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SECTION 1 - BASIC PROBABILITY CONCEPTS

PROBABILITY SPACES AND EVENTS

Sample point and sample space: A sample point is the simple outcome of a random experiment. The probability space (also called sample space) is the collection of all possible sample points related to a specified experiment. When the experiment is performed, one of the sample points will be the outcome. The probability space is the "full set" of possible outcomes of the experiment.

Mutually exclusive outcomes: Outcomes are mutually exclusive if they cannot occur simultaneously. They are also referred to as **disjoint** outcomes.

Exhaustive outcomes: Outcomes are exhaustive if they combine to be the entire probability space, or equivalently, if at least one of the outcomes must occur whenever the experiment is performed.

Event: Any collection of sample points, or any subset of the probability space is referred to as an event. We say that "event A has occurred" if the experimental outcome was one of the sample points in A.

Union of events A and B: $A \cup B$ denotes the union of events A and B, and consists of all sample points that are in either A or B.

Union of events $A_1, A_2, ..., A_n$: $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$ denotes the union of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are in at least one of the A_i's. This definition can be extended to the union of infinitely many events.

Intersection of events $A_1, A_2, ..., A_n$: $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$ denotes the intersection of the events $A_1, A_2, ..., A_n$, and consists of all sample points that are simultaneously in all of the A_i's. ($A \cap B$ is also denoted $A \cdot B$ or AB).

Mutually exclusive events $A_1, A_2, ..., A_n$: Two events are mutually exclusive if they have no sample points in common, or equivalently, if they have **empty intersection**. Events $A_1, A_2, ..., A_n$ are mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$, where $\emptyset$ denotes the empty set with no sample points. Mutually exclusive events cannot occur simultaneously.

Exhaustive events $B_1, B_2, ..., B_n$: If $B_1 \cup B_2 \cup \cdots \cup B_n = S$, the entire probability space, then the events $B_1, B_2, ..., B_n$ are referred to as exhaustive events.

Complement of event A: The complement of event A consists of all sample points in the probability space that are **not in A**. The complement is denoted $\overline{A}$, $\sim A$, A' or A^c and is equal to $\{x : x \notin A\}$. When the underlying random experiment is performed, to say that the complement of A has occurred is the same as saying that A has not occurred.

Subevent (or subset) A of event B: If event B contains all the sample points in event A, then A is a subevent of B, denoted $A \subset B$. The occurrence of event A implies that event B has occurred.

Partition of event A: Events $C_1, C_2, ..., C_n$ form a partition of event A if $A = \bigcup_{i=1}^{n} C_i$ and the C_i's are mutually exclusive.

DeMorgan's Laws:

(i) $(A \cup B)' = A' \cap B'$, to say that $A \cup B$ has not occurred is to say that A has not occurred <u>and</u> B has not occurred ; this rule generalizes to any number of events;
$\left(\bigcup_{i=1}^{n} A_i\right)' = (A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n' = \bigcap_{i=1}^{n} A_i'$

(ii) $(A \cap B)' = A' \cup B'$, to say that $A \cap B$ has not occurred is to say that either A has not occurred <u>or</u> B has not occurred (or both have not occurred) ; this rule generalizes to any number of events, $\left(\bigcap_{i=1}^{n} A_i\right)' = (A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n' = \bigcup_{i=1}^{n} A_i'$

Indicator function for event A: The function $I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is the indicator function for event A, where x denotes a sample point. $I_A(x)$ is 1 if event A has occurred.

Basic set theory was reviewed in Section 0 of these notes. The Venn diagrams presented there apply in the sample space and event context presented here.

Example 1-1:

Suppose that an "experiment" consists of tossing a six-faced die. The **probability space** of outcomes consists of the set $\{1,2,3,4,5,6\}$, each number being a **sample point** representing the number of spots that can turn up when the die is tossed. The outcomes 1 and 2 (or more formally, $\{1\}$ and $\{2\}$) are an example of **mutually exclusive** outcomes, since they cannot occur simultaneously on one toss of the die. The collection of all the outcomes (sample points) 1 to 6 are **exhaustive** for the experiment of tossing a die since one of those outcomes must occur. The collection $\{2,4,6\}$ represents the **event** of tossing an even number when tossing a die. We define the following events.

$A=\{1,2,3\}=$ "a number less than 4 is tossed" ,

$B=\{2,4,6\}=$ "an even number is tossed" ,

$C=\{4\}=$ "a 4 is tossed" ,

$D=\{2\}=$ "a 2 is tossed" .

Then (i) $A\cup B=\{1,2,3,4,6\}$,

(ii) $A\cap B=\{2\}$,

(iii) A and C are mutually exclusive since $A\cap C=\emptyset$ (when the die is tossed it is not possible for both A and C to occur),

(iv) $D\subset B$,

(v) $A'=\{4,5,6\}$ (complement of A) ,

(vi) $B'=\{1,3,5\}$,

(vii) $A\cup B=\{1,2,3,4,6\}$, so that $(A\cup B)'=\{5\}=A'\cap B'$ (this illustrates one of DeMorgan's Laws). □

Some rules concerning operations on events:

(i) $A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)$ and
$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n)$ for any events $A, B_1, B_2, ..., B_n$

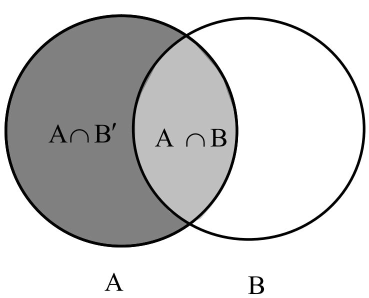
(ii) If $B_1, B_2, ..., B_n$ are exhaustive events $\left(\bigcup_{i=1}^{n} B_i = S\text{, the entire probability space}\right)$,
then for any event A,

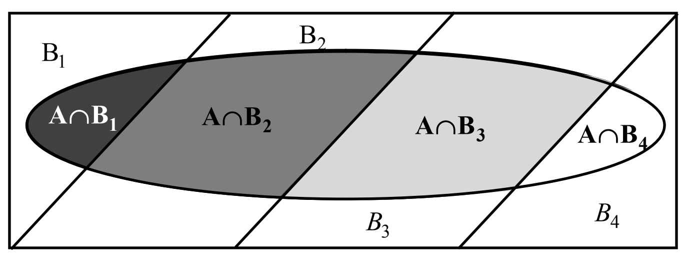
$$A = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) .$$

If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive events, then they form a **partition of the probability space**. For example, the events $B_1 = \{1,2\}$, $B_2 = \{3,4\}$ and $B_3 = \{5,6\}$ form a partition of the probability space for the outcomes of tossing a single die.

The general idea of a partition is illustrated in the diagram below. As a special case of a partition, if B is any event, then B and B' form a partition of the probability space. We then get the following identity for any two events A and B:

$\boldsymbol{A = A \cap (B \cup B') = (A \cap B) \cup (A \cap B')}$; note also that $A \cap B$ and $A \cap B'$ form a partition of event A.





(iii) For any event A, $A \cup A' = S$, the entire probability space, and $A \cap A' = \emptyset$

(iv) $A \cap B' = \{x : x \in A \text{ and } x \notin B\}$ is sometimes denoted $A - B$, and consists of all sample points that are in event A but not in event B

(v) If $A \subset B$ then $A \cup B = B$ and $A \cap B = A$.

PROBABILITY

Probability function for a discrete probability space: A discrete probability space (or sample space) is a set of a finite or countable infinite number of sample points. $P[a_i]$ or p_i denotes the probability that sample point (or outcome) a_i occurs. There is some underlying "random experiment" whose outcome will be one of the a_i's in the probability space. Each time the experiment is performed, one of the a_i's will occur. The probability function P must satisfy the following two conditions:

(i) $0 \leq P[a_i] \leq 1$ for each a_i in the sample space, and
(ii) $P[a_1] + P[a_2] + \cdots = \sum_{\text{all } i} P[a_i] = 1$ (total probability for a probability space is always 1).

This definition applies to both finite and infinite probability spaces.

Tossing an ordinary die is an example of an experiment with a finite probability space $\{1, 2, 3, 4, 5, 6\}$. An example of an experiment with an infinite probability space is the tossing of a coin until the first head appears. The toss number of the first head could be any positive integer, 1, or 2, or 3, ... The probability space for this experiment is the infinite set of positive integers $\{1, 2, 3, ...\}$ since the first head could occur on any toss starting with the first toss. The notion of discrete random variable covered later is closely related to the notion of discrete probability space and probability function.

Uniform probability function: If a probability space has a finite number of sample points, say k points, $a_1, a_2, ..., a_k$, then the probability function is said to be **uniform** if each sample point has the same probability of occurring ; $P[a_i] = \frac{1}{k}$ for each $i = 1, 2, ..., k$. Tossing a fair die would be an example of this, with $k = 6$.

Probability of event *A*: An event A consists of a subset of sample points in the probability space. In the case of a discrete probability space, the probability of A is
$P[A] = \sum_{a_i \in A} P[a_i]$, the sum of $P[a_i]$ over all sample points in event A.

Example 1-2: In tossing a "fair" die, it is assumed that each of the six faces has the same chance of $\frac{1}{6}$ of turning up. If this is true, then the probability function $P(j) = \frac{1}{6}$ for $j = 1, 2, 3, 4, 5, 6$ is a uniform probability function on the sample space $\{1, 2, 3, 4, 5, 6\}$.

