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Contents

Chapter 1 Survival Distributions

OBJECTIVES

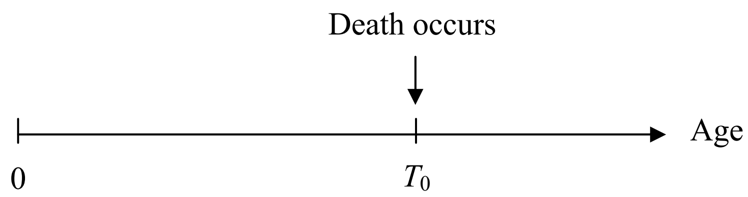
1. To define future lifetime random variables
2. To specify survival functions for future lifetime random variables
3. To define actuarial symbols for death and survival probabilities and develop relationships between them
4. To define the force of mortality

In Exam FM, you valued cash flows that are paid at some known future times. In Exam MLC, by contrast, you are going to value cash flows that are paid at some unknown future times. Specifically, the timings of the cash flows are dependent on the future lifetime of the underlying individual. These cash flows are called life contingent cash flows, and the study of these cash flows is called life contingencies.

It is obvious that an important part of life contingencies is the modeling of future lifetimes. In this chapter, we are going to study how we can model future lifetimes as random variables. A few simple probability concepts you learnt in Exam P will be used.

1. 1 Age-at-death Random Variable

Let us begin with the age-at-death random variable, which is denoted by T_0. The definition of T_0 can be easily seen from the diagram below.



The age-at-death random variable can take any value within $[0, \infty)$. Sometimes, we assume that no individual can live beyond a certain very high age. We call that age the limiting age, and denote it by ω. If a limiting age is assumed, then T_0 can only take a value within $[0, \omega]$.

We regard T_0 as a continuous random variable, because it can, in principle, take any value on the interval $[0, \infty)$ if there is no limiting age or $[0, \omega]$ if a limiting age is assumed. Of course, to model T_0, we need a probability distribution. The following notation is used throughout this study guide (and in the examination).

– $F_0(t) = \Pr(T_0 \le t)$ is the (cumulative) distribution function of T_0.

– $f_0(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} F_0(t)$ is the probability density function of T_0. For a small interval Δt, the product $f_0(t)\Delta t$ is the (approximate) probability that the age at death is in between t and $t + \Delta t$.

In life contingencies, we often need to calculate the probability that an individual will survive to a certain age. This motivates us to define the survival function:

$$S_0(t) = \Pr(T_0 > t) = 1 - F_0(t).$$

Note that the subscript "0" indicates that these functions are specified for the age-at-death random variable (or equivalently, the future lifetime of a person age 0 now).

Not all functions can be regarded as survival functions. A survival function must satisfy the following requirements:

1. $S_0(0) = 1$. This means every individual can live at least 0 years.
2. $S_0(\omega) = 0$ or $\lim_{t \to \infty} S_0(t) = 0$. This means that every individual must die eventually.
3. $S_0(t)$ is monotonically decreasing. This means that, for example, the probability of surviving to age 80 cannot be greater than that of surviving to age 70.

Summing up, $f_0(t)$, $F_0(t)$ and $S_0(t)$ are related to one another as follows.

FORMULA

Relations between $f_0(t)$, $F_0(t)$ and $S_0(t)$

$$f_0(t) = \frac{d}{dt}F_0(t) = -\frac{d}{dt}S_0(t), \tag{1.1}$$

$$S_0(t) = \int_t^\infty f_0(u)du = 1 - \int_0^t f_0(u)du = 1 - F_0(t), \tag{1.2}$$

$$\Pr(a < T_0 \le b) = \int_a^b f_0(u)du = F_0(b) - F_0(a) = S_0(a) - S_0(b). \tag{1.3}$$

Note that because T_0 is a continuous random variable, $\Pr(T_0 = c) = 0$ for any constant c. Now, let us consider the following example.

Example **1.1** [Structural Question]

You are given that $S_0(t) = 1 - t/100$ for $0 \le t \le 100$.

(a) Verify that $S_0(t)$ is a valid survival function.

(b) Find expressions for $F_0(t)$ and $f_0(t)$.

(c) Calculate the probability that T_0 is greater than 30 and smaller than 60.

Solution

(a) First, we have $S_0(0) = 1 - 0/100 = 1$.

Second, we have $S_0(100) = 1 - 100/100 = 0$.

Third, the first derivative of $S_0(t)$ is $-1/100$, indicating that $S_0(t)$ is non-increasing.

Hence, $S_0(t)$ is a valid survival function.

(b) We have $F_0(t) = 1 - S_0(t) = t/100$, for $0 \le t \le 100$.

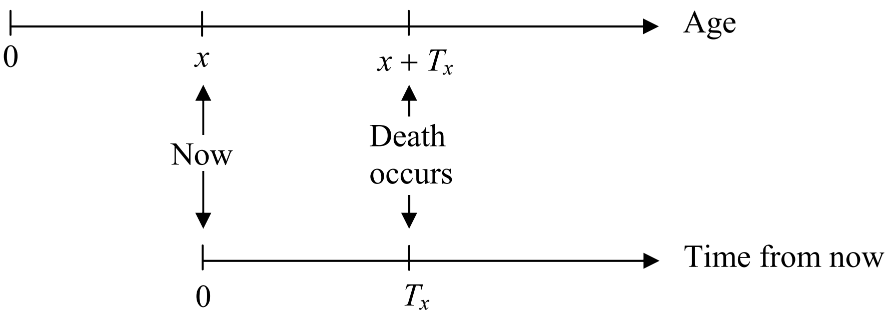
Also, we have and $f_0(t) = \frac{d}{dt}F_0(t) = 1/100$, for $0 \le t \le 100$.

(c) $\Pr(30 < T_0 < 60) = S_0(30) - S_0(60) = (1 - 30/100) - (1 - 60/100) = 0.3$.

[END]

1. 2 Future Lifetime Random Variable

Consider an individual who is age x now. Throughout this text, we use (x) to represent such an individual. Instead of the entire lifetime of (x), we are often more interested in the future lifetime of (x). We use T_x to denote the future lifetime random variable for (x). The definition of T_x can be easily seen from the diagram below.



[Note: For brevity, we may only display the portion starting from age x (i.e., time 0) in future illustrations.]

If there is no limiting age, T_x can take any value within $[0, \infty)$. If a limiting age is assumed, then T_x can only take a value within $[0, \omega - x]$. We have to subtract x because the individual has attained age x at time 0 already.

We let $S_x(t)$ be the survival function for the future lifetime random variable. The subscript "x" here indicates that the survival function is defined for a life who is age x now. It is important to understand that when modeling the future lifetime of (x), we always know that the individual is alive at age x. Thus, we may evaluate $S_x(t)$ as a conditional probability:

$$\begin{aligned} S_x(t) &= \Pr(T_x > t) = \Pr(T_0 > x + t \mid T_0 > x) \\ &= \frac{\Pr(T_0 > x + t \cap T_0 > x)}{\Pr(T_0 > x)} = \frac{\Pr(T_0 > x + t)}{\Pr(T_0 > x)} = \frac{S_0(x+t)}{S_0(x)}. \end{aligned}$$

The third step above follows from the equation $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$, which you learnt in Exam P.

FORMULA

Survival Function for the Future Lifetime Random Variable

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{1.4}$$

With $S_x(t)$, we can obtain $F_x(t)$ and $f_x(t)$ by using

$$F_x(t) = 1 - S_x(t) \quad \text{and} \quad f_x(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_x(t),$$

respectively.

Example 1.2 [Structural Question]

You are given that $S_0(t) = 1 - t/100$ for $0 \le t \le 100$.

(a) Find expressions for $S_{10}(t)$, $F_{10}(t)$ and $f_{10}(t)$.

(b) Calculate the probability that an individual age 10 now can survive to age 25.

(c) Calculate the probability that an individual age 10 now will die within 15 years.

Solution

(a) In this part, we are asked to calculate functions for an individual age 10 now (i.e., $x = 10$). Here, $\omega = 100$ and therefore these functions are defined for $0 \le t \le 90$ only.

First, we have $S_{10}(t) = \dfrac{S_0(10+t)}{S_0(10)} = \dfrac{1-(10+t)/100}{1-10/100} = 1 - \dfrac{t}{90}$, for $0 \le t \le 90$.

Second, we have $F_{10}(t) = 1 - S_{10}(t) = t/90$, for $0 \le t \le 90$.

Finally, we have $f_{10}(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} F_{10}(t) = \dfrac{1}{90}$.

