ we simulate X as follows. Let r denote the pseudouniform $(0,1)$ value that will be used for simulation.

If $0 \le r < .125$ simulate $X = 0$, if $.125 \le r < .5$ simulate $X = 1$,
if $.5 \le r < .875$ simulate $X = 2$, and if $.875 \le r < 1.0$ simulate $X = 3$.

We must check that the simulated distribution replicates the original probabilities. This is guaranteed since the probability of a subinterval of $(0,1)$ is simply the length of the subinterval.

Therefore, $P[0 \le r < .125] = .125 = P[X = 0]$, $P[.125 \le r < .5] = .375 = P[X = 1]$,
etc. This is a properly defined simulation. □

We can generalize the procedure used in Example SI1-2.

Suppose that a discrete random variable X has probability function $P[X = x_j] = p_j$ for $j = 0, 1, 2, ...$
where $p_0 + p_1 + p_2 + \cdots = 1$.

We can simulate a value of X in the following way. Given a pseudouniform number U from $(0,1)$, find the integer j such that $F(x_{j-1}) = \sum_{i=0}^{j-1} p_i \le U < \sum_{i=0}^{j} p_i = F(x_j)$.

Then the simulated value of X is x_j .

For instance, if $U < p_0$ then x_0 is the simulated value of X,

if $p_0 = F(x_0) \le U < F(x_1) = p_0 + p_1$ then x_1 is the simulated value of X,
if $p_0 + p_1 = F(x_1) \le U < F(x_2) = p_0 + p_1 + p_2$ then x_2 is the simulated value of X, etc.

Note that $\sum_{i=0}^{j-1} p_i = P[X \le x_{j-1}] = F_X(x_{j-1})$, and $\sum_{i=0}^{j} p_j = P[X \le x_j] = F_X(x_j)$, so that
if $F(x_{j-1}) \le U < F(x_j)$, then the simulated value of X is x_j. This method is referred to as the **inversion method** of simulation. This procedure is sometimes referred to with the phrase **small pseudouniform random numbers correspond to small simulated values**.

Example SI1-3:
Of a group of three independent lives with medical expense insurance, the number having medical expenses during a year is distributed according to a binomial distribution with $p = 0.9$ and $n = 3$. The amount, X, of medical expenses for any person, once expenses occur, has the following distribution:

x	100	$10,000$
$f(x)$	0.9	0.1

Each year, the insurance company pays the total medical expenses for the group in excess of 5,000. Use the pseudouniform $(0,1)$ random numbers, 0.01 and 0.20, in the order given, to generate the number of claims for each of two years. Use the following pseudouniform $(0,1)$ (pseudo)random numbers, in the order given, to generate the amount of each claim: 0.80 , 0.95 , 0.70 , 0.96 , 0.54 , 0.01 .

Calculate the total amount that the insurance company pays for the two years.

Solution:
The probabilities for the given binomial distribution are

$p_0 = P[N = 0] = (.1)^3 = .001,$
$p_1 = P[N = 1] = 3(.9)(.1)^2 = .027,$
$p_2 = P[N = 2] = 3(.9)^2(.1) = .243,$
$p_3 = P[N = 3] = (.9)^3 = .729$, and
$p_0 + p_1 = .028$, $p_0 + p_1 + p_2 = .271$

The pseudouniform $(0,1)$ number .01 satisfies $p_0 = .001 \leq .01 < .028 = p_0 + p_1$.

Therefore, the simulated number of medical expenses for the first year is 1. The pseudouniform number .20 satisfies $p_0 + p_1 = .028 \leq .20 < .271 = p_0 + p_1 + p_2$, so the simulated number of medical expenses for the second year is 2.

For the claim amount distribution, $F_X(100) = .9$, $F_X(10,000) = 1$.

In the first year, there is one medical expense. Since the first pseudouniform number to be used for simulating expenses is $.8 < F_X(100)$, the simulated expense in the first year is 100.

There are two expenses in the second year.

The first is simulated using the pseudouniform number .95, and since $F_X(100) = .9 \leq .95 < 1 = F_X(10,000)$, the simulated expense is 10,000.

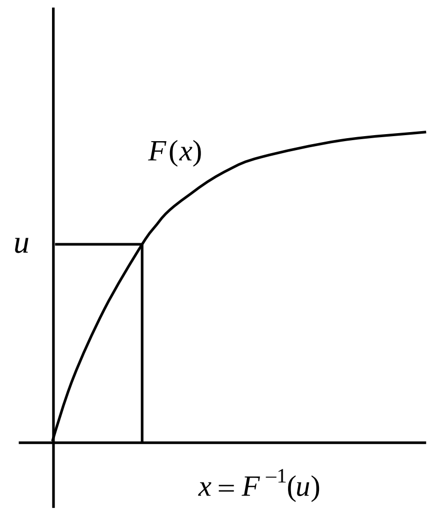
The second expense in the second year is simulated from the pseudouniform number .7, and since $.7 \leq .9 = F_X(100)$, this simulated expense is 100.

The total expense for the second year is 10,100. Each year the insurer pays total medical expenses in excess of 5000.

The insurer pays 0 in the first year, since the total medical expense was 100, and the insurer pays $10,100 - 5000 = 5,100$ in the second year. □

SI-1.2 Simulation of a Continuous Random Variable

The inversion method for a continuous random variable
Given the continuous random variable X with cdf $F_X(x)$, a value of X can be simulated from a random number $U = u$ from the pseudouniform distribution on $(0,1)$ in the following way. Solve the equation $u = F_X(x)$ for x; the solution is the simulated value of X. The solution of the equation is sometimes written $x = F_X^{-1}(u)$. This is illustrated in the graph below.



Example SI1-4:
You are given a random variable X with the following probability density function:
$f(x) = \begin{cases} 24x^2 & \text{if } 0 \le x \le .5 \\ 0, & \text{otherwise} \end{cases}$.

Let v be a pseudouniform random number between 0 and 1. Use the inversion method to determine a random observation from the distribution of X, given $v = 0.125$.

