Mahler's Guide to

Frequency Distributions

Exam C

prepared by
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Study Aid 2015-C-**1**

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Mahler's Guide to Frequency Distributions

Information in bold or sections whose title is in bold are more important for passing the exam. Larger bold type indicates it is extremely important.
Information presented in italics (or sections whose title is in italics) should not be needed to directly answer exam questions and should be skipped on first reading. It is provided to aid the reader's overall understanding of the subject, and to be useful in practical applications.

Highly Recommended problems are double underlined.
Recommended problems are underlined.[1]

Solutions to the problems in each section are at the end of that section.

[1] Note that problems include both some written by me and some from past exams. The latter are copyright by the CAS and SOA, and are reproduced here solely to aid students in studying for exams. The solutions and comments are solely the responsibility of the author; the CAS and SOA bear no responsibility for their accuracy. While some of the comments may seem critical of certain questions, this is intended solely to aid you in studying and in no way is intended as a criticism of the many volunteers who work extremely long and hard to produce quality exams. In some cases I've rewritten these questions in order to match the notation in the current Syllabus.

Past Exam Questions by Section of this Study Aid[2]

	Course 3	Course 3	Course 3	Course 3	Course 3	Course 3	**CAS 3**	**SOA 3**	**CAS 3**
Section	Sample	5/00	11/00	5/01	11/01	11/02	11/03	11/03	5/04
1									
2									
3							14		
4									16
5									
6							18		
7									
8									28
9									
10									
11				25		28			32
12									
13									
14		37							
15									
16			2	16 36	30	27			26
17			13						
18									
19	12	4		3 15	27	5	15		
20									

The CAS/SOA did not release the 5/02 and 5/03 exams.
From 5/00 to 5/03, the Course 3 Exam was jointly administered by the CAS and SOA.
Starting in 11/03, the CAS and SOA gave separate exams. (See the next page.)

[2] Excluding any questions that are no longer on the syllabus.

	CAS 3	**SOA 3**	**CAS 3**	**SOA M**	**CAS 3**	**SOA M**	**CAS 3**	**CAS 3**	**SOA M**	**4/C**
Section	11/04	11/04	5/05	5/05	11/05	11/05	5/06	11/06	11/06	5/07
1										
2										
3	22 24	8	15							
4	23			39	24			32		
5										
6	21		28				32	23 24 31	22	
7										
8										
9										
10								25		
11			16	19			31			
12										
13										
14										
15						27				
16						18	35		30	
17		32						19	39	
18										
19			10							
20										

The SOA did not release its 5/04 and 5/06 exams.

This material was moved to Exam 4/C in 2007.

The CAS/SOA did not release the 11/07 and subsequent exams.

Section 1, Introduction

This Study Aid will review what a student needs to know about the frequency distributions in Loss Models. Much of the first seven sections you should have learned on Exam P.

In actuarial work, frequency distributions are applied to the number of losses, the number of claims, the number of accidents, the number of persons injured per accident, etc.

Frequency Distributions are discrete functions on the nonnegative integers: 0, 1, 2, 3, ...

There are three named frequency distributions you should know:
Binomial, with special case Bernoulli
Poisson
Negative Binomial, with special case Geometric.

Most of the information you need to know about each of these distributions is shown in Appendix B, attached to the exam. Nevertheless, since they appear often in exam questions, it is desirable to know these frequency distributions well, particularly the Poisson Distribution.

In addition, one can make up a frequency distribution.
How to work with such unnamed frequency distributions is discussed in the next section.

In later sections, the important concepts of Compound Distributions and Mixed Distributions will be discussed.[3]
The most important case of a mixed frequency distribution is the Gamma-Poisson frequency process.

[3] Compound Distributions are mathematically equivalent to Aggregate Distributions, which are discussed in "Mahler's Guide to Aggregate Distributions."

Section 2, Basic Concepts

The probability density function[4] f(i) can be non-zero at either a finite or infinite number of points. In the former case, the probability density function is determined by a table of its values at these finite number of points.

The f(i) can take on any values provided they satisfy $0 \le f(i) \le 1$ and $\sum_{i=0}^{\infty} f(i) = 1$.

For example:

Number of Claims	Probability Density Function	Cumulative Distribution Function
0	0.1	0.1
1	0.2	0.3
2	0	0.3
3	0.1	0.4
4	0	0.4
5	0	0.4
6	0.1	0.5
7	0	0.5
8	0	0.5
9	0.1	0.6
10	0.3	0.9
11	0.1	1
Sum	1	

The Distribution Function[5] is the cumulative sum of the probability density function:

$$F(j) = \sum_{i=0}^{j} f(i).$$

In the above example, F(3) = f(0) + f(1) + f(2) + f(3) = 0.1 + 0.2 + 0 + 0.1 = 0.4.

[4] **<u>Loss Models</u> calls the probability density function of frequency the "probability function" or p.f. and uses the notation p_k for f(k), the density at k.**

[5] Also called the cumulative distribution function.

Moments:

One can calculate the moments of such a distribution.

For example, the first moment or mean is:
(0)(0.1) + (1)(0.2) + (2)(0) + (3)(0.1) + (4)(0) + (5)(0) + (6)(0.1) + (7)(0) + (8)(0) + (9)(0.1) + (10)(0.3) + (11)(0.1) = 6.1.

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims
0	0.1	0	0
1	0.2	0.2	0.2
2	0	0	0
3	0.1	0.3	0.9
4	0	0	0
5	0	0	0
6	0.1	0.6	3.6
7	0	0	0
8	0	0	0
9	0.1	0.9	8.1
10	0.3	3	30
11	0.1	1.1	12.1
Sum	1	6.1	54.9

$E[X] = \Sigma i\ f(i)$ = Average of X = 1st moment about the origin = 6.1.

$E[X^2] = \Sigma i^2\ f(i)$ = Average of X^2 = 2nd moment about the origin = 54.9.

The second moment is:
$(0^2)(0.1) + (1^2)(0.2) + (2^2)(0) + (3^2)(0.1) + (4^2)(0) + (5^2)(0) + (6^2)(0.1) + (7^2)(0) + (8^2)(0) + (9^2)(0.1) + (10^2)(0.3) + (11^2)(0.1) = 54.9$.

Mean = E[X] = 6.1.

Variance = second central moment = $E[(X - E[X])^2] = E[X^2] - E[X]^2$ = 17.69.

Standard Deviation = Square Root of Variance = 4.206.

The **mean** is the average or expected value of the random variable. For the above example, the mean is 6.1 claims.

In general means add; E[X+Y] = E[X] + E[Y]. Also multiplying a variable by a constant multiplies the mean by the same constant; E[kX] = kE[X].
The mean is a linear operator: E[aX + bY] = aE[X] + bE[Y].

The mean of a frequency distribution can also be computed as a sum of its survival functions:[6]

$$E[X] = \sum_{i=0}^{\infty} \text{Prob}[X > i] = \sum_{i=0}^{\infty} \{1 - F(i)\}.$$

Mode and Median:

The mean differs from the **mode** which represents the value most likely to occur. The mode is the point at which the density function reaches its maximum. The mode for the above example is 10 claims.

For a discrete distribution, take the 100pth percentile as the first value at which F(x) ≥ p.[7]
The 80th percentile for the above example is 10; F(9) = 0.6, F(10) = 0.9.

The median is the 50th percentile. For frequency distributions, and other discrete distributions, the median is the first value at which the distribution function is greater than or equal to 0.5. The median for the above example is 6 claims; F(6) = 0.5.

Definitions:

Exposure Base: The basic unit of measurement upon which premium is determined.

For example, the exposure base could be car-years, $100 of payrolls, number of insured lives, etc. The rate for Workers' Compensation Insurance might be $3.18 per $100 of payroll, with $100 of payroll being one exposure.

Frequency: The number of losses or number of payments random variable, (unless indicated otherwise) stated per exposure unit.

For example the frequency could be the number of losses per (insured) house-year.

Mean Frequency: Expected value of the frequency.

For example, the mean frequency might be 0.03 claims per insured life per year.

[6] This is analogous to the situation for a continuous loss distributions; the mean of a Loss Distribution can be computed as the integral of its survival function.
[7] Definition 3.6 in Loss Models. $F(\pi_p-) \le p \le F(\pi_p)$.

Problems:

Use the following frequency distribution for the next 5 questions:

Number of Claims	Probability
0	0.02
1	0.04
2	0.14
3	0.31
4	0.36
5	0.13

2.1 (1 point) What is the mean of the above frequency distribution?

A. less than 3
B. at least 3.1 but less than 3.2
C. at least 3.2 but less than 3.3
D. at least 3.3 but less than 3.4
E. at least 3.4

2.2 (1 point) What is the mode of the above frequency distribution?

A. 2 B. 3 C. 4 D. 5 E. None of the above.

2.3 (1 point) What is the median of the above frequency distribution?

A. 2 B. 3 C. 4 D. 5 E. None of the above.

2.4 (1 point) What is the standard deviation of the above frequency distribution?

A. less than 1.1
B. at least 1.1 but less than 1.2
C. at least 1.2 but less than 1.3
D. at least 1.3 but less than 1.4
E. at least 1.4

2.5 (1 point) What is the 80th percentile of the above frequency distribution?

A. 2 B. 3 C. 4 D. 5 E. None of A, B, C, or D.

2.6 (1 point) The number of claims, N, made on an insurance portfolio follows the following distribution:

n	Pr(N=n)
0	0.7
1	0.2
2	0.1

What is the variance of N?
A. less than 0.3
B. at least 0.3 but less than 0.4
C. at least 0.4 but less than 0.5
D. at least 0.5 but less than 0.6
E. at least 0.6

Use the following information for the next 8 questions:
V and X are each given by the result of rolling a six-sided die.
V and X are independent of each other.
Y= V + X.
Z = 2X.
Hint: The mean of X is 3.5 and the variance of X is 35/12.

2.7 (1 point) What is the mean of Y?
A. less than 7.0
B. at least 7.0 but less than 7.1
C. at least 7.1 but less than 7.2
D. at least 7.2 but less than 7.3
E. at least 7.4

2.8 (1 point) What is the mean of Z?
A. less than 7.0
B. at least 7.0 but less than 7.1
C. at least 7.1 but less than 7.2
D. at least 7.2 but less than 7.3
E. at least 7.4

2.9 (1 point) What is the standard deviation of Y?
A. less than 2.0
B. at least 2.0 but less than 2.3
C. at least 2.3 but less than 2.6
D. at least 2.9 but less than 3.2
E. at least 3.2

<u>2.10</u> (1 point) What is the standard deviation of Z?
A. less than 2.0
B. at least 2.0 but less than 2.3
C. at least 2.3 but less than 2.6
D. at least 2.9 but less than 3.2
E. at least 3.2

2.11 (1 point) What is the probability that Y = 8?
A. less than .10
B. at least .10 but less than .12
C. at least .12 but less than .14
D. at least .14 but less than .16
E. at least .16

2.12 (1 point) What is the probability that Z = 8?
A. less than .10
B. at least .10 but less than .12
C. at least .12 but less than .14
D. at least .14 but less than .16
E. at least .16

2.13 (1 point) What is the probability that X = 5 if Y≥10?
A. less than .30
B. at least .30 but less than .32
C. at least .32 but less than .34
D. at least .34 but less than .36
E. at least .36

2.14 (1 point) What is the expected value of X if Y≥10?
A. less than 5.0
B. at least 5.0 but less than 5.2
C. at least 5.2 but less than 5.4
D. at least 5.4 but less than 5.6
E. at least 5.6

2.15 (3 points) N is uniform and discrete from 0 to b; Prob[N = n] = 1/(b+1), n = 0, 1, 2, ... , b.

$N \wedge 10 \equiv$ Minimum[N, 10].

If E[$N \wedge 10$] = 0.875 E[N], determine b.

A. 13 B. 14 C. 15 D. 16 E. 17

2.16 (2 points) What is the variance of the following distribution?

Claim Count:	0	1	2	3	4	5	> 5
Percentage of Insureds:	60.0%	24.0%	9.8%	3.9%	1.6%	0.7%	0%

A. 0.2 B. 0.4 C. 0.6 D. 0.8 E. 1.0

2.17 (3 points) N is uniform and discrete from 1 to S; Prob[N = n] = 1/S, n = 1, 2, ... , S.
Determine the variance of N, as a function of S.

2.18 (4, 5/88, Q.31) (1 point) The following table represents data observed for a certain class of insureds. The regional claims office is being set up to service a group of 10,000 policyholders from this class.

Number of Claims n	Probability of a Policyholder Making n Claims in a Year
0	0.84
1	0.07
2	0.05
3	0.04

If each claims examiner can service a maximum of 500 claims in a year, and you want to staff the office so that you can handle a number of claims equal to two standard deviations more than the mean, how many examiners do you need?
A. 5 or less B. 6 C. 7 D. 8 E. 9 or more

2.19 (4B, 11/99, Q.7) (2 points) A player in a game may select one of two fair, six-sided dice.
Die A has faces marked with 1, 2, 3, 4, 5, and 6. Die B has faces marked with 1, 1, 1, 6, 6, and 6.
If the player selects Die A, the payoff is equal to the result of one roll of Die A. If the player selects Die B, the payoff is equal to the mean of the results of n rolls of Die B.
The player would like the variance of the payoff to be as small as possible.
Determine the smallest value of n for which the player should select Die B.
A. 1 B. 2 C. 3 D. 4 E. 5

2.20 (1, 11/01, Q.32) (1.9 points) The number of injury claims per month is modeled by a random variable N with $P[N = n] = \frac{1}{(n+1)\ (n+2)}$, where $n \geq 0$.

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.
(A) 1/3 (B) 2/5 (C) 1/2 (D) 3/5 (E) 5/6

Solutions to Problems:

2.1. D. mean = (0)(.02) + (1)(.04) + (2)(.14) + (3)(.31) + (4)(.36) + (5)(.13) = **3.34**.
Comment: Let S(n) = Prob[N > n] = survival function at n.
S(0) = 0.98. S(1) = .94

$E[N] = \sum_{0}^{\infty} S(i) = .98 + .94 + .80 + .49 + .13 + 0 = 3.34.$

2.2. C. f(4) = 36% which is the greatest value attained by the probability density function, therefore the mode is **4**.

2.3. B. Since F(2) = 0.2 < 0.5 and F(3) = 0.51 ≥ 0.5 the median is **3**.

Number of Claims	Probability	Distribution
0	2%	2%
1	4%	6%
2	14%	20%
3	31%	51%
4	36%	87%
5	13%	100%

2.4. B. Variance = (second moment) - $(\text{mean})^2$ = 12.4 - 3.34^2 = 1.244.

Standard Deviation = $\sqrt{1.244}$ = **1.116**.

2.5. C. Since F(3) = 0.51 < 0.8 and F(4) = 0.87 ≥ 0.8, the 80th percentile is **4**.

2.6. C. Mean = (.7)(0) + (.2)(1) + (.1)(2) = .4.
Variance = $(.7)(0 - .4)^2 + (.2)(1 - .4)^2 + (.1)(2 - .4)^2$ = **0.44**.
Alternately, Second Moment = $(.7)(0^2) + (.2)(1^2) + (.1)(2^2)$ = .6. Variance = .6 - $.4^2$ = **0.44**.

2.7. B. E[Y] = E[V + X] = E[V] + E[X] = 3.5 + 3.5 = **7**.

2.8. B. E[Z] = E[2X] = 2 E{X] = (2)(3.5) = **7**.

2.9. C. Var[Y] = Var[V+X] = Var[V]+V[X] = (35/12)+(35/12) = 35/6 = 5.83.
Standard Deviation[Y] = $\sqrt{5.83}$ = **2.41**.

2.10. E. Var[Z] = Var[2X] = 2^2Var[X] = (4)(35/12) = 35/3 = 11.67.
Standard Deviation[Z] = $\sqrt{11.67}$ = **3.42**.

2.11. C. For Y = 8 we have the following possibilities: V =2, X=6; V=3, X=5; V=4;X=4; V=5; X=3; V=6, X=2. Each of these has a (1/6)(1/6) = 1/36 chance, so the total chance that Y =8 is 5/36 = **0.139**.

Comment: The distribution function for Y is:

y	2	3	4	5	6	7	8	9	10	11	12
f(y)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

2.12. E. Z = 8 when X = 4, which has probability 1/6.

Comment: The distribution function for Z is:

z	2	4	6	8	10	12
f(z)	1/6	1/6	1/6	1/6	1/6	1/6

Note that even though Z has the same mean as Y, it has a significantly different distribution function. This illustrates the difference between adding the result of several independent identically distributed variables, and just multiplying a single result by a constant. (If the variable has a finite variance), the Central Limit applies to the prior situation, but not the latter. The sum of N independent dice starts to look like a Normal Distribution as N gets large.