(b) The probability that an individual age 10 now can survive to age 25 is given by

$$\Pr(T_{10} > 15) = S_{10}(15) = 1 - \frac{15}{90} = \frac{5}{6}.$$

(c) The probability that an individual age 10 now will die within 15 years is given by

$$\Pr(T_{10} \le 15) = F_{10}(15) = 1 - S_{10}(15) = \frac{1}{6}.$$

[END]

Chapter 2 Life Tables

OBJECTIVES

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2. 1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For x = 1, 2, …., the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0\, S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$ {}_t p_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x},$$

which means that we can calculate ${}_t p_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by ${}_t d_x$. It immediately follows that ${}_t d_x = l_x - l_{x+t}$.

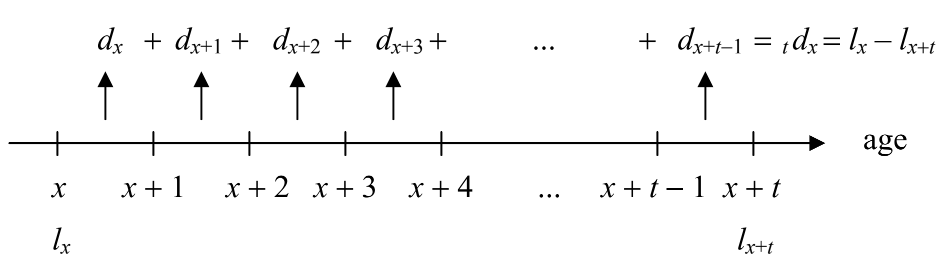
We can then calculate ${}_t q_x$ and ${}_{m|n} q_x$ by the following two relations:

$$ {}_t q_x = \frac{{}_t d_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \qquad {}_{m|n} q_x = \frac{{}_n d_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}.$$

When $t = 1$, we can omit the subscript t and write ${}_1 d_x$ as d_x. By definition, we have

$$ {}_t d_x = d_x + d_{x+1} + \ldots + d_{x+t-1}.$$

Graphically,



Also, when $t = 1$, we have the following relations:

$$d_x = l_x - l_{x+1},\ \ p_x = \frac{l_{x+1}}{l_x}\text{, and } q_x = \frac{d_x}{l_x}.$$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities ${}_tp_x$ and death probabilities ${}_tq_x$ for all integral values of t and x easily.

FORMULA

Life Table Functions

$$ {}_tp_x = \frac{l_{x+t}}{l_x} \tag{2.1}$$

$$ {}_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \ldots + d_{x+t-1} \tag{2.2}$$

$$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} \tag{2.3}$$

Exam questions are often based on the Illustrative Life Table, which is, of course, provided in the examination. To obtain a copy of this table, download the most updated Exam MLC syllabus. On the last page of the syllabus, you will find a link to Exam MLC Tables, which encompass the Illustrative Life Table. You may also download the table directly from

http://www.soa.org/files/edu/edu-2008-spring-mlc-tables.pdf.

The Illustrative Life Table contains a lot of information. For now, you only need to know and use the first three columns: x, l_x, and $1000q_x$. For example, to obtain the value q_{60}, simply use the column labeled $1000q_x$. You should obtain $1000q_{60} = 13.76$, which means $q_{60} = 0.01376$. It is also possible, but more tedious, to calculate q_{60} using the column labeled l_x; we have $q_{60} = 1 - l_{61} / l_{60} = 0.01376$.

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have ${}_5p_{60} = l_{65} / l_{60} = 7533964 / 8188074 = 0.92011$ and ${}_5q_{60} = 1 - {}_5p_{60} = 1 - 0.92011 =$

0.07989. Here, you should not base your calculations on the column labeled $1000q_x$, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1

You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) ${}_5p_{20}$

(b) q_{24}

(c) ${}_{4|1}q_{20}$

Solution

(a) $${}_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{95650.15}{96178.01} = 0.994512.$$

(b) $$q_{24} = \frac{d_{24}}{l_{24}} = \frac{112.7102}{95762.86} = 0.001177.$$

(c) $${}_{4|1}q_{20} = \frac{{}_1d_{24}}{l_{20}} = \frac{112.7102}{96178.01} = 0.001172.$$

[END]

Example 2.2 [Structural Question]

You are given:

(i) $S_0(x) = 1 - \dfrac{x}{100}, \quad 0 \le x \le 100$

(ii) $l_0 = 100$

(a) Find an expression for l_x for $0 \le x \le 100$.

(b) Calculate q_2.

(c) Calculate ${}_3q_2$.

Solution

(a) $l_x = l_0 S_0(x) = 100 - x$.

(b) $q_2 = \dfrac{l_2 - l_3}{l_2} = \dfrac{98-97}{98} = \dfrac{1}{98}$.

(c) ${}_3q_2 = \dfrac{l_2 - l_5}{l_2} = \dfrac{98-95}{98} = \dfrac{3}{98}$.

[END]

In Exam MLC, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3

For a certain population of 20 years old, you are given:

(i) 2/3 of the population are nonsmokers, and 1/3 of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over [0, 80).

(iii) The future lifetime of a smoker is uniformly distributed over [0, 50).

Calculate ${}_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution

The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}l_{20}$ nonsmokers and $\frac{1}{3}l_{20}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 40 is 1 – 20/80 = 3/4. For smokers, the proportion of individuals who can survive to age 40 is 1 – 20/50 = 3/5. At age 40, there are $\frac{3}{4}\frac{2}{3}l_{20} = 0.5l_{20}$ nonsmokers and $\frac{3}{5}\frac{1}{3}l_{20} = 0.2l_{20}$ smokers. Hence, among those who can survive to age 40, 5/7 are nonsmokers and 2/7 are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over [0, 60). This means that the probability for a nonsmoker to survive from age 40 to age 45 is 1 – 5/60 = 11/12.
- For a smoker at age 40, the remaining lifetime is uniformly distributed over [0, 30). This means that the probability for a smoker to survive from age 40 to age 45 is 1 – 5/30 = 5/6.

Finally, for the whole population, we have

$$ {}_5p_{40} = \frac{5}{7}\times\frac{11}{12}+\frac{2}{7}\times\frac{5}{6}=\frac{25}{28}. $$

[END]

2. 2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of ${}_tp_x$ and ${}_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a <u>fractional age assumption</u>.

Chapter 15 Participating Insurance

OBJECTIVES

1. To understand the key features of par policies
2. To describe how dividends on a par policy are computed
3. To describe how bonuses on a par policy are computed, and how they can be expressed as bonus rates
4. To calculate the profit vector of a par policy

Participating insurances (also known as par policies in the US and with profit policies in the UK) are insurance policies that pay dividends. The dividends are a portion of the insurance company's profits and are paid to the policyholder as if he or she were a stockholder. When claims are low and the company's investments perform well, dividends rise. However, in many cases dividends are not guaranteed and they may not be paid if the insurance company cannot make a profit.

In this short chapter we briefly discuss participating insurances and we shall stress on the following aspects: (1) the calculation of dividends and bonus, and (2) the options that are available to policyholders.

15. 1 Dividends

In Chapter 13 we have discussed the calculation of annual profit in great detail. Recall that for a single decrement model, in policy year $h + 1$, for a policy that is in force,

Total reserve
$= {}_hV$ at time h

$-$ Death benefit $q_{x+h}b_{h+1}$
$-$ Settlement expense $q_{x+h}E_{h+1}$
$-$ Total reserve needed $= (p_{x+h})\,{}_{h+1}V$

time

h $\qquad$ $h + 1$

$+$ Contract premium G_h
$-$ fraction of premium paid for expense c_hG_h
$-$ per policy expense e_h

The expected profit that emerges at the end of the $(h + 1)$th policy year per policy in force at the start of the year is then

$$\Pr_{h+1} = [{}_hV + G_h(1 - c_h) - e_h](1 + i_{h+1}) - (b_{h+1} + E_{h+1})q_{x+h} - p_{x+h}\,{}_{h+1}V\,.$$

The actual profit can be computed using realized interest rate, mortality and expenses.

For non-participating policies, the actual profits belong to the insurance company. The profits from a participating policy, however, are shared between the policyholder and the insurance company, usually in pre-specified proportions. In the official textbook *Actuarial Mathematics for Life Contingent Risks*, the following definitions are used:

- The term “dividend” is used when the profit share is distributed in the form of cash (or cash equivalent, such as a reduction of premium).
- The term “bonus” is used when the profit share is distributed in the form of additional insurance, such as increased death benefits, or extra term life insurance. Bonus in this case is the (additional) face amount of insurance that can be purchased.