Solution:
The cdf of X is $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 24t^2\,dt = 8x^3$ for $0 \le x \le .5$, and $F(x) = 1$ for $x \ge .5$.
Given v pseudouniform on $(0,1)$, the simulated value of X using the inversion method is the value of x in the solution of $v = F(x)$, so that $.125 = 8x^3$, from which we get $x = .25$. □

It is understood when applying the inversion method, that given a pseudouniform number u, we solve for x in the equation $u = F(x)$ to get the simulated value x. This may also be referred to with the phrase **small random numbers correspond to small simulated values**. It is also possible the may see the phrase "small random numbers correspond to large simulated values. If that is the case, then we solve for x in the equation $u = 1 - F(x)$. This will result in a different simulated numerical value x, but it will be a statistically valid simulation.

Example SI1-5:
You are to use the inversion method to simulate three values of a random variable with the following distribution function: $F(0)=0, \quad F(1)=0.4, \quad F(2)=1.0$

$F(x)$ is linear in the interval $[0,1]$ and in the interval $[1,2]$.

You are to use the following pseudorandom numbers from the pseudouniform distribution on $(0,1)$: 0.2, 0.4, 0.7.
Calculate the mean of the three simulated numbers.

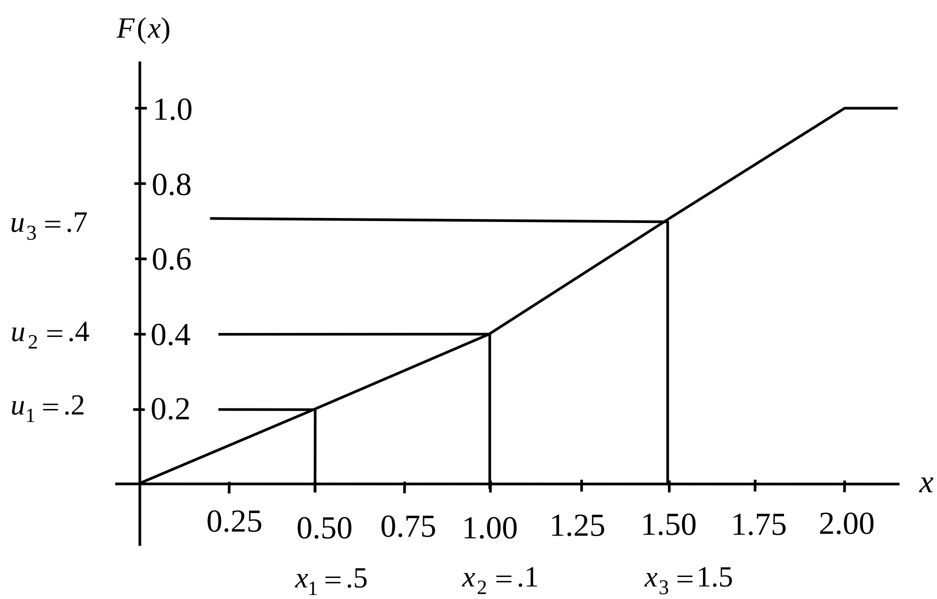
Solution:

The cdf is $F(x)=\begin{cases} 0 & x<0 \\ .4x & 0\le x\le 1 \\ .6x-.2 & 1<x\le 2 \\ 1 & x>2 \end{cases}$.

We have constructed the cdf by finding the equation of the lines between successive point in the distribution. For instance, since $F(0)=0$ and $F(1)=.4$, and since we know that $F(x)$ is a linear function for $0\le x\le 1$. it must be true that $F(x)=.4x$ for $0\le x\le 1$ (we can use the "two-point" formula for finding the equation of a straight line; in this case the two (x,y) -points are $(0,0)$ and $(1,.4)$). We can use the same reasoning to find $F(x)$ for $1\le x\le 2$.

For $0\le x\le 1$, $F(x)$ ranges from 0 to .4, and for $1<x\le 2$, $F(x)$ ranges from .4 to 1. In order to apply the inversion method, and solve for x from $u=F(x)$, we must use the appropriate "piece" of $F(x)$ from this piecewise formulation of $F(x)$. If $0<u\le .4$, we use $F(x)=.4x$, and if $.4<u\le 1$, we use $F(x)=.6x-.2$. The first pseudouniform $(0,1)$ number is $u_1=.2$, therefore, the first simulated value of X is x_1, the solution of $.2=u_1=F(x_1)=.4x_1 \rightarrow x_1=.5$.
Then, $u_2=.4 \rightarrow .4=F(x_2)=.4x_2 \rightarrow x_2=1$, and $u_3=.7 \rightarrow .7=F(x_3)=.6x_3-.2 \rightarrow x_3=1.5$.
The mean of x_1,x_2,x_3 is $\frac{.5+1+1.5}{3}=1$.

The graph below illustrates the relationship between the pseudouniform $(0,1)$ values and the corresponding simulated x-values.



□

SI-1.3 Simulation of Some Specific Random Variables

Simulation of a Uniform Distribution on the Interval (a, b)
Simulation of the uniform distribution on (a,b) using the inversion method is quite straightforward. If X has a uniform distribution on (a,b), then

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Given a pseudorandom number u, we solve for x from the equation $u = F_X(x) = \frac{x-a}{b-a}$.
This results in $x = (b-a) \times u + a$. The simple interpretation of this is that we have "stretched" (or "compressed") the interval from length 1 to length $b-a$, and we have "shifted" it by a distance of a.

Simulation of a Normal Random Variable
Suppose that W has a normal distribution with mean μ and standard deviation σ. We can simulate W by first simulating a standard normal variable Z, and then $\sigma Z + \mu$ is a valid simulated value of W. The inversion method can be used to simulate standard normal Z.

There are two alternative methods to the inversion method that we consider for simulating standard normal variables.

1. Box-Muller Method: Given two independent pseudorandom numbers u_1 and u_2, the values of z_1 and z_2 are independent standard normal variables if $z_1 = \sqrt{-2\,ln\,u_1}\ cos(2\pi\,u_2)$ and $z_2 = \sqrt{-2\,ln\,u_1}\ sin(2\pi\,u_2)$.

2. Polar Method: Given two independent pseudorandom uniform numbers u_1 and u_2, calculate $x_1 = 2u_1 - 1$ and $x_2 = 2u_2 - 1$, and then calculate $w = x_1^2 + x_2^2$. If $w \geq 1$, then start over with two new pseudorandom numbers. Calculate $y = \sqrt{-2\,ln(w)\,/\,w}$. Then $z_1 = x_1 \cdot y$ and $z_2 = x_2 \cdot y$ are independent standard normal random variables.

Example SI1-6:
Given the two independent pseudorandom numbers $u_1 = 0.87$ and $u_2 = .44$, find the simulated pair of independent standard normal random variables using each of the three methods outlined above.