The event "an even number is tossed" is $A = \{2, 4, 6\}$, and has probability
$P[A] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$. □

Continuous probability space: An experiment can result in an outcome which can be any real number in some interval. For example, imagine a simple game in which a pointer is spun randomly and ends up pointing at some spot on a circle. The angle from the vertical (measured clockwise) is between 0 and 2π. The probability space is the interval $(0, 2\pi]$, the set of possible angles that can occur. We regard this as a continuous probability space. In the case of a continuous probability space (an interval), we describe probability by assigning probability to subintervals rather than individual points. If the spin is "fair", so that all points on the circle are equally likely to occur, then intuition suggests that the probability assigned to an interval would be the fraction that the interval is of the full circle. For instance, the probability that the pointer ends up between "3 O'clock" and "9 O'clock" (between $\pi/2$ or 90° and $3\pi/2$ or 270° from the vertical) would be .5, since that subinterval is one-half of the full circle. The notion of a continuous random variable, covered later in this study guide, is related to a continuous probability space.

Some rules concerning probability:

(i) $\boldsymbol{P[S] = 1}$ **if $\boldsymbol{S}$ is the entire probability space** (when the underlying experiment is performed, some outcome must occur with probability 1; for instance $S = \{1, 2, 3, 4, 5, 6\}$ for the die toss example).

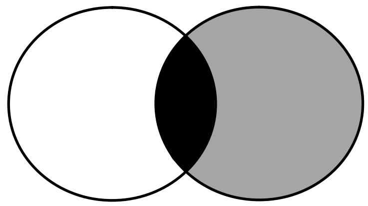
(ii) $\boldsymbol{P[\emptyset] = 0}$ (the probability of no face turning up when we toss a die is 0).

(iii) **If events $\boldsymbol{A_1, A_2, ..., A_n}$ are mutually exclusive (also called disjoint) then**
$$\boldsymbol{P[\bigcup_{i=1}^{n} A_i] = P[A_1 \cup A_2 \cup \cdots \cup A_n] = P[A_1] + P[A_2] + \cdots + P[A_n] = \sum_{i=1}^{n} P[A_i]}\ .$$
This extends to infinitely many mutually exclusive events. This rule is similar to the rule discussed in Section 0 of this study guide, where it was noted that the number of elements in the union of mutually disjoint sets is the sum of the numbers of elements in each set. When we have mutually exclusive events and we add the event probabilities, there is no double counting.

(iv) For any event A, $\boldsymbol{0 \leq P[A] \leq 1}$.

(v) If $A \subset B$ then $P[A] \leq P[B]$.

(vi) For any events A, B and C, $\boldsymbol{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$.
This relationship can be explained as follows. We can formulate $A \cup B$ as the union of three mutually exclusive events as follows: $A \cup B = (A \cap B') \cup (A \cap B) \cup (B \cap A')$.
This is expressed in the following Venn diagram.



$A \cap B'$ $A \cap B$ $B \cap A'$

Since these are mutually exclusive events, it follows that
$P[A \cup B] = P[A \cap B'] + P[A \cap B] + P[B \cap A']$.
From the Venn diagram we see that $A = (A \cap B') \cup (A \cap B)$, so that
$\boldsymbol{P[A] = P[A \cap B'] + P[A \cap B]}$, and we also see that $P[B \cap A'] = P[B] - P[A \cap B]$.
It then follows that
$P[A \cup B] = (P[A \cap B'] + P[A \cap B]) + P[B \cap A'] = P[A] + P[B] - P[A \cap B]$.

We subtract $P[A \cap B]$ because $P[A] + P[B]$ counts $P[A \cap B]$ twice. $P[A \cup B]$ is the probability that at least one of the two events A, B occurs. This was reviewed in Section 0, where a similar rule was described for counting the number of elements in $A \cup B$.

For the union of three sets we have
$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

(vii) For any event A, $\boldsymbol{P[A'] = 1 - P[A]}$.

(viii) For any events A and B, $\boldsymbol{P[A] = P[A \cap B] + P[A \cap B']}$
(this was mentioned in (vi), it is illustrated in the Venn diagram above).

(ix) For exhaustive events $B_1, B_2, ..., B_n$, $P[\bigcup_{i=1}^{n} B_i] = 1$.
If $B_1, B_2, ..., B_n$ are exhaustive and mutually exclusive, they form a **partition** of the entire probability space, and for any event A,
$$\boldsymbol{P[A] = P[A \cap B_1] + P[A \cap B_2] + \cdots + P[A \cap B_n] = \sum_{i=1}^{n} P[A \cap B_i]} .$$

(x) If P is a uniform probability function on a probability space with k points, and if event A consists of m of those points, then $P[A] = \frac{m}{k}$.

(xi) **The words "percentage" and "proportion" are used as alternatives to "probability"**.
As an example, if we are told that the percentage or proportion of a group of people that are of a certain type is 20%, this is generally interpreted to mean that a randomly chosen person from the group has a 20% probability of being of that type. This is the "long-run frequency" interpretation of probability. As another example, suppose that we are tossing a fair die. In the long-run frequency interpretation of probability, to say that the probability of tossing a 1 is $\frac{1}{6}$ is the same as saying that if we repeatedly toss the die, the proportion of tosses that are 1's will approach $\frac{1}{6}$.

(xii) for any events $A_1, A_2, ..., A_n$, $P[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} P[A_i]$, with equality holding if and only if the events are mutually exclusive

Example 1-3: Suppose that $P[A \cap B] = .2$, $P[A] = .6$, and $P[B] = .5$.
Find $P[A' \cup B']$, $P[A' \cap B']$, $P[A' \cap B]$ and $P[A' \cup B]$.

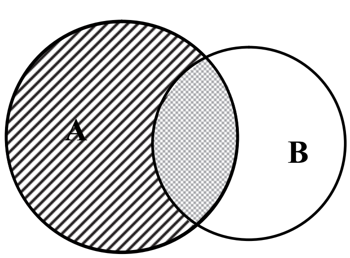
Solution: Using probability rules we get the following.
$P[A' \cup B'] = P[(A \cap B)'] = 1 - P[A \cap B] = .8$.

$P[A \cup B] = P[A] + P[B] - P[A \cap B] = .6 + .5 - .2 = .9$
$\rightarrow P[A' \cap B'] = P[(A \cup B)'] = 1 - P[A \cup B] = 1 - .9 = .1$.

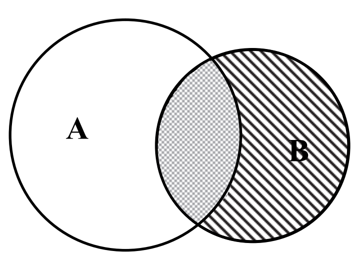
$P[B] = P[B \cap A] + P[B \cap A'] \rightarrow P[A' \cap B] = .5 - .2 = .3$.

$P[A' \cup B] = P[A'] + P[B] - P[A' \cap B] = .4 + .5 - .3 = .6$.

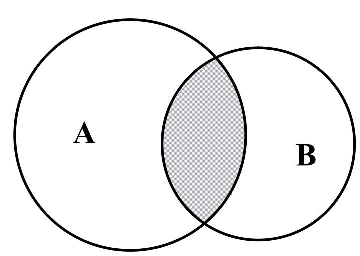
The following Venn diagrams illustrate the various combinations of intersections, unions and complements of the events A and B.



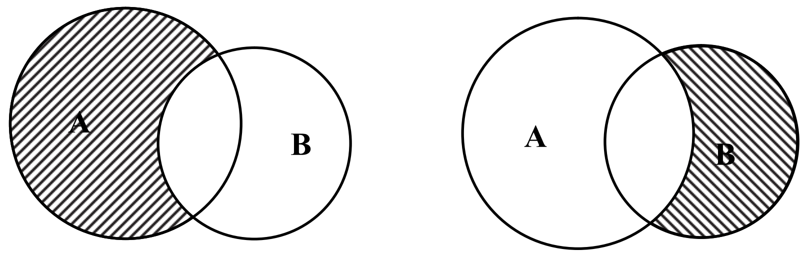
$P[A] = .6$



$P[B] = .5$

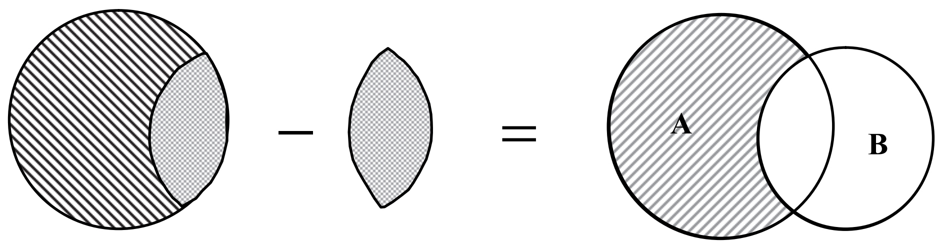


$P[A \cap B] = .2$

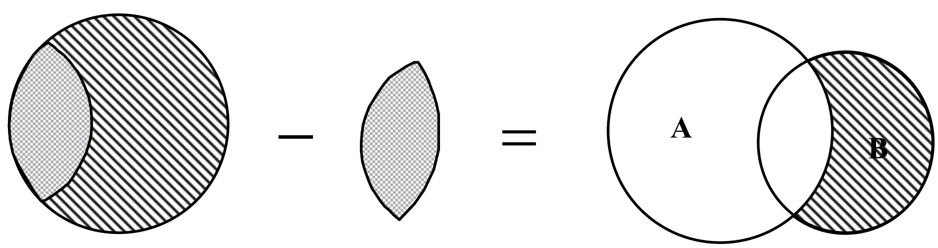


$P[A \cap B'] = .4$ $\qquad$ $P[A' \cap B] = .3$

From the following Venn diagrams we see that $P[A \cap B'] = P[A] - P[A \cap B] = .6 - .2 = .4$ and $P[A' \cap B] = P[B] - P[A \cap B] = .5 - .2 = .3$.