In the US, it is common for policyholders of participating insurances to be given choices about the distribution, with premium reduction being a standard option. For single-premium policy, the policyholder may receive the cash dividend.

Since the case for cash dividend or premium refund is simpler than additional death benefit, let us discuss dividends first. To begin with, we look at an example with no surrender.

Example **15.1** [Structural Question]

A life aged 50 purchases a fully discrete participating whole life contract. The sum insured is 100,000. You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) The premium is 2500.

(iii) The insurer holds net premium reserves.

(iv) Net premium reserves are based on an interest rate of 6% and mortality following the Illustrative Life Table is used for computing the reserve. The reserves and death probabilities in the first 4 years are:

k	q_{49+k}	${}_kV$
1	0.005920	1406.19
2	0.006422	2856.55
3	0.006972	4350.89
4	0.007576	5888.85

(v) Pre-contract expense is 1200, assumed incurred at $t = 0$, and premium expenses of 50 are incurred with each premium payment, excluding the first.

Calculate the expected emerging profit Pr_k, $k = 0$, 1, 2, 3 and 4, assuming an investment return of 8% under the contractual arrangements in (a), (b) and (c):

(a) The participating contract pays no dividends during the first 4 years.

(b) From the second policy year onwards, 80% of profit that emerges each year is distributed to all policyholders as a cash dividend. Surviving policyholders would receive the dividend as cash, and beneficiaries of the policyholders who die during the year would receive the dividend as additional death benefit. No dividend is payable in the first policy year. No dividend is declared if the emerging profit is negative.

(c) From the second policy year onwards, surviving policyholders receive 80% of the profit per policy as cash dividends. The beneficiaries of the policyholders who die during the year would <u>not</u> receive the dividend as additional death benefit.

(d) Consider the arrangement in (b). How much premiums during the first 5 policy years would a surviving policyholders need to pay if the dividends are not distributed as cash but as a premium refund?

(e) Consider the arrangement in (c). How much of the profit that emerges from the third year is distributed as cash?

Solution

(a) When the contract pays no dividends during the first 4 years, the profit would not be shared, and hence we can use the same methodology as illustrated in Chapter 13:

$\mathrm{Pr}_0 = -1200$

$\mathrm{Pr}_1 = 2500 \times 1.08 - 100000 \times 0.005920 - 0.99408 \times 1406.19 = 710.135$

$\mathrm{Pr}_2 = (1406.19 + 2500 - 50) \times 1.08 - 100000 \times 0.006422 - 0.993578 \times 2856.55 = 684.280$

$\mathrm{Pr}_3 = (2856.55 + 2500 - 50) \times 1.08 - 100000 \times 0.006972 - 0.993028 \times 4350.89 = 713.318$

$\mathrm{Pr}_4 = (4350.89 + 2500 - 50) \times 1.08 - 100000 \times 0.007576 - 0.992424 \times 5888.85 = 743.165$

(b) When 80% of the emerging profit (except for the first policy year) are distributed, the profits calculated in (a) would be denoted by Pr_{k-}, and the dividends are

$$\mathrm{Div}_k = 0.8\mathrm{Pr}_{k-}, \quad \text{for } k > 1.$$

Since the dividends have to be distributed no matter if the policyholder survives (distributed as dividends) or dies (distributed as additional death benefit payable at the end of the year)

$$\mathrm{Pr}_k = \mathrm{Pr}_{k-} - \mathrm{Div}_k = 0.2\mathrm{Pr}_{k-}, \quad \text{for } k > 1.$$

k	Div_k	Pr_k
1	0	710.135
2	547.424	136.856
3	570.6544	142.6636
4	594.532	148.633

(c) The expected profits generated per policy are the Pr_k's in (a). The dividend is distributed only to surviving policyholders at the end of the year. So

$$\mathrm{Pr}_k = \mathrm{Pr}_{k-} - p_{49+k}\,\mathrm{Div}_k, \quad \text{for } k > 1.$$

k	Pr_k
1	710.135
2	140.372
3	146.642
4	153.137

(d) In this case the premium payable at time k would be deducted by the dividend. So,

k	Premium
0	2500
1	2500
2	1952.576
3	1929.3456
4	1905.468

(e) The percentage is 1 – 146.372 / 713.318 = 1 – 20.52% = 79.48%, which is less than 80%. This is because there are no cash distributed for policies who terminate during year 3 and the whole profit would be grabbed by the insurer.

[END]

Example 15.2

Consider the policy in Example 15.1 again, but with the following profit sharing mechanism. From the second policy year onwards, the insurer distributes a cash dividend to holders of policies which are in force at the year end. The cash dividend is determined using 80% of the emerging surplus at the year end, if the surplus is positive.

Project the cash dividend for this policy at the end of year 3 for surviving policies.

Solution

The goal of the insurer is to distribute 80% of the emerging surplus. Since only surviving policies would get cash dividends and policies that terminate get nothing, the surviving policies would get more than 80% of the emerging surplus.

Let the cash dividend to be paid be Div_3. The profit after distribution of cash dividend is

$$713.318 - \text{Div}_3.$$

We want

$$0.2 = \frac{713.318 - p_{52}\text{Div}_3}{713.318} = 1 - \frac{p_{52}}{713.318}\text{Div}_3$$

and hence $\text{Div}_3 = \dfrac{0.8 \times 713.318}{p_{52}} = \dfrac{570.6544}{0.993028} = 574.661$. This is 574.661 / 713.318 = 80.56% of the profits before distribution of the cash dividends.

[END]

Note that in the above example, the dividend for a police in force at the end of the year is calculated from $\text{Div}_3 = 0.8\text{Pr}_{3-} / p_{52}$, while in Example 15.1(c), we have $\text{Div}_3 = 0.8\text{Pr}_{3-}$ for surviving policies and hence the expected dividends to be paid for a policy that is in police at the beginning of the year is $0.8\text{Pr}_{3-} \times p_{53}$. A slight change in wording can give quite different answer.

You can continue with the calculation of Example 15.1 and 15.2 for $k = 5$, 6, and so on, and find all the Pr_k's for k being sufficiently large so that the remaining profits would be negligibly small. If you do so for Example 15.1, you can verify that the NPV in (b) is 869.4 under a risk adjusted rate of 10%. However, without the help of a spreadsheet program it is next to impossible to compute all Pr_k's and then obtain the profit signature and the associated profit measures. In Exam MLC, it is more likely that they give you the reserves, expenses and mortality for two to three years and ask you to compute the dividend, just like the following example:

Example **15.3**

A life aged 60 purchases a fully discrete participating whole life contract. The sum insured is 100,000. You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) The premium is 4000.

(iii) The insurance company holds full preliminary term reserves.

(iv) Full preliminary term reserves are based on an interest rate of 6% and mortality following the Illustrative Life Table is used for computing the reserve.

(v) Premium expenses of 50 are incurred with each premium payment, excluding the first.

(vi) Annual profits are computed assuming reserves and premiums earn 8% interest per year.

Calculate the dividend payable 22 years after the issuance of the policy for the following contractual agreements:

(a) The insurance company would distribute 90% of profit that emerges at the end of each year from the fifteenth policy year onwards to policyholders (or his / her beneficiaries) as a cash dividend no matter if the policyholder survives or not.

(b) The insurance company would distribute 90% of profit that emerges at the end of each year from the fifteenth policy year onwards is distributed to surviving policyholders as a cash dividend.

Solution

We first compute ${}_{21}V$ and ${}_{22}V$. Because a full preliminary term reserve is used,

$$ {}_{21}V = 100000\,{}_{20}V_{61} = 100000\left(1-\frac{\ddot{a}_{81}}{\ddot{a}_{61}}\right) = 100000\left(1-\frac{5.6533}{10.9041}\right) = 48154.4$$

$$ {}_{22}V = 100000\,{}_{21}V_{61} = 100000\left(1-\frac{\ddot{a}_{82}}{\ddot{a}_{61}}\right) = 100000\left(1-\frac{5.4063}{10.9041}\right) = 50419.6$$

Also, $q_{81} = 0.08764$.

So, $\text{Pr}_{22-} = (48154.4 + 4000 - 50) \times 1.08 - 0.08764 \times 100000 - 0.91236 \times 50419.6 = 1507.917$.

(a) The dividend payable is $0.9\text{Pr}_{22-} = 1357.13$.

(b) Let the dividend be Div_{22}. We need

$$0.1 = 1 - \frac{p_{81}}{\text{Pr}_{22-}}\text{Div}_{22}$$

and hence $\text{Div}_{22} = \dfrac{0.9 \times \text{Pr}_{22-}}{p_{81}} = \dfrac{1357.13}{0.91236} = 1487.49$.