Solution:
Inverse Method: We solve for z_1 from the equation $\Phi(z_1) = u_1 = .87$ using the standard normal table to get $z_1 = 1.13$, and in a similar way, we get $z_2 = \Phi^{-1}(.44) = -.15$.

Box-Muller Method: $z_1 = \sqrt{-2\,ln(.87)}\ cos(2\pi \times .44) = -.49$,
$z_2 = \sqrt{-2\,ln(.87)}\ sin(2\pi \times .44) = .10$.

Polar Method: $x_1 = .74$, $x_2 = -.12$, $w = (.74)^2 + (-.12)^2 = .562 < 1$.
$y = \sqrt{-2\,ln(.562)} = 1.1525$, $z_1 = .853$, $z_2 = -.138$. □

Note that if we wish to simulate the lognormal random variable Y with parameters μ and σ, we would first simulate a standard normal Z, and then the simulated value of the lognormal would be $Y = e^{\mu + \sigma Z}$.

Simulation of the exponential distribution
The exponential distribution with mean θ has cdf $F_X(x) = 1 - e^{-x/\theta}$. Given the value $U = u$ from the uniform distribution over $(0,1)$, the simulated value of X based on the standard form of the inversion method (for which small random numbers correspond to small simulated values) is the solution of $u = F_X(x) = 1 - e^{-x/\theta}$, so that $x = -\theta\, log(1-u)$.
When we simulate the exponential distribution using the inversion method, this is the default approach, unless indicated otherwise, as in the next paragraph.

The basic form of the inversion method of simulation results in small pseudouniform numbers corresponding to small simulated values. The reason for this is that given pseudouniform number u, the simulated value of x for the random variable X is found by solving for x from $u = F_X(x)$. The distribution function $F_X(x)$ is a non-decreasing function, so larger x-values correspond to larger u-values. A variation on the inversion method is one for which large x-values correspond to small u-values. In this case, the simulated value of x is found by solving the equation $u = 1 - F_X(x)$. In general, this will result in a different numerical value for the simulated value of x than the value for the basic inversion method. But this is still a valid simulation method. It has occasionally arisen on an exam question. If the question states "large pseudorandom numbers correspond to small simulated values" or something similar, then this indicates that this approach is to be used. If we apply this simulation approach to the exponential distribution, we would solve the equation $u = 1 - F(x) = e^{-x/\theta}$, so that $x = -\theta\, ln\, u$. As mentioned already, this would result in a different numerical simulated value than the standard form of the inversion method, but it still would be a statistically valid simulation (and would be slightly simpler to apply algebraically).

Simulation of the Erlang (gamma) distribution
The gamma distribution with parameters n and θ, where n is an integer ≥ 1 can be simulated in the following way (in the Exam C Tables, this distribution would be described as the gamma distribution with parameters $\alpha = n$ and θ). With n being an integer ≥ 1, this gamma random variable is the sum of n independent exponential random variables, each with mean θ. We would simulate n independent exponentials and add them to get the gamma X. From n pseudouniform $(0,1)$ variables $u_1, u_2, ..., u_n$, using the approach in the paragraph above to simulate an exponential with mean θ, we get
$X = -\theta\, ln\, u_1 - \theta\, ln\, u_2 - \cdots - -\theta\, ln\, u_n = -\theta\, log(u_1 u_2 \cdots u_n)$, the simulated gamma.

When α is an integer ≥ 1, the gamma distribution is also referred to as an Erlang distribution.

Simulation of an $(a,b,0)$ random variable
Recall that the $(a,b,0)$ class consists of the Poisson, binomial and negative binomial distributions. We can simulate them by first obtaining a sequence of simulated exponential values $s_0,\ s_1,\ s_2,\ \ldots$, where each s_j is a simulated exponential random variable with mean $\frac{1}{\lambda_j}$. The λ_j values depend on which $(a,b,0)$ distribution we are trying to simulate, described below. We keep track of the running total of the s-values: $t_0 = s_0$, $t_1 = t_0 + s_1$, $t_2 = t_1 + s_2$, $\ldots$, $t_k = t_{k-1} + s_k$. We keep simulating s-values until $t_k > 1$, and when that happens, the simulated value of the $(a,b,0)$ random variable in question is k.

The λ_j values are determined as follows:

Poisson Distribution with mean λ: $\lambda_j = \lambda$ for all j.

Binomial Distribution with parameters m and q: $\lambda_j = c + dj$, where $d = ln(1-q)$ and $c = -md$.

Negative Binomial with parameters r and β: $\lambda_j = c + dj$, where $d = ln(1+\beta)$ and $c = rd$.

The theoretical justification for the validity of this simulation method will not be provided here. There are references in the "Loss Models" book.

Example SI1-7:
You are given the following sequence of pseudorandom numbers:
$.59\,,\ .33\,,\ .89\,,\ .68\,,\ .45\,,\ .19\,,\ .72\,,\ .08$
Use them in applying the simulation method for $(a,b,0)$ distributions outlined above in the following cases.
(a) Simulate a Poisson random variable with a mean of 4.
(b) Simulate a binomial random variable with $m=40$ and $q=.1$.
(c) Simulate a negative binomial random variable with $r=40$ and $\beta=.1$.

Solution:
(a) We wish to simulate a Poisson random variable with a mean of $\lambda=4$. We simulate the sequence of s-values. For the Poisson, all λ_j values are equal to the Poisson mean, which is $\lambda=4$ in this case.
s_0 is simulated from an exponential distribution with a mean of $\frac{1}{\lambda_0}=\frac{1}{10}=.1$.
$s_0=\frac{-ln(1-u_0)}{\lambda}=\frac{-ln(1-.59)}{4}=.2229$, $t_0=s_0=.2229$,
$s_1=\frac{-ln(1-.33)}{4}=.1001$, $t_1=t_0+s_1=.3230$, $s_2=\frac{-ln(1-.89)}{4}=.5518$, $t_2=t_1+s_1=.8748$,
$s_3=\frac{-ln(1-.68)}{4}=.2849$, $t_3=t_2+s_3=1.1597>1$.
Since t_3 is the first t-value greater than 1, the simulated Poisson variable has a value of 3.