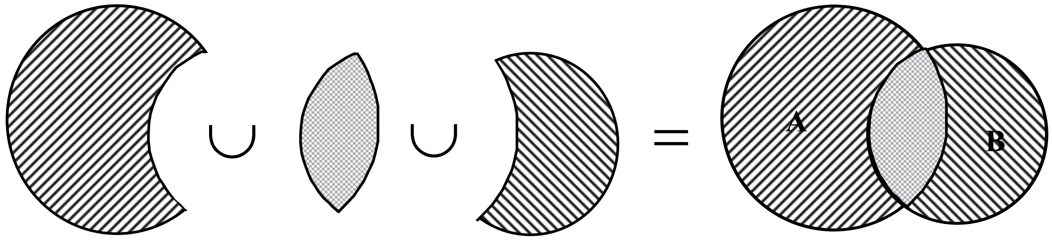


$P[A]$	$-$	$P[A \cap B]$	$=$	$P[A \cap B']$
.6	$-$	.2	$=$	.4



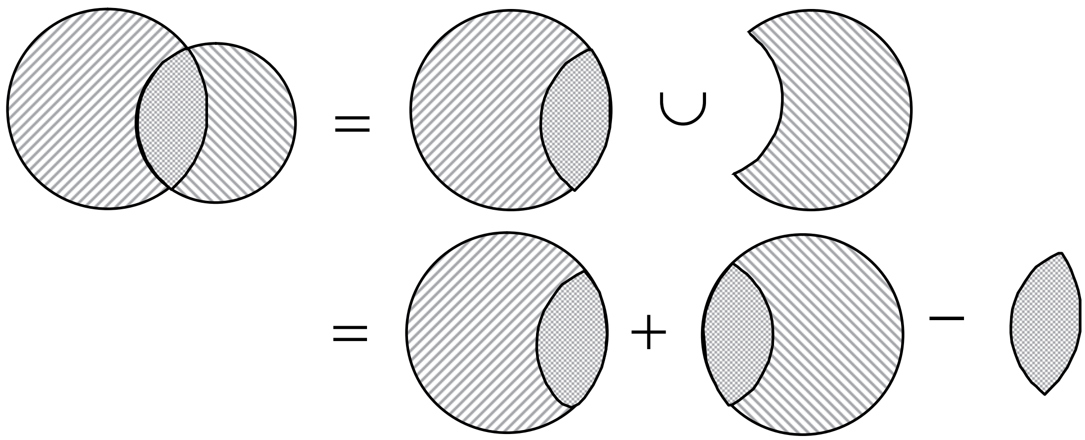
$P[B]$	$-$	$P[A \cap B]$	$=$	$P[A' \cap B]$
.5	$-$	.2	$=$	.3

The following Venn diagrams shows how to find $P[A \cup B]$.



$P[A \cap B']$	$+$	$P[A \cap B]$	$+$	$P[A' \cap B]$	$=$	$P[A \cup B]$
.4	$+$	.2	$+$	.3	$=$	.9

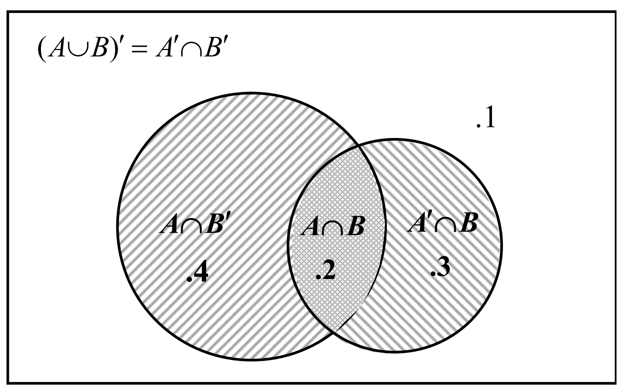
The relationship $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ is explained in the following Venn



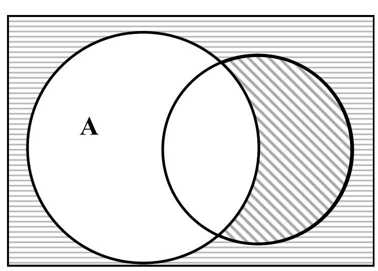
$$P[A \cup B] \quad = \quad P[A] \quad + \quad P[B] \quad - \quad P[A \cap B]$$

$$= .6 + .5 - .2 = .9$$

The components of the events and their probabilities are summarized in the following diagram.



We can represent a variety of events in Venn diagram form and find their probabilities from the component events described in the previous diagram. For instance, the complement of A is the combined shaded region in the following Venn diagram, and the probability is $P[A'] = .3 + .1 = .4$. We can get this probability also from $P[A'] = 1 - P[A] = 1 - .6 = .4$.



Another efficient way of summarizing the probability information for events A and B is in the form of a table.

	$P[A] = .6$ (given)	$P[A']$
$P[B] = .5$ (given)	$P[A \cap B] = .2$ (given)	$P[A' \cap B]$
$P[B']$	$P[A \cap B']$	$P[A' \cap B']$

Complementary event probabilities can be found from $P[A'] = 1 - P[A] = .4$ and $P[B'] = 1 - P[B] = .5$. Also, across each row or along each column, the "intersection probabilities" add up to the single event probability at the end or top:

$P[B] = P[A \cap B] + P[A' \cap B] \rightarrow .5 = .2 + P[A' \cap B] \rightarrow P[A' \cap B] = .3$,
$P[A] = P[A \cap B] + P[A \cap B'] \rightarrow .6 = .2 + P[A \cap B'] \rightarrow P[A \cap B'] = .4$, and
$P[A'] = P[A' \cap B] + P[A' \cap B'] \rightarrow .4 = .3 + P[A' \cap B'] \rightarrow P[A' \cap B'] = .1$ or
$P[B'] = P[A \cap B'] + P[A' \cap B'] \rightarrow .5 = .4 + P[A' \cap B'] \rightarrow P[A' \cap B'] = .1$.

These calculations can be summarized in the next table.

		$P[A] = .6$ (given)	$\Rightarrow$	$P[A'] = 1 - P[A] = .4$
		$\Uparrow$		$\Uparrow$
$P[B] = .5$	$\Leftarrow$	$P[A \cap B] = .2$	$+$	$P[A' \cap B] = \mathbf{.3}$
		$+$		$+$
$P[B'] = \mathbf{.5}$	$\Leftarrow$	$P[A \cap B'] = \mathbf{.4}$	$+$	$P[A' \cap B'] = \mathbf{.1}$

□

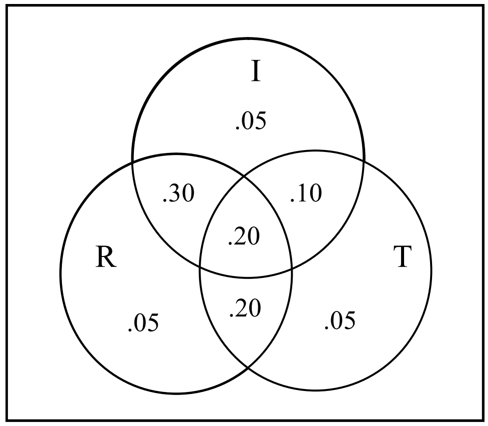
Example 1-4:

A survey is made to determine the number of households having electric appliances in a certain city. It is found that 75% have radios (R), 65% have electric irons (I), 55% have electric toasters (T), 50% have (IR), 40% have (RT), 30% have (IT), and 20% have all three. Find the probability that a household has at least one of these appliances.

Solution:

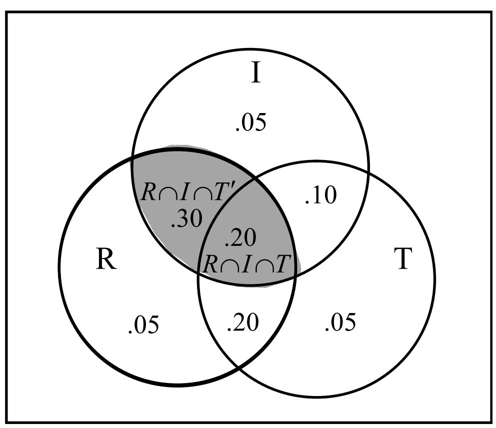
$$
\begin{aligned}
P[R \cup I \cup T] &= P[R] + P[I] + P[T] \\
&\quad - P[R \cap I] - P[R \cap T] - P[I \cap T] + P[R \cap I \cap T] \\
&= .75 + .65 + .55 - .5 - .4 - .3 + .2 = .95 .
\end{aligned}
$$

The following diagram deconstructs the three events.



The entries in this diagram were calculated from the "inside out". For instance, since $P(R \cap I) = .5$ (given) , and since $P(R \cap I \cap T) = .2$ (also given) , it follows that
$P(R \cap I \cap T') = .3$, since
$.5 = P(R \cap I) = P(R \cap I \cap T) + P(R \cap I \cap T') = .2 + P(R \cap I \cap T')$
(this uses the rule $P(A) = P(A \cap B) + P(A \cap B')$, where $A = R \cap I$ and $B = T$).

This is illustrated in the following diagram.



The value ".05" that is inside the diagram for event R refers to $P(R \cap I' \cap T')$ (the proportion who have radios but neither irons nor toasters). This can be found in the following way.