[END]

The examples above are not realistic because they do not consider the possibility of lapses. However, lapse is very important because the premiums for such policies are high (see the end of this section for the explanation.) To model lapses, we use the framework introduced in Chapter 8: deaths (d) occur continuously throughout the year, and lapses (w) may happen immediately before a premium payment. Let

$$q'^{(d)}_{x+h} \quad \text{and} \quad q'^{(w)}_{x+h}$$

be the independent rates of decrement. Then assuming annual premiums are payable at the beginning of each year, we have

$$q^{(d)}_{x+h} = q'^{(d)}_{x+h}, \quad q^{(w)}_{x+h} = p'^{(d)}_{x+h} q'^{(w)}_{x+h}, \qquad p^{(\tau)}_{x+h} = p'^{(d)}_{x+h}(1 - q'^{(w)}_{x+h}),$$

and the profit before dividend is

Exam MLC Mock Test 7

SECTION A – Multiple Choice

****BEGINNING OF EXAMINATION****

1. You are given:

(i) The force of mortality follows Gompertz's law with $B = 0.00005$ and $c = 1.1$.

(ii) $f(t, t + k)$ is the forward interest rate, contracted at time 0, effective from time t to $t + k$.

(iii) The following forward interest rates:

t	0	1	2
$f(t, 3)$	0.05	0.03	0.06

Calculate $\ddot{s}_{60:\overline{3}|}$.

(A) 3.40

(B) 3.41

(C) 3.42

(D) 3.43

(E) 3.44

2. You are given:

(i) $A_x = 0.3$

(ii) $A_{x:\overline{n}|} = 0.55$

(iii) $\bar{A}_x = 0.315$

(iv) $\bar{A}_{x:\overline{n}|} = 0.56$

(v) Deaths are uniformly distributed over each year of age.

Find $A_{\stackrel{1}{x}:\overline{n}|}$.

(A) 0.35

(B) 0.38

(C) 0.41

(D) 0.44

(E) 0.47

3. A certain policy providing a death benefit of 1,000 at the end of the year of death is issued with a gross premium of 25. Let ${}_tAS$ be the asset share at time t. You are given:

(i) $i = 0.03$

(ii) ${}_{10}AS = 170$

(iii) Percent of premium expense for the 11$^{\text{th}}$ premium is 0.1.

(iv) Expense per 1,000 incurred at the beginning of the 11$^{\text{th}}$ year is 2.5.

(v) The probability of decrement by death in the 11$^{\text{th}}$ year is 0.003

(vi) The probability of decrement by withdrawal in the 11$^{\text{th}}$ year is 0.1

(vii) The cash value in the 11$^{\text{th}}$ year is ${}_{11}CV = 170$.

Find ${}_{11}AS$.

(A) 169

(B) 171

(C) 185

(D) 190

(E) 196

4. You are given:

(i) $i = 0.06$

(ii) $\ddot{a}_{60} = 10.50$

(iii) $\mu_{60} = 0.015$

(iv) Using Woolhouse's formula with three terms, the value of $\ddot{a}_{60}^{(2)}$ is γ_1.

(v) Assuming uniform distribution of deaths over each year of age, the value of $\ddot{a}_{60}^{(2)}$ is γ_2.

Find $10000(\gamma_1 - \gamma_2)$.

(A) −6

(B) −3

(C) 0

(D) 3

(E) 6

5. For a fully discrete whole life insurance of 1000 on (30), you are given:

(i) $_kV$ is the net premium reserve at the end of year k.

(ii) $_{10}V = 180$

(iii) $A_{30} = 0.35065$

(iv) $v = 0.95$

(v) $q_{40} = 0.02$, $q_{41} = 0.03$

Calculate $_{12}V$.

(A) 202

(B) 218

(C) 222

(D) 228

(E) 240

Exam MLC Mock Test 7

SECTION B – Written Answer

1. (*8 points*) Let T_0 be a continuous random variable whose possible values are in $(0, \infty)$.

(a) (*3 points*) Suppose that T_0 has a unique mode at $x_0 > 0$.

Show that at x_0, $\mu_x' = \mu_x^2$.

(b) (*5 points*) Consider the force of mortality

$$\mu_x = A + \frac{Bc^x}{1 + Dc^x} \text{ for } x > 0.$$

Here $A \geq 0$, B, $D > 0$, and $c > 1$.

(i) Find $S_0(x)$.

(ii) What is the mode of the distribution of T_0 when $A = 0$?

2. (*8 points*) For two independent future lifetime random variables T_x and T_y, you are given:

- ${}_tp_x = 1 - t^2q_x,\ 0 \le t \le 1$
- ${}_tp_y = 1 - t^2q_y,\ 0 \le t \le 1$
- $q_x = 0.080$ and $q_y = 0.004$

(a) (*2 points*) For $0 < t < 1$, derive an expression for $f_{xy}(t)$ in terms of t.

(b) (*2 points*) Calculate the probability that (y) will die within 6 months and will be predeceased by (x).

(c) (*4 points*) Let the force of interest be 8%. Calculate

$$1000\,(\overline{IA})^{2}_{xy:\overline{1}|}\,,$$

the expected present value of \$1000t dollars payable when x dies at time t, if the death of x happens within 1 year, and if x dies after y.

3. (*8 points*) Consider a 3-year term universal life insurance on (50). The death benefit is \$80,000 plus the account value at the end of the year of death. The death benefit is payable at the end of the year of death. There is no corridor factor requirement. You profit test the contract using the following basis:

- Premiums of \$3,000 each are paid at the beginning of years 1, 2 and 3.
- The projected account values at the end of years 1, 2 and 3 are \$1,950, \$4,100 and \$6,400, respectively.
- Incurred expenses are \$200 at inception, \$50 plus 1% of premium at renewal, \$100 on surrender, and \$100 on death.
- The surrender charge is \$800 for all durations.
- The insurer's earned rate of interest is 10% for all three years.
- Mortality experience is 120% of the Illustrative Life Table.
- Surrenders occur at year ends only. The surrender rates for years 1, 2 and 3 are 10%, 20% and 100%, respectively.
- The insurer holds the full account value as reserve for the contract.

Find the profit vector.

Solutions to Mock Test 7

Section A

1.	A	11.	C
2.	A	12.	B
3.	E	13.	E
4.	E	14.	D
5.	B	15.	D
6.	D	16.	C
7.	B	17.	D
8.	C	18.	B
9.	E	19.	B
10.	A	20.	C

1. [Chapter 12] Answer: (A)

We need to compute $\ddot{s}_{60:\overline{3}|} = \dfrac{1+f(2,3)}{p_{62}} + \dfrac{(1+f(1,3))^2}{{}_2p_{61}} + \dfrac{(1+f(0,3))^3}{{}_3p_{60}}$. For Gompertz's law,

$$ {}_tp_x = \exp\left(-\int_x^{x+t}(Bc^u)\mathrm{d}u\right) = \exp\left(-B\frac{c^{x+t}-c^x}{\ln c}\right). $$

We have

$$ p_{62} = \exp\left(-0.00005\times\frac{1.1^{63}-1.1^{62}}{\ln 1.1}\right) = 0.980858 $$

$$ {}_2p_{61} = \exp\left(-0.00005\times\frac{1.1^{63}-1.1^{61}}{\ln 1.1}\right) = 0.963774 $$

$$ {}_3p_{60} = \exp\left(-0.00005\times\frac{1.1^{63}-1.1^{60}}{\ln 1.1}\right) = 0.948502 $$

So the answer is $\dfrac{1.06}{0.980858} + \dfrac{1.03^2}{0.963774} + \dfrac{1.05^3}{0.948502} = 3.402$.

2. [Chapter 3] Answer: (A)

We need to find ${}_nE_x$. From (i), (iii) and (v), we get $\dfrac{i}{\delta} = \dfrac{\bar{A}_x}{A_x} = \dfrac{0.315}{0.3}$.