N times a single die has a flat distribution similar to that of X or Z, regardless of N.

2.13. C. If Y ≥ 10, then we have the possibilities V=4, X=6; V=5, X =5; V=5, X=6; V=6, X=4; V=6 ,X=5; V=6, X=6. Out of these 6 equally likely probabilities, for 2 of them X=5. Therefore if Y≥10, there is a 2/6 = **0.333** chance that X = 5.

Alternately, Prob[Y ≥ 10] = 6/36 = 1/6.

Prob[X = 5 ad Y ≥ 10] = 2/36 = 1/18.

Prob[X = 5 | Y ≥ 10] = (1/18) / (1/6) = **1/3**.

Comment: This is an example of a conditional distribution.

The distribution of f(x | y ≥10) is:

x	4	5	6
f(x | y ≥10)	1/6	2/6	3/6

The distribution of f(x | y =10) is:

x	4	5	6
f(x | y =10)	1/3	1/3	1/3

2.14. C. The distribution of f(x | y ≥10) is:

x	4	5	6
f(x | y ≥10)	1/6	2/6	3/6

(1/6)(4) + (2/6)(5) + (3/6)(6) = 32 / 6 = **5.33**.

2.15 C. E[N] = (0 + 1 + 2 + ... + b)/(b + 1) = {b(b+1)/2}/(b + 1).
For b ≥ 10, $E[N \wedge 10]$ = {0 + 1 + 2 + ... 9 + (b-9)(10)}/(b + 1) = (45 + 10b - 90)/(b + 1).
$E[N \wedge 10]$ = 0.875 E[N]. ⇒ 10b - 45 = .875b(b+1)/2. ⇒ $.875b^2 - 19.125b + 90 = 0$.
$b = (19.125 \pm \sqrt{19.125^2 - (4)(0.875)(90)})/1.75 = (19.125 \pm 7.125)/1.75$ = 15 or 6.857.
However, b has to be integer and at least 10, so b = **15**.
Comment: The limited expected value is discussed in "Mahler's Guide to Loss Distributions."
If b = 15, then there are 6 terms that enter the limited expected value as 10:
$E[N \wedge 10]$ = (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 10 + 10 + 10 +10 + 10}/16 = 105/16.
E[N] = (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 +13 + 14 + 15}/16 = 15/2.
Their ratio is 0.875.

2.16. E. Mean = 0.652 and the variance = $1.414 - 0.652^2$ = **0.989.**

Number of Claims	A Priori Probability	Number Times Probability	Number Squared Times Probability
0	0.60000	0.00000	0.00000
1	0.24000	0.24000	0.24000
2	0.09800	0.19600	0.39200
3	0.03900	0.11700	0.35100
4	0.01600	0.06400	0.25600
5	0.00700	0.03500	0.17500
Sum	1	0.652	1.41400

2.17. E[N] = (1 + 2 + ... + S)/S = {S(S+1)/2}/S = (S + 1)/2.
$E[N^2] = (1^2 + 2^2 + ... + S^2)/S$ = {S(S+1)(2S + 1)/6}/S = (S + 1)(2S + 1)/6.
$Var[N] = E[N^2] - E[N]^2 = (S + 1)(2S + 1)/6 - \{(S + 1)/2\}^2 = \{(S + 1)/12\}\{2(2S + 1) - 3(S + 1)\}$
= {(S + 1)/12}(S - 1) = **$(S^2 - 1)/12$**.
Comment: For S = 6, a six-sided die, Var[N] = 35/12.

2.18. C. The first moment is: (.84)(0) +(.07)(1) +(.05)(2) +(.04)(3) = 0.29.
The 2nd moment is: $(.84)(0^2) + (.07)(1^2) + (.05)(2^2) + (.04)(3^2)$ = 0.63. Thus the variance is:
$0.63 - 0.29^2$ = .5459 for a single policyholder. For 10,000 independent policyholders, the variance of the sum is (10000)(.5459) = 5459. The standard deviation is: $\sqrt{5459}$ = 73.9.
The mean number of claims is (10000)(.29) = 2900. Adding two standard deviations one gets 3047.8. This requires **7** claims handlers (since 6 can only handle 3000 claims.)

2.19. C. Both Die A and Die B have a mean of 3.5.

The variance of Die A is: $(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) / 6 = 35/12$.

The variance of Die B is: $2.5^2 = 6.25$.

The variance of an average of n rolls of Die B is 6.25/n. We want 6.25 /n < 35/12.
Thus n > (6.25)(12/35) = 2.14. Thus the smallest n is **3**.

2.20. B. Prob[N ≥ 1 I N ≤ 4] = Prob[1 ≤ N ≤ 4]/Prob[N ≤ 4] =
(1/6 + 1/12 + 1/20 + 1/30)/(1/2 + 1/6 + 1/12 + 1/20 + 1/30) = 20/50 = **2/5**.
Comment: *For integer a and b, such that 0 < a < b,*

$$\sum_{k=a}^{k=b-1} 1/k = (b-a) \sum_{n=0}^{\infty} 1/\{(n+a)(n+b)\}.$$

Therefore, $\{(b-a) / \sum_{k=a}^{k=b-1} 1/k\} / \{(n+a)(n+b)\}$, $n \ge 0$, *is a frequency distribution.*

This is a heavy-tailed distribution without a finite mean.
If b = a + 1, then $f(n) = a/\{(n+a)(n+a+1)\}$, $n \ge 0$.
In this question, a =1, b = 2, and $f(n) = 1/\{(n+1)(n+2)\}$, $n \ge 0$.

Section 3, Binomial Distribution

Assume one has five independent lives, each of which has a 10% chance of dying over the next year. What is the chance of observing two deaths? This is given by the product of three factors. The first is the chance of death to the power two. The second factor is the chance of not dying to the power 3 = 5 - 2. The final factor is the ways to pick two lives out of five, or the binomial coefficient of:

$$\binom{5}{2} = \frac{5!}{2!\ 3!} = 10.$$

The chance of observing two deaths is:

$$0.1^2\ 0.9^3 \binom{5}{2} = 7.29\%.$$

The chance of observing other numbers of deaths in this case is:

Number of Deaths	Chance of Observation	Binomial Coefficient
0	59.049%	1
1	32.805%	5
2	7.290%	10
3	0.810%	10
4	0.045%	5
5	0.001%	1
Sum	1	

This is a just an example of a Binomial distribution, for q = 0.1 and m = 5.

For the binomial distribution: $\mathbf{f(x) = m!\ q^x (1-q)^{m-x} / \{x!(m-x)!\}}$ **x = 0, 1, 2, 3,...., m.**

Note that the binomial density function is only positive for $x \le m$; there are at most m claims. The Binomial has two parameters m and q. m is the maximum number of claims and q is the chance of success.[8]

Written in terms of the binomial coefficient the Binomial density function is:

$$\mathbf{f(x) = \binom{m}{x} q^x\ (1-q)^{m-x},\quad x = 0, 1, 2, 3,..., m.}$$

[8] I will use the notation in Loss Models and the tables attached to your exam. Many of you are familiar with the notation in which the parameters for the Binomial Distribution are n and p rather than m and q as in Loss Models.

Bernoulli Distribution:

The Bernoulli is a distribution with q chance of 1 claim and 1-q chance of 0 claims. There are only two possibilities: either a success or a failure. The Bernoulli is a special case of the Binomial for m = 1.

The mean of the Bernoulli is q. The second moment of the Bernoulli is $(0^2)(1\text{-}q) + (1^2)q = q$.

Therefore the variance is $q - q^2 = q(1\text{-}q)$.

Binomial as a Sum of Independent Bernoullis:

The example of five independent lives was the sum of five variables each of which was a Bernoulli trial with chance of a claim 10%. In general, **the Binomial can be thought of as the sum of the results of m independent Bernoulli trials, each with a chance of success q**. Therefore, the sum of two independent Binomial distributions with the same chance of success q, is another Binomial distribution; if X is Binomial with parameters q and m_1, while Y is Binomial with parameters q and m_2, then X+Y is Binomial with parameters q and $m_1 + m_2$.

Mean and Variance:

Since the Binomial is a sum of the results of m identical Bernoulli trials, the mean of the Binomial is m times the mean of a Bernoulli, which is mq.
The mean of the Binomial is mq.

Similarly the variance of a Binomial is m times the variance of the corresponding Bernoulli, which is mq(1-q).
The variance of a Binomial is mq(1-q).

For the case m = 5 and q = 0.1 presented previously:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	59.049%	0.00000	0.00000	0.00000
1	32.805%	0.32805	0.32805	0.32805
2	7.290%	0.14580	0.29160	0.58320
3	0.810%	0.02430	0.07290	0.21870
4	0.045%	0.00180	0.00720	0.02880
5	0.001%	0.00005	0.00025	0.00125
Sum	1	0.50000	0.70000	1.16000

The mean is: 0.5 = (5)(0.1) = mq.
The variance is: $E[X^2] - E[X]^2 = 0.7 - 0.5^2 = 0.45 = (5)(0.1)(0.9) = mq(1\text{-}q)$.

<u>Properties of the Binomial Distribution</u>:

Since $0 < q < 1$: $mq(1-q) < mq$.
Therefore, **the variance of any Binomial is less than its mean.**

A Binomial Distribution with parameters m and q, is the sum of m independent Bernoullis, each with parameter q. Therefore, **if one sums independent Binomials with the same q, then one gets another Binomial, with the same q parameter and the sum of their m parameters.**

Exercise: X is a Binomial with q = 0.4 and m = 8. Y is a Binomial with q = 0.4 and m = 22.
Z is a Binomial with q = 0.4 and m = 17. X, Y, and Z are independent of each other.
What form does X + Y + Z have?
[Solution: X + Y + Z is a Binomial with q = 0.4 and m = 8 + 22 + 17 = 47.]

Specifically, the sum of n independent identically distributed Binomial variables, with the same parameters q and m, is a Binomial with parameters q and nm.

Exercise: X is a Binomial with q = 0.4 and m = 8.
What is the form of the sum of 25 independent random draws from X?
[Solution: A random draw from a Binomial Distribution with q = 0.4 and m = (25)(8) = 200.]

Thus if one had 25 exposures, each of which had an independent Binomial frequency process with q = 0.4 and m = 8, then the portfolio of 25 exposures has a Binomial frequency process with q = 0.4 and m = 200.

<u>Thinning a Binomial</u>:

If one selects only some of the claims, in a manner independent of frequency, then if all claims are Binomial with parameters m and q, the selected claims are also Binomial with parameters m and q' = q(expected portion of claims selected).

For example, assume that the number of claims is given by a Binomial Distribution with m = 9 and q = 0.3. Assume that on average 1/3 of claims are large.
Then the number of large losses is also Binomial, but with parameters m = 9 and q = 0.3/3 = 0.1.
The number of small losses is also Binomial, but with parameters m = 9 and q = (0.3)(2/3) = 0.2.[9]

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[9] *The number of small and large losses are not independent; in the case of a Binomial they are negatively correlated. In the case of a Poisson, they are independent.*

Binomial Distribution

Support: x = 0, 1, 2, 3, ..., **m**. Parameters: $1 > q > 0$, $m \geq 1$. m is integer.

m = 1 is a Bernoulli Distribution.

D. f. : *$F(x) = 1 - \beta(x+1, m-x ; q) = \beta(m-x, x+1 ; 1-q)$ Incomplete Beta Function*

P. d. f. : $$\mathbf{f(x) = \frac{m!\ q^x\ (1-q)^{m-x}}{x!\ (m-x)!} = \binom{m}{x} q^x\ (1-q)^{m-x}}.$$

Mean = mq

Variance = mq(1-q) **Variance / Mean = 1 - q < 1.**

Coefficient of Variation = $\sqrt{\frac{1 - q}{m\,q}}$.

Skewness = $\frac{1 - 2q}{\sqrt{mq(1 - q)}}$.

Kurtosis = $3 + \frac{1}{mq(1 - q)} - \frac{6}{m}$.

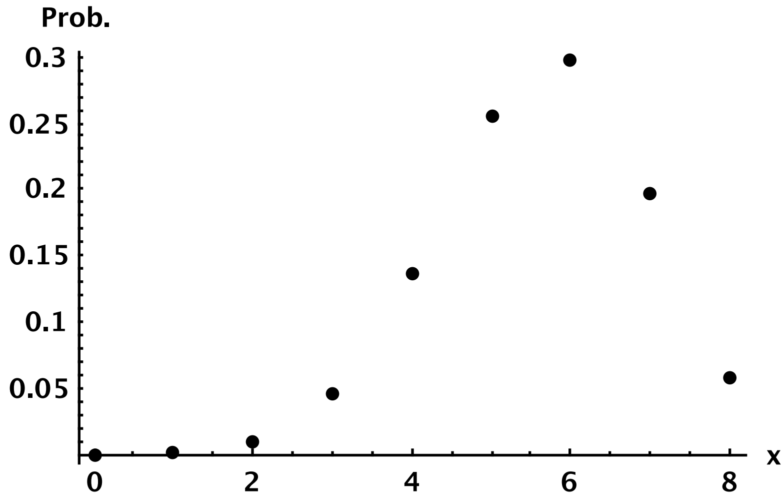
Mode = largest integer in mq + q (if mq + q is an integer, then f(mq + q) = f(mq + q- 1)
and both mq + q and mq + q - 1 are modes.)

Probability Generating Function: $P(z) = \{1 + q(z-1)\}^m$

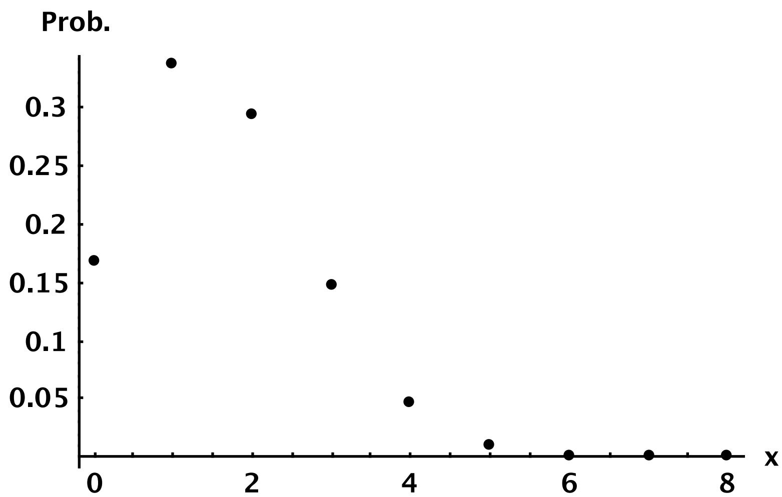
$f(x+1)/f(x) = a + b/(x+1)$, $a = -q/(1-q)$, $b = (m+1)q/(1-q)$, $f(0) = (1-q)^m$.

Moment Generating Function: $M(s) = (qe^s + 1-q)^m$

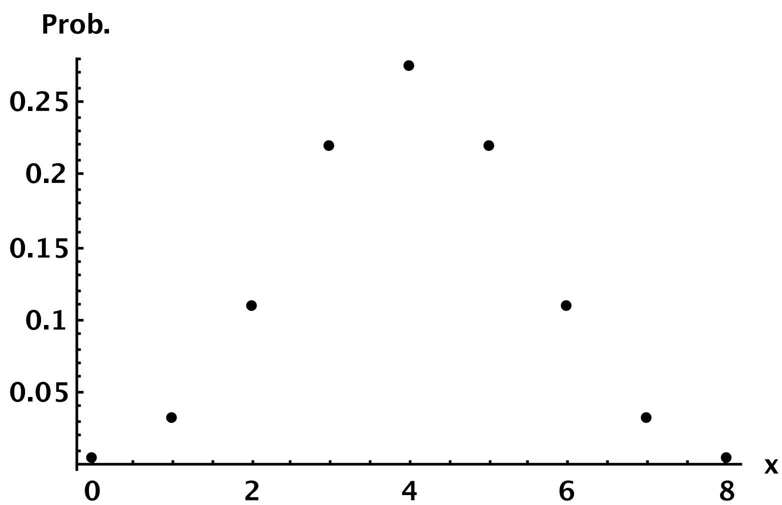
Binomial Distribution with m = 8 and q = 0.7:



Binomial Distribution with m = 8 and q = 0.2:



Binomial Distribution with m = 8 and q = 0.5:



Binomial Coefficients:

The binomial coefficient of x out of n trials is:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!} = \frac{n(n-1)(n-2)\ ...\ (n+1-x)}{x(x-1)(x-2)\ ...\ (1)} = \frac{\Gamma(n+1)}{\Gamma(x+1)\ \Gamma(n+1-x)}.$$

Below are some examples of Binomial Coefficients:

n	x=0	x=1	x=2	x=3	x=4	x=5	x=6	x=7	x=8	x=9	x=10	x=11
2	1	2	1									
3	1	3	3	1								
4	1	4	6	4	1							
5	1	5	10	10	5	1						
6	1	6	15	20	15	6	1					
7	1	7	21	35	35	21	7	1				
8	1	8	28	56	70	56	28	8	1			
9	1	9	36	84	126	126	84	36	9	1		
10	1	10	45	120	210	252	210	120	45	10	1	
11	1	11	55	165	330	462	462	330	165	55	11	1

It is interesting to note that the entries in a row sum to 2^n.