First we find $P(R \cap I')$:
$.75 = P(R) = P(R \cap I) + P(R \cap I') = .5 + P(R \cap I') \rightarrow P(R \cap I') = .25.$

$P(R\cap I')$ is the proportion with radios but not irons; this is the ".05" inside R combined with the ".2" in the lower triangle inside $R\cap T$. Then we find $P(R\cap I'\cap T)$:

$$.4 = P(R\cap T) = P(R\cap I\cap T) + P(R\cap I'\cap T)$$
$$= .2 + P(R\cap I'\cap T) \rightarrow P(R\cap I'\cap T) = .2\,.$$

Finally we find $P(R\cap I'\cap T')$:

$$.25 = P(R\cap I') = P(R\cap I'\cap T) + P(R\cap I'\cap T')$$
$$= .2 + P(R\cap I'\cap T') \rightarrow P(R\cap I'\cap T') = .05\,.$$

The other probabilities in the diagram can be found in a similar way. Notice that $P(R\cup I\cup T)$ is the sum of the probabilities of all the disjoint pieces inside the three events, $P(R\cup I\cup T) = .05 + .05 + .05 + .1 + .2 + .3 + .2 = .95$.

We can also use the rule

$P(R\cup I\cup T) = P(R) + P(I) + P(T) - P(R\cap I) - P(R\cap T) - P(I\cap T) + P(R\cap I\cap T)$

$= .75 + .65 + .55 - .5 - .4. - 3 + .2 = .95$.

Either way, this implies that 5% of the households have none of the three appliances. □

It is possible that information is given in terms of numbers of units in each category rather than proportion of probability of each category that was given in Example 1-4.

Example 1-5:

In a survey of 120 students, the following data was obtained.

60 took English, 56 took Math, 42 took Chemistry, 34 took English and Math, 20 took Math and Chemistry, 16 took English and Chemistry, 6 took all three subjects.

Find the number of students who took

(i) none of the subjects,

(ii) Math, but not English or Chemistry,

(iii) English and Math but not Chemistry.

Solution:

Since $E\cap M$ has 34 and $E\cap M\cap C$ has 6 , it follows that $E\cap M\cap C'$ has 28 .

The other entries are calculated in the same way (very much like the previous example).

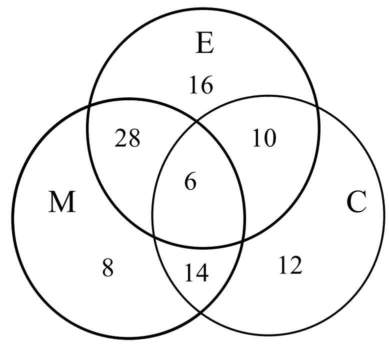
(i) The total number of students taking any of the three subjects is $E\cup M\cup C$, and is $16 + 28 + 6 + 10 + 8 + 14 + 12 = 94$. The remaining 26 (out of 120) students are not taking any of the three subjects (this could be described as the set $E'\cap M'\cap C'$).

(ii) $M\cap E'\cap C'$ has 8 students .

(iii) $E\cap M\cap C'$ has 28 students .

Example 1-5 continued

The following diagram illustrates how the numbers of students can be deconstructed.



□

PROBLEM SET 1

Basic Probability Concepts

1. A survey of 1000 people determines that 80% like walking and 60% like biking, and all like at least one of the two activities. What is the probability that a randomly chosen person in this survey likes biking but not walking?

A) 0 B) .1 C) .2 D) .3 E) .4

2. (SOA) Among a large group of patients recovering from shoulder injuries, it is found that 22% visit both a physical therapist and a chiropractor, whereas 12% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

A) 0.26 B) 0.38 C) 0.40 D) 0.48 E) 0.62

3. (SOA) An insurer offers a health plan to the employees of a large company. As part of this plan, the individual employees may choose exactly two of the supplementary coverages A, B, and C, or they may choose no supplementary coverage. The proportions of the company's employees that choose coverages A, B, and C are $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$, respectively. Determine the probability that a randomly chosen employee will choose no supplementary coverage.

A) 0 B) $\dfrac{47}{144}$ C) $\dfrac{1}{2}$ D) $\dfrac{97}{144}$ E) $\dfrac{7}{9}$

4. (SOA) An auto insurance company has 10,000 policyholders.
Each policyholder is classified as
(i) young or old;
(ii) male or female; and
(iii) married or single.
Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

A) 280 B) 423 C) 486 D) 880 E) 896

5. (SOA) The probability that a visit to a primary care physician's (PCP) office results in neither lab work nor referral to a specialist is 35% . Of those coming to a PCP's office, 30% are referred to specialists and 40% require lab work. Determine the probability that a visit to a PCP's office results in both lab work and referral to a specialist.

A) 0.05 B) 0.12 C) 0.18 D) 0.25 E) 0.35

6. (SOA) You are given $P[A \cup B] = 0.7$ and $P[A \cup B'] = 0.9$. Determine $P[A]$.

A) 0.2 B) 0.3 C) 0.4 D) 0.6 E) 0.8

7. (SOA) A survey of a group's viewing habits over the last year revealed the following information:

(i) 28% watched gymnastics
(ii) 29% watched baseball
(iii) 19% watched soccer
(iv) 14% watched gymnastics and baseball
(v) 12% watched baseball and soccer
(vi) 10% watched gymnastics and soccer
(vii) 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

A) 24 B) 36 C) 41 D) 52 E) 60

8. (SOA) Under an insurance policy, a maximum of five claims may be filed per year by a policyholder. Let p_n be the probability that a policyholder files n claims during a given year, where $n = 0, 1, 2, 3, 4, 5$. An actuary makes the following observations:

   i) $p_n \geq p_{n+1}$ for $n = 0, 1, 2, 3, 4$.

   ii) The difference between p_n and p_{n+1} is the same for $n = 0, 1, 2, 3, 4$.

   iii) Exactly 40% of policyholders file fewer than two claims during a given year.

   Calculate the probability that a random policyholder will file more than three claims during a given year.

   A) 0.14 B) 0.16 C) 0.27 D) 0.29 E) 0.33

9. (SOA) The probability that a member of a certain class of homeowners with liability and property coverage will file a liability claim is 0.04, and the probability that a member of this class will file a property claim is 0.10. The probability that member of this class will file a liability claim but not a property claim is 0.01.

   Calculate the probability that a randomly selected member of this class of homeowners will not file a claim of either type.

   A) 0.850 B) 0.860 C) 0.864 D) 0.870 E) 0.890

10. (SOA) A mattress store sells only, king, queen and twin-size mattresses. Sales records at the store indicate that one-fourth as many queen-size mattresses are sold as king and twin-size mattresses combined. Records also indicate that three times as many king-size mattresses are sold as twin-size mattresses.

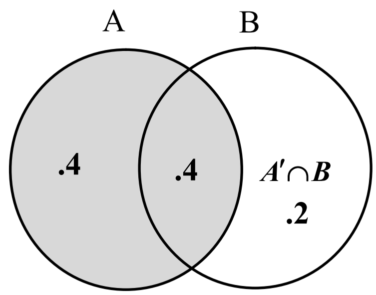
    Calculate the probability that the next mattress sold is either king or queen-size

    A) 0.12 B) 0.13 C) 0.080 D) 0.85 E) 0.95

PROBLEM SET 1 SOLUTIONS

1. Let $A =$ "like walking" and $B =$ "like biking" . We use the interpretation that "percentage" and "proportion" are taken to mean "probability".

We are given $P(A) = .8$, $P(B) = .6$ and $P(A \cup B) = 1$.
From the diagram below we can see that since $A \cup B = A \cup (B \cap A')$ we have
$P(A \cup B) = P(A) + P(A' \cap B) \rightarrow P(A' \cap B) = .2$ is the proportion of people who like biking but (and) not walking . In a similar way we get $P(A \cap B') = .4$.



An algebraic approach is the following. Using the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get $1 = .8 + .6 - P(A \cap B) \rightarrow P(A \cap B) = .4$. Then, using the rule $P(B) = P(B \cap A) + P(B \cap A')$, we get $P(B \cap A') = .6 - .4 = .2$. Answer: C

2. C - chiropractor visit ; T - therapist visit.
We are given $P(C \cap T) = .22$, $P(C' \cap T') = .12$, $P(C) = P(T) + .14$.
$88 = 1 - P(C' \cap T') = P(C \cup T) = P(C) + P(T) - P(C \cap T)$
$= P(T) + .14 + P(T) - .22 \rightarrow P(T) = .48$. Answer: D

3. Since someone who chooses coverage must choose exactly two supplementary coverages, in order for someone to choose coverage A, they must choose either A-and-B or A-and-C. Thus, the proportion of $\frac{1}{4}$ of individuals that choose A is
$P[A \cap B] + P[A \cap C] = \frac{1}{4}$ (where this refers to the probability that someone chosen at random in the company chooses coverage A). In a similar way we get

Then, $(P[A \cap B] + P[A \cap C]) + (P[B \cap A] + P[B \cap C]) + (P[C \cap A] + P[C \cap B])$
$= 2(P[A \cap B] + P[A \cap C] + P[B \cap C]) = \frac{1}{4} + \frac{1}{3} + \frac{5}{12} = 1$.

It follows that $P[A \cap B] + P[A \cap C] + P[B \cap C] = \frac{1}{2}$.

3. continued

This is the probability that a randomly chosen individual chooses some form of coverage, since if someone who chooses coverage chooses exactly two of A,B, and C. Therefore, the probability that a randomly chosen individual does not choose any coverage is the probability of the complementary event, which is also $\frac{1}{2}$. Answer: C

4. We identify the following subsets of the set of 10,000 policyholders:

$Y =$ young, with size 3000 (so that $Y' =$ old has size 7000) ,

$M =$ male, with size 4600 (so that $M' =$ female has size 5400), and

$C =$ married, with size 7000 (so that $C' =$ single has size 3000).