On the other hand, $\dfrac{\bar{A}^1_{x:\overline{n}|}}{A^1_{x:\overline{n}|}} = \dfrac{i}{\delta}$. So,

$$ \frac{\bar{A}_{x:\overline{n}|} - {}_nE_x}{A_{x:\overline{n}|} - {}_nE_x} = \frac{0.315}{0.3} \quad \Rightarrow \quad \frac{0.56 - {}_nE_x}{0.55 - {}_nE_x} = \frac{0.315}{0.3} $$

$$0.55(0.315) - 0.56(0.3) = (0.315 - 0.3)\,{}_nE_x$$
$${}_nE_x = 0.35$$

3. [Chapter 9] Answer: (E)

   By the recursive relation for asset shares,

$${}_{11}AS = \frac{(170 + 0.9 \times 25 - 2.5) \times 1.03 - 1000 \times 0.003 - 170 \times 0.1}{1 - 0.003 - 0.1} = \frac{175.7}{0.897} = 195.875\,.$$

4. [Chapter 4] Answer: (E)

   Woolhouse's formula for whole life annuities is given by

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)\,.$$

   Therefore, $\gamma_1 = 10.5 - \frac{2-1}{2\times 2} - \frac{2^2-1}{12\times 2^2}(\ln(1.06) + 0.015) = 10.245421$.

   Assuming uniform distribution of deaths over each year of age, we have

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)\,.$$

   From Exam MLC Tables, we obtain $\alpha(2) = 1.00021$, $\beta(2) = 0.25739$. Therefore,

$$\gamma_2 = 1.00021 \times 10.5 - 0.25739 = 10.244815.$$

   Finally, $10000(\gamma_1 - \gamma_2) = 100000(10.245421 - 10.244815) = 6$. Hence, the answer is (E).

5. [Chapter 6] Answer: (B)

   From statements (iii) and (iv), we can back out the net annual premium:

$$d = 1 - v = 0.05,$$

$$1000P_{30} = \frac{1000dA_{30}}{1 - A_{30}} = \frac{50 \times 0.35065}{1 - 0.35065} = 27\,.$$

   By Fackler's accumulation formula,

$${}_{11}V = \frac{({}_{10}V + \pi)(1+i) - 1000q_{40}}{p_{40}} = \frac{(180 + 27)/0.95 - 20}{0.98} = 201.9334\,,$$

$${}_{12}V = \frac{({}_{11}V + \pi)(1+i) - 1000q_{41}}{p_{41}} = \frac{(201.9334 + 27)/0.95 - 30}{0.97} = 217.51\,.$$

6. [Chapter 11] Answer: (D)

   We need to compute ${}_nq_{xy}^{1}$. Since the number is attached to (x), we condition on T_x, which has a density of

$$f_x(t) = \begin{cases} \dfrac{1}{100-x} & 0 < t < 100-x \\ 0 & t \geq 100-x \end{cases}.$$

As a result,

$$\begin{aligned} {}_n q_{xy}^{\;1} &= \int_0^n {}_t p_y f_x(t)\mathrm{d}t = \int_0^{100-x} {}_t p_y \frac{1}{100-x}\mathrm{d}t + \int_{100-x}^n {}_t p_y (0)\mathrm{d}t \\ &= \frac{1}{100-x}\int_0^{100-x} e^{-\mu t}\mathrm{d}t = \frac{1}{(100-x)\mu}(1-e^{-\mu(100-x)}). \end{aligned}$$

7. [Chapter 11] Answer: (B)

We calculate ${}_{6|}q_{50}$, ${}_{6|}q_{60}$, and ${}_{6|}q_{50:60}$.

Firstly, ${}_{6|}q_{50} = 0.9 \times 0.03 = 0.027$. Secondly, ${}_{6|}q_{60} = 0.8 \times 0.05 = 0.04$.

Thirdly ${}_{6|}q_{50:60} = {}_6p_{50:60} \times q_{56:66} = (0.9 \times 0.8) \times (1 - 0.97 \times 0.95) = 0.05652$.

Finally, by symmetric relation, the answer is $0.027 + 0.04 - 0.05652 = 0.01048$.

8. [Chapter 8] Answer: (C)

From (iii), decrement 1 is SUDD. So $\mu_{x+0.75}^{(1)} = \dfrac{q_x'^{(1)}}{1-0.75q_x'^{(1)}}$ holds.

From (i),

$$\begin{aligned} \frac{4}{197} &= \frac{q_x'^{(1)}}{1-0.75q_x'^{(1)}} \\ 197q_x'^{(1)} &= 4 - 3q_x'^{(1)} \\ q_x'^{(1)} &= 0.02 \end{aligned}$$

Now we can apply the formula for SUDD (Equation (8.6)) to obtain

$${}_{0.75}q_x^{(1)} = q_x'^{(1)}\left(0.75 - \frac{0.75^2}{2}(q_x'^{(2)} + q_x'^{(3)}) + \frac{0.75^3}{3}q_x'^{(2)}q_x'^{(3)}\right) = 0.0148598.$$

9. [Chapter 7 + 9] Answer: (E)

The anticipated probability of survival $p_{x+1}^{(\tau)}$ is $1 - 0.011 - 0.15 = 0.839$.

The expected profit for a policy that is in force at time 2 is

$$(61.7 + 75 \times 0.94 - 2)(1.04) - 1000 \times 0.011 - 10 \times 0.15 - 136.51 \times 0.839 = 8.37611.$$

The actual probability of survival is $1 - 0.015 - 0.2 = 0.785$.

The actual profit for the policy is

$$(61.7 + 75 \times 0.95 - 3)(1.12) - 1000 \times 0.015 - 10 \times 0.20 - 136.51 \times 0.785 = 21.38365.$$

Since there are $2355 / 0.785 = 3000$ policy in force at the beginning of the year, the total gain is $3000 \times (21.38365 - 8.37611) = 39022.62$.

10. [Chapter 10] Answer: (A)

(I) The statement is incorrect. The exit rate for state 0 is 0.02 + 0.03 + 0.01 = 0.06. By Equation (10.2),

$$ {}_h p_{x+t}^{00} = 1 - 0.06h + o(h). $$

(II) States 1 and 3 are absorbing states. The Q matrix of the CTMC is

$$ \begin{bmatrix} -0.06 & 0.02 & 0.03 & 0.01 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & -0.07 & 0.06 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

KFE says $\dfrac{\mathrm{d}}{\mathrm{d}t}({}_t\mathbf{P}_x) = {}_t\mathbf{P}_x\, Q_{x+t}$. For ${}_t p_x^{20}$, we look at

$$ \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} {}_t p_x^{00} & {}_t p_x^{01} & {}_t p_x^{02} & {}_t p_x^{03} \\ 0 & 1 & 0 & 0 \\ {}_t p_x^{20} & {}_t p_x^{21} & {}_t p_x^{22} & {}_t p_x^{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}_t p_x^{00} & {}_t p_x^{01} & {}_t p_x^{02} & {}_t p_x^{03} \\ 0 & 1 & 0 & 0 \\ {}_t p_x^{20} & {}_t p_x^{21} & {}_t p_x^{22} & {}_t p_x^{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.06 & 0.02 & 0.03 & 0.01 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0 & -0.07 & 0.06 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

As a result, $\dfrac{\mathrm{d}\,{}_t p_x^{20}}{\mathrm{d}t} = -0.06\,{}_t p_x^{20} + 0.01\,{}_t p_x^{22}$, and hence statement (ii) is also incorrect.

(III) The differential equation satisfied by ${}_t p_x^{01}$ is

$$ \frac{\mathrm{d}\,{}_t p_x^{01}}{\mathrm{d}t} = 0.02\,{}_t p_x^{00}. $$

If ${}_t p_x^{01} = \dfrac{1 - e^{-0.06t}}{3}$ is the solution, we can put it into the differential equation above to obtain

$$ 0.02 e^{-0.06t} = 0.02\,{}_t p_x^{00} $$

which would gives us ${}_t p_x^{00} = e^{-0.06t} = {}_t p_x^{\overline{00}}$. However, since state 0 can be re-entered from state 2, ${}_t p_x^{00} > {}_t p_x^{\overline{00}}$ and this means that ${}_t p_x^{01} = \dfrac{1 - e^{-0.06t}}{3}$ is not the correct solution. Thus, statement (III) is incorrect.

11. [Chapter 4] Answer: (C)

We first calculate the mean of Y, which is actually $\bar{a}_{x:\overline{10|}}$.