(b) We begin with $d=ln(1-q)=ln\,.9=-.105361$ and $c=-md=-40(-.105361)=4.2144$.
Then $\lambda_0=c=4.2144$, and $s_0=\frac{-ln(1-.59)}{4.2144}=.2116$, $t_0=s_0=.2116$.
Then, $\lambda_1=c+d=4.1091$ and $s_1=\frac{-ln(1-.33)}{4.1091}=.0975$, $t_1=t_0+s_1=.3090$.
Then, $\lambda_2=c+2d=4.0037$ and $s_2=\frac{-ln(1-.89)}{4.0037}=.5513$, $t_2=t_1+s_2=.8603$.
Then, $\lambda_3=c+3d=3.8983$ and $s_3=\frac{-ln(1-.68)}{3.8983}=.2923$, $t_3=t_2+s_3=1.1526>1$.
Since t_3 is the first t-value greater than 1, the simulated Binomial variable has a value of 3.

(c) We begin with $d=ln(1+\beta)=ln\,1.1=.095311$ and $c=rd=40(.095311)=3.8124$. Then
$\lambda_0=c=3.8124$, and $s_0=\frac{-ln(1-.59)}{3.8124}=.2339$, $t_0=s_0=.2339$.
Then, $\lambda_1=c+d=3.9077$ and $s_1=\frac{-ln(1-.33)}{3.9077}=.1025$, $t_1=t_0+s_1=.3364$.
Then, $\lambda_2=c+2d=4.0303$ and $s_2=\frac{-ln(1-.89)}{4.0030}=.5514$, $t_2=t_1+s_2=.8878$.
Then, $\lambda_3=c+3d=4.0983$ and $s_3=\frac{-ln(1-.68)}{4.0983}=.2780$, $t_3=t_2+s_3=1.1658>1$.
Since t_3 is the first t-value greater than 1, the simulated Negative Binomial variable has a value of 3. □

SI-1.4 Simulation of a Distribution That Is Partly Discrete and Partly Continuous

If a distribution is partly continuous and partly discrete, then the cdf $F_X(x)$ increases continuously on the continuous region of X, and there are discrete jumps in $F_X(x)$ at the points of probability of X, with the size of the jump being the amount of probability at the point. Given pseudouniform $(0,1)$ number u, if u is in a region where F is continuous, then we solve $F_X(x) = u$ to get the simulated value of X. If u is inside a "jump" interval, then the simulated value of x is the discrete point where the "jump" occurs.

Example SI1-8:
A random variable X has distribution function $F(x)$ with:

(i) $F(x) = 0$ for $x < 0$ (ii) $F(0) = \frac{1}{2}$ (iii) $F'(x) = x$ for $0 < x < 1$,

where F' is the derivative of F. Three sample values $X_1,\ X_2,\ X_3$ are simulated from the distribution of X by applying the inversion method to the three observations from the uniform distribution on $(0,1)$. The observations are 0.25 , 0.625 and 0.52 . Determine the simulated values.

Solution:
From the definition of $F(x)$, we see that X has a discrete point of probability at $X = 0$, with $P[X=0] = .5$. Then, $F(x) = \frac{1}{2}x^2 + c$ (antiderivative of $F'(x) = x$) for $x > 0$.

Since $F(0) = \frac{1}{2}$, it follows that $c = \frac{1}{2}$ for $F(x)$ to continue from $\frac{1}{2}$ to 1 as x goes from 0 to 1.

The inversion method is applied as follows. If $0 < u \leq \frac{1}{2}$, then the simulated value of X is 0 (this is where there is a jump in the graph of $F(x)$), and if $\frac{1}{2} < u < 1$, then the simulated value of X is the solution of $u = F(x) = \frac{1}{2}x^2 + \frac{1}{2}$.

From the three given pseudouniform values, $u_1 = .25$ results in a simulated value of $x_1 = 0$, $u_2 = .625 = \frac{1}{2}x_2^2 + \frac{1}{2} \rightarrow x_2 = .5,$ and $u_3 = .52 = \frac{1}{2}x_3^2 + \frac{1}{2} \rightarrow x_3 = .2$. □

SI-1.5 Simulation of a Mixture Distribution

The inversion method of simulation can be applied to any distribution, including a mixture distribution. The application of the method may require numerical approximation. The following example illustrates this.

Example SI1-9:
Suppose that X is a mixture of three exponential random variables, with means 1 , 4 and 5, and with mixing weights .5 , .25 and .25, respectively. Apply the inversion method using a pseudorandom uniform number of .3 to simulate X.

Solution:
The cdf of X is $F_X(x) = .5(1-e^{-x}) + .25(1-e^{-.25x}) + .25(1-e^{-.2x})$.
Given a pseudorandom uniform number $u = .3$, to simulate X we solve for x from the equation $u = .5(1-e^{-x}) + .25(1-e^{-.25x}) + .25(1-e^{-.2x})$.
There is no algebraic solution to this equation. It would have to be solved by numerical approximation (the solution is $x = .631$). □

An alternative method for simulating a finite mixture distribution
There is an alternate procedure that can be applied to simulate a finite mixture distribution like the one in Example SI1-9. The procedure is as follows.

Step 1 - create a finite discrete distribution based on the number of mixing distributions, using the mixing weights as the probabilities for the respective mixture components;
in Example SI1-9 there are three mixing distributions, so we create the three point random variable $I = \{1, 2, 3\}$, and assign a probability of .5 to $I = 1$, which is associated with the exponential random variable with mean 1, and we assign $P(I = 2) = .25$, associated with the exponential with mean 4, and $P(I = 3) = .25$, associated with the exponential with mean 5

Step 2 - simulate a value for I; this will require a pseudorandom uniform value, say u

Step 3 - once the value of I is determined, that indicates which of the mixture components is to be simulated; using a new pseudorandom number, say v, we simulate the appropriate mixture component distribution

We would apply this procedure to the mixture distribution in Example SI1-9 as follows. We define a 3-point random variable on the integers $I = \{1, 2, 3\}$ using the mixing weights as probabilities: $P(I = 1) = .5$, $P(I = 2) + .25$ and $P(I = 3) = .25$. We then simulate I as follows: given pseudorandom uniform u, if $u \leq .5$, then $I = 1$, if $.5 < u \leq .75$ then $I = 2$ and if $u > .75$ then $I = 3$. Once we have determine the value of I, we then use a new (independent) pseudorandom uniform number, say v, to simulate the exponential random variable represented by that value of I. This method requires two independent pseudorandom uniform numbers. For example, if $u = .55$ and $v = .79$, we simulate $I = 2$ because $.5 < .55 \leq .75$. Then we use $v = .79$ to simulate the exponential random variable with mean 4 (that was the one identified as Y_2, since $I = 2$). We find x from $.79 = 1 - e^{-.25x}$, so that $X = 6.24$.