We are also given that $Y \cap M$ has size 1320 , $M \cap C$ has size 3010 ,

$Y \cap C$ has size 1400 , and $Y \cap M \cap C$ has size 600 .

We wish to find the size of the subset $Y \cap M' \cap C'$.

We use the following rules of set theory:

(i) if two finite sets are disjoint (have no elements in common, also referred to as empty intersection), then the total number of elements in the union of the two sets is the sum of the numbers of elements in each of the sets;

(ii) for any sets A and B, $A = (A \cap B) \cup (A \cap B')$, and $A \cap B$ and $A \cap \overline{B}$ are disjoint.

Applying rule (ii), we have $Y = (Y \cap M) \cup (Y \cap M')$. Applying rule (i), it follows that the size of $Y \cap M'$ must be $3000 - 1320 = 1680$.

We now apply rule (ii) to $Y \cap C$ to get $Y \cap C = (Y \cap C \cap M) \cup (Y \cap C \cap M')$.

Applyng rule (i), it follows that $Y \cap C \cap M'$ has size $1400 - 600 = 800$.

Now applying rule (ii) to $Y \cap M'$ we get $Y \cap M' = (Y \cap M' \cap C) \cup (Y \cap M' \cap C')$.

Applying rule (i), it follows that $Y \cap M' \cap C'$ has size $1680 - 800 = 880$.

Within the "Young" category, which we are told is 3000, we can summarize the calculations in the following table. This is a more straightforward solution.

	Married 1400 (given)	Single $1600 = 3000 - 1400$
Male 1320 (given)	$\boxed{600 \text{ (given)}}$	$\boxed{720 = 1320 - 600}$
Female $1680 =$ $3000 - 1320$	$\boxed{800 = 1400 - 600}$	$\boxed{880 = 1600 - 720}$

Answer: D

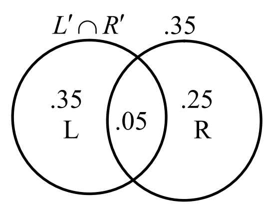
5. We identify events as follows:

L : lab work needed
R : referral to a specialist needed

We are given $P[L' \cap R'] = .35$, $P[R] = .3$, $P[L] = .4$. It follows that

$P[L \cup R] = 1 - P[L' \cap R'] = .65$, and then since

$P[L \cup R] = P[L] + P[R] - P[L \cap R]$, we get $P[L \cap R] = .3 + .4 - .65 = .05$.



These calculations can be summarized in the following table

	L , .4 given	L' , .6 $.6 = 1 - .4$	
R , .3 given	$L \cap R$ $.05 = .4 - .35$	$L' \cap R$ $.25 = .3 - .05$	
R' , .7 $.7 = 1 - .3$	$L \cap R'$ $.35 = .7 - .35$	$L' \cap R'$, .35 given	Answer: A

6. $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, $P[A \cup B'] = P[A] + P[B'] - P[A \cap B']$.

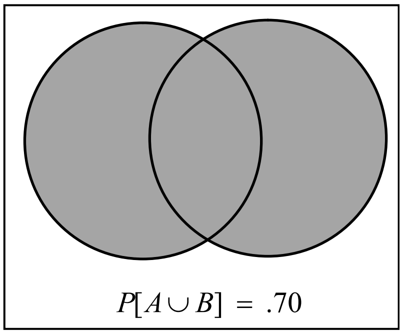
We use the relationship $P[A] = P[A \cap B] + P[A \cap B']$. Then

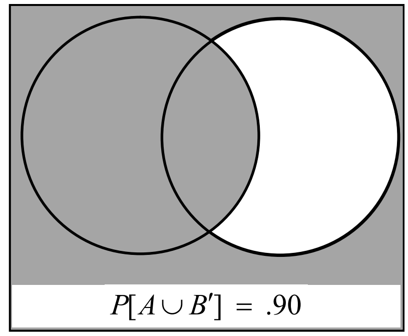
$$P[A \cup B] + P[A \cup B'] = P[A] + P[B] - P[A \cap B] + P[A] + P[B'] - P[A \cap B']$$
$$= 2P[A] + 1 - P[A] = P[A] + 1 \text{ (since } P[B] + P[B'] = 1) .$$

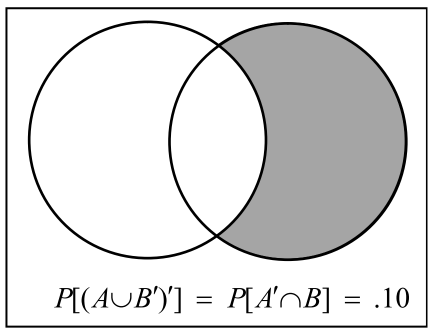
Therefore, $.7 + .9 = P[A] + 1$ so that $P[A] = .6$.

6. continued

An alternative solution is based on the following Venn diagrams.







In the third diagram, the shaded area is the complement of that in the second diagram (using De Morgan's Law, we have $(A \cup B')' = A' \cap B'' = A' \cap B$). Then it can be seen from diagrams 1 and 3 that $A = (A \cup B) - (A' \cap B)$, so that
$P[A] = P[A \cup B] - P[A' \cap B] = .7 - .1 = .6$. Answer: D

7. We identify the following events:
G - watched gymnastics , B - watched baseball , S - watched soccer .
We wish to find $P[G' \cap B' \cap S']$. By DeMorgan's rules we have
$P[G' \cap B' \cap S'] = 1 - P[G \cup B \cup S]$.
We use the relationship

$$P[G \cup B \cup S] = P[G] + P[B] + P[S] - \big(P[G \cap B] + P[G \cap S] + P[B \cap S]\big) + P[G \cap B \cap S] .$$

We are given $P[G] = .28$, $P[B] = .29$, $P[S] = .19$,
$P[G \cap B] = .14$, $P[G \cap S] = .10$, $P[B \cap S] = .12$, $P[G \cap B \cap S] = .08$.
Then $P[G \cup B \cup S] = .48$ and $P[G' \cap B' \cap S'] = 1 - .48 = .52$. Answer: D

8. The probability in question is $p_4 + p_5$. We know that $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 1$.
We are given that $p_0 - p_1 = p_1 - p_2 = p_2 - p_3 = p_3 - p_4 = p_4 - p_5$,
and $p_0 + p_1 = .4$. If we let t be equal to the common difference $p_n - p_{n+1}$, then
$p_1 = p_0 + t$, $p_2 = p_0 + 2t$, $p_3 = p_0 + 3t$, $p_4 = p_0 + 4t$ and $p_5 = p_0 + 5t$.
Then $p_0 + p_1 + p_2 + p_3 + p_4 + p_5 = 6p_0 + 15t = 1$. When we combine this equation with $p_0 + p_1 = 2p_0 + t = .4$, we can solve the two equations to get $p_0 = \frac{5}{24}$, and $t = -\frac{1}{60}$.
Then $p_4 = p_0 + 4t = \frac{17}{120}$, and $p_5 = p_0 + 5t = \frac{15}{120}$, so that $p_4 + p_5 = \frac{32}{120} = .27$. Answer: C

9. We define the following events: $L =$ file a liability claim , $P =$ file a property claim.
We are given $P(L) = 0.04$, $P(P) = 0.10$, $P(L \cap P') = 0.01$. We wish to find $P(L' \cap P')$.
$P(P') = 1 - P(P)0 = 90$ and $0.90 = P(P') = P(L \cap P') + P(L' \cap P') = 0.01 + P(L' \cap P')$.
It follows that $P(L' \cap P') = 0.89$. Answer: E

10. We define T to be the event that the next mattress sold is twin-size, and similarly we define K and Q as the events that the next mattress sold is king-size and queen-size, respectively. We define $P(T) = c$. Then $P(K) = 3c$ and $P(Q) = \frac{1}{4} \times [P(K) + P(T)] = c$.
Since $P(T) + P(Q) + P(K) = 1$, we have $5c = 1$, so that $c = 0.2$.
Then $P(K \cup Q) = 4c = 0.8$. Answer: C

SECTION 8 - JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS

Joint distribution of random variables X and Y

A random variable X is a numerical outcome that results from some random experiment, such as the number that turns up when tossing a die. It is possible that an experiment may result in two or more numerical outcomes. A simple example would be the numbers that turn up when tossing two dice. X could be the number that turns up on the first die and Y could be the number on the second die. Another example could be the following experiment. A coin is tossed and if the outcome is head then toss one die, and if the outcome is tails then toss two dice. We could set $X = 1$ for a head and $X = 2$ for a tail and $Y =$ total on the dice thrown. In both of the examples just described, we have a pair of random variables X and Y, that result from the experiment. X and Y might be unrelated or independent of one another (as in the example of the toss of two independent dice), or they might be related to each other (as in the coin-dice example).

We describe the probability distribution of two or more random variables together as a **joint distribution**. As in the case of a single discrete random variable, we still describe probabilities for each possible pair of outcomes for a pair of discrete random variables. In the case of a pair of random variables X and Y, there would be probabilities of the form $P[(X = x) \cap (Y = y)]$ for each pair (x, y) of possible outcomes. For a pair of continuous random variables X and Y, there would be a density function to describe density over a two dimensional region.

A joint distribution of two random variables has a probability function or probability density function $f(x, y)$ that is a function of two variables (sometimes denoted $f_{X,Y}(x, y)$). It is defined over a two-dimensional region. For joint distributions of continuous random variables X and Y, the region of probability (the probability space) is usually a rectangle or triangle in the x-y plane.

If X and Y are discrete random variables, then $f(x, y) = P[(X = x) \cap (Y = y)]$ is the joint probability function, and it must satisfy

(i) $0 \leq f(x, y) \leq 1$ and

(ii) $\sum_x \sum_y f(x, y) = 1$.