Under constant force of mortality (over age x to x + 10),

$$ \bar{a}_{x:\overline{10|}} = \frac{1}{\mu + \delta}(1 - e^{-10(\mu+\delta)}) = \frac{1 - e^{-1}}{0.1} = 6.32121. $$

To calculate probabilities involving Y, we first write down the precise definition of Y as a function of T_x:

$$Y = \begin{cases} \dfrac{1-e^{-0.08T_x}}{0.08} & T_x \le 10 \\ \dfrac{1-e^{-0.8}}{0.08} = 6.88 & T_x > 10 \end{cases}$$

The event "$Y \le 0.75 \times 6.32121 = 4.7409$" is equivalent to

$$\frac{1-e^{-0.08T_x}}{0.08} \le 4.7409 \qquad \text{or} \qquad T_x \le 5.9608.$$

This means the required probability is

$$\Pr(T_x \le 5.9608) = 1 - e^{-5.9608 \times 0.02} = 0.1124.$$

12. [Chapters 9 + 16] Answer: (B)

The salary over age (63, 64) is 75,000 × 3.643 / 3.589 = 76,128.45.
The salary over age (64, 65) is 75,000 × 3.698 / 3.589 = 77,277.79.

The death benefit payable at age 62.5 is 75,000 × 3 = 225,000. The probability for this to occur is 312 / 42680.

The death benefit payable at age 63.5 is 76,128.45 × 3 = 228,385.35. The probability for this to occur is 284 / 42860.

The death benefit payable at age 64.5 is 77,277.79 × 3 = 231,833.37. The probability for this to occur is 215 / 42860.

So the APV of the death benefit is

$$\frac{225{,}000}{1.06^{1/2}} \times \frac{312}{42680} + \frac{228385.35}{1.06^{3/2}} \times \frac{284}{42680} + \frac{231833.37}{1.06^{5/2}} \times \frac{215}{42680} = 3999.64.$$

13. [Chapter 9] Answer: (E)

Let the net premium be π and the contract premium be G. Decrement due to death is denoted by (1) and decrement due to withdrawal is denoted by (2).

Since ${}_0V = 0$, the recursion relation for net premium reserves says

$$1.1\pi = 36 p_x^{(\tau)} + 1000 q_x^{(1)} + 36(0.4) q_x^{(2)}.$$

Since ${}_0AS = 0$, the recursion relation for asset shares says

$$1.1(0.5G) = 18 p_x^{(\tau)} + 1000 q_x^{(1)} + 36(0.4) q_x^{(2)}$$

or

$$1.1G = 36 p_x^{(\tau)} + 2000 q_x^{(1)} + 72(0.4) q_x^{(2)}$$

Upon subtraction, we get

$$1.1(G-\pi) = 1000q_x^{(1)} + 36(0.4)q_x^{(2)}$$

$$1.1(9.36) = 100q_x^{(2)} + 14.4q_x^{(2)}$$

On solving, we get $q_x^{(2)} = 0.09$.

14. [Chapter 3] Answer: (D)

The benefit function is $e^{0.05t}$ during the first year ($0 < t < 1$), $2e^{0.05t}$ during the second year ($1 \le t < 2$), and $3e^{0.05t}$ during the third year ($2 \le t < 3$), and so on, up to the tenth year.

The APV of this insurance is

$$\int_0^{10} b_t v^t \mathrm{f}_x(t)\mathrm{d}t = \int_0^{10} \lfloor t+1 \rfloor e^{0.05t} e^{-0.04t} 0.01e^{-0.01t}\mathrm{d}t = 0.01\int_0^{10} \lfloor t+1 \rfloor \mathrm{d}t$$
$$= 0.01(1+2+3+\ldots+10) = 0.01\left(\frac{11\times 10}{2}\right) = 0.55$$

15. [Chapter 12] Answer: (D)

Let Z be the present value random variable for a discrete whole life insurance of \$1 on (x). It is known that $\mathrm{Var}(Z) = ({}^2A_{50} - (A_{50})^2)$.

We can express the random variable U as follows:

$$U = \frac{1}{N}\sum_{i=1}^{N} MZ_i ,$$

where Z_i's are mutually independent and are identically distributed as Z. It follows that

$$\mathrm{Var}(U) = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} MZ_i\right) = \frac{1}{N^2}\mathrm{Var}\left(\sum_{i=1}^{N} MZ_i\right)$$
$$= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}(MZ_i) = \frac{1}{N^2}\sum_{i=1}^{N} M^2\mathrm{Var}(Z_i)$$
$$= \frac{M^2 N\mathrm{Var}(Z)}{N^2} = \frac{M^2\mathrm{Var}(Z)}{N}.$$

Hence, $\mathrm{Var}(U) = M^2({}^2A_{50} - (A_{50})^2)/N$.

16. [Chapter 10] Answer: (C)

The EPV of the benefit is $5000\int_0^5 e^{-0.03t}({}_t p_x^{\overline{00}} \mu_{x+t}^{02} + {}_t p_x^{01} \mu_{x+t}^{12})\mathrm{d}t$.

The first part of the EPV comes from a direct transition from state 0 to 2, without going through state 1. The second part of the EPV comes from a transition from state 0 to state 1 and then to state 2.

Since $v_{x+t}^{0} = 0.2 + 0.08t + 0.05 + 0.05t = 0.25 + 0.13t$,

$${}_t p_x^{\overline{00}} = \exp\left(-\int_0^t (0.25 + 0.13s)\mathrm{d}s\right) = \exp(-0.25t - 0.065t^2)$$

So, $g(1.5) = 5000e^{-0.045}(e^{-0.52125} \times 0.125 + 0.1087 \times 0.18375) = 450.26$.

17. [Chapter 11 + 14] Answer: (D)

The account value at the end of month 12 is

$$\text{AV}_{12} = (1800 + 95 \times 0.9 - 8 - \frac{10\times 2.8}{1.004}) \times 1.004 = 1857.01.$$

So, the cash surrender value is 1857.01 − 400 = 1457.01.

The insured is age 61 when he surrenders.

The expected present value of the annuity is

$$\begin{aligned}&0.4P\ddot{a}_{61:51} + 0.6P\ddot{a}_{\overline{61:51}}\\&= P(0.4\ddot{a}_{61:51} + 0.6\ddot{a}_{51} + 0.6\ddot{a}_{61} - 0.6\ddot{a}_{61:51})\\&= P(0.6\times 13.0803 + 0.6\times 10.9041 - 0.2\times 9.9409)\\&= 12.40246\end{aligned}$$

So the annuity has an annual payment of $P = \dfrac{1457.01}{12.40246} = 117.5$.

18. [Chapter 3, 4 and 5] Answer: (B)

For the standard population, $A_{50} = \dfrac{0.005}{1.05} + \dfrac{0.995}{1.05}\times 0.4 = 0.3838095$

So, $0.001\,P = \dfrac{dA_{50}}{1-A_{50}} = 0.0296607$.

For this particular life,

$$A_{50}{*} = \frac{0.0025}{1.05} + \frac{0.9975}{1.05}\times 0.4 = 0.3823810,$$

$$\ddot{a}_{50}{*} = \frac{1-A_{50}{*}}{d} = 12.97$$

So, $\text{E}({}_0L) = 1000(0.3823810 - 0.0296607 \times 12.97) = -0.0023 \times 1000 = -2.3$.

19. [Chapter 7] Answer: (B)

This is like the set up of FPT reserve, just that the valuation premiums during the first two years completely offset the cost of insurances during the first two years. As a result,

$${}_{12}V^{\text{mod}} = 5000\,{}_{10}V_{32} = 5000\left(1 - \frac{\ddot{a}_{42}}{\ddot{a}_{32}}\right) = 5000\left(1 - \frac{14.5510}{15.6831}\right) = 361.$$

20. [Chapter 12] Answer: (C)

From (iv), at 4% interest,

$$\ddot{a}_{80:\overline{4}|} = 4.3686 - \frac{870}{1000}\frac{1}{1.04^4} = 3.62492$$

$$A^{1}_{80:\overline{3}|} = 0.1655 - \frac{50}{1000}\frac{1}{1.04^4} - \frac{55}{1000}\frac{1}{1.04^5} = 0.077554$$

Now we add back the APV of the cash flows on or after time 4:

$$\ddot{a}_{80:\overline{5}|} = 3.62492 + 0.87 / 1.05^4 = 4.34067$$

$$A^{1}_{80:\overline{5}|} = 0.077554 + 0.05 / 1.05^4 + 0.055 / 1.07^5 = 0.1579034$$

The net annual premium is 10000 × 0.1579034 / 4.34067 = 363.8.

Section B

1. [Chapter 1]

(a) $f_0(x) = S_0(x)\mu_x$. By product rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f_0(x) &= \mu_x\frac{\mathrm{d}}{\mathrm{d}x}S_0(x) + S_0(x)\frac{\mathrm{d}}{\mathrm{d}x}\mu_x \\ &= \mu_x[-S_0(x)\mu_x] + S_0(x)\mu_x' \\ &= S_0(x)(\mu_x' - \mu_x^2)\end{aligned}$$

$S_0(x) > 0$ for any $0 < x < \infty$. At point x_0, $f_0'(x)$ must have to be zero. This gives the result desired.