For example, 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6.

Also note that for x=0 or x=n the binomial coefficient is one.

The entries in a row can be computed from the previous row. For example, the entry 45 in the row n =10 is the sum of 9 and 36 the two entries above it and to the left. Similarly, 120 = 36+84.

Note that: $\binom{n}{x} = \binom{n}{n-x}$.

For example,

$$\binom{11}{5} = \frac{11!}{5!\ (11-5)!} = \frac{39{,}916{,}800}{(120)\ (720)} = 462 = \frac{11!}{6!\ (11-6)!} = \binom{11}{6}.$$

Using the Functions of the Calculator to Compute Binomial Coefficients:

Using the TI-30X-IIS, the binomial coefficient $\binom{n}{i}$ can be calculated as follows:

n
PRB
▸
nCr
Enter
i
Enter

For example, in order to calculate $\binom{10}{3} = \frac{10!}{3!\ 7!} = 120$:

10
PRB
▸
nCr
Enter
3
Enter

Using instead the BA II Plus Professional, in order to calculate $\binom{10}{3} = \frac{10!}{3!\ 7!} = 120$:

10
2nd
nCr
3
=

The TI-30XS Multiview calculator saves time doing repeated calculations using the same formula.

For example constructing a table of the densities of a Binomial distribution, with m = 5 and q = 0.1:[10]

$f(x) = \binom{5}{x} 0.1^x \, 0.9^{5-x}$.

table
$y = (5 \text{ nCr } x) * .1^x * .9^{(5-x)}$
Enter
Start = 0
Step = 1
Auto
OK

x = 0 y = 0.59049
x = 1 y = 0.32805
x = 2 y = 0.07290
x = 3 y = 0.00810
x = 4 y = 0.00045
x = 5 y = 0.00001

[10] Note that to get Binomial coefficients hit the prb key and select nCr.

Relation to the Beta Distribution:

The binomial coefficient looks almost like 1 over a complete Beta function.[11]
The incomplete Beta distribution for integer parameters can be used to compute the sum of terms from the Binomial Distribution.[12]

$$\beta(a,b;x) = \sum_{i=a}^{i=a+b-1} \binom{a+b-1}{i} x^i (1-x)^{a+b-(i+1)} .$$

For example, $\beta(6, 9; 0.3) = 0.21948 = \sum_{i=6}^{i=14} \binom{14}{i} 0.3^i\, 0.7^{14-i}$.

By taking appropriate differences of two Betas one can get any sum of binomials terms.
For example:

$\binom{n}{a} q^a (1-q)^{n-a} = \beta(a, n-(a-1) ; q) - \beta(a+1. n-a ; q).$

For example, $\binom{10}{3} 0.2^3\, 0.8^7 = (120)\, 0.2^3\, 0.8^7 = 0.20133 = \beta(3, 8 ; 0.2) - \beta(4, 7; 0.2)$

$\beta(a, b ; x) = 1 - \beta(b, a ; 1-x) = F_{2a, 2b}$ [bx / {a(1-x)}] where F is the distribution function of the F-distribution with 2a and 2b degrees of freedom.

For example, $\beta(4,7; .607) = .950 = F_{8,14}$ [(7)(.607) /{(4)(.393)}] = $F_{8,14}$ [2.70].[13]

[11] The complete Beta Function is defined as $\Gamma(a)\Gamma(b) / \Gamma(a+b)$.
It is the divisor in front of the incomplete Beta function and is equal to the integral from 0 to 1 of $x^{a-1}(1-x)^{b-1}$.
[12] For a discussion of the Beta Distribution, see "Mahler's Guide to Loss Distributions". On the exam you should either compute the sum of binomial terms directly or via the Normal Approximation. Note that the use of the Beta Distribution is an exact result, not an approximation. See for example the Handbook of Mathematical Functions, by Abramowitz, et. al.
[13] If one did an F-Test with 8 and 14 degrees of freedom, then there would be a 5% chance that the value exceeds 2.7.

Problems:

Use the following information for the next seven questions:
One observes 9 independent lives, each of which has a 20% chance of death over the coming year.

3.1 (1 point) What is the mean number of deaths over the coming year?
A. less than 1.8
B. at least 1.8 but less than 1.9
C. at least 1.9 but less than 2.0
D. at least 2.0 but less than 2.1
E. at least 2.1

3.2 (1 point) What is the variance of the number of deaths observed over the coming year?
A. less than 1.5
B. at least 1.5 but less than 1.6
C. at least 1.6 but less than 1.7
D. at least 1.7 but less than 1.8
E. at least 1.8

3.3 (1 point) What is the chance of observing 4 deaths over the coming year?
A. less than 7%
B. at least 7% but less than 8%
C. at least 8% but less than 9%
D. at least 9% but less than 10%
E. at least 10%

3.4 (1 point) What is the chance of observing no deaths over the coming year?
A. less than 13%
B. at least 13% but less than 14%
C. at least 14% but less than 15%
D. at least 15% but less than 16%
E. at least 16%

3.5 (3 points) What is the chance of observing 6 or more deaths over the coming year?
A. less than .1%
B. at least .1% but less than .2%
C. at least .2% but less than .3%
D. at least .3% but less than .4%
E. at least .4%

3.6 (1 point) What is the median number of deaths per year?
A. 0 B. 1 C. 2 D. 3 E. None of A, B, C, or D

3.7 (1 point) What is the mode of the distribution of deaths per year?
A. 0 B. 1 C. 2 D. 3 E. None of A, B, C, or D

3.8 (1 point) Assume that each year that Joe starts alive, there is a 20% chance that he will die over the coming year. What is the chance that Joe will die over the next 5 years?
A. less than 67%
B. at least 67% but less than 68%
C. at least 68% but less than 69%
D. at least 69% but less than 70%
E. at least 70%

3.9 (2 points) One insures 10 independent lives for 5 years. Assume that each year that an insured starts alive, there is a 20% chance that he will die over the coming year.
What is the chance that 6 of these 10 insureds will die over the next 5 years?
A. less than 20%
B. at least 20% but less than 21%
C. at least 21% but less than 22%
D. at least 22% but less than 23%
E. at least 23%

3.10 (1 point) You roll 13 six-sided dice. What is the chance of observing exactly 4 sixes?
A. less than 10%
B. at least 10% but less than 11%
C. at least 11% but less than 12%
D. at least 12% but less than 13%
E. at least 13%

3.11 (1 point) You roll 13 six-sided dice. What is the average number of sixes observed?
A. less than 1.9
B. at least 1.9 but less than 2.0
C. at least 2.0 but less than 2.1
D. at least 2.1 but less than 2.2
E. at least 2.2

3.12 (1 point) You roll 13 six-sided dice.
What is the mode of the distribution of the number of sixes observed?
A. 1 B. 2 C. 3 D. 4 E. None of A, B, C, or D

3.13 (3 point) You roll 13 six-sided dice.
What is the median of the distribution of the number of sixes observed?
A. 1 B. 2 C. 3 D. 4 E. None of A, B, C, or D

3.14 (1 point) You roll 13 six-sided dice. What is the variance of the number of sixes observed?
A. less than 1.9
B. at least 1.9 but less than 2.0
C. at least 2.0 but less than 2.1
D. at least 2.1 but less than 2.2
E. at least 2.2

3.15 (2 point) The number of losses is Binomial with q = 0.4 and m = 90.
The sizes of loss are Exponential with mean 50, $F(x) = 1 - e^{-x/50}$.
The number of losses and the sizes of loss are independent.
What is the probability of seeing exactly 3 losses of size greater than 100?
A. 9% B. 11% C. 13% D. 15% E. 17%

3.16 (2 points) Total claim counts generated from Policy A follow a Binomial distribution with parameters m = 2 and q = 0.1. Total claim counts generated from Policy B follow a Binomial distribution with parameters m = 2 and q = 0.6. Policy A is independent of Policy B.
For the two policies combined, what is the probability of observing 2 claims in total?
A. 32% B. 34% C. 36% D. 38% E. 40%

3.17 (2 points) Total claim counts generated from a portfolio follow a Binomial distribution with parameters m = 9 and q = 0.1. Total claim counts generated from another independent portfolio follow a Binomial distribution with parameters m = 15 and q = 0.1.
For the two portfolios combined, what is the probability of observing exactly 4 claims in total?
A. 11% B. 13% C. 15% D. 17% E. 19%

3.18 (3 points) The number of losses follows a Binomial distribution with m = 6 and q = 0.4.
Sizes of loss follow a Pareto Distribution with $\alpha = 4$ and $\theta = 50{,}000$.
There is a deductible of 5000, and a coinsurance of 80%.
Determine the probability that there are exactly two payments of size greater than 10,000.
A. 11% B. 13% C. 15% D. 17% E. 19%

Use the following information for the next two questions:

- A state holds a lottery once a week.
- The cost of a ticket is 1.
- 1,000,000 tickets are sold each week.
- The prize is 1,000,000.
- The chance of each ticket winning the prize is 1 in 1,400,000, independent of any other ticket.
- In a given week, there can be either no winner, one winner, or multiple winners.
- If there are multiple winners, each winner gets a 1,000,000 prize.
- The lottery commission is given a reserve fund of 2,000,000 at the beginning of the year.
- In any week where no prize is won, the lottery commission sends its receipts of 1 million to the state department of revenue.
- In any week in which prize(s) are won, the lottery commission pays the prize(s) from receipts and if necessary the reserve fund.
- If any week there is insufficient money to pay the prizes, the lottery commissioner must call the governor of the state, in order to ask the governor to authorize the state department of revenue to provide money to pay owed prizes and reestablish the reserve fund.

3.19 (3 points) What is the probability that the lottery commissioner has to call the governor the first week?

A. 0.5% B. 0.6% C. 0.7% D. 0.8% E. 0.9%

3.20 (4 points) What is the probability that the lottery commissioner does <u>not</u> have to call the governor the first year (52 weeks)?

A. 0.36% B. 0.40% C. 0.44% D. 0.48% E. 0.52%

3.21 (3 points) The number of children per family follows a Binomial Distribution m = 4 and q = 0.5. For a child chosen at random, how many siblings (brothers and sisters) does he have on average?

A. 1.00 B. 1.25 C. 1.50 D. 1.75 E. 2.00

3.22 (2, 5/85, Q.2) (1.5 points) Suppose 30 percent of all electrical fuses manufactured by a certain company fail to meet municipal building standards. What is the probability that in a random sample of 10 fuses, exactly 3 will fail to meet municipal building standards?

A. $\binom{10}{3}(0.3^7)(0.7^3)$ B. $\binom{10}{3}(0.3^3)(0.7^7)$ C. $10\,(0.3^3)(0.7^7)$

D. $\sum_{i=0}^{3}\binom{10}{i}(0.3^i)(0.7^{10-i})$ E. 1

3.23 (160, 11/86, Q.14) (2.1 points) In a certain population ${}_{40}p_{25} = 0.9$.

From a random sample of 100 lives at exact age 25, the random variable X is the number of lives who survive to age 65. Determine the value one standard deviation above the mean of X.

(A) 90 (B) 91 (C) 92 (D) 93 (E) 94

3.24 (160, 5/91, Q.14) (1.9 points)

From a study of 100 independent lives over the interval (x, x+1], you are given:

(i) The underlying mortality rate, q_x, is 0.1.

(ii) l_{x+s} is linear over the interval.

(iii) There are no unscheduled withdrawals or intermediate entrants.

(iv) Thirty of the 100 lives are scheduled to end observation, all at age x + 1/3.

(v) D_x is the random variable for the number of observed deaths.

Calculate $\mathrm{Var}(D_x)$.

(A) 6.9 (B) 7.0 (C) 7.1 (D) 7.2 (E) 7.3

3.25 (2, 2/96, Q.10) (1.7 points) Let X_1, X_2, and X_3, be independent discrete random variables with probability functions

$P[X_i = k] = \binom{n_i}{k} p^k (1-p)^{n_i - k}$ for i = 1, 2, 3, where $0 < p < 1$.

Determine the probability function of $S = X_1 + X_2 + X_3$, where positive.

A. $\binom{n_1 + n_2 + n_3}{s} p^s (1-p)^{n_1 + n_2 + n_3 - s}$

B. $\sum_{i=1}^{3} \frac{n_i}{n_1 + n_2 + n_3} \binom{n_i}{s} p^s (1-p)^{n_i - s}$

C. $\prod_{i=1}^{3} \binom{n_i}{s} p^s (1-p)^{n_i - s}$

D. $\sum_{i=1}^{3} \binom{n_i}{s} p^s (1-p)^{n_i -}$

E. $\binom{n_1 n_2 n_3}{s} p^s (1-p)^{n_1 n_2 n_3 - s}$

3.26 (2, 2/96, Q.44) (1.7 points) The probability that a particular machine breaks down on any day is 0.2 and is independent of the breakdowns on any other day.
The machine can break down only once per day.
Calculate the probability that the machine breaks down two or more times in ten days.
A. 0.0175 B. 0.0400 C. 0.2684 D. 0.6242 E. 0.9596

3.27 (4B, 11/96, Q.23) (2 points) Two observations are made of a random variable having a binomial distribution with parameters $m = 4$ and $q = 0.5$.
Determine the probability that the sample variance is zero.
A. 0
B. Greater than 0, but less than 0.05
C. At least 0.05, but less than 0.15
D. At least 0.15, but less than 0.25
E. At least 0.25

3.28 (Course 1 Sample Exam, Q.40) (1.9 points) A small commuter plane has 30 seats. The probability that any particular passenger will not show up for a flight is 0.10, independent of other passengers. The airline sells 32 tickets for the flight. Calculate the probability that more passengers show up for the flight than there are seats available.
A. 0.0042 B. 0.0343 C. 0.0382 D. 0.1221 E. 0.1564

3.29 (1, 5/00, Q.40) (1.9 points)
A company prices its hurricane insurance using the following assumptions:
(i) In any calendar year, there can be at most one hurricane.
(ii) In any calendar year, the probability of a hurricane is 0.05 .
(iii) The number of hurricanes in any calendar year is independent of the number of hurricanes in any other calendar year.
Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period.
(A) 0.06 (B) 0.19 (C) 0.38 (D) 0.62 (E) 0.92

3.30 (1, 5/01, Q.13) (1.9 points) A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). What is the probability that at least 9 participants complete the study in one of the two groups, but not in both groups?
(A) 0.096 (B) 0.192 (C) 0.235 (D) 0.376 (E) 0.469

3.31 (1, 5/01, Q.37) (1.9 points) A tour operator has a bus that can accommodate 20 tourists. The operator knows that tourists may not show up, so he sells 21 tickets. The probability that an individual tourist will not show up is 0.02, independent of all other tourists.
Each ticket costs 50, and is non-refundable if a tourist fails to show up. If a tourist shows up and a seat is not available, the tour operator has to pay 100, the ticket cost plus a penalty of 50, to the tourist. What is the expected revenue of the tour operator?
(A) 935 (B) 950 (C) 967 (D) 976 (E) 985

3.32 (1, 11/01, Q.27) (1.9 points) A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year, independent of any other employee.
Determine the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.
(A) 24 (B) 30 (C) 40 (D) 60 (E) 120

3.33 (CAS3, 11/03, Q.14) (2.5 points) The Independent Insurance Company insures 25 risks, each with a 4% probability of loss. The probabilities of loss are independent.
On average, how often would 4 or more risks have losses in the same year?
A. Once in 13 years
B. Once in 17 years
C. Once in 39 years
D. Once in 60 years
E. Once in 72 years

3.34 (CAS3, 11/04, Q.22) (2.5 points) An insurer covers 60 independent risks.
Each risk has a 4% probability of loss in a year.
Calculate how often 5 or more risks would be expected to have losses in the same year.
A. Once every 3 years
B. Once every 7 years
C. Once every 11 years
D. Once every 14 years
E. Once every 17 years

3.35 (CAS3, 11/04, Q.24) (2.5 points) A pharmaceutical company must decide how many experiments to run in order to maximize its profits.