If X and Y are continuous random variables, then $f(x, y)$ must satisfy

(i) $f(x, y) \geq 0$ and

(ii) $\int_{-\infty}^{\infty} f(x, y)\, dy\, dx = 1$.

In the two dice example described above, if the two dice are tossed independently of one another then $f(x,y) = P[(X=x)\cap(Y=y)] = P[X=x]\times P[Y=y] = \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$ for each pair with $x = 1,2,3,4,5,6$ and $y = 1,2,3,4,5,6$. The coin-die toss example above is more complicated because the number of dice tossed depends on whether the toss is head or tails. If the coin toss is a head then $X = 1$ and $Y = 1,2,3,4,5,6$ so

$$f(1,y) = P[(X=1)\cap(Y=y)] = \frac{1}{2}\times\frac{1}{6} = \frac{1}{12} \text{ for } y = 1,2,3,4,5,6 .$$

If the coin toss is tail then $X = 2$ and $Y = 2,3,\ldots,12$ with

$$f(2,2) = P[(X=2)\cap(Y=2)] = \frac{1}{2}\times\frac{1}{36} = \frac{1}{72} \ ,$$
$$f(2,3) = P[(X=2)\cap(Y=3)] = \frac{1}{2}\times\frac{2}{36} = \frac{1}{36} \ \text{, etc.}$$

It is possible to have a joint distribution in which one variable is discrete and one is continuous, or either has a mixed distribution. The joint distribution of two random variables can be extended to a joint distribution of any number of random variables.

If A is a subset of two-dimensional space, then $P[(X,Y)\in A]$ is the summation (discrete case) or double integral (continuous case) of $f(x,y)$ over the region A.

Example 8-1:

X and Y are discrete random variables which are jointly distributed with the probability function $f(x,y)$ defined in the following table:

		X		
		-1	0	1
	1	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$
Y	0	$\frac{1}{9}$	0	$\frac{1}{6}$
	-1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$

From this table we see, for example, that $P[X=0, Y=-1] = f(0,-1) = \frac{1}{9}$.

Find (i) $P[X+Y=1]$, (ii) $P[X=0]$ and (iii) $P[X<Y]$.

Solution:

(i) We identify the (x,y)-points for which $X+Y=1$, and the probability is the sum of $f(x,y)$ over those points. The only x,y combinations that sum to 1 are the points $(0,1)$ and $(1,0)$. Therefore, $P[X+Y=1] = f(0,1)+f(1,0) = \frac{1}{9}+\frac{1}{6} = \frac{5}{18}$.

(ii) We identify the (x,y)-points for which $X=0$. These are $(0,-1)$ and $(0,1)$ (we omit $(0,0)$ since there is no probability at that point). $P[X=0] = f(0,-1)+f(0,1) = \frac{1}{9}+\frac{1}{9} = \frac{2}{9}$

(iii) The (x,y)-points satisfying $X<Y$ are $(-1,0)$, $(-1,1)$ and $(0,1)$. Then $P[X<Y] = f(-1,0)+f(-1,1)+f(0,1) = \frac{1}{9}+\frac{1}{18}+\frac{1}{9} = \frac{5}{18}$. □

Example 8-2:

Suppose that $f(x,y) = K(x^2+y^2)$ is the density function for the joint distribution of the continuous random variables X and Y defined over the unit square bounded by the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, find K. Find $P[X+Y \geq 1]$.

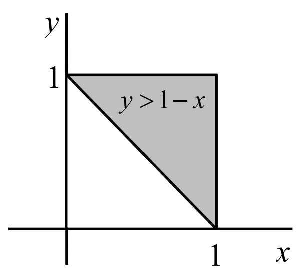
Solution:

In order for $f(x,y)$ to be a properly defined joint density, the (double) integral of the density function over the region of density must be 1, so that

$$1 = \int_0^1 \int_0^1 K(x^2+y^2)\,dy\,dx = K \cdot \tfrac{2}{3} \;\Rightarrow\; K = \tfrac{3}{2}$$
$$\Rightarrow\; f(x,y) = \tfrac{3}{2}(x^2+y^2) \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1.$$

In order to find the probability $P[X+Y \geq 1]$, we identify the two dimensional region representing $X+Y \geq 1$. This is generally found by drawing the boundary line for the inequality, which is $x+y=1$ (or $y = 1-x$) in this case, and then determining which side of the line is represented in the inequality. We can see that $x+y \geq 1$ is equivalent to $y \geq 1-x$.

This is the shaded region in the graph below.



The probability $P[X+Y \geq 1]$ is found by integrating the joint density over the two-dimensional region. It is possible to represent two-variable integrals in either order of integration. In some cases one order of integration is more convenient than the other. In this case there is not much advantage of one direction of integration over the other.

$$P[X+Y \geq 1] = \int_0^1 \int_{1-x}^1 \tfrac{3}{2}(x^2+y^2)\,dy\,dx = \int_0^1 \tfrac{1}{2}(3x^2y + y^3\Big|_{y=1-x}^{y=1})\,dx$$
$$= \int_0^1 \tfrac{1}{2}(3x^2 + 1 - 3x^2(1-x) - (1-x)^3)\,dx = \tfrac{3}{4}.$$

Reversing the order of integration, we have $x \geq 1-y$, so that

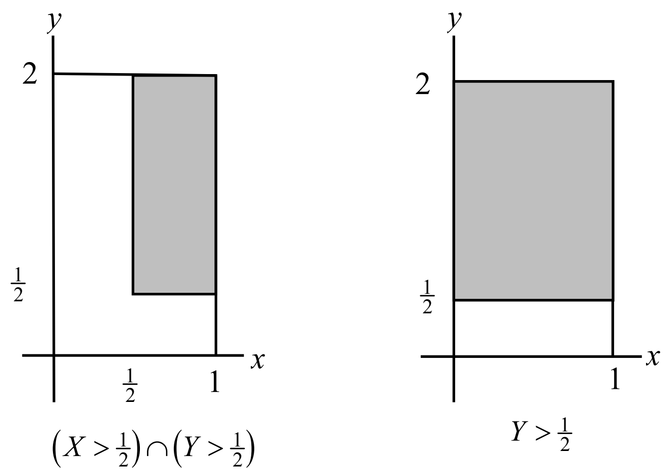
$$P[X+Y \geq 1] = \int_0^1 \int_{1-y}^1 \tfrac{3}{2}(x^2+y^2)\,dx\,dy = \tfrac{3}{4}.$$ □

Example 8-3:

Continuous random variables X and Y have a joint distribution with density function
$f(x,y) = x^2 + \frac{xy}{3}$ for $0 < x < 1$ and $0 < y < 2$.
Find the conditional probability $P[X > \frac{1}{2} | Y > \frac{1}{2}]$.

Solution:

We use the usual definition $P[A|B] = \frac{P[A \cap B]}{P[B]}$.
$P[X > \frac{1}{2} | Y > \frac{1}{2}] = \frac{P[(X > \frac{1}{2}) \cap (Y > \frac{1}{2})]}{P[Y > \frac{1}{2}]}$.
These regions are described in the following diagram



$$P[(X > \tfrac{1}{2}) \cap (Y > \tfrac{1}{2})] = \int_{1/2}^{1} \int_{1/2}^{2} [x^2 + \tfrac{xy}{3}] \, dy \, dx = \tfrac{43}{64}.$$

$$P[Y > \tfrac{1}{2}] = \int_{1/2}^{2} [\int_0^1 f(x,y) \, dx] \, dy = \int_{1/2}^{2} \int_0^1 [x^2 + \tfrac{xy}{3}] \, dx \, dy = \tfrac{13}{16}$$

$$\to P[X > \tfrac{1}{2} | Y > \tfrac{1}{2}] = \tfrac{43/64}{13/16} = \tfrac{43}{52}. \qquad \square$$

Cumulative distribution function of a joint distribution: If random variables X and Y have a joint distribution, then the cumulative distribution function is $F(x,y) = P[(X \le x) \cap (Y \le y)]$.
In the continuous case, $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t) \, dt \, ds$,
and in the discrete case, $F(x,y) = \sum_{s=-\infty}^{x} \sum_{t=-\infty}^{y} f(s,t)$. In the continuous case, $\frac{\partial^2}{\partial x \, \partial y} F(x,y) = f(x,y)$

Example 8-4:

The cumulative distribution function for the joint distribution of the continuous random variables X and Y is $F(x,y) = (.2)(3x^3 y + 2x^2 y^2)$, for $0 \le x \le 1$ and $0 \le y \le 1$. Find $f(\frac{1}{2}, \frac{1}{2})$.

Solution: $f(x,y) = \frac{\partial^2}{\partial x \, \partial y} F(x,y) = (.2)(9x^2 + 8xy) \to f(\frac{1}{2}, \frac{1}{2}) = \frac{17}{20}$. $\square$

PRACTICE EXAM 7

1. A study of the relationship between blood pressure and cholesterol level showed the following results for people who took part in the study:

   (a) of those who had high blood pressure, 50% had a high cholesterol level, and
   (b) of those who had high cholesterol level, 80% had high blood pressure.

   Of those in the study who had at least one of the conditions of high blood pressure or high cholesterol level, what is the proportion who had both conditions?