(b) (i) $\int_0^x \mu_t \mathrm{d}t = Ax + \int_0^x \frac{Bc^t}{1+Dc^t}\mathrm{d}t = Ax + \frac{B}{D\ln c}[\ln(1+Dc^t)]_0^x = Ax + \frac{B}{D\ln c}\ln\frac{1+Dc^x}{1+D}$.

So, $S_0(x) = \exp\left(-Ax - \frac{B}{D\ln c}\ln\frac{1+Dc^x}{1+D}\right)$.

(ii) It is easy to see that $\mu_x' = \frac{Bc^x \ln c}{(1+Dc^x)^2}$ and hence

$$\begin{aligned}\mu_x' - \mu_x^2 &= \frac{Bc^x\ln c}{(1+Dc^x)^2} - \frac{B^2c^{2x}}{(1+Dc^x)^2} \\ &= \frac{(B\ln c)c^x - B^2c^{2x}}{(1+Dc^x)^2} \\ &= \frac{Bc^x(\ln c - Bc^x)}{(1+Dc^x)^2}\end{aligned}$$

For $\frac{\mathrm{d}}{\mathrm{d}x}f_0(x) = 0$, $\ln c - Bc^x = 0$, and hence $x_0 = \frac{\ln(\ln c) - \ln B}{\ln c}$.

2. [Chapter 11]

(a) $\mathrm{F}_{xy}(t) = {}_tq_{xy} = 1 - {}_tp_x\,{}_tp_y = t^2q_x + t^2q_y - t^4\,q_xq_y = 0.084t^2 - 0.00032t^4$,

and by differentiation, $\mathrm{f}_{xy}(t) = 0.168t - 0.00128t^3$.

(b) $\int_0^{0.5} {}_tq_x \mathrm{f}_y(t)\mathrm{d}t = \int_0^{0.5} {}_tq_x \mathrm{d}({}_tq_y) = \int_0^{0.5} 0.08t^2 \mathrm{d}(0.004t^2) = 0.00064\int_0^{0.5} t^3 \mathrm{d}t = 0.00001$

(c) $1000\int_0^1 e^{-0.08t}t\,{}_tq_y \mathrm{f}_x(t)\mathrm{d}t = 1000\int_0^1 e^{-0.08t}t^3 q_y 2tq_x \mathrm{d}t = 0.64\int_0^1 e^{-0.08t}t^4 \mathrm{d}t$

$$\int_0^1 t^4 e^{-kt}\mathrm{d}t = -\frac{1}{k}\int_0^1 t^4 \mathrm{d}(e^{-kt}) = \frac{-e^{-k}}{k} + \frac{4}{k}\int_0^1 t^3 e^{-kt}\mathrm{d}t = \frac{-e^{-k}}{k} - \frac{4}{k^2}\int_0^1 t^3 \mathrm{d}(e^{-kt})$$

$$= \frac{-e^{-k}}{k} - \frac{4e^{-k}}{k^2} + \frac{12}{k^2}\int_0^1 t^2 e^{-kt}\mathrm{d}t = \frac{-e^{-k}}{k} - \frac{4e^{-k}}{k^2} + \frac{12}{k^3}\int_0^1 t^2 \mathrm{d}(e^{-kt})$$

$$= \frac{-e^{-k}}{k} - \frac{4e^{-k}}{k^2} - \frac{12e^{-k}}{k^3} + \frac{24}{k^3}\int_0^1 te^{-kt}\mathrm{d}t = \frac{-e^{-k}}{k} - \frac{4e^{-k}}{k^2} - \frac{12e^{-k}}{k^3} + \frac{24}{k^3}\frac{\frac{1-e^{-k}}{k} - e^{-k}}{k}$$

(where we have used $\int_0^1 te^{-kt}\mathrm{d}t = (\bar{I}\bar{a})_{\overline{1}|k} = \dfrac{\bar{a}_{\overline{1}|k} - e^{-k}}{k}$ in the last step)

so that $\int_0^1 e^{-0.08t}t^4 \mathrm{d}t = 0.187113$, and hence the EPV is 0.11975.

3. [Chapter 14]

At time 0
Initial expenses: 200
Pr_0: −200

At time 1
AV_0: 0
AV_1: 1950
P_1: 3000
E_1: 0 (since initial expenses are incorporated in Pr_0)
EDB_1: $1.2 \times 0.00592 \times (80000 + 1950 + 100) = 582.8832$
[Note: This is a specified amount plus the account value (Type B) policy. In the absence of any corridor factor requirement, the death benefit is \$80000 plus the time-1 account value of \$1950. The value 0.00592 is obtained from the Illustrative Life Table.]
ESB_1: $(1 - 1.2 \times 0.00592) \times 0.1 \times (1950 - 800 + 100) = 124.112$.
[Note: The cash value CV_1 is the account value $AV_1 = 1950$ less the surrender charge $SC_1 = 800$.]
EAV_1: $(1 - 1.2 \times 0.00592)(1 - 0.1)(1950) = 1742.5325$
Pr_1: $(0 + 3000 - 0)(1.1) - 582.8832 - 124.112 - 1742.5325 = 850.4723$

The answer is thus (A). Of course you can continue to calculate Pr_2 and Pr_3:

At time 2
AV_1: 1950
AV_2: 4100
P_2: 3000
E_2: $50 + 0.01 \times 3000 = 80$
EDB_2: $1.2 \times 0.00642 \times (80000 + 4100 + 100) = 648.6768$

ESB$_2$: $(1 - 1.2 \times 0.00642) \times 0.2 \times (4100 - 800 + 100) = 674.7613$
EAV$_2$: $(1 - 1.2 \times 0.00642)(1 - 0.2)(4100) = 3254.7309$
Pr$_2$: $(1950 + 3000 - 80)(1.1) - 648.6768 - 674.7613 - 3254.7309 = 778.831$

At time 3
AV$_2$: 4100
AV$_3$: 6400
P_3: 3000
E_3: $50 + 0.01 \times 3000 = 80$
EDB$_3$: $1.2 \times 0.00697 \times (80000 + 6400 + 100) = 723.486$
ESB$_3$: $(1 - 1.2 \times 0.00697) \times 1 \times (6400 - 800 + 100) = 5652.3252$
EAV$_3$: $(1 - 1.2 \times 0.00697)(1 - 1)(6400) = 0$
[Note: The surrender rate for $t = 3$ is 100%, which means all policyholders surrender at the end of the term. Hence, EAV$_3$ must be zero.]
Pr$_3$: $(4100 + 3000 - 80)(1.1) - 723.486 - 5652.3252 - 0 = 1346.1888$

Therefore, the profit vector is given by $(-200, 850, 779, 1346)'$.

4. [Chapter 2]

We are given:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
84			0.30445	86
85			q_{87}	87
86			0.35360	88
87			0.38020	89

By using the information from statements (i) and (ii), we obtain

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
84			0.30445	86
85		0.152225	q_{87}	87
86	0.152225	$0.5q_{87}$	0.35360	88
87	$0.5q_{87}$	0.1768	0.38020	89

The trick to solve this question is to make use of the fact that of $l_{[86]}$ and $l_{[87]}$ would evolve to l_{89}. Firstly,

$$l_{89} = l_{[87]}\, p_{[87]}\, p_{[87]+1} = l_{[87]}\,(1 - 0.5q_{87})(1 - 0.1768).$$

Similarly,

$$l_{89} = l_{[86]}\, p_{[86]}\, p_{[86]+1}\, p_{88} = 1000 \times (1 - 0.152225)(1 - 0.5q_{87})(1 - 0.35360)$$

As a result,

$$1000 \times (1 - 0.152225)(1 - 0.5q_{87})(1 - 0.35360) = l_{[87]}\,(1 - 0.5q_{87})(1 - 0.1768)$$

On solving, we get $l_{[87]} = \dfrac{1000(1-0.152225)(1-0.35360)}{1-0.1768} = 665.697$.