- The company will receive a grant of $1 million if one or more of its experiments is successful.
- Each experiment costs $2,900.
- Each experiment has a 2% probability of success, independent of the other experiments.
- All experiments are run simultaneously.
- Fixed expenses are $500,000.
- Ignore investment income.

The company performs the number of experiments that maximizes its expected profit.
Determine the company's expected profit before it starts the experiments.
A. 77,818 B. 77,829 C. 77,840 D. 77,851 E. 77,862

3.36 (SOA3, 11/04, Q.8 & 2009 Sample Q.124) (2.5 points)
For a tyrannosaur with a taste for scientists:
(i) The number of scientists eaten has a binomial distribution with $q = 0.6$ and $m = 8$.
(ii) The number of calories of a scientist is uniformly distributed on (7000, 9000).
(iii) The numbers of calories of scientists eaten are independent, and are independent of the number of scientists eaten.
Calculate the probability that two or more scientists are eaten and exactly two of those eaten have at least 8000 calories each.
(A) 0.23 (B) 0.25 (C) 0.27 (D) 0.30 (E) 0.3

3.37 (CAS3, 5/05, Q.15) (2.5 points) A service guarantee covers 20 television sets.
Each year, each set has a 5% chance of failing. These probabilities are independent.
If a set fails, it is replaced with a new set at the end of the year of failure.
This new set is included under the service guarantee.
Calculate the probability of no more than 1 failure in the first two years.
A. Less than 40.5%
B. At least 40.5%, but less than 41.0%
C. At least 41.0%, but less than 41.5%
D. At least 41.5%, but less than 42.0%
E. 42.0% or more

Solutions to Problems:

3.1. B. Binomial with q =0 .2 and m = 9. Mean = (9)(.2) = **1.8**.

3.2. A. Binomial with q = 0.2 and m = 9. Variance = (9)(.2)(1-.2) = **1.44**.

3.3. **A.** Binomial with q = 0.2 and m = 9. f(4) = 9!/(4! 5!) $.2^4 .8^5$ = **6.61%**.

3.4. B. Binomial with q = 0.2 and m = 9. f(0) = 9!/(0! 9!) $.2^0 .8^9$ = **13.4%**.

3.5. D. Binomial with q = 0.2 and m = 9.
The chance of observing different numbers of deaths is:

Number of Deaths	Chance of Observation	Binomial Coefficient
0	13.4218%	1
1	30.1990%	9
2	30.1990%	36
3	17.6161%	84
4	6.6060%	126
5	1.6515%	126
6	0.2753%	84
7	0.0295%	36
8	0.0018%	9
9	0.0001%	1

Adding the chances of having 6, 7, 8, or 9 claims the answer is **0.307%**.
Alternately one can add the chances of having 0, 1, 2, 3, 4 or 5 claims and subtract this sum from unity.
Comment: Although you should not do so for the exam, one could also answer this question using the Incomplete Beta Function. The chance of more than x claims is β(x+1, m-x; q).

The chance of more than 5 claims is: β(5+1, 9-5; .2) = β(6, 4; .2) = 0.00307.

3.6. C. For a discrete distribution such as we have here, employ the convention that the median is the first value at which the distribution function is greater than or equal to .5.
F(1) = 0.134 + 0.302 = 0.436 < 50%, F(2) = 0.134 + 0.302 + 0.302 = 0.738 > 50%,
and therefore the median is **2**.

3.7. E. The mode is the value at which f(n) is a maximum; f(1) = .302 = f(2) and both 1 and 2 are modes. Alternately, in general for the Binomial the mode is the largest integer in mq + q; the largest integer in 2 is 2, but when mq + q is an integer both it and the integer one less are modes.
Comment: This is a somewhat unfair question. While it seems to me that E is the best single answer, one could also argue for B or C. If you are unfortunate enough to have an apparently unfair question on your exam, do not let it upset you while taking the exam.

3.8. B. The chance that Joe is alive at the end of 5 years is $(1-.2)^5$ = .32768. Therefore, the chance that he died is 1 - 0.32768 = **0.67232**.

3.9. D. Based on the solution of the previous problem, for each life the chance of dying during the five year period is 0.67232. Therefore, the number of deaths for the 10 independent lives is Binomial with m = 10 and q = 0.67232.

$f(6) = ((10!)/\{(6!)(4!)\})\ (0.67232^6)\ (0.32768^4) = (210)(0.0924)(0.01153) =$ **0.224**.

The chances of other numbers of deaths are as follows:

Number of Deaths	Chance of Observation	Binomial Coefficient
0	0.001%	1
1	0.029%	10
2	0.270%	45
3	1.479%	120
4	5.312%	210
5	13.078%	252
6	22.360%	210
7	26.216%	120
8	20.171%	45
9	9.197%	10
10	1.887%	1
Sum	1	1024

3.10. B. The chance of observing a six on an individual six-sided die is 1/6. Assuming the results of the dice are independent, one has a Binomial distribution with q =1/6 and m =13.

$f(4) = 13!/(4!\ 9!)\ (1/6)^4(5/6)^9 =$ **10.7%**.

3.11. D, 3.12. B, & 3.13. B. Binomial with q =1/6 and m =13. Mean = (1/6)(13) = **2.17**.
For the Binomial the mode is the largest integer in mq + q = (13)(1/6) + (1/6) = 2.33; the largest integer in 2.33 is **2**. Alternately compute all of the possibilities and 2 is the most likely.
F(1) = .336 <.5 and F(2) = .628 ≥ .5, therefore the median is **2**.

Number of Deaths	Chance of Observation	Binomial Coefficient	Cumulative Distribution
0	9.3463879%	1	9.346%
1	24.3006085%	13	33.647%
2	29.1607302%	78	62.808%
3	21.3845355%	286	84.192%
4	10.6922678%	715	94.885%
5	3.8492164%	1287	98.734%
6	1.0264577%	1716	99.760%
7	0.2052915%	1716	99.965%
8	0.0307937%	1287	99.996%
9	0.0034215%	715	100.000%
10	0.0002737%	286	100.000%
11	0.0000149%	78	100.000%
12	0.0000005%	13	100.000%
13	0.0000000%	1	100.000%
Sum	1	8192	

3.14. A. Binomial with q =1/6 and m =13. Variance = (13)(1/6)(1-1/6) = **1.806**.

3.15. D. S(100) = $e^{-100/50}$ = .1353. Therefore, thinning the original Binomial, the number of large losses is Binomial with m = 90 and q = (.1353)(.4) = (.05413).
f(3) = {(90)(89)(88)/3!} ($.05413^3$)$(1 - .05413)^{87}$ = **.147**.

3.16. D. Prob[2 claims in total] =
Prob[A = 0]Prob[B = 2] + Prob[A= 1]Prob[B = 1] + Prob[B = 2]Prob[A = 0] =
($.9^2$)($.6^2$) + {(2)(.1)(.9)}{(2)(.6)(.4)} + ($.1^2$)($.4^2$) = **37.96%**.
Comment: The sum of A and B is not a Binomial distribution, since their q parameters differ.

3.17. B. For the two portfolios combined, total claim counts follow a Binomial distribution with parameters m = 9 + 15 = 24 and q = 0.1.

$$f(4) = \binom{24}{4} q^4(1-q)^{20} = \{(24)(23)(22)(21)/4!\}(.1^4)(.9^{20}) = \mathbf{12.9\%}.$$

3.18. B. A payment is of size greater than 10,000 if the loss is of size greater than:
10000/.8 + 5000 = 17,500.

Probability of a loss of size greater than 17,500 is: $\{50/(50 + 17.5)\}^4$ = 30.1%.

The large losses are Binomial with m = 6 and q = (.301)(0.4) = 0.1204.

$f(2) = \binom{6}{2} .1204^2 (1 - .1204)^4 =$ **13.0%**.

Comment: An example of thinning a Binomial.

3.19. B. The number of prizes is Binomial with m = 1 million and q = 1/1,400,000.

$f(0) = (1 - 1/1400000)^{1000000} = 48.95\%$.

$f(1) = 1000000(1 - 1/1400000)^{999999} (1/1400000) = 34.97\%$.

$f(2) = \{(1000000)(999999)/2\}(1 - 1/1400000)^{999998} (1/1400000)^2 = 12.49\%$.

$f(3) = \{(1000000)(999999)(999998)/6\}(1 - 1/1400000)^{999997} (1/1400000)^3 = 2.97\%$.

n	f(n)
0	48.95%
1	34.97%
2	12.49%
3	2.97%
4	0.53%
5	0.08%
6	0.01%
Sum	100.00%

The first week, the lottery has enough money to pay 3 prizes,
(1 million in receipts + 2 million in the reserve fund.)
The probability of more than 3 prizes is: 1 - (48.95% + 34.97% + 12.49% + 2.97%) = **0.62%**.

3.20. C. Each week there is a .4895 + .3497 = .8392 chance of no need for the reserve fund.
Each week there is a .1249 chance of a 1 million need from the reserve fund.
Each week there is a .0297 chance of a 2 million need from the reserve fund.
Each week there is a .0062 chance of a 3 million or more need from the reserve fund.
The governor will be called if there is at least 3 weeks with 2 prizes each (since each such week depletes the reserve fund by 1 million), or if there is 1 week with 2 prizes plus 1 week with 3 prizes, or if there is a week with 4 prizes.
Prob[Governor not called] = Prob[no weeks with more than 1 prize] +
Prob[1 week @2, no weeks more than 2] + Prob[2 weeks @2, no weeks more than 2] +
Prob[0 week @2, 1 week @3, no weeks more than 3] =

$(.8392^{52}) + (52)(.1249)(.8392^{51}) + ((52)(51)/2)(.1249^2)(.8392^{50}) + (52)(.0297)(.8392^{51}) =$

.00011 + .00085 + .00323 + .00020 = **0.00439.**

Comment: The lottery can not keep receipts from good weeks in order to build up the reserve fund.

3.21. C. Let n be the number of children in a family.
The probability that the child picked is in a family of size n is proportional to the product of the size of family and the proportion of families of that size: n f(n).

Thus, Prob[child is in a family of size n] = n f(n) / Σ n f(n) = n f(n) / E[N].

For n > 0, the number of siblings the child has is n - 1.

Thus the mean number of siblings is: $\frac{\sum_1 n\ f(n)\ (n-1)}{E[N]} = \frac{\sum_1 (n^2 - n)\ f(n)}{E[N]} = \frac{E[N^2] - E[N]}{E[N]} =$

$$\frac{E[N^2]}{E[N]} - 1 = \frac{Var[N] + E[N]^2}{E[N]} - 1 = \frac{Var[N]}{E[N]} + E[N] - 1 = \frac{mq(1-q)}{mq} + mq - 1 = 1 - q + mq - 1$$

= (m - 1)q = (3)(0.5) = **1.5**.

Alternately, assume for example 10,000 families.

Number of children	Binomial Density	Number of Families	Number of Children	Number of Siblings	Product of # of Children Times # of Siblings
0	0.0625	625	0	0	0
1	0.2500	2,500	2,500	0	0
2	0.3750	3,750	7,500	1	7,500
3	0.2500	2,500	7,500	2	15,000
4	0.0625	625	2,500	3	7,500
Total	1.0000	10,000	20,000	6	30,000

Mean of number of siblings for a child chosen at random is: 30,000 / 20,000 = **1.5**.
Comment: The average size family has two children; each of these children has one sibling.
However, a child chosen at random is more likely to be from a large family.

3.22. B. A Binomial Distribution with m = 10 and q = 0.3. $f(3) = \binom{10}{3} (0.3^3)\ (0.7^7)$.

3.23. D. The number of people who survive is Binomial with m = 100 and q = 0.9.
Mean = (100)(0.9) = 90. Variance = (100)(0.9)(0.1) = 9. Mean + Standard Deviation = **93**.

3.24. E. The number of deaths is sum of two Binomials, one with m = 30 and q = 0.1/3, and the other with m = 70 and q = 0.1.
The sum of their variances is: (30)(0.03333)(1 - 0.03333) + (70)(0.1)(0.9) = 0.967 + 6.3 = **7.267**.

3.25. A. Each X_i is Binomial with parameters n_i and p.

The sum is Binomial with parameters $n_1 + n_2 + n_3$ and p.

3.26. D. Binomial with m = 10 and q = 0.2. 1 - f(0) - f(1) = 1 - $.8^{10}$ - (10)(.2)($.8^9$) = **0.624**.

3.27. E. The sample variance is the average squared deviation from the mean; thus the sample variance is positive unless all the observations are equal. In this case, the sample variance is zero if and only if the two observations are equal. For this Binomial the chance of observing a given number of claims is:

number of claims:	0	1	2	3	4
probability:	1/16	4/16	6/16	4/16	1/16

Thus the chance that the two observations are equal is:

$(1/16)^2 + (4/16)^2 + (6/16)^2 + (4/16)^2 + (1/16)^2$ = 70/256 = **.273.**

Comment: For example, the chance of 3 claims is: ((m!)/{(3!)((m-3)!)}) $q^3(1-q)$ =

((4!)/{(3!)((1!)}) $.5^3(1-.5)$ = 4/16.

3.28. E. The number of passengers that show up for a flight is Binomial with m = 32 and q = 0.90. Prob[more show up than seats] = f(31) + f(32) = 32(0.1)(0.9^{31}) + 0.9^{32} = **0.1564**.

3.29. E. The number of hurricanes is Binomial with m = 20 and q = 0.05.

Prob[< 3 hurricanes] = f(0) + f(1) + f(2) = 0.95^{20} + 20(0.05)(0.95^{19}) + 190(0.05^2)(0.95^{18}) = **0.9245**.

3.30. E. Each group is Binomial with m = 10 and q = 0.8.

Prob[at least 9 complete] = f(9) + f(10) = 10(.2)(0.8^9) + 0.8^{10} = 0.376.

Prob[one group has at least 9 and one group does not] = (2)(.376)(1 - 0.376) = **0.469**.

3.31. E. The bus driver collects (21)(50) = 1050 for the 21 tickets he sells. However, he may be required to refund 100 to one passenger if all 21 ticket holders show up. The number of tourists who show up is Binomial with m = 21 and q = 0.98.

Expected penalty is: 100 f(21) = 100(0.98^{21}) = 65.425.

Expected revenue is: (21)(50) - 65.425 = **984.6**.

3.32. D. The fund will be inadequate if there are more than 120/C payments.
The number of payments is Binomial with m = 20 and q = .02.

x	f	F
0	0.66761	0.66761
1	0.27249	0.94010
2	0.05283	0.99293
3	0.00647	0.99940

There is a 1 - .94010 = 5.990% chance of needing more than one payment.
There is a 1 - .992930 = 0.707% chance of needing more than two payments.

Thus we need to require that the fund be able to make two payments. 120/C = 2. ⇒ C = **60**.

3.33. D. This is the sum of 25 independent Bernoullis, each with q = .04.
The number of losses per year is Binomial with m = 25 and q = .04.

$f(0) = (1 - q)^m = (1 - 0.04)^{25} = 0.3604$.

$f(1) = mq(1 - q)^{m-1} = (25)(0.04)(1 - 0.04)^{24} = 0.3754$.

$f(2) = \{m(m-1)/2!\}q^2(1 - q)^{m-2} = (25)(24/2)(0.04^2)(1 - 0.04)^{23} = 0.1877$.

$f(3) = \{m(m-1)(m-2)/3!\}q^3(1 - q)^{m-3} = (25)(24)(23/6)(0.04^3)(1 - 0.04)^{22} = 0.0600$.

Prob[at least 4] = 1 - {f(0) + f(1) + f(2) + f(3)} = 1 - 0.9835 = 0.0165.
4 or more risks have losses in the same year on average once in: 1/0.0165 = **60.6 years**.

3.34. C. A Binomial Distribution with m = 60 and q = .04.

$f(0) = 0.96^{60} = 0.08635$. $f(1) = (60)(0.04)0.96^{59} = 0.21588$.

$f(2) = \{(60)(59)/2\}(0.04^2)0.96^{58} = 0.26535$. $f(3) = \{(60)(59)(58)/6\}(0.04^3)0.96^{57} = 0.21376$.

$f(4) = \{(60)(59)(58)(57)/24\}(0.04^4)0.96^{56} = 0.12692$.

1 - f(0) - f(1) - f(2) - f(3) - f(4) = 1 - 0.08635 - 0.21588 - 0.26535 - 0.21376 - 0.12692 = 0.09174.
1/0.09174 = **Once every 11 years**.