   A) $\frac{1}{3}$ B) $\frac{4}{9}$ C) $\frac{5}{9}$ D) $\frac{2}{3}$ E) $\frac{7}{9}$

2. A study of international athletes shows that of the two performance-enhancing steroids Dianabol and Winstrol, 5% of athletes use Dianabol and not Winstrol, 2% use Winstrol and not Dianabol, and 1% use both. A breath test has been developed to test for the presence of the these drugs in an athlete. Past use of the test has resulted in the following information regarding the accuracy of the test. Of the athletes that are using both drugs, the test indicates that 75% are using both drugs, 15% are using Dianabol only and 10% are using Winstrol only. In 80% of the athletes that are using Dianabol but not Winstrol, the test indicates they are using Dianabol but not Winstrol, and for the other 20% the test indicates they are using both drugs. In 60% of the athletes that are using Winstrol but not Dianabol, the test indicates that they are using Winstrol only, and for the other 40% the test indicates they are using both drugs. For all athletes that are using neither Dianabol nor Winstrol, the test always indicates that they are using neither drug.

   Of those athletes who test positive for Dianabol but not Winstrol, find the percentage that are using both drugs.

   A) 1.2% B) 2.4% C) 3.6% D) 4.8% E) 6.0%

3. The random variable N has the following characteristics:

   (i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.
   (ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

   Which of the following is a correct expression for Prob$(N = 2)$?

   A) $0.125p^2$
   B) $0.375 + 0.125p$
   C) $0.375 + 0.125p^2$
   D) $0.375 - 0.125p^2$
   E) $0.375 - 0.125p$

4. An insurance company does a study of claims that arrive at a regional office. The study focuses on the days during which there were at most 2 claims. The study finds that for the days on which there were at most 2 claims, the average number of claims per day is 1.2 . The company models the number of claims per day arriving at that office as a Poisson random variable. Based on this model, find the probability that at most 2 claims arrive at that office on a particular day.

A) .62 B) .64 C) .66 D) .68 E) .70

5. An actuarial trainee working on loss distributions encounters a special distribution. The student reads a discussion of the distribution and sees that the density of X is $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$ on the region $X > \theta$, where α and θ must both be > 0, and the mean is $\frac{\alpha\theta}{\alpha-1}$ if $\alpha > 1$.

The student is analyzing loss data that is assumed to follow such a distribution, but the values of α and θ are not specified, although it is known that $\theta < 200$. The data shows that the average loss for all losses is 180, and the average loss for all losses that are above 200 is 300.

Find the median of the loss distribution.

A) Less than 100
B) At least 100, but less than 120
C) At least 120, but less than 140
D) At least 140, but less than 160
E) At least 160

6. An insurance claims administrator verifies claims for various loss amounts.
For a loss claim of amount x, the amount of time spent by the administrator to verify the claim is uniformly distributed between 0 and $1 + x$ hours. The amount of each claim received by the administrator is uniformly distributed between 1 and 2. Find the average amount of time that an administrator spends on a randomly arriving claim.

A) 1.125 B) 1.250 C) 1.375 D) 1.500 E) 1.625

7. A husband and wife have a health insurance policy. The insurer models annual losses for the husband separately from the wife. X is the annual loss for the husband and Y is the annual loss for the wife. X has a uniform distribution on the interval $(0, 5)$ and Y has a uniform distribution on the interval $(0, 7)$, and X and Y are independent. The insurer applies a deductible of 2 to the combined annual losses, and the insurer pays a maximum of 8 per year. Find the expected annual payment made by the insurer for this policy.

A) 2 B) 3 C) 4 D) 5 E) 6

PRACTICE EXAM 7 - SOLUTIONS

1. We will use B to denote the event that a randomly chosen person in the study has high blood pressure, and C will denote the event high cholesterol level.
The information given tells us that $P(C|B) = .50$ and $P(B|C) = .80$.
We wish to find $P(B \cap C|B \cup C)$. This is

$$\begin{aligned}\frac{P[(B\cap C)\cap(B\cup C)]}{P(B\cup C)} &= \frac{P[B\cap C)]}{P(B)+P(C)-P(B\cap C)} \\ &= \frac{1}{\frac{P(B)+P(C)-P(B\cap C)}{P(B\cap C)}} = \frac{1}{[\frac{1}{P(C|B)}+\frac{1}{P(C|B)}-1]} \\ &= \frac{1}{\frac{1}{.5}+\frac{1}{.8}-1} = \frac{1}{2.25} = \frac{4}{9} \ .\end{aligned}$$

Answer: B

2. We define the following events:
D - the athlete uses Dianabol
W - the athlete uses Winstrol
TD - the test indicates that the athlete uses Dianabol
TW - the test indicates that the athlete uses Winstrol

We are given the following probabilities
$P(D \cap W') = .05$, $P(D' \cap W) = .02$, $P(D \cap W) = .01$,
$P(TD \cap TW|D \cap W) = .75$, $P(TD \cap TW'|D \cap W) = .15$, $P(TD' \cap TW|D \cap W) = .1$,
$P(TD \cap TW|D \cap W') = .2$, $P(TD \cap TW'|D \cap W') = .8$,
$P(TD \cap TW|D' \cap W) = .4$, $P(TD' \cap TW|D' \cap W) = .6$.

We wish to find $P(D \cap W|TD \cap TW') = \frac{P(D\cap W\cap TD\cap TW')}{P(TD\cap TW')}$.
The numerator is
$P(D \cap W \cap TD \cap TW') = P(TD \cap TW'|D \cap W) \cdot P(D \cap W) = (.15)(.01) = .0015$.

The denominator is

$$\begin{aligned}P(TD \cap TW') = \ & P(TD \cap TW' \cap D \cap W) + P(TD \cap TW' \cap D' \cap W) \\ & + P(TD \cap TW' \cap D \cap W') + P(TD \cap TW' \cap D' \cap W')\end{aligned}$$

We have used the rule $P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots$, where $B_1, B_2, ...$ forms a partition. The partition in this case is $B_1 = D \cap W$, $B_2 = D' \cap W$, $B_3 = D \cap W'$, $B_4 = D' \cap W'$, since an athlete must be using both, one or neither of the drugs.

2. continued

We have just seen that $P(TD \cap TW' \cap D \cap W) = .0015$.

In a similar way, we have

$P(TD \cap TW' \cap D' \cap W) = P(TD \cap TW' | D' \cap W) \cdot P(D' \cap W) = (0)(.02) = 0$, and

$P(TD \cap TW' \cap D \cap W') = P(TD \cap TW' | D \cap W') \cdot P(D \cap W') = (.8)(.05) = .04$, and

$P(TD \cap TW' \cap D' \cap W') = P(TD \cap TW' | D' \cap W') \cdot P(D' \cap W') = (0)(.92) = 0$

(note that $P(D' \cap W') = 1 - P(D \cup W) = 1 - P(D \cap W') - P(D' \cap W) - P(D \cap W) = .92$

Then, $P(D \cap W | TD \cap TW') = \frac{.0015}{.0015+0+.04+0} = .036$, 3.6% . Answer: C

3. $P(N=2) = pP(N_1 = 2) + (1-p)P(N_2 = 2) = p(.5)^2 + (1-p)6(.5)^4 = .375 - .125p$.
We have used the binomial probabilities $\binom{m}{k} q^k (1-q)^{m-k}$. Answer: E

4. Suppose that the mean number of claims per day arriving at the office is λ.

Let X denote the number of claims arriving in one day.

Then the probability of at most 2 claims in one day is $P(X \le 2) = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2}$.

The conditional probability of 0 claims arriving on a day given that there are at most 2 for the day is

$$P(X=0|X \le 2) = \frac{P(X=0)}{P(X \le 2)} = \frac{e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{1}{1+\lambda+\frac{\lambda^2}{2}} .$$

The conditional probability of 1 claim arriving on a day given that there are at most 2 for the day is

$$P(X=1|X \le 2) = \frac{P(X=1)}{P(X \le 2)} = \frac{\lambda e^{-\lambda}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}} .$$

The conditional probability of 2 claims arriving on a day given that there are at most 2 for the day is

$$P(X=2|X \le 2) = \frac{P(X=2)}{P(X \le 2)} = \frac{\frac{\lambda^2 e^{-\lambda}}{2}}{e^{-\lambda}+\lambda e^{-\lambda}+\frac{\lambda^2 e^{-\lambda}}{2}} = \frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}} .$$

The expected number of claims per day, given that there were at most 2 claims per day is

$$(0)\left(\frac{1}{1+\lambda+\frac{\lambda^2}{2}}\right) + (1)\left(\frac{\lambda}{1+\lambda+\frac{\lambda^2}{2}}\right) + (2)\left(\frac{\frac{\lambda^2}{2}}{1+\lambda+\frac{\lambda^2}{2}}\right) = \frac{\lambda+\lambda^2}{1+\lambda+\frac{\lambda^2}{2}} .$$

We are told that this is 1.2 .

Therefore $\lambda + \lambda^2 = (1.2)(1 + \lambda + \frac{\lambda^2}{2})$, which becomes the quadratic equation $.4\lambda^2 - .2\lambda - 1.2 = 0$. Solving the equation results in $\lambda = 2$ or -1.5 , but we ignore the negative root. The probability of at most 2 claims arriving at the office on a particular day is $P(X \le 2) = e^{-2} + 2e^{-2} + \frac{2^2 e^{-2}}{2} = .6767$. Answer: D

5. The distribution function will be $F(y) = \int_\theta^y f(x)\,dx = \int_\theta^y \frac{\alpha\theta^\alpha}{x^{\alpha+1}}\,dx = 1 - \frac{\theta^\alpha}{y^\alpha}$.
The median m occurs where $F(m) = \frac{1}{2}$. If α and θ were known, we could find the median.
The average loss for all losses is $\frac{\alpha\theta}{\alpha-1} = 180$, but both θ and α are not known.
The conditional distribution of loss amount x given that $X > 200$ is
$f(x|X > 200) = \frac{f(x)}{P(X>200)} = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \Big/ \frac{\theta^\alpha}{200^\alpha} = \frac{\alpha 200^\alpha}{x^{\alpha+1}}$.
This random variable has a mean of $\frac{200\alpha}{\alpha-1}$. We are given that this mean is 300,
so $\frac{200\alpha}{\alpha-1} = 300$, and therefore $\alpha = 3$.
Then, from $\frac{\alpha\theta}{\alpha-1} = 180$, we get $\frac{3\theta}{2} = 180$, so that $\theta = 120$.
The median m satisfies the relation $\frac{1}{2} = F(m) = 1 - \frac{\theta^\alpha}{m^\alpha} = 1 - \left(\frac{120}{m}\right)^3$, so that $m = 151.2$.
Answer: D

6. $X =$ amount of loss claim, uniformly distributed on $(1,2)$, so $f_X(x) = 1$ for $1 < x < 2$.
$Y =$ amount of time spent verifying claim.
We are given that the conditional distribution of Y given $X = x$ is uniform on $(0, 1+x)$,
so $f(y|x) = \frac{1}{1+x}$ for $0 < y < 1+x$.