Suggested Solutions to MLC April 2015

Section A: Multiple Choice Questions

1.	B	11.	D
2.	C	12.	B
3.	B	13.	B
4.	D	14.	A
5.	A	15.	E
6.	C	16.	D
7.	C	17.	A
8.	E	18.	D
9.	C	19.	E
10.	D	20.	C

1. [Chapter 2] Answer: (B)

Let I_i = 1 if the ith members who are age 35 now is alive 30 years after the clue is established, and 0 otherwise. Similarly, let $J_i = 1$ if the ith members who are age 45 now is alive 30 years after the clue is established, and 0 otherwise. Then

$$N = \sum_{i=1}^{1000} (I_i + J_i).$$

Since the I_i's and J_i's are independent, and follow B(1, ${}_{30}p_{35}$) and B(1, ${}_{30}p_{45}$), respectively,

$$\begin{aligned} \mathrm{E}(N) &= \sum_{i=1}^{1000} [\mathrm{E}(I_i) + \mathrm{E}(J_i)] \\ &= 1000[\mathrm{E}(I_1) + \mathrm{E}(J_1)] \\ &= 1000({}_{10}p_{35} + {}_{10}p_{45}) \end{aligned} \qquad \begin{aligned} \mathrm{Var}(N) &= \sum_{i=1}^{1000} [\mathrm{Var}(I_i) + \mathrm{Var}(J_i)] \\ &= 1000[\mathrm{Var}(I_1) + \mathrm{Var}(J_1)] \\ &= 1000({}_{10}p_{35\ 10}q_{35} + {}_{10}p_{45\ 10}q_{45}) \end{aligned}$$

Since ${}_{30}p_{35} = l_{65} / l_{35} = 7533964 / 9420657 = 0.799728$
and ${}_{30}p_{45} = l_{75} / l_{45} = 5396081 / 9164051 = 0.588831$,

$$\mathrm{E}(N) = 1388.559 \text{ and } \mathrm{Var}(N) = 402.2721.$$

By normal approximation (without the continuity correction),

$$\frac{n - 1388.559}{\sqrt{402.2721}} \geq 1.645.$$

So $n \geq 1421.553$ and the least integer value of n is 1422.

2. [Chapter 10] Answer: (C)

The probability of the event A_s that the life healthy at 50 would be disabled first time (counting from age 50) for a period of at least 1 year, starting age 50 + s, is

$$\Pr(A_s) = {}_s p_{50}^{\overline{00}} (\mu_{50+s}^{01} \, \mathrm{d}s) p_{50+s}^{\overline{11}} = e^{-0.05s} 0.02 e^{-0.11} \mathrm{d}s .$$

Noting that the events A_s above are mutually exclusive for different values of s (because of the condition disabled "first time"), the required probability is obtained by integrating (actually the same as summing if ds is not involved in the probability above) s from 0 to 14 (not 15, because we need the extra one year to fulfil the 1 year disability):

$$\int_0^{14} e^{-0.05s} 0.02 e^{-0.11} \mathrm{d}s = 0.02 e^{-0.11} \frac{1 - e^{-0.05 \times 14}}{0.05} = 0.18039.$$

3. [Chapter 11] Answer: (B)

Noting that transition from state 0 to state 1 or state 2 are irreversible, ${}_{10} p_{30:30}^{00} = {}_{10} p_{30:30}^{\overline{00}}$.

To this end, the exit rate for state 0 is

$$v_{30+t:30+t}^{0} = \mu_{30+t:30+t}^{01} + \mu_{30+t:30+t}^{02} = 0.02 + 0.0007 \times 1.075^{30+t} .$$

$$\int_0^{10} v_{30+t:30+t}^{0} \mathrm{d}t = 0.02 \times 10 + \frac{0.0007 \times 1.075^{30}}{\ln 1.075} (1.075^{10} - 1) = 0.289912,$$

and hence the answer required is $e^{-0.289912} = 0.74833$.

4. [Chapter 3] Answer: (D)

Rewrite the random variable as $Z = \begin{cases} 0 \\ v^{T_x} \\ v^{T_x} \\ v^{T_x} \end{cases} + \begin{cases} 0 \\ 0 \\ v^{T_x} \\ v^{T_x} \end{cases} - \begin{cases} 0 \\ 0 \\ 0 \\ 2v^{T_x} \end{cases} = Z_1 + Z_2 - Z_3 .$

(If you wonder why we use this particular decomposition, the hint is that in the options given, most of them, except (C), starts with ${}_{10|}\overline{A}_x$ or its equivalent. So we first create the present value random variable that corresponds to this APV, and then build up the 2 in the layer $20 < T_x < 30$, and finally subtract the extra part for $T_x \geq 30$.)

Hence, the correct expression for E(Z) is ${}_{10|}\overline{A}_x + {}_{20|}\overline{A}_x - 2\,{}_{30|}\overline{A}_x$.

(A): It misses the coefficient "2" for the last term.
(B): The expression for (B) is equivalent to $\overline{A}_x + {}_{20|}\overline{A}_x - 2\,{}_{30|}\overline{A}_x$. Hence the first term is not correct.
(C): The first term is not correct.
(D): This is equivalent to the expression above.
(E): This expression can be rearranged as

$${}_{10}E_x \overline{A}_{x+10} + {}_{20}E_x \overline{A}_{x+20} - {}_{10}E_x \, {}_{10}E_{x+20} \overline{A}_{x+30} = {}_{10|}\overline{A}_x + {}_{20|}\overline{A}_x - {}_{10}E_x \, {}_{10|}\overline{A}_{x+20} ,$$

and hence the last term is not correct.

Section B: Written Answer

1. [Chapter 10]

(a) Let $v(t)$ be the present value (at time 0) of 1 dollar payable at time t.
Let $I(A)$ be 1 if event A occurs, and 0 otherwise. Then $\mathrm{E}(I(A) \mid B) = \Pr(A \mid B)$.

For any state j,

$$\begin{aligned}\bar{a}^{0j}_{x:\overline{10}|} &= \mathrm{E}\left[\int_0^{10} v(t) I(Y(x+t)=j)\mathrm{d}t \,\middle|\, Y(x)=0\right] \\ &= \int_0^{10} v(t)\mathrm{E}[I(Y(x+t)=j) \mid Y(x)=0]\mathrm{d}t \\ &= \int_0^{10} v(t)\,{}_tp_x^{0j}\,\mathrm{d}t.\end{aligned}$$

The sum required is

$$\sum_{j=0}^{2} \bar{a}^{0j}_{x:\overline{10}|} = \sum_{j=0}^{2}\int_0^{10} v(t)\,{}_tp_x^{0j}\,\mathrm{d}t = \int_0^{10} v(t)\left(\sum_{j=0}^{2} {}_tp_x^{0j}\right)\mathrm{d}t = \int_0^{10} v(t)\mathrm{d}t = \bar{a}_{\overline{10}|}.$$

(b) Using (a), we get $\bar{a}^{01}_{x:\overline{10}|} = \bar{a}_{\overline{10}|} - \bar{a}^{00}_{x:\overline{10}|} - \bar{a}^{02}_{x:\overline{10}|} = \dfrac{1-e^{-1}}{0.1} - 4.49 - 1.36 = 0.4712056.$

EPV of all benefits = EPV of continuous disability benefits + EPV of death benefit
$= 1000\bar{a}^{01}_{x:\overline{10}|} + 10000\bar{A}^{02}_{x:\overline{10}|}$
$= 1000 \times 0.4712056 + 10000 \times 0.3871 = 4342.206$

The net premium rate is $\dfrac{4342.206}{4.49} = 967.08365 \approx 967.$

(c) By Thiele's differential equation for reserves,

$$\frac{\mathrm{d}}{\mathrm{d}t}{}_tV^{(0)} = \pi_t^0 + \delta_t\,{}_tV^{(0)} - \sum_{j=1}^{2}(b_t^{(j)} + {}_tV^{(j)} - {}_tV^{(0)})\mu^{0j}_{x+t}.$$

At time 3, when the state is 0, the premium rate is 967. The transition benefit to state 1 is 0, while the transition benefit to state 2 is the death benefit, which is 10000. So,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}{}_tV^{(0)} &= 967 + 0.1\times1304.54 - (0+7530.09-1304.54)\times0.04 - (10000+0-1304.54)\times0.06 \\ &= 170.16\end{aligned}$$

(d) ${}_{10}p_x^{\overline{00}} = \exp\left(-\int_0^{10} v^0_{x+t}\mathrm{d}t\right) = \exp\left[-\int_0^{10}(0.04+0.02t)\mathrm{d}t\right] = e^{-0.4-0.01\times100} = e^{-1.4}$

Let P^* be the new net premium rate.
To get the refund of premium, no any transition can occur within 10 years. This means

EPV of all benefits is the original EPV of benefits plus the EPV of refund of premiums
$= 4342.206 + 10P^*\, e^{-0.1\times10} \times {}_{10}p_x^{\overline{00}}$
$= 4342.206 + 0.90718P^*$

By the equivalence principle,

$$4342.206 + 0.90718P^* = 4.49P^*$$