3.35. A. Assume n experiments are run. Then the probability of no successes is 0.98^n.

Thus the probability of at least one success is: $1 - 0.98^n$.

Expected profit is:

$(1,000,000)(1 - 0.98^n) - 2900n - 500,000 = 500,000 - (1,000,000)0.98^n - 2900n$.

Setting the derivative with respect to n equal to zero:

$0 = -\ln(0.98)(1,000,000)0.98^n - 2900. \Rightarrow 0.98^n = .143545. \Rightarrow n = 96.1$.

Taking n = 96, the expected profit is **77,818**.
Comment: For n = 97, the expected profit is 77,794.

3.36. D. (9000 - 8000)/(9000 - 7000) = 1/2. Half the scientists are “large”.
Therefore, thinning the original Binomial, the number of large scientist is Binomial with m = 8 and q = 0.6/2 = 0.3. f(2) = {(8)(7)/2} $(.7^6)(.3^2)$ = **0.2965**.
Alternately, this is a compound frequency distribution, with primary distribution a Binomial with q = 0.6 and m = 8, and secondary distribution a Bernoulli with q = 1/2 (half chance a scientist is large.)
One can use the Panjer Algorithm. For the primary Binomial Distribution,
a = -q/(1-q) = -.6/.4 = -1.5. b = (m+1)q/(1-q) = (9)(1.5) = 13.5. P(z) = $\{1 + q(z-1)\}^m$.
c(0) = $P_p(s(0)) = P_p(.5) = \{1 + (.6)(.5-1)\}^8$ = .057648.

$$c(x) = \frac{1}{1 - a\, s(0)} \sum_{j=1}^{x} (a + jb/x)\, s(j)\, c(x-j) = (1/1.75) \sum_{j=1}^{x} (-1.5 + j13.5/x)\, s(j)\, c(x-j).$$

c(1) = (1/1.75)(-1.5 + 13.5) s(1) c(0) = (1/1.75)(12)(1/2)(.057648) = .197650.
c(2) = (1/1.75){(-1.5 + 13.5/2) s(1) c(1) + (-1.5 + (2)13.5/2) s(2) c(0)} =
(1/1.75){(5.25)(1/2)(.197650) + (12)(0)(.057648)} = **0.296475**.
Alternately, one can list all the possibilities:

Number of Scientist	Binomial Probability	Given the number of Scientist, the Probability that exactly two are large	Extension
0	0.00066	0	0.00000
1	0.00786	0	0.00000
2	0.04129	0.25	0.01032
3	0.12386	0.375	0.04645
4	0.23224	0.375	0.08709
5	0.27869	0.3125	0.08709
6	0.20902	0.234375	0.04899
7	0.08958	0.1640625	0.01470
8	0.01680	0.109375	0.00184
Sum	1.00000		**0.29648**

For example if 6 scientist have been eaten, then the chance that exactly two of them are large is:
(0.5^6) 6! / (4!2!) = 0.234375. In algebraic form, this solution is:

$$\sum_{n=2}^{n=8} \frac{8!}{n!\,(8-n)!}\, 0.6^n\, 0.4^{8-n}\, \frac{n!}{2!\,(n-2)!}\, 0.5^n = (1/2) \sum_{n=2}^{n=8} \frac{8!}{(n-2)!\,(8-n)!}\, 0.3^n\, 0.4^{8-n}$$

$$= (1/2)(8)(7)(0.3^2) \sum_{i=0}^{i=6} \frac{6!}{i!\,(6-i)!}\, 0.3^i\, 0.4^{6-i} = (28)(0.09)(0.3 + 0.4)^6 = \mathbf{0.2965}.$$

<u>Comment</u>: The Panjer Algorithm (Recursive Method) is discussed in
“Mahler’s Guide to Aggregate Distributions.”

"two or more scientists are eaten and exactly two of those eaten have at least 8000 calories each"

⇔ exactly two "large" scientists are eaten as well as some unknown number of "small" scientists.

At least 2 claims of which exactly two are large.

⇔ exactly 2 large claims and some unknown number of small claims.

3.37. A. One year is Binomial Distribution with m = 20 and q = 0.05.
The years are independent of each other.
Therefore, the number of failures over 2 years is Binomial Distribution with m = 40 and q = 0.05.
Prob[0 or 1 failures] = $0.95^{40} + (40)(0.95^{39})(0.05)$ = **39.9%**.
<u>Comment</u>: In this question, when a TV fails it is replaced. Therefore, we can have a failure in both years for a given customer. A somewhat different question than asked would be,
assuming each customer owns one set, calculate the probability that no more than one customer suffers a failure during the two years.
For a given customer, the probability of no failure in the first two years is: $0.95^2 = 0.9025$.
The probability of 0 or 1 customer suffering a failure is:
$0.9025^{20} + (20)(0.0975)(0.9025^{19})$ = 40.6%.

Section 4, Poisson Distribution

The Poisson Distribution is the single most important frequency distribution to study for the exam.[14]
The density function for the Poisson is:

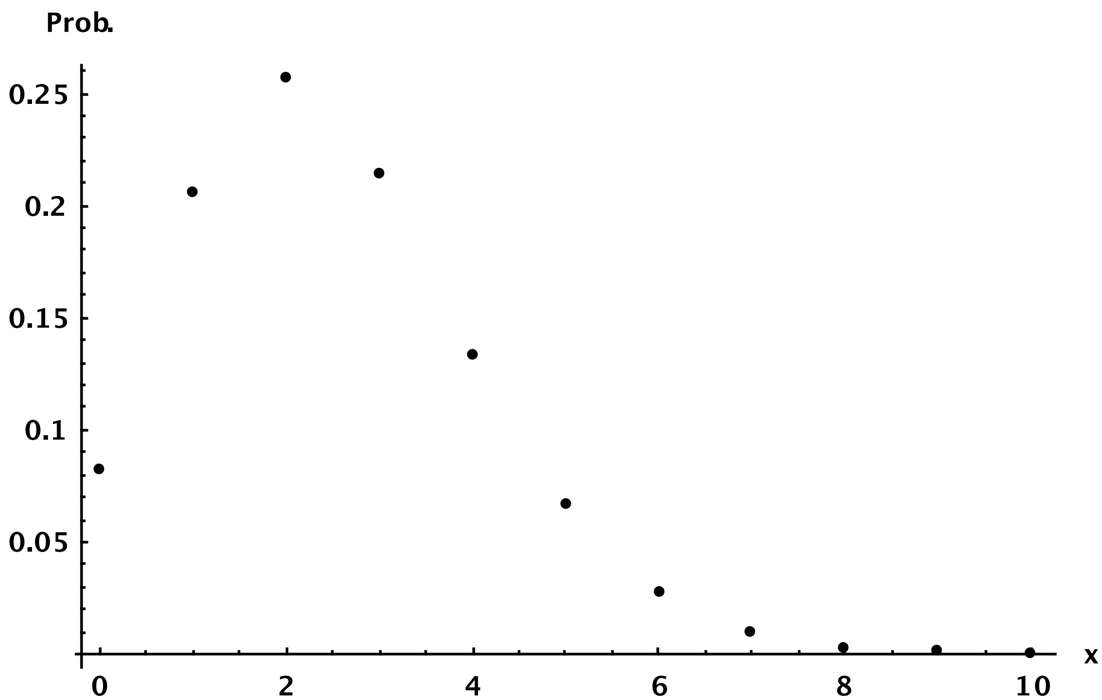
$f(x) = \lambda^x e^{-\lambda} / x!, x \geq 0.$

Note that unlike the Binomial, the Poisson density function is positive for all $x \geq 0$; there is no limit on the possible number of claims. The Poisson has a single parameter λ.

The Distribution Function is 1 at infinity since $\lambda^x / x!$ is the series for e^λ.

For example, here's a Poisson for $\lambda = 2.5$:

n	0	1	2	3	4	5	6	7	8	9	10
f(n)	0.082	0.205	0.257	0.214	0.134	0.067	0.028	0.010	0.003	0.001	0.000
F(n)	0.082	0.287	0.544	0.758	0.891	0.958	0.986	0.996	0.999	1.000	1.000



For example, the chance of 4 claims is: $f(4) = \lambda^4 e^{-\lambda} / 4! = 2.5^4 e^{-2.5} / 4! = .1336$.

Remember, there is a small chance of a very large number of claims.
For example, $f(15) = 2.5^{15} e^{-2.5} / 15! = 6 \times 10^{-8}$.
Such large numbers of claims can contribute significantly to the higher moments of the distribution.

[14] The Poisson comes up among other places in the Gamma-Poisson frequency process, to be discussed in a subsequent section.

Let's calculate the first two moments for this Poisson distribution with λ= 2.5:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Distribution Function
0	0.08208500	0.00000000	0.00000000	0.08208500
1	0.20521250	0.20521250	0.20521250	0.28729750
2	0.25651562	0.51303124	1.02606248	0.54381312
3	0.21376302	0.64128905	1.92386716	0.75757613
4	0.13360189	0.53440754	2.13763017	0.89117802
5	0.06680094	0.33400471	1.67002357	0.95797896
6	0.02783373	0.16700236	1.00201414	0.98581269
7	0.00994062	0.06958432	0.48709021	0.99575330
8	0.00310644	0.02485154	0.19881233	0.99885975
9	0.00086290	0.00776611	0.06989496	0.99972265
10	0.00021573	0.00215725	0.02157252	0.99993837
11	0.00004903	0.00053931	0.00593244	0.99998740
12	0.00001021	0.00012257	0.00147085	0.99999762
13	0.00000196	0.00002554	0.00033196	0.99999958
14	0.00000035	0.00000491	0.00006875	0.99999993
15	0.00000006	0.00000088	0.00001315	0.99999999
16	0.00000001	0.00000015	0.00000234	1.00000000
17	0.00000000	0.00000002	0.00000039	1.00000000
18	0.00000000	0.00000000	0.00000006	1.00000000
19	0.00000000	0.00000000	0.00000001	1.00000000
20	0.00000000	0.00000000	0.00000000	1.00000000
Sum	1.00000000	2.50000000	8.75000000	

The mean is 2.5 = λ. The variance is: $E[X^2] - E[X]^2 = 8.75 - 2.5^2 = 2.5 = \lambda$.

In general, **the mean of the Poisson is λ and the variance is λ.**

In this case the mode is 2, since f(2) = .2565, larger than any other value of the probability density function. In general, the mode of the Poisson is the largest integer in λ.[15] This follows from the fact that for the Poisson f(x+1) / f(x) = λ / (x+1). Thus for the Poisson the mode is less than or equal to the mean λ.

The median in this case is 2, since F(2) = .544 ≥ .5, while F(1) = .287 < .5. The median as well as the mode are less than the mean, which is typical for distributions skewed to the right.

[15] If λ is an integer then f(λ) = f(λ-1), and both λ and λ-1 are modes.

Claim Intensity, Derivation of the Poisson:

Assume one has a claim intensity of ξ. The chance of having a claim over an extremely small period of time Δt is approximately $\xi(\Delta t)$. (The claim intensity is analogous to the force of mortality in Life Contingencies.) If the claim intensity is a constant over time and the chance of having a claim in any interval is independent of the chance of having a claim in any other disjoint interval, then the number of claims observed over a period time t is given by a Poisson Distribution, with parameter ξt.

A Poisson is characterized by a constant independent claim intensity and vice versa. For example, if the chance of a claim each month is 0.1%, and months are independent of each other, the distribution of number of claims over a 5 year period (60 months) is Poisson with mean = 6%. For the Poisson, the parameter λ = mean =
(claim intensity)(total time covered). Therefore, if for example one has a Poisson in each of five years with parameter λ, then over the entire 5 year period one has a Poisson with parameter 5λ.

Adding Poissons:

The sum of two independent variables each of which is Poisson with parameters λ_1 and λ_2 is also Poisson, with parameter $\lambda_1 + \lambda_2$.[16] This follows from the fact that for a very small time interval the chance of a claim is the sum of the chance of a claim from either variable, since they are independent.[17] If the total time interval is one, then the chance of a claim from either variable over a very small time interval Δt is $\lambda_1 \Delta t + \lambda_2 \Delta t = (\lambda_1 + \lambda_2)\Delta t$. Thus the sum of the variables has constant claim intensity $(\lambda_1 + \lambda_2)$ over a time interval of one, and is therefore a Poisson with parameter $\lambda_1 + \lambda_2$.

For example, the sum of a two independent Poisson variables with means 3% and 5%
is a Poisson variable with mean 8%. So if a portfolio consists of one risk Poisson with mean 3% and one risk Poisson with mean 5%, the number of claims observed for the whole portfolio is Poisson with mean 8%.

[16] See Theorem 6.1 in Loss Models.

[17] This can also be shown from simple algebra, by summing over i + j = k the terms $(\lambda^i e^{-\lambda} / i!) (\mu^j e^{-\mu} / j!) = e^{-\lambda+\mu} (\lambda^i \mu^j / i!\, j!)$. By the Binomial Theorem, these terms sum to $e^{-\lambda+\mu} (\lambda+\mu)^k / k!$.

Exercise: Assume one had a portfolio of 25 exposures. Assume each exposure has an independent Poisson frequency process, with mean 3%. What is the frequency distribution for the claims from the whole portfolio?
[Solution: A Poisson Distribution with mean: (25)(3%) = 0.75.]

If one has a large number of independent events each with a small probability of occurrence, then the number of events that occurs has approximately a constant claims intensity and is thus approximately Poisson Distributed. Therefore the Poisson Distribution can be useful in modeling such situations.

Thinning a Poisson:[18]

Sometimes one selects only some of the claims. This is sometimes referred to as "thinning" the Poisson distribution. For example, **if frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a Poisson.**

For example, assume that we have a Poisson with mean frequency of 30 and that the size of loss distribution is such that 20% of the losses are greater than $1 million (in constant dollars). Then the number of losses observed greater than $1 million (in constant dollars) is also Poisson but with a mean of (20%)(30) = 6. Similarly, losses observed smaller than $1 million (in constant dollars) is also Poisson, but with a mean of (80%)(30) = 24.

Exercise: Frequency is Poisson with $\lambda = 5$.
Sizes of loss are Exponential with mean = 100. $F(x) = 1 - e^{-x/100}$.
Frequency and severity are independent.
What is the distribution of the number of losses of size less than 50?
What is the distribution of the number of losses of size more than 200?
What is the distribution of the number of losses of size between 50 and 200?
[Solution: For the Exponential, $F(50) = 1 - e^{-50/100} = 0.393$.
Thus the number of small losses are Poisson with mean: 5 F(50) = (5)(0.393) = 1.97.
For the Exponential, $1 - F(200) = e^{-200/100} = 0.135$.
Thus the number of large losses are Poisson with mean: (0.135)(5) = 0.68.
For the Exponential, $F(200) - F(50) = e^{-0.5} - e^{-2} = 0.471$.
Thus the number of medium sized losses are Poisson with mean: (0.471)(5) = 2.36.
Comment: As will be discussed, these three Poisson Distributions are independent of each other.]

[18] See Theorem 6.2 in Loss Models.
See for example, 3, 5/00, Q.2.

In this example, the total number of losses are Poisson and therefore has a constant independent claims intensity of 5. Since frequency and severity are independent, the large losses also have a constant independent claims intensity of 5 {1 - F(200)}, which is therefore Poisson with this mean. Similarly, the small losses have constant independent claims intensity of 5F(50) and therefore are Poisson. Also, these two processes are independent of each other.

If in this example we had a deductible of 200, then only losses of size greater than 200 would result in a (non-zero) payment. Loss Models refers to the number of payments as N^P, in contrast to N^L the number of losses.[19] In this example, N^L is Poisson with mean 5, while for a 200 deductible N^P is Poisson with mean: 5 {1 - F(200)} = (5)(0.135) = 0.68.

Thinning a Poisson based on size of loss is a special case of decomposing Poisson frequencies. The key idea is that there is some way to divide the claims up into mutually exclusive types that are independent. **Then each type is also Poisson, and the Poisson Distributions for each type are independent.**

Exercise: Claim frequency follows a Poisson Distribution with a mean of 20% per year. 1/4 of all claims involve attorneys. If attorney involvement is independent between different claims, what is the probability of getting 2 claims involving attorneys in the next year?
[Solution: Claims with attorney involvement are Poisson with mean frequency 20%/4 = 5%.
Thus $f(2) = (0.05)^2 e^{-0.05} / 2! = 0.00119$.]