We wish to find $E[Y]$. The joint density of X and Y is
$f(x,y) = f(y|x) \cdot f_X(x) = \frac{1}{1+x}$ for $0 < y < 1+x$ and $1 < x < 2$.

There are a couple of ways to find $E[Y]$:

(i) $E[Y] = \int\int y\, f(x,y)\,dy\,dx$ or $E[Y] = \int\int y\, f(x,y)\,dx\,dy$, with careful setting of the integral limits, or

(ii) $E[Y] = \int y\, f_Y(y)\,dy$, where $f_Y(y)$ is the pdf of the marginal distribution of Y.
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot$

(iii) The double expectation rule, $E[Y] = E[\,E[Y|X]\,]$.

If we apply the first approach for method (i), we get
$E[Y] = \int_1^2 \int_0^{1+x} y \cdot \frac{1}{1+x}\,dy\,dx = \int_1^2 \frac{(1+x)^2}{2(1+x)}\,dy = \int_1^2 \frac{1+x}{2}\,dx = \frac{5}{4}$.

If we apply the second approach for method (i), we must split the double integral into
$E[Y] = \int_0^2 \int_1^2 y \cdot \frac{1}{1+x}\,dx\,dy + \int_2^3 \int_{y-1}^2 y \cdot \frac{1}{1+x}\,dx\,dy$

The first integral becomes $\int_0^2 y\, ln\left(\frac{3}{2}\right)dy = 2\, ln\left(\frac{3}{2}\right)$.
The second integral becomes $\int_2^3 y\,[ln 3 - ln\, y]\,dy = \frac{5}{2}\, ln 3 - \int_2^3 y\, ln\, y\, dy$.
The integral $\int_2^3 y\, ln\, y\, dy$ is found by integration by parts.

6. continued

Let $\int y\,ln\,y\,dy = A$.

Let $u = y$ and $dv = ln\,y\,dy$, then $v = y\,lny - y$ (antiderivative of $ln\,y$), and then

$A = \int y\,ln\,y\,dy = y(y\,lny - y) - \int(y\,ln\,y - y)\,dy = y^2\,ln\,y - y^2 - A + \frac{y^2}{2}$,

so that $A = \int y\,ln\,y\,dy = \frac{1}{2}y^2\,ln\,y - \frac{y^2}{4}$.

Then $\int_2^3 y\,ln\,y\,dy = \frac{1}{2}y^2\,ln\,y - \frac{y^2}{4}\Big|_2^3 = \frac{9}{2}\,ln3 - \frac{9}{4} - (\frac{4}{2}\,ln\,2 - 1) = \frac{9}{2}\,ln3 - 2\,ln2 - \frac{5}{4}$.

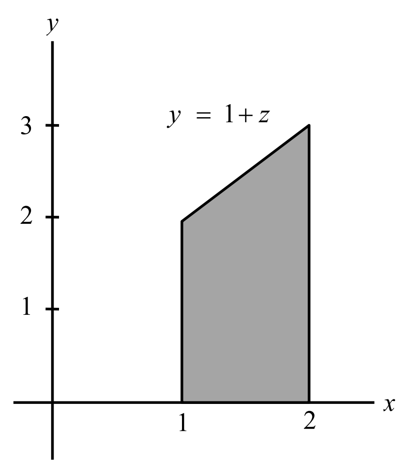
Finally, $E[Y] = 2\,ln\left(\frac{3}{2}\right) + \frac{5}{2}\,ln3 - \int_2^3 y\,ln\,y\,dy$

$= 2\,ln\,3 - 2\,ln\,2 + \frac{5}{2}\,ln3 - (\frac{9}{2}\,ln3 - 2\,ln2 - \frac{5}{4}) = \frac{5}{4}$.

The first order of integration for method (i) was clearly the more efficient one.

(ii) This method is equivalent to the second approach in method

(i) because we find $f_Y(y)$ from the relationship $f_Y(y) = \int f(x,y)\,dx$. The two-dimensional region of probability for the joint distribution is $1 < x < 2$ and $0 < y < 1 + x$. This is illustrated in the graph below



For $0 < y < 2$, $f_Y(y) = \int_1^2 f(x,y)\,dx = \int_1^2 \frac{1}{1+x}\,dx = ln\left(\frac{3}{2}\right)$,

and for $2 \le x < 3$, $f_Y(y) = \int_{y-1}^2 f(x,y)\,dx = \int_{y-1}^2 \frac{1}{1+x}\,dx = ln\,3 - ln\,y$.

Then $E[Y] = \int_0^2 y\,ln\left(\frac{3}{2}\right)dy + \int_2^3 y\,[ln3 - ln\,y]\,dy$, which is the same as the second part of method (i).

(iii) According to the double expectation rule, for any two random variables U and W, we have $E[U] = E[\,E[U|W]\,]$. Therefore, $E[Y] = E[\,E[Y|X]\,]$.

We are told that the conditional distribution of Y given $X = x$ is uniform on the interval $(0, 1+x)$, so $E[Y|X] = \frac{1+X}{2}$.

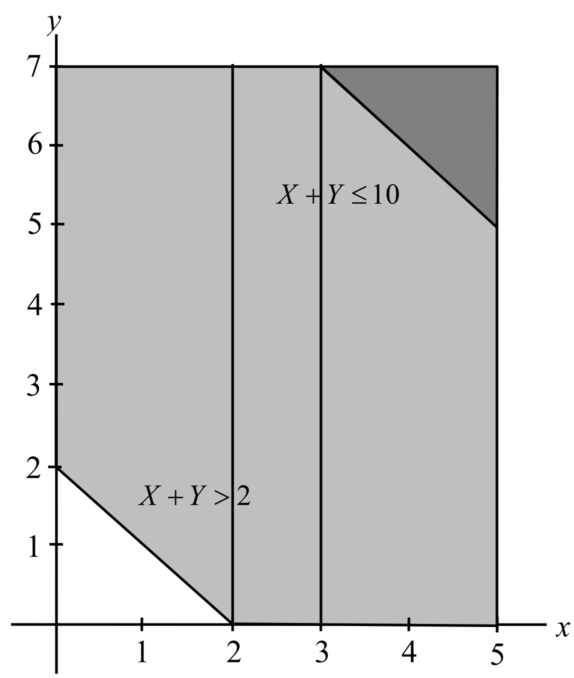
Then $E[\,E[Y|X]\,] = E[\frac{1+X}{2}] = \frac{1}{2} + \frac{1}{2}E[X] = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{2}\right) = \frac{5}{4}$, since X is uniform on $(1,2)$ and X has mean $\frac{3}{2}$. Answer: B

7. The joint distribution of X and Y has pdf $f(x,y) = \frac{1}{5} \cdot \frac{1}{7} = \frac{1}{35}$ on the rectangle $0 < x < 5$ and $0 < y < 7$. The insurer pays $X+Y-2$ if the combined loss $X+Y$ is > 2.

The maximum payment of 8 is reached if $X+Y-2 \geq 8$, or equivalently, if $X+Y \geq 10$.

Therefore, the insurer pays $X+Y-2$ if $2 < X+Y \leq 10$ (the lighter shaded region in the diagram below) , and the insurer pays 8 if $X+Y > 10$ (the darker shaded region in the diagram below). The expected amount paid by the insurer is a combination of two integrals:
$\int\int (x+y-2) \cdot \frac{1}{35}\, dy\, dx$, where the integral is taken over the region $2 < x+y \leq 10$ (the lightly shaded region), plus
$\int\int 8 \cdot \frac{1}{35}\, dy\, dx$, where the integral is taken over the region $X+Y > 10$ (the darker region).

The second integral is $\frac{8}{35} \cdot (2) = \frac{16}{35}$, since the area of the darkly shaded triangle is 2 (it is a 2×2 right triangle).



The first integral can be broken into three integrals:

$$\int_0^2 \int_{2-x}^7 (x+y-2) \cdot \frac{1}{35}\, dy\, dx \;+\; \int_2^3 \int_0^7 (x+y-2) \cdot \frac{1}{35}\, dy\, dx + \int_3^5 \int_0^{10-x} (x+y-2) \cdot \frac{1}{35}\, dy\, dx$$
$$= \frac{1}{35} \cdot \left[\int_0^2 \frac{(x+5)^2}{2}\, dx \;+\; \int_2^3 \frac{7(2x+3)}{2}\, dx \;+\; \int_3^5 \frac{60+4x-x^2}{2}\, dx \right]$$
$$= \frac{1}{35} \cdot \left[\frac{109}{3} + 28 + \frac{179}{3} \right] = \frac{124}{35} .$$

The total expected insurance payment is $\frac{16}{35} + \frac{124}{35} = \frac{140}{35} = 4$. Answer: C