SI-1.6 Simulation In A Life Contingent Insurance or Multiple Decrement Context

A simple example of simulation in a life-contingent context would be to simulate the number of deaths in a particular time period from a cohort of lives. For each age in the group, the number of deaths would be a binomial random variable based on the number of individuals at risk at a particular age, and the mortality probability for that age. The binomial with a large m can be inconvenient to simulate directly with table lookup, but a normal approximation could be applied.

Another example would be to simulate the year in which a particular death takes place. With reference to a life table, the distribution function of K_x, complete years until death has the following cdf $F_{K(x)}(k) = P(T_x \leq k) = {}_{k+1}q_x$. Given a pseudorandom number u we would simulate K_x in the usual method for a discrete random variable, as follows:
if ${}_kq_x \leq u < {}_{k+1}q_x$ then the simulate value of K_x is k.
For instance, if $u < q_x$ ($= {}_1q_x)$ then the simulated value of K_x is 0.

In a multiple decrement context, there are competing risks. For a cohort at age x, the decrements might be death, disability and withdrawal in an insurance context. Each of these would have an associated probability of decrement for the year, $q_x^{(d)}$ (death), $q_x^{(i)}$ (disability), and $q_x^{(w)}$ (withdrawal). The number of individuals out of a cohort who "leave" due to the various decrements forms a multinomial distribution. We simulate the numbers of decrements in a sequential way. Suppose that there are n individuals in the cohort. We choose a decrement and simulate an ordinary binomial random variable based on that n and the decrement probability.

Suppose we simulate deaths first. We then subtract the simulated number of deaths from n, and simulate the next decrement again as an ordinary binomial. Suppose the next decrement we simulate is disability. We simulate a binomial with $n - d_x^{(d)}$ trials and with probability
$\frac{q_x^{(i)}}{1-q_x^{(d)}}$ (not just simply the disability decrement probability $q_x^{(i)}$, but rather the **conditional disability probability given that there is no longer a death decrement**).
After we simulate the disabilities, we continue in a similar way to simulate the number of withdrawals. We can simulate the decrements in any order, as long as we use conditional probabilities in successive steps. The following example illustrates this.

Example SI-10:
A group of 1000 healthy insured lives at age (x) are subject to the following three decrement probabilities in year of age (x): $q_x^{(d)} = .1$ (death) , $q_x^{(i)} = .2$ (disability) , $q_x^{(w)} = .25$ (withdrawal).
Using the normal approximation and pseudouniform numbers $u_1 = .4$, $u_2 = .1$, $u_3 = .7$ simulate the decrements for the year in the following order: deaths, disabilities and then withdrawals.

Solution:
The number of deaths in the year has a binomial distribution with mean $1000 \times .1 = 100$ and variance $1000 \times .1 \times .9 = 90$. We simulate a normal random variable with those parameters. The simulated value is $d_x^{(d)} = 100 + \sqrt{90} \times \Phi^{-1}(.4) = 100 + \sqrt{90} \times (-.253) = 98$ (rounded to nearest integer).

Given 98 deaths from the original group of 1000, the number of disabilities has a binomial distribution based on $1000 - 98 = 902$ trials, with probability $q = \frac{q_x^{(i)}}{1-q_x^{(d)}} = \frac{.2}{1-.1} = \frac{2}{9}$, so the mean is $902 \times \frac{2}{9} = 200.4$ and the variance is $902 \times \frac{2}{9} \times \frac{7}{9} = 155.9$. We simulate a normal random variable with those parameters. The simulated value is
$d_x^{(i)} = 200.4 + \sqrt{155.9} \times \Phi^{-1}(.1) = 200.4 + \sqrt{155.9} \times (-1.282) = 184$ (rounded to nearest integer).

Given 98 deaths and 184 disabilities from the original group of 1000, the number of withdrawals has a binomial distribution based on $1000 - 98 - 184 = 718$ trials, with probability
$q = \frac{q_x^{(w)}}{1-q_x^{(d)}-q_x^{(i)}} = \frac{.25}{1-.1-.2} = \frac{5}{14}$, so the mean is $718 \times \frac{5}{14} = 256.4$ and the variance is
$718 \times \frac{5}{14} \times \frac{9}{14} = 164.8$. We simulate a normal random variable with those parameters. The simulated value is $d_x^{(w)} = 256.4 + \sqrt{164.8} \times \Phi^{-1}(.7) = 256.4 + \sqrt{164.8} \times (.524) = 263$ (rounded to nearest integer). □

The simulations in Example SI1-10 could have been done in a different order, and likely would have resulted in different numerical values for deaths, disabilities and withdrawals. The decrements can be simulated in any order. As long as the conditional binomial distributions are used for successive simulations, the method is valid.

Another application of simulation that we consider is simulation of the present value random variable for benefits in a life contingent context. The following example provides an illustration.

Example SI-11:
You are using the inversion method to simulate Z, the present value random variable for a special two-year term insurance on (70). You are given:

(i) (70) is subject to only two causes of death, with

k	${}_{k\|}q_{70}^{(1)}$	${}_{k\|}q_{70}^{(2)}$
0	0.10	0.10
1	0.10	0.50

(ii) Death benefits, payable at the end of the year of death, are:

During year	Benefit for Cause 1	Benefit for Cause 2
1	1000	1100
1	1`00	1100

(iii) $i = 0.06$

(iv) For this trial your random number, from the uniform distribution on $[0, 1]$, is 0.35 .

Calculate the simulated value of Z for this trial.