Derivation of Results for Thinning Poissons:[20]

If losses are Poisson with mean λ, and one selects a portion, t, of the losses in a manner independent of the frequency, then the selected losses are also Poisson but with mean λt.

$$\text{Prob[\# selected losses = n]} = \sum_{m=n}^{\infty} \text{Prob[m total \# losses] Prob[n of m losses are selected]}$$

$$= \sum_{m=n}^{\infty} \frac{e^{-\lambda} \lambda^m}{m!} \frac{t^n (1-t)^{m-n} m!}{n! (m-n)!} = \frac{e^{-\lambda} t^n \lambda^n}{n!} \sum_{m=n}^{\infty} \frac{\lambda^{m-n} (1-t)^{m-n}}{(m-n)!} =$$

$$\frac{e^{-\lambda} t^n \lambda^n}{n!} \sum_{i=0}^{\infty} \lambda^i (1-t)^i / i! = \frac{e^{-\lambda} t^n \lambda^n}{n!} e^{\lambda(1-t)} = e^{-\lambda t}(t\lambda)^n/n! = f(n) \text{ for a Poisson with mean } t\lambda.$$

In a similar manner, the number not selected follows a Poisson with mean $(1-t)\lambda$.

[19] I do not regard this notation as particularly important, although it is possible that it will be used in an exam question. See Section 8.6 of Loss Models.
[20] I previously discussed how these results follow from the constant, independent claims intensity.

Prob[# selected losses = n | # not selected losses = j] =
Prob[total # = n + j and # not selected losses = j] / Prob[# not selected losses = j] =

$$\frac{\text{Prob[total \# = n + j] Prob[\# not selected losses = j | total \# = n + j]}}{\text{Prob[\# not selected losses = j]}} =$$

$$\frac{\frac{e^{-\lambda}\lambda^{n+j}}{(n+j)!}\ \frac{(1-t)^j t^n (n+j)!}{n!\ j!}}{\frac{e^{-(1-t)\lambda}\{(1-t)\lambda\}^j}{j!}} = e^{-\lambda t}(t\lambda)^n/n! =$$

f(n) for a Poisson with mean $t\lambda$ = Prob[# selected losses = n].
Thus the number selected and the number not selected are independent. They are independent Poisson distributions. The same result follows when dividing into more than 2 disjoint subsets.

Effect of Exposures:[21]

Assume one has 100 exposures with independent, identically distributed frequency distributions. If each one is Poisson, then so is the sum, with mean 100λ. If we change the number of exposures to for example 150, then the sum is Poisson with mean 150λ, or 1.5 times the mean in the first case. In general, as the exposures change, the distribution remains Poisson with the mean changing in proportion.

Exercise: The total number of claims from a portfolio of private passenger automobile insured has a Poisson Distribution with $\lambda = 60$. If next year the portfolio has only 80% of the current exposures, what is its frequency distribution?
[Solution: Poisson with $\lambda = (.8)(60) = 48$.]

This same result holds for a Compound Frequency Distribution, to be discussed subsequently, with a primary distribution that is Poisson.

[21] See Section 7.4 of Loss Models, not on the syllabus.

Poisson Distribution

Support: x = 0, 1, 2, 3, ... Parameters: $\lambda > 0$

D. f. : *$F(x) = 1 - \Gamma(x+1 ; \lambda)$* *Incomplete Gamma Function*[22]

P. d. f. : **$f(x) = \lambda^x e^{-\lambda} / x!$**

Mean = λ **Variance = λ** **Variance / Mean = 1.**

Coefficient of Variation = $\frac{1}{\sqrt{\lambda}}$ Skewness = $\frac{1}{\sqrt{\lambda}}$ = CV. *Kurtosis = $3 + 1/\lambda = 3 + CV^2$.*

Mode = largest integer in λ (if λ is an integer then both λ and λ-1 are modes.)

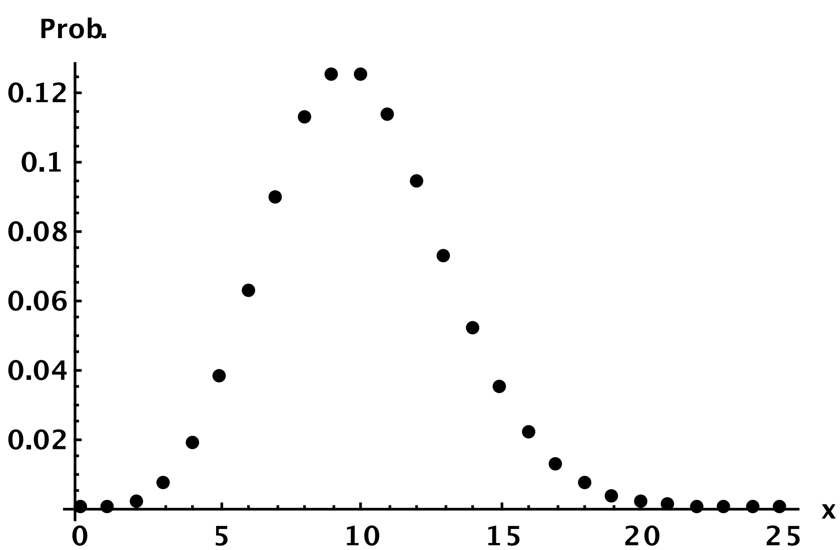
nth Factorial Moment = λ^n.

Probability Generating Function: **$P(z) = e^{\lambda(z-1)}$**, $\lambda > 0$.

Moment Generating Function: $M(s) = \exp[\lambda(e^s - 1)]$.

$f(x+1) / f(x) = a + b / (x+1)$, $a = 0$, $b = \lambda$, $f(0) = e^{-\lambda}$.

A Poisson Distribution for $\lambda = 10$:



[22] x+1 is the shape parameter of the Incomplete Gamma which is evaluated at the point λ. Thus one can get the sum of terms for the Poisson Distribution by using the Incomplete Gamma Function.

Problems:

Use the following information for the next five questions:

The density function for n is: $f(n) = 6.9^n e^{-6.9} / n!$, $n = 0, 1, 2, ...$

4.1 (1 point) What is the mean of the distribution?

A. less than 6.9
B. at least 6.9 but less than 7.0
C. at least 7.0 but less than 7.1
D. at least 7.1 but less than 7.2
E. at least 7.2

4.2 (1 point) What is the variance of the distribution?

A. less than 6.9
B. at least 6.9 but less than 7.0
C. at least 7.0 but less than 7.1
D. at least 7.1 but less than 7.2
E. at least 7.2

4.3 (2 points) What is the chance of having less than 4 claims?

A. less than 9%
B. at least 9% but less than 10%
C. at least 10% but less than 11%
D. at least 11% but less than 12%
E. at least 12%

4.4 (2 points) What is the mode of the distribution?

A. 5 B. 6 C. 7 D. 8 E. None of the Above.

4.5 (2 points) What is the median of the distribution?

A. 5 B. 6 C. 7 D. 8 E. None of the Above.

4.6 (2 points) The male drivers in the State of Grace each have their annual claim frequency given by a Poisson distribution with parameter equal to 0.05.
The female drivers in the State of Grace each have their annual claim frequency given by a Poisson distribution with parameter equal to 0.03.
You insure in the State of Grace 20 male drivers and 10 female drivers.
Assume the claim frequency distributions of the individual drivers are independent.
What is the chance of observing 3 claims in a year?
A. less than 9.6%
B. at least 9.6% but less than 9.7%
C. at least 9.7% but less than 9.8%
D. at least 9.8% but less than 9.9%
E. at least 9.9 %

4.7 (2 points) Assume that the frequency of hurricanes hitting the State of Windiana is given by a Poisson distribution, with an average annual claim frequency of 82%. Assume that the losses in millions of constant 1998 dollars from such a hurricane are given by a Pareto Distribution with $\alpha = 2.5$ and $\theta = 400$ million. Assuming frequency and severity are independent, what is chance of two or more hurricanes each with more than $250 million (in constant 1998 dollars) of loss hitting the State of Windiana next year?
(There may or may not be hurricanes of other sizes.)
A. less than 2.1%
B. at least 2.1% but less than 2.2%
C. at least 2.2% but less than 2.3%
D. at least 2.3% but less than 2.4%
E. at least 2.4%

Use the following information for the next 3 questions:
The claim frequency follows a Poisson Distribution with a mean of 10 claims per year.

4.8 (2 points) What is the chance of having more than 5 claims in a year?
A. 92% B. 93% C. 94% D. 95% E. 96%

4.9 (2 points) What is the chance of having more than 8 claims in a year?
A. 67% B. 69% C. 71% D. 73% E. 75%

4.10 (1 point) What is the chance of having 6, 7, or 8 claims in a year?
A. 19% B. 21% C. 23% D. 25% E. 27%

4.11 (2 points) You are given the following:
- Claims follow a Poisson Distribution, with a mean of 27 per year.
- The size of claims are given by a Weibull Distribution with $\theta = 1000$ and $\tau = 3$.
- Frequency and severity are independent.

Given that during a year there are 7 claims of size less than 500, what is the expected number of claims of size greater than 500 during that year?
(A) 20 (B) 21 (C) 22 (D) 23 (E) 24

4.12 (1 point) Frequency follows a Poisson Distribution with $\lambda = 7$.
20% of losses are of size greater than $50,000.
Frequency and severity are independent.
Let N be the number of losses of size greater than $50,000.
What is the probability that N = 3?
A. less than 9%
B. at least 9% but less than 10%
C. at least 10% but less than 11%
D. at least 11% but less than 12%
E. at least 12%

4.13 (1 point) N follows a Poisson Distribution with $\lambda = 0.1$. What is Prob[N = 1 | N ≤ 1]?
A. 8% B. 9% C. 10% D. 11% E. 12%

4.14 (1 point) N follows a Poisson Distribution with $\lambda = 0.1$. What is Prob[N = 1 | N ≥ 1]?
A. 91% B. 92% C. 93% D. 94% E. 95%

4.15 (2 points) N follows a Poisson Distribution with $\lambda = 0.2$. What is E[1/(N+1)]?
A. less than 0.75
B. at least 0.75 but less than 0.80
C. at least 0.80 but less than 0.85
D. at least 0.85 but less than 0.90
E. at least 0.90

4.16 (2 points) N follows a Poisson Distribution with $\lambda = 2$. What is E[N | N > 1]?
A. 2.6 B. 2.7 C. 2.8 D. 2.9 E. 3.0

4.17 (2 points) The total number of claims from a book of business with 500 exposures has a Poisson Distribution with $\lambda = 27$. Next year, this book of business will have 600 exposures. Next year, what is the probability of this book of business having a total of 30 claims?

A. 5.8% B. 6.0% C. 6.2% D. 6.4% E. 6.6%

Use the following information for the next two questions:

N follows a Poisson Distribution with $\lambda = 1.3$. Define $(N-j)_+ = n-j$ if $n \geq j$, and 0 otherwise.

4.18 (2 points) Determine $E[(N-1)_+]$.

A. 0.48 B. 0.51 C. 0.54 D. 0.57 E. 0.60

4.19 (2 points) Determine $E[(N-2)_+]$.

A. 0.19 B. 0.20 C. 0.21 D. 0.22 E. 0.23

4.20 (2 points) The total number of non-zero payments from a policy with a $500 deductible follows a Poisson Distribution with $\lambda = 3.3$.

The ground up losses follow a Weibull Distribution with $\tau = 0.7$ and $\theta = 2000$.

If this policy instead had a $1000 deductible, what would be the probability of having 4 non-zero payments?

A. 14% B. 15% C. 16% D. 17% E. 18%

4.21 (3 points) The number of major earthquakes that hit the state of Allshookup is given by a Poisson Distribution with 0.05 major earthquakes expected per year.

- Allshookup establishes a fund that will pay 1000/major earthquake.
- The fund charges an annual premium, payable at the start of each year, of 60.
- At the start of this year (before the premium is paid) the fund has 300.
- Claims are paid immediately when there is a major earthquake.
- If the fund ever runs out of money, it immediately ceases to exist.
- Assume no investment income and no expenses.

What is the probability that the fund is still functioning in 40 years?

A. Less than 40%
B. At least 40%, but less than 41%
C. At least 41%, but less than 42%
D. At least 42%, but less than 43%
E. At least 43%

4.22 (2 points) You are given the following:

- A business has bought a collision policy to cover its fleet of automobiles.
- The number of collision losses per year follows a Poisson Distribution.
- The size of collision losses follows an Exponential Distribution with a mean of 600.
- Frequency and severity are independent.
- This policy has an ordinary deductible of 1000 per collision.
- The probability of no payments on this policy during a year is 74%.

Determine the probability that of the collision losses this business has during a year, exactly three of them result in no payment on this policy.

(A) 8% (B) 9% (C) 10% (D) 11% (E) 12%

4.23 (2 points) A Poisson Distribution has a coefficient of variation 0.5.
Determine the probability of exactly seven claims.

(A) 4% (B) 5% (C) 6% (D) 7% (E) 8%

4.24 (2 points) X and Y are each Poisson random variables.
Sample values of X and Y are drawn together in pairs.

$Z_i = X_i + Y_i$.

E[Z] = 10. Var[Z] = 12.

Find the covariance of X and Y.

A. 0 B. 0.5 C. 1.0 D. 1.5 E. 2.0

4.25 (2 points) The size of losses prior to the effect of any deductible follow a Pareto Distribution with $\alpha = 3$ and $\theta = 1000$.

With a deductible of 100, the number of non-zero payments is Poisson with mean 5.
If instead there is a deductible of 250, what is the probability of exactly 3 non-zero payments?

(A) 16% (B) 18% (C) 20% (D) 22% (E) 24%

4.26 (2 points) The number of losses is Poisson with mean 10.
Loss sizes follow an Exponential Distribution with mean 400.
The number of losses and the size of those losses are independent.
What is the probability of exactly 5 losses of size less than 200 and exactly 6 losses of size greater than 200?

(A) 2.0% (B) 2.5% (C) 3.0% (D) 3.5% (E) 4.0%

4.27 (2, 5/83, Q.4) (1.5 points) If $\overline{X}$ is the mean of a random sample of size n from a Poisson distribution with parameter λ, then which of the following statements is true?

A. $\overline{X}$ has a Normal distribution with mean λ and variance λ.

B. $\overline{X}$ has a Normal distribution with mean λ and variance λ/n.

C. $\overline{X}$ has a Poisson distribution with parameter λ.

D. $n\overline{X}$ has a Poisson distribution with parameter λ^n.

E. $n\overline{X}$ has a Poisson distribution with parameter nλ.

4.28 (2, 5/83, Q.28) (1.5 points) The number of traffic accidents per week in a small city has a Poisson distribution with mean equal to 3.
What is the probability of exactly 2 accidents in 2 weeks?

A. $9e^{-6}$ B. $18e^{-6}$ C. $25e^{-6}$ D. $4.5e^{-3}$ E. $9.5e^{-3}$

4.29 (2, 5/83, Q.45) (1.5 points) Let X have a Poisson distribution with parameter λ = 1.
What is the probability that X ≥ 2, given that X ≤ 4?

A. 5/65 B. 5/41 C. 17/65 D. 17/41 E. 3/5

4.30 (2, 5/85, Q.9) (1.5 points) The number of automobiles crossing a certain intersection during any time interval of length t minutes between 3:00 P.M. and 4:00 P.M. has a Poisson distribution with mean t. Let W be time elapsed after 3:00 P.M. before the first automobile crosses the intersection. What is the probability that W is less than 2 minutes?

A. $1 - 2e^{-1} - e^{-2}$ B. e^{-2} C. $2e^{-1}$ D. $1 - e^{-2}$ E. $2e^{-1} + e^{-2}$

4.31 (2, 5/85, Q.16) (1.5 points) In a certain communications system, there is an average of 1 transmission error per 10 seconds. Let the distribution of transmission errors be Poisson.
What is the probability of more than 1 error in a communication one-half minute in duration?

A. $1 - 2e^{-1}$ B. $1 - e^{-1}$ C. $1 - 4e^{-3}$ D. $1 - 3e^{-3}$ E. $1 - e^{-3}$

4.32 (2, 5/88, Q.49) (1.5 points) The number of power surges in an electric grid has a Poisson distribution with a mean of 1 power surge every 12 hours.
What is the probability that there will be no more than 1 power surge in a 24-hour period?

A. $2e^{-2}$ B. $3e^{-2}$ C. $e^{-1/2}$ D. $(3/2)e^{-1/2}$ E. $3e^{-1}$

4.33 (4, 5/88, Q.48) (1 point) An insurer's portfolio is made up of 3 independent policyholders with expected annual frequencies of 0.05, 0.1, and 0.15.
Assume that each insured's number of claims follows a Poisson distribution.
What is the probability that the insurer experiences fewer than 2 claims in a given year?
A. Less than 0.9
B. At least 0.9, but less than 0.95
C. At least 0.95, but less than 0.97
D. At least 0.97, but less than 0.99
E. Greater than 0.99

4.34 (2, 5/90, Q.39) (1.7 points) Let X, Y, and Z be independent Poisson random variables with E(X) = 3, E(Y) = 1, and E(Z) = 4. What is $P[X + Y + Z \leq 1]$?
A. $13e^{-12}$ B. $9e^{-8}$ C. $(13/12)e^{-1/12}$ D. $9e^{-1/8}$ E. $(9/8)e^{-1/8}$

4.35 (4B, 5/93, Q.1) (1 point) You are given the following:
• A portfolio consists of 10 independent risks.
• The distribution of the annual number of claims for each risk in the portfolio is given by a Poisson distribution with mean $\mu = 0.1$.
Determine the probability of the portfolio having more than 1 claim per year.
A. 5% B. 10% C. 26% D. 37% E. 63%

4.36 (4B, 11/94, Q.19) (3 points) The density function for a certain parameter, α, is

$f(\alpha) = 4.6^{\alpha}e^{-4.6} / \alpha!$, $\alpha = 0, 1, 2, ...$

Which of the following statements are true concerning the distribution function for α?
1. The mode is less than the mean.
2. The variance is greater than the mean.
3. The median is less than the mean.
A. 1 B. 2 C. 3 D. 1, 2 E. 1, 3

4.37 (4B, 5/95, Q.9) (2 points) You are given the following:
• The number of claims for each risk in a group of identical risks follows a Poisson distribution.
• The expected number of risks in the group that will have no claims is 96.
• The expected number of risks in the group that will have 2 claims is 3.
Determine the expected number of risks in the group that will have 4 claims.
A. Less than .01
B. At least .01, but less than .05
C. At least .05, but less than .10
D. At least .10, but less than .20
E. At least .20

4.38 (2, 2/96, Q.21) *(1.7 points) Let X be a Poisson random variable with E(X) = ln(2).*
Calculate E[cos(πx)].
A. 0 B. 1/4 C. 1/2 D. 1 E. 2ln(2)

4.39 (4B, 11/98, Q.1) (1 point) You are given the following:
- The number of claims follows a Poisson distribution.
- Claim sizes follow a Pareto distribution.

Determine the type of distribution that the number of claims with sizes greater than 1,000 follows.
A. Poisson B. Pareto C. Gamma D. Binomial E. Negative Binomial

4.40 (4B, 11/98, Q.2) (2 points) The random variable X has a Poisson distribution with mean n - 1/2, where n is a positive integer greater than 1. Determine the mode of X.
A. n-2 B. n-1 C. n D. n+1 E. n+2

4.41 (4B, 11/98, Q.18) (2 points) The number of claims per year for a given risk follows a distribution with probability function $p(n) = \lambda^n e^{-\lambda} / n!$, n = 0, 1,..., $\lambda > 0$.
Determine the smallest value of λ for which the probability of observing three or more claims during two given years combined is greater than 0.1.
A. Less than 0.7
B. At least 0.7, but less than 1.0
C. At least 1.0, but less than 1.3
D. At least 1.3, but less than 1.6
E. At least 1.6

4.42 (4B, 5/99, Q.8) (3 points) You are given the following:
- Each loss event is either an aircraft loss or a marine loss.
- The number of aircraft losses has a Poisson distribution with a mean of 0.1 per year. Each loss is always 10,000,000.
- The number of marine losses has a Poisson distribution with a mean of 0.2 per year. Each loss is always 20,000,000.
- Aircraft losses occur independently of marine losses.
- From the first two events each year, the insurer pays the portion of the combined losses that exceeds 10,000,000.

Determine the insurer's expected annual payments.
A. Less than 1,300,000
B. At least 1,300,000, but less than 1,800,000
C. At least 1,800,000, but less than 2,300,000
D. At least 2,300,000, but less than 2,800,000
E. At least 2,800,000

4.43 (IOA 101, 4/00, Q.5) (2.25 points) An insurance company's records suggest that experienced drivers (those aged over 21) submit claims at a rate of 0.1 per year, and inexperienced drivers (those 21 years old or younger) submit claims at a rate of 0.15 per year.
A driver can submit more than one claim a year.
The company has 40 experienced and 20 inexperienced drivers insured with it.
The number of claims for each driver can be modeled by a Poisson distribution, and claims are independent of each other.
Calculate the probability the company will receive three or fewer claims in a year.

4.44 (1, 5/00, Q.24) (1.9 points) An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims.
If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?
(A) $1/\sqrt{3}$ (B) 1 (C) $\sqrt{2}$ (D) 2 (E) 4

4.45 (3, 5/00, Q.2) (2.5 points) Lucky Tom finds coins on his way to work at a Poisson rate of 0.5 coins/minute. The denominations are randomly distributed:
(i) 60% of the coins are worth 1;
(ii) 20% of the coins are worth 5; and
(iii) 20% of the coins are worth 10.
Calculate the conditional expected value of the coins Tom found during his one-hour walk today, given that among the coins he found exactly ten were worth 5 each.
(A) 108 (B) 115 (C) 128 (D) 165 (E) 180

4.46 (1, 11/00, Q.23) (1.9 points) A company buys a policy to insure its revenue in the event of major snowstorms that shut down business. The policy pays nothing for the first such snowstorm of the year and 10,000 for each one thereafter, until the end of the year.
The number of major snowstorms per year that shut down business is assumed to have a Poisson distribution with mean 1.5.
What is the expected amount paid to the company under this policy during a one-year period?
(A) 2,769 (B) 5,000 (C) 7,231 (D) 8,347 (E) 10,578

4.47 (3, 11/00, Q.29) (2.5 points) Job offers for a college graduate arrive according to a Poisson process with mean 2 per month. A job offer is acceptable if the wages are at least 28,000.
Wages offered are mutually independent and follow a lognormal distribution,
with $\mu = 10.12$ and $\sigma = 0.12$.
Calculate the probability that it will take a college graduate more than 3 months to receive an acceptable job offer.
(A) 0.27 (B) 0.39 (C) 0.45 (D) 0.58 (E) 0.61

4.48 (1, 11/01, Q.19) (1.9 points) A baseball team has scheduled its opening game for April 1.
If it rains on April 1, the game is postponed and will be played on the next day that it does not rain.
The team purchases insurance against rain. The policy will pay 1000 for each day, up to 2 days, that the opening game is postponed. The insurance company determines that the number of consecutive days of rain beginning on April 1 is a Poisson random variable with mean 0.6.
What is the standard deviation of the amount the insurance company will have to pay?
(A) 668 (B) 699 (C) 775 (D) 817 (E) 904

4.49 (CAS3, 11/03, Q.31) (2.5 points) Vehicles arrive at the Bun-and-Run drive-thru at a Poisson rate of 20 per hour. On average, 30% of these vehicles are trucks.
Calculate the probability that at least 3 trucks arrive between noon and 1:00 PM.
A. Less than 0.80
B. At least 0.80, but less than 0.85
C. At least 0.85, but less than 0.90
D. At least 0.90, but less than 0.95
E. At least 0.95

4.50 (CAS3, 5/04, Q.16) (2.5 points) The number of major hurricanes that hit the island nation of Justcoast is given by a Poisson Distribution with 0.100 storms expected per year.

- Justcoast establishes a fund that will pay 100/storm.
- The fund charges an annual premium, payable at the start of each year, of 10.
- At the start of this year (before the premium is paid) the fund has 65.
- Claims are paid immediately when there is a storm.
- If the fund ever runs out of money, it immediately ceases to exist.
- Assume no investment income and no expenses.

What is the probability that the fund is still functioning in 10 years?
A. Less than 60%
B. At least 60%, but less than 61%
C. At least 61%, but less than 62%
D. At least 62%, but less than 63%
E. At least 63%

4.51 (CAS3, 11/04, Q.17) (2.5 points) You are given:

- Claims are reported at a Poisson rate of 5 per year.
- The probability that a claim will settle for less than $100,000 is 0.9.

What is the probability that no claim of $100,000 or more is reported during the next 3 years?
A. 20.59% B. 22.31% C. 59.06% D. 60.65% E. 74.08%

4.52 (CAS3, 11/04, Q.23) (2.5 points) Dental Insurance Company sells a policy that covers two types of dental procedures: root canals and fillings.
There is a limit of 1 root canal per year and a separate limit of 2 fillings per year.
The number of root canals a person needs in a year follows a Poisson distribution with $\lambda = 1$,
and the number of fillings a person needs in a year is Poisson with $\lambda = 2$.
The company is considering replacing the single limits with a combined limit of 3 claims per year, regardless of the type of claim.
Determine the change in the expected number of claims per year if the combined limit is adopted.
A. No change
B. More than 0.00, but less than 0.20 claims
C. At least 0.20, but less than 0.25 claims
D. At least 0.25, but less than 0.30 claims
E. At least 0.30 claims

4.53 (SOA M, 5/05, Q.5) (2.5 points)
Kings of Fredonia drink glasses of wine at a Poisson rate of 2 glasses per day.
Assassins attempt to poison the king's wine glasses. There is a 0.01 probability that any given glass is poisoned. Drinking poisoned wine is always fatal instantly and is the only cause of death. The occurrences of poison in the glasses and the number of glasses drunk are independent events.
Calculate the probability that the current king survives at least 30 days.
(A) 0.40 (B) 0.45 (C) 0.50 (D) 0.55 (E) 0.60

4.54 (CAS3, 11/05, Q.24) (2.5 points) For a compound loss model you are given:

- The claim count follows a Poisson distribution with $\lambda = 0.01$.
- Individual losses are distributed as follows:

x	F(x)
100	0.10
300	0.20
500	0.25
600	0.40
700	0.50
800	0.70
900	0.80
1,000	0.90
1,200	1.00

Calculate the probability of paying at least one claim after implementing a $500 deductible.
A. Less than 0.005
B. At least 0.005, but less than 0.010
C. At least 0.010, but less than 0.015
D. At least 0.015, but less than 0.020
E. At least 0.020

4.55 (CAS3, 11/05, Q.31) (2.5 points) The Toronto Bay Leaves attempt shots in a hockey game according to a Poisson process with mean 30. Each shot is independent.
For each attempted shot, the probability of scoring a goal is 0.10.
Calculate the standard deviation of the number of goals scored by the Bay Leaves in a game.
A. Less than 1.4
B. At least 1.4, but less than 1.6
C. At least 1.6, but less than 1.8
D. At least 1.8, but less than 2.0
E. At least 2.0

4.56 (CAS3, 11/06, Q.32) (2.5 points) You are given:

- Annual frequency follows a Poisson distribution with mean 0.3.
- Severity follows a normal distribution with F(100,000) = 0.6.

Calculate the probability that there is at least one loss greater than 100,000 in a year.
A. Less than 11 %
B. At least 11%, but less than 13%
C. At least 13%, but less than 15%
D. At least 15%, but less than 17%
E. At least 17%

4.57 (SOA M, 11/06, Q.9) (2.5 points) A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts are 1, 2, 3,… without limit.

The probability that any given payout is equal to i is $1/2^i$. Payouts are independent.

Calculate the probability that there are no payouts of 1, 2, or 3 in a given 20 minute period.

(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

4.58 (CAS3L, 5/09, Q.8) (2.5 points) Bill receives mail at a Poisson rate of 10 items per day.
The contents of the items are randomly distributed:

- 50% of the items are credit card applications.
- 30% of the items are catalogs.
- 20% of the items are letters from friends.

Bill has received 20 credit card applications in two days.
Calculate the probability that for those same two days, he receives at least 3 letters from friends and exactly 5 catalogs.
A. Less than 6%
B. At least 6%, but less than 10%
C. At least 10%, but less than 14%
D. At least 14%, but less than 18%
E. At least 18%

4.59 (CAS3L, 5/09, Q.9) (2.5 points) You are given the following information:

- Policyholder calls to a call center follow a homogenous Poisson process with $\lambda = 250$ per day.
- Policyholders may call for 3 reasons: Endorsement, Cancellation, or Payment.
- The distribution of calls is as follows:

Call Type	Percent of Calls
Endorsement	50%
Cancellation	10%
Payment	40%

Using the normal approximation with continuity correction, calculate the probability of receiving more than 156 calls in a day that are either endorsements or cancellations.
A. Less than 27%
B. At least 27%, but less than 29%
C. At least 29%, but less than 31%
D. At least 31%, but less than 33%
E. At least 33%

4.60 (CAS3L, 11/09, Q.11) (2.5 points) You are given the following information:

- Claims follow a compound Poisson process.
- Claims occur at the rate of $\lambda = 10$ per day.
- Claim severity follows an exponential distribution with $\theta = 15{,}000$.
- A claim is considered a large loss if its severity is greater than 50,000.

What is the probability that there are exactly 9 large losses in a 30-day period?

A. Less than 5%

B. At least 5%, but less than 7.5%

C. At least 7.5%, but less than 10%

D. At least 10%, but less than 12.5%

E. At least 12.5%

Solutions to Problems:

4.1. B. This is a Poisson distribution with a parameter of 6.9. The mean is therefore **6.9**.

4.2. B. This is a Poisson distribution with a parameter of 6.9.
The variance is therefore **6.9**.

4.3. A. One needs to sum the chances of having 0, 1, 2, and 3 claims:

n	0	1	2	3
f(n)	0.001	0.007	0.024	0.055
F(n)	0.001	0.008	0.032	**0.087**

For example, $f(3) = 6.9^3 e^{-6.9} / 3! = (328.5)(.001008)/6 = .055$.

4.4. B. The mode is the value at which f(n) is a maximum; f(6) = .151 and the mode is therefore **6**.

n	0	1	2	3	4	5	**6**	7	8
f(n)	0.001	0.007	0.024	0.055	0.095	0.131	0.151	0.149	0.128

Alternately, in general for the Poisson the mode is the largest integer in the parameter; the largest integer in 6.9 is 6.

4.5. C. For a discrete distribution such as we have here, employ the convention that the median is the first value at which the distribution function is greater than or equal to .5.
$F(7) \geq 50\%$ and $F(6) < 50\%$, and therefore the median is **7**.

n	0	1	2	3	4	5	6	**7**	8
f(n)	0.001	0.007	0.024	0.055	0.095	0.131	0.151	0.149	0.128
F(n)	0.001	0.008	0.032	0.087	0.182	0.314	0.465	0.614	0.742

4.6. E. The sum of Poisson variables is a Poisson with the sum of the parameters.
The sum has a Poisson parameter of (20)(.05) + (10)(.03) = 1.3.
The chance of three claims is $(1.3^3)e^{-1.3} / 3! =$ **9.98%**.

4.7. E. For the Pareto Distribution, $S(x) = 1 - F(x) = \{\theta/(\theta+x)\}^\alpha$.

$S(250) = \{400/(400+250)\}^{2.5} = 0.2971$.

Thus the distribution of hurricanes with more than $250 million of loss is Poisson with mean frequency of (82%)(.2971) = 24.36%.

The chance of zero such hurricanes is $e^{-0.2436} = 0.7838$.

The chance of one such hurricane is: $(0.2436)e^{-0.2436} = 0.1909$.

The chance of more than one such hurricane is: 1 - (0.7838 + 0.1909) = **0.0253**.

4.8. B. $f(n) = e^{-\lambda}\lambda^n / n! = e^{-10}\, 10^n / n!$

n	0	1	2	3	4	5
f(n)	0.0000	0.0005	0.0023	0.0076	0.0189	0.0378
F(n)	0.0000	0.0005	0.0028	0.0103	0.0293	0.0671

Thus the chance of having more than 5 claims is 1 - .0671 = **.9329**.
<u>Comment:</u> *Although one should not do so on the exam, one can also solve this using the Incomplete Gamma Function. The chance of having more than 5 claims is the Incomplete Gamma with shape parameter 5+1 =6 at the value 10: $\Gamma(6;10)$ = .9329.*

4.9. A. $f(n) = e^{-\lambda}\lambda^n / n! = e^{-10}\, 10^n / n!$

n	0	1	2	3	4	5	6	7	8
f(n)	0.0000	0.0005	0.0023	0.0076	0.0189	0.0378	0.0631	0.0901	0.1126
F(n)	0.0000	0.0005	0.0028	0.0103	0.0293	0.0671	0.1301	0.2202	0.3328

Thus the chance of having more than 8 claims is 1 - .3328 = **.6672**.
<u>Comment:</u> *The chance of having more than 8 claims is the incomplete Gamma with shape parameter 8+1 = 9 at the value 10: $\Gamma(9;10)$ = 0.6672.*

4.10. E. One can add up: f(6) + f(7) + f(8) = 0.0631 + 0.0901 + 0.1126 = **0.2657**.
Alternately, one can use the solutions to the two previous questions.
F(8) - F(5) = {1-F(5)} - {1-F(8)} = 0.9239 - 0.6672 = 0.2657.
<u>Comment</u>: *Prob[6, 7, or 8 claims] = $\Gamma(6;10)$ - $\Gamma(9;10)$ = 0.9239 - 0.6672 = 0.2657.*

4.11. E. The large and small claims are independent Poisson Distributions. Therefore, the observed number of small claims has no effect on the expected number of large claims.
S(500) = $\exp(-(500/1000)^3)$ = 0.8825. Expected number of large claims is: (27)(0.8825) = **23.8**.