Solution:
To apply the inversion method, we must determine the distribution function of Z. This requires finding the possible present value amounts that occur under the various circumstances of death by cause 1 or 2 in year 1 or 2, or survival of the two years. There are five possible present value of benefit amounts:

	Event	Present value at issue	Probability
1	the insured survives the two year term	0	
2	death by cause 1 in year 1	$1000v = 943.4$	$q_{70}^{(1)} = 0.10$
3	death by cause 2 in year 1	$1100v = 1037.7$	$q_{70}^{(2)} = 0.10$
4	death by cause 1 in year 2	$1100v^2 = 979.0$	${}_{1\|}q_{70}^{(1)} = 0.10$
5	death by cause 2 in year 2	$1200v^2 = 1068.0$	${}_{1\|}q_{70}^{(2)} = 0.50$

Once we have determined the possible values for Z, we construct the distribution function for Z.
The probability and distribution function of Z is

Z :	0	943.4	979.0	1037.7	1068.0
Prob.	.2	.1	.1	.1	.5
F	.2	.3	.4	.5	1.0

According to the inversion method applied to a discrete, given the uniform random number u, we must find the two successive points of the distribution function F containing u. With $u = .35$, we see that $F(943.4) = .3 \leq .35 < .4 = F(979.0)$. The inversion method then indicates that the simulated value is the upper end of the interval for Z; so in this case, the simulated value of Z is 979. □

SI-1.7 Using Simulation to Estimate a Mean or a Probability

Suppose that we are simulating a random variable X which has mean μ and variance σ^2. Suppose we are trying to estimate μ, and we want to determine the number of simulated values of X needed, say n, so that $P[\,|\overline{X}-\mu| \leq .05\mu\,] = .9$. This is the same idea that was considered in limited fluctuation credibility. We saw that the number of (simulated) observations of X needed is $n \geq (\frac{1.645}{.05})^2 \cdot \frac{\sigma^2}{\mu^2}$. We can change the "$.05\mu$" factor to "$.01\mu$", and if we do so, then we change .05 to .01 in the denominator of the right side of the inequality. We can change the probability to .95 from .9, and if we do so, then we change 1.645 to 1.96 on the right side of the inequality. We usually do not know μ or σ^2, so as the successive values of X are simulated, we calculate updated estimates of μ and σ^2, and continue to simulate X's until the inequality $n \geq (\frac{1.645}{.05})^2 \cdot \frac{s^2}{\overline{x}^2}$ is satisfied.

We can apply a similar sort of idea to estimating a probability. Suppose that p is the unknown probability of success for an experiment. Our objective is to estimate p to within 1% of the actual value of p with a probability of 95%. We will estimate p with $Q_n = \frac{P_n}{n}$, where $P_n =$ #successes in n simulated trials. Q_n is the proportion of the simulated experiments out of n that are successes. We want $P[|Q_n - p| \leq .01p] \geq .95$. With a relatively large n, Q_n is approximately normal, so we can apply a similar approach as in the previous paragraph. This results in the following inequality $n \geq (\frac{1.96}{.01})^2 \cdot \frac{n-P_n}{P_n}$. p could be any probability, for instance $F_X(t)$, in which case, $P_n =$ #simulated X's that are $\leq t$ in n simulated trials.

PRACTICE EXAMS

ACTEX EXAM C/4 - PRACTICE EXAM 1

1. X has a Weibull distribution with parameters τ and θ. Find the density function $g(z)$ of the random variable $Z = 1 - e^{-(X/\theta)^\tau}$.

A) $\frac{1}{\theta}$ for $0 < z < \theta$ B) $\frac{1}{\tau}$ for $0 < z < \tau$ C) e^{-z} for $z > 0$
D) $\frac{1}{\theta^\tau} e^{-z/\theta^\tau}$ for $z > 0$ E) 1 for $0 < z < 1$

2. A portfolio of risks models the annual loss of an individual risk as having an exponential distribution with a mean of Λ. For a randomly selected risk from the portfolio, the value of Λ has an inverse gamma distribution with a mean of 40 and a standard deviation of 20. For a randomly chosen risk, find the probability that the annual loss for that risk is greater than 20.

A) .524 B) .544 C) .564 D) .584 E) .604

3. You are given the following:
- Losses follow a distribution (prior to the application of any deductible) with mean 2000.
- The loss elimination ratio (LER) at a deductible of 1000 is 0.3.
- 60 percent of the losses (in number) are less than the deductible of 1000.

Determine the average size of a loss that is less than the deductible of 1000.

A) Less than 300 B) At least 300 nut less than 320 C) At least 320 but less than 340
D) At least 340 but less than 360 E) At least 360

4. A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts are 1,2,3,... without limit. The probability that any given payout is equal to i is $\frac{1}{2^i}$. Payouts are independent. Calculate the probability that there are no payouts of 1, 2, or 3 in a given 20 minute period.

A) 0.08 B) 0.13 C) 0.18 D) 0.23 E) 0.28

5. Zoom Buy Tire Store, a nationwide chain of retail tire stores, sells 2,000,000 tires per year of various sizes and models. Zoom Buy offers the following road hazard warranty:
"If a tire sold by us is irreparably damaged in the first year after purchase, we'll replace it free, regardless of the cause."

The average annual cost of honoring this warranty is \$10,000,000, with a standard deviation of \$40,000. Individual claim counts follow a binomial distribution, and the average cost to replace a tire is \$100. All tires are equally likely to fail in the first year, and tire failures are independent. Calculate the standard deviation of the placement cost per tire.

A) Less than \$60 B) At least \$60, but less than \$65
C) At least \$65, but less than \$70 D) At least \$70, but less than \$75 E) At least \$75

6. You are given the following random sample: 7 , 12 , 15 , 19 , 26 , 27 , 29 , 29 , 30 , 33 , 38 , 53

Determine the method of percentile matching estimate of $P[X > 30]$ using the 25th and 75th smoothed empirical percentiles for a distribution with cdf $F(x) = 1 - \frac{1}{1+(\frac{x}{\theta})^\alpha}$.

A) Less than .30 B) At least .30 but less than .32 C) At least .32 but less than .34
D) At least .34 but less than .36 E) At least .36

7. Suppose a 3-year data set is divided into a year-by-year count of new entrants, deaths and right-censored observations:

$$n_0 = 1000\,,\ d_0 = 20\,,\ w_0 = 30\,,$$
$$n_1 = 200\,,\ d_1 = 10\,,\ w_1 = 20\,,$$
$$n_2 = 200\,,\ d_2 = 15\,,\ w_2 = 30\,.$$

A is the estimate of $S(2)$ using the approximation for large data sets if truncation is at the beginning of each interval and censoring is at the end of each interval and B is the estimate of $S(2)$ using the approximation for large data sets if truncation and censoring occur at mid-interval.

Actuarial exposure is used for both estimates. Find A/B .