4.12. D. Frequency of large losses follows a Poisson Distribution with λ = (20%)(7) = 1.4.
$f(3) = 1.4^3\, e^{-1.4}/3!$ = **11.3%**.

4.13. B. Prob[N = 1 | N ≤ 1] = Prob[N=1]/Prob[N ≤ 1] = $\lambda e^{-\lambda}/(e^{-\lambda} + \lambda e^{-\lambda}) = \lambda/(1 + \lambda)$ = **0.0909**.

4.14. E. Prob[N = 1 | N ≥ 1] = Prob[N = 1]/Prob[N ≥ 1] = $\lambda e^{-\lambda}/(1 - e^{-\lambda}) = \lambda/(e^{\lambda} - 1)$ = **0.9508**.

4.15. E. $E[1/(N+1)] = \sum_0^{\infty} f(n)/(n+1) = \sum_0^{\infty} (e^{-.2}\ .2^n/n!)/(n+1) = (e^{-.2}/.2) \sum_0^{\infty} 0.2^{n+1}/(n+1)!$

$= (e^{-.2}/.2) \sum_1^{\infty} 0.2^i/i! = (e^{-.2}/.2)\{\sum_0^{\infty} 0.2^i/i! - 0.2^0/0!\} = (e^{-0.2}/.2)(e^{.2} - 1) = (1 - e^{-.2})/.2 =$ **0.906**.

Comment: The densities of a Poisson with $\lambda = 0.2$ add to one. $\Rightarrow \sum_0^{\infty} 0.2^i/i! = e^{-0.2}$.

4.16. D. $E[N] = P[N = 0]0 + P[N = 1]1 + P[N > 1]E[N \mid N > 1]$.
$2 = 2e^{-2} + (1 - e^{-2} - 2e^{-2})E[N \mid N > 1]$. $E[N \mid N > 1] = (2 - 2e^{-2})/(1 - e^{-2} - 2e^{-2}) =$ **2.91**.

Alternately, $E[N \mid N > 1] = \dfrac{\sum_{n=2}^{\infty} n\ e^{-\lambda}\ \lambda^n\ /\ n!}{\sum_{n=2}^{\infty} e^{-\lambda}\ \lambda^n\ /\ n!} = \dfrac{\sum_{n=2}^{\infty} e^{-\lambda}\ \lambda^n\ /\ (n-1)!}{\text{Prob}[N\ >\ 1]} = \dfrac{\lambda \sum_{i=1}^{\infty} e^{-\lambda}\ \lambda^i\ /\ i!}{1\ -\ e^{-\lambda}\ -\ \lambda e^{-\lambda}}$

$= \dfrac{\lambda\,(1\ -\ e^{-\lambda})}{1\ -\ e^{-\lambda}\ -\ \lambda e^{-\lambda}}$. Plugging in $\lambda = 2$, the result is **2.91**.

4.17. E. Next year the frequency is Poisson with $\lambda = (600/500)(27) = 32.4$.
$f(30) = e^{-32.4}\ 32.4^{30}/30! =$ **6.63%**.

4.18. D. $E[N \mid N \ge 1]\ \text{Prob}[N \ge 1] + 0\ \text{Prob}[N = 0] = E[N] = \lambda. \Rightarrow E[N \mid N \ge 1]\ \text{Prob}[N \ge 1] = \lambda$.
$E[(N-1)_+] = E[N - 1 \mid N \ge 1]\ \text{Prob}[N \ge 1] + 0\ \text{Prob}[N = 0] = (E[N \mid N \ge 1] - 1)\ \text{Prob}[N \ge 1] =$
$E[N \mid N \ge 1]\ \text{Prob}[N \ge 1] - \text{Prob}[N \ge 1] = \lambda - (1 - e^{-\lambda}) = \lambda + e^{-\lambda} - 1 = 1.3 + e^{-1.3} - 1 =$ **0.5725**.

Alternately, $E[(N-1)_+] = \sum_{n=1}^{\infty} (n-1)\ f(n) = \sum_{n=1}^{\infty} n\ f(n) - \sum_{n=1}^{\infty} f(n) = E[N] - \text{Prob}[N \ge 1] = \lambda + e^{-\lambda} - 1$.

Alternately, $E[(N-1)_+] = E[N] - E[N \wedge 1] = \lambda - \text{Prob}[N \ge 1] = \lambda + e^{-\lambda} - 1$.

Alternately, $E[(N-1)_+] = E[(1-N)_+] + E[N] - 1 = \text{Prob}[N = 0] + \lambda - 1 = e^{-\lambda} + \lambda - 1$.

Comment: For the last two alternate solutions, see “Mahler’s Guide to Loss Distributions.”

4.19. B. E[N | N ≥ 2] Prob[N ≥ 2] + 1 Prob[N = 1] + 0 Prob[N = 0] = E[N] = λ.

$\Rightarrow$ E[N | N ≥ 2] Prob[N ≥ 2] = $\lambda - \lambda e^{-\lambda}$.

$E[(N-2)_+]$ = E[N - 2 | N ≥ 2] Prob[N ≥ 2] + 0 Prob[N < 2] = (E[N | N ≥ 2] - 2) Prob[N ≥ 2] =

E[N | N ≥ 2] Prob[N ≥ 2] - 2Prob[N ≥ 2] = $\lambda - \lambda e^{-\lambda} - 2(1 - e^{-\lambda} - \lambda e^{-\lambda}) = \lambda + 2e^{-\lambda} + \lambda e^{-\lambda} - 2$ =

$1.3 + 2e^{-1.3} + 1.3e^{-1.3} - 2 =$ **0.199**.

Alternately, $E[(N-2)_+] = \sum_{n=2}^{\infty} (n-2) f(n) = \sum_{n=2}^{\infty} n f(n) - 2\sum_{n=2}^{\infty} f(n) = E[N] - \lambda e^{-\lambda} - 2\,\text{Prob}[N \geq 2]$ =

$= \lambda - \lambda e^{-\lambda} - 2(1 - e^{-\lambda} - \lambda e^{-\lambda}) = \lambda + 2e^{-\lambda} + \lambda e^{-\lambda} - 2$.

Alternately, $E[(N-2)_+] = E[N] - E[N \wedge 2] = \lambda$ - (Prob[N = 1] + 2Prob[N ≥ 2]) = $\lambda + 2e^{-\lambda} + \lambda e^{-\lambda} - 2$.

Alternately, $E[(N-2)_+] = E[(2-N)_+] + E[N] - 2$ = 2Prob[N = 0] + Prob[N = 1] + λ - 2 =

$2e^{-\lambda} + \lambda e^{-\lambda} + \lambda - 2$.

Comment: For the last two alternate solutions, see "Mahler's Guide to Loss Distributions."

4.20. A. For the Weibull, S(500) = $\exp[-(500/2000)^{.7}]$ = .6846.

S(1000) = $\exp[-(1000/2000)^{.7}]$ = .5403. Therefore, with the $1000 deductible, the non-zero payments are Poisson, with λ = (.5403/.6846)(3.3) = 2.60. f(4) = $e^{-2.6}\, 2.6^4/4!$ = **14.1%**.

4.21. D. In the absence of losses, by the beginning of year 12, the fund would have:
300 + (12)(60) = 1020 > 1000.
In the absence of losses, by the beginning of year 29, the fund would have:
300 + (29)(60) = 2040 > 2000.
Thus in order to survive for 40 years there have to be 0 events in the first 11 years,
at most one event during the first 28 years, and at most two events during the first 40 years.
Prob[survival through 40 years] =
Prob[0 in first 11 years]{Prob[0 in next 17 years]Prob[0, 1, or 2 in final 12 years] +
Prob[1 in next 17 years]Prob[0 or 1 in final 12 years]} =

$e^{-0.55}\{(e^{-0.85})(e^{-0.6} + 0.6e^{-0.6} + 0.6^2e^{-0.6}/2) + (0.85e^{-0.85})(e^{-0.6} + 0.6e^{-0.6})\} = 3.14e^{-2} =$ **0.425**

Comment: Similar to CAS3, 5/04, Q.16.

4.22. C. The percentage of large losses is: $e^{-1000/600} = 18.89\%$.

Let λ be the mean of the Poisson distribution of all losses.

Then the large losses, those of size greater than 1000, are also Poisson with mean 0.1889λ.

74% = Prob[no payments] = Prob[0 large losses] = $\exp[-0.1889\lambda]$. $\Rightarrow \lambda = 1.59$.

The small losses, those of size less than 1000, are also Poisson with mean:
(1 - 0.1889) (1.59) = 1.29.

Prob[3 small losses] = $1.29^3\ e^{-1.29} / 6$ = **9.8%.**

Comment: Only those losses of size greater than 1000 result in a payment.
There can be any number of small losses without affecting the payments.

4.23. C. The coefficient of variation is the ratio of the standard deviation to the mean, which for a Poisson Distribution is: $\sqrt{\lambda}\ / \lambda = 1/\sqrt{\lambda}$. $1/\sqrt{\lambda} = 0.5$. $\Rightarrow \lambda = 4$. $\Rightarrow f(3) = 4^7\ e^{-4} / 7!$ = **5.95%.**

4.24. C. 10 = E[Z] = E[X + Y] = E[X] + E[Y] = Var[X] + Var[Y].
12 = Var[Z] = Var[X + Y] = Var[X] + Var[Y] + 2 Cov[X, Y] = 10 + 2Cov[X, Y].
Therefore, Cov[X, Y] = (12 - 10)/2 = **1**.

4.25. D. The survival function of the Pareto is: $S(x) = \{1000/(1000+x)\}^3$.
S(100) = 0.7513. S(250) = 0.5120.
Thus with a deductible of 250 the number of non-zero payments is Poisson with mean:
(5)(0.5120/0.7513) = 3.407.

Probability of exactly 3 non-zero payments is: $(3.407^3)\ e^{-3.407} / 6$ = **21.8%.**

4.26. B. For the Exponential, F(200) = 1 - exp[-200/400] = 0.3935.
Thus the small loses are Poisson with mean (0.3935)(10) = 3.935,
the large loses are Poisson with mean (1 - 0.3935)(10) = 6.065,
and the number of small and large losses are independent.

Probability of 5 small losses is: $(3.935^5)\ e^{-3.935} / 5! = 0.1537$..

Probability of 6 large losses is: $(6.065^6)\ e^{-6.065} / 6! = 0.1606$.

Probability of 5 small losses and 6 large losses is: (0.1537)(0.1606) = **2.47%.**

4.27. E. $n\bar{X}$, the sum of n Poissons each with mean λ, is a Poisson with mean $n\lambda$.

Comment: $\bar{X}$ can be non-integer, and therefore it cannot have a Poisson distribution.

As $n \to \infty$, $\bar{X} \to$ a Normal distribution with mean λ and variance λ/n.

4.28. B. Over two weeks, the number of accidents is Poisson with mean 6.
$f(2) = e^{-\lambda}\lambda^2/2 = e^{-6}\ 6^2/2 =$ **$18e^{-6}$**.

4.29. C. Prob[X ≥ 2 | X ≤ 4] = {f(2) + f(3) + f(4)}/{f(0) + f(1) + f(2) + f(3) + f(4)} =
$e^{-1}(1/2 + 1/6 + 1/24)/\{e^{-1}(1 + 1 + 1/2 + 1/6 + 1/24)\} = (12 + 4 + 1)/(24 + 24 + 12 + 4 + 1)$
= **17/65**.

4.30. D. Prob[W ≤ 2] = 1 - Prob[no cars by time 2] = **$1 - e^{-2}$**.

4.31. C. Prob[0 errors in 30 seconds] = $e^{-30/10} = e^{-3}$. Prob[1 error in 30 seconds] = $3e^{-3}$.
Prob[more than one error in 30 seconds] = **$1 - 4e^{-3}$**.

4.32. B. Prob[0 or 1 surges] = $e^{-24/12} + 2e^{-2} =$ **$3e^{-2}$**.

4.33. C. The sum of three independent Poissons is also a Poisson, whose mean is the sum of the individual means. Thus the portfolio of three insureds has a Poisson distribution with mean 0.05 + 0.10 + 0.15 = 0.30. For a Poisson distribution with mean θ, the chance of zero claims is $e^{-\theta}$ and that of 1 claim is $\theta e^{-\theta}$. Thus the chance of fewer than 2 claims is: $(1+\theta)e^{-\theta}$.
Thus for this portfolio, the chance of 2 or fewer claims is: $(1+.3)e^{-.3} =$ **0.963**.

4.34. B. X + Y + Z is Poisson with $\lambda = 3 + 1 + 4 = 8$. $f(0) + f(1) = e^{-8} + 8e^{-8} =$ **$9e^{-8}$**.

4.35. C. The sum of independent Poissons is a Poisson, with a parameter the sum of the individual Poisson parameters. In this case the portfolio is Poisson with a parameter = (10)(.1) = 1.
Chance of zero claims is e^{-1}. Chance of one claim is $(1)e^{-1}$.
Chance of more than one claim is: $1 - (e^{-1} + e^{-1}) =$ **0.264**.

4.36. E. This is a Poisson distribution with a parameter of 4.6. The mean is therefore 4.6. The mode is the value at which $f(\alpha)$ is a maximum; f(4) = .188 and the mode is 4. Therefore statement **1 is true**. For the Poisson the variance equals the mean, and therefore statement **2 is false**.
For a discrete distribution such as we have here, the median is the first value at which the distribution function is greater than or equal to .5. F(4) > 50%, and therefore the median is 4 and less than the mean of 4.6. Therefore statement **3 is true**.

n	0	1	2	3	4	5	6	7	8
f(n)	0.010	0.046	0.106	0.163	0.188	0.173	0.132	0.087	0.050
F(n)	0.010	0.056	0.163	0.326	0.513	0.686	0.818	0.905	0.955

<u>Comment</u>: For a Poisson with parameter λ, the mode is the largest integer in λ. In this case $\lambda = 4.6$ so the mode is 4. Items 1 and 3 can be answered by computing enough values of the density and adding them up. Alternately, since the distribution is skewed to the right (has positive skewness), both the peak of the curve and the 50% point are expected to be to the left of the mean. The median is less affected by the few extremely large values than is the mean, and therefore for curves skewed to the right the median is smaller than the mean. For curves skewed to the right, the largest single probability most commonly occurs at less than the mean, but this is not true of all such curves.

4.37. B. Assume we have R risks in total. The Poisson distribution is given by:
$f(n) = e^{-\lambda}\lambda^n / n!$, n=0,1,2,3,... Thus for n=0 we have $R\ e^{-\lambda} = 96$. For n = 2 we have $R\ e^{-\lambda}\lambda^2 / 2 = 3$. By dividing these two equations we can solve for $\lambda = (6/96)^{.5} = 1/4$.
The number of risks we expect to have 4 claims is: $R\ e^{-\lambda}\lambda^4 / 4! = (96)(1/4)^4 / 24 =$ **0.0156**.

4.38. B. cos(0) = 1. $\cos(\pi) = -1$. $\cos(2\pi) = 1$. $\cos(3\pi) = -1$.
$E[\cos(\pi x)] = \Sigma\ (-1)^x f(x) = e^{-\lambda}\{1 - \lambda + \lambda^2/2! - \lambda^3/3! + \lambda^4/3! - \lambda^5/5! + ...\} =$
$e^{-\lambda}\{1 + (-\lambda) + (-\lambda)^2/2! + (-\lambda)^3/3! + (-\lambda)^4/3! + (-\lambda)^5/5! + ...\} = e^{-\lambda}e^{-\lambda} = e^{-2\lambda}$.
For $\lambda = \ln(2)$, $e^{-2\lambda} = 2^{-2} =$ **1/4**.

4.39. A. If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a **Poisson.**

4.40. B. The mode of the Poisson with mean λ is the largest integer in λ.
The largest integer in n - 1/2 is **n-1.**
Alternately, for the Poisson f(x)/ f(x-1) = λ/x. Thus