A) Less than 1.000 B) At least 1.000 but less than 1.020
C) At least 1.020 but less than 1.040 D) At least 1.040 but less than 1.060
E) At least 1.060

8. Claim sizes of 10 or greater are described by a single parameter Pareto distribution, with parameter α. A sample of claim sizes is as follows: 10 12 14 18 21 25

Calculate the method of moments estimate for α for this sample.

A) Less than 2.0 B) At least 2.0, but less than 2.1 C) At least 2.1, but less than 2.2
D) At least 2.2, but less than 2.3 E) At least 2.3

9. Let X_1, X_2, X_3 be independent Poisson random variables with means θ, 2θ, and 3θ respectively. What is the maximum likelihood estimator of θ based on sample values $x_1, x_2,$ and x_3 from the distributions of X_1, X_2 and X_3, respectively,

A) $\frac{1}{2}\,\overline{x}$ B) $\overline{x}$ C) $\frac{x_1+2x_2+3x_3}{6}$ D) $\frac{3x_1+2x_2+x_3}{6}$ E) $\frac{6x_1+3x_2+2x_3}{11}$

10. You are given the following random sample of 12 data points from a population distribution X:
7 , 15 , 15 , 19 , 26 , 27 , 29 , 29 , 30 , 33 , 38 , 53

Suppose that the distribution variance is 100.

Determine the bias in the biased form of the sample variance as an estimator of the distribution variance.

A) Less than -5 B) At least -5 but less than 0 C) At least 0 but less than 5
D) At least 5 but less than 10 E) At least 10

ACTEX EXAM C/4 - PRACTICE EXAM 1 SOLUTIONS

1. $Z = g(X) = 1 - e^{-(X/\theta)^\tau} \rightarrow X = \theta[-ln(1-Z)]^{1/\tau} = k(Z)$.

$$\begin{aligned} f_Z(z) &= f_X(k(z)) \cdot k'(z) \\ &= \frac{\tau(\theta[-ln(1-z)]^{1/\tau})^{\tau-1} e^{-(\theta[-ln(1-z)]^{1/\tau}/\theta)^\tau}}{\theta^\tau} \cdot \theta \frac{1}{\tau}[-ln(1-z)]^{\frac{1}{\tau}-1} \cdot \frac{1}{1-z} \\ &= 1 \text{ for } 0 < z < 1 . \end{aligned}$$

Z has a uniform distribution on $(0,1)$. Answer: E

2. Given $\Lambda = \lambda$, the annual loss X has an exponential distribution with mean λ and Λ has an inverse gamma distribution. X is a continuous mixture distribution of an "exponential over an inverse gamma".

Suppose that the inverse gamma distribution of Λ has parameters α and θ. We are given that the mean and standard deviation of Λ are 20 and 10. Therefore, the variance of Λ is 100 and the 2nd moment of Λ is $E[\Lambda^2] = Var[\Lambda] + (E[\Lambda])^2 = 20^2 + 40^2 = 2000$.
The mean of an inverse gamma is $\frac{\theta}{\alpha-1}$. and the 2nd moment is $\frac{\theta^2}{(\alpha-2)(\alpha-1)}$.
From the two equations $\frac{\theta}{\alpha-1} = 40$ and $\frac{\theta^2}{(\alpha-2)(\alpha-1)} = 2000$,
we get $\frac{2000}{40^2} = \frac{\theta^2}{(\alpha-2)(\alpha-1)} \Big/ (\frac{\theta}{\alpha-1})^2 = \frac{\alpha-1}{\alpha-2}$. Then solving for α results in $\alpha = 6$.
Substituting back into $\frac{\theta}{\alpha-1} = 40$, we get that $\theta = 200$.

When we have a continuous mixture distribution for X over Λ, the pdf, expected values and probabilities for the marginal distribution of X can be found by conditioning over Λ.

The conditional pdf of X given $\Lambda = \lambda$ is $f(x|\Lambda = \lambda) = \frac{1}{\lambda}e^{-x/\lambda}$ and the pdf of the inverse gamma distribution of Λ is $f_\Lambda(\lambda) = \frac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1} \cdot \Gamma(\alpha)}$.
From the calculated parameter values, we have $f_\Lambda(\lambda) = \frac{200^6 e^{-200/\lambda}}{\lambda^7 \cdot \Gamma(6)}$.
We can find the probability $P(X > 20)$ by conditioning over λ:
$P(X > 20) = \int_0^\infty P(X > 20|\lambda) \cdot f_\Lambda(\lambda)\, d\lambda = \int_0^\infty e^{-20/\lambda} \cdot \frac{200^6 e^{-200/\lambda}}{\lambda^7 \cdot \Gamma(6)}\, d\lambda = \frac{200^6}{\Gamma(6)} \cdot \int_0^\infty \frac{e^{-220/\lambda}}{\lambda^7}\, d\lambda$.

We know that the inverse gamma pdf must integrate to 1 (as any pdf must), so that $\int_0^\infty \frac{\theta^\alpha e^{-\theta/\lambda}}{\lambda^{\alpha+1} \cdot \Gamma(\alpha)}\, d\lambda = 1$. It follows that $\int_0^\infty \frac{e^{-\theta/\lambda}}{\lambda^{\alpha+1}}\, d\lambda = \frac{\Gamma(\alpha)}{\theta^\alpha}$.
Therefore, $\int_0^\infty \frac{e^{-220/\lambda}}{\lambda^7}\, d\lambda = \int_0^\infty \frac{e^{-220/\lambda}}{\lambda^{6+1}}\, d\lambda = \frac{\Gamma(6)}{220^6}$, and $P(X > 20) = \frac{200^6}{\Gamma(6)} \cdot \frac{\Gamma(6)}{220^6} = .564$.

It is possible to show in general that if the conditional distribution of X given Λ is exponential with mean Λ, and if Λ has an inverse gamma distribution with parameters α and θ, then the marginal (unconditional) distribution of X is (two-parameter) Pareto with parameters α and θ.

In this example, the marginal distribution of X would be Pareto with parameters $\alpha = 6$ and $\theta = 100$, and $P(X > x) = (\frac{\theta}{\theta+x})^\alpha$, so that $P(X > 20) = (\frac{200}{220})^6$. Answer: C

3. We are to find $E[X|X<1000] = \int_0^{1000} x \cdot \frac{f(x)}{F(1000)}\,dx = \frac{1}{.6}\int_0^{1000} x\,f(x)\,dx$, since we are given $F(1000) = .6$ and since the conditional density function of X given that $X<c$ is $\frac{f(x)}{F(c)}$.
We are also given $E[X] = 2000$, and $LER = \frac{E[X\wedge 1000]}{E[X]} = .3$.
Therefore, $E[X\wedge 1000] = 600$.

But $E[X\wedge 1000] = \int_0^{1000} x\,f(x)\,dx + 1000[1-F(1000)] = \int_0^{1000} x\,f(x)\,dx + 400$, so that $\int_0^{1000} x\,f(x)\,dx = 200$. Then $E[X|X<1000] = \frac{1}{.6}(200) = 333.33$. Answer: C

4. When a payout occurs, it is 1, 2 or 3 with probability $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8}$.
The number of payouts that are 1, 2 or 3 follows a Poisson process with an hourly rate of $5\times\frac{7}{8}=\frac{35}{8}$. The expected number of payouts that are 1, 2 or 3 in 20 minutes, say N, has a Poisson distribution with mean $\frac{35/8}{3}=\frac{35}{24}$. The probability that there are no payouts of 1, 2, or 3 in a given 20 minute period is the probability that $N=0$, which is $e^{-35/24}=.233$. Answer: D

5. We denote by X the warranty claim that arises from the sale of a tire with a mean of $E[X]$ and variance $Var[X]$. The total claim for the year is $S=\sum_{i=1}^{2{,}000{,}000} X_i$, where X_i is the warranty claim that arises from the sale of the ith tire (we are told that 2,000,000 tires are sold).
$E[S] = 2,000,000E[X]$, and since tire failures are independent of one another, $Var[S] = 2,000,000Var[X]$. We are given that $E[S] = 10,000,000$, so that $E[X] = 5$, and we are given $Var[S] = 40,000^2$ so that $Var[X] = \frac{40{,}000^2}{2{,}000{,}000} = 800$.

When a tire is replaced, the cost of replacement is the random variable Y. We are given $E[Y] = 100$, and we are asked to find the standard deviation of Y. We are told that individual claim counts follow a binomial distribution. We interpret this as saying that the tire will either fail during the year with probability p or the tire will not fail, with probability $1-p$.

For each tire sold, the warranty cost X has distribution $X = \begin{cases} Y & \text{prob. } p \\ 0 & \text{prob. } 1-p \end{cases}$.
Then $E[X] = 5 = p\cdot E[Y] = 100p$, so that $p = .05$.

Above we found that $Var[X] = 800$, so that $E[X^2] - 5^2 = 800$ and then $E[X^2] = 825$.

It is also true that $E[X^2] = 825 = p\cdot E[Y^2] = .05\,E[Y^2]$, so that $E[Y^2] = 16,500$.

Then, $Var[Y] = E[Y^2] - (E[Y])^2 = 16,500 - 100^2 = 6,500$.

Finally, the standard deviation of Y is $\sqrt{6,500} = 80.6$. Answer: E

6. The smoothed empirical estimate of the 25th percentile is 16, and of the 75th percentile is 32.25. The cdf is $F(x) = 1 - \frac{1}{1+(\frac{x}{\theta})^\alpha}$. Using the empirical estimates, we get two equations: $.25 = F(16) = 1 - \frac{1}{1+(\frac{16}{\theta})^\alpha}$, $.75 = F(32.25) = 1 - \frac{1}{1+(\frac{32.25}{\theta})^\alpha}$.

Then, $1 + (\frac{16}{\theta})^\alpha = \frac{1}{.75} = \frac{4}{3}$ and $1 + (\frac{32.25}{\theta})^\alpha = \frac{1}{.25} = 4$, so that $(\frac{32.25}{\theta})^\alpha \Big/ (\frac{16}{\theta})^\alpha = 3/(\frac{1}{3}) = 9$. Then $(\frac{32.25}{16})^\alpha = 9$, so that $\alpha = \frac{ln(9)}{ln(32.25/16)} = 3.135$, and $\theta = 22.715$. The estimated value of $P[X > 30]$ is $1 - F(30) = \frac{1}{1+(\frac{30}{22.715})^{3.135}} = .295$.
Answer: A

7. With truncation at the start and censoring at the end of each interval, the exposures for the first two intervals are $e_0 = 1000$, $e_1 = 1150$. Then $A = A = \left(1 - \frac{20}{1000}\right)\left(1 - \frac{10}{1150}\right) = .9715$.
With truncation and censoring at the middle of each interval, the exposures for the first two intervals are $e_0 = \frac{1000-30}{2} = 485$, $e_1 = 950 + \frac{200-20}{2} = 1040$. Then $B = \left(1 - \frac{20}{485}\right)\left(1 - \frac{10}{1040}\right) = .9495$. Then $A/B = 1.023$. Answer: C

8. "Claim sizes are 10 or greater" indicates that $\theta = 10$. The mean of the single parameter Pareto with $\theta = 10$ is $\geq \frac{\alpha\theta}{\alpha-1} = \frac{10\alpha}{\alpha-1}$. The sample mean is $\frac{100}{6}$. According to the method of moments, we set $\frac{10\alpha}{\alpha-1} = \frac{100}{6}$ and solve for α. The resulting value is $\alpha = 2.5$. Answer: E

9. The likelihood function is
$L = f_1(x_1) \cdot f_2(x_2) \cdot f_3(x_3) = \frac{e^{-\theta}\theta^{x_1}}{x_1!} \cdot \frac{e^{-2\theta}(2\theta)^{x_2}}{x_2!} \cdot \frac{e^{-3\theta}(3\theta)^{x_3}}{x_3!} = \frac{e^{-6\theta}\,\theta^{x_1+x_2+x_3} \cdot 2^{x_2} \cdot 3^{x_3}}{x_1!\,x_2!\,x_3!}$.
$ln\,L = -6\theta + (x_1+x_2+x_3) \cdot ln\,\theta + c$ (where c does not depend on θ).
Setting $\frac{d}{d\theta}\,ln\,L = 0$ results in $\theta = \frac{x_1+x_2+x_3}{6} = \frac{\overline{x}}{2}$. Answer: A

10. The bias in the estimator is $E[\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2] - 100$. We know that $E[\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2] = 100$, since $\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$ is an unbiased estimator of $Var[X]$.
Therefore, $E[\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2] - 100 = E[\frac{n-1}{n} \cdot \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2] - 100$
$= \frac{11}{12}(100) - 100 = -8.33$. Answer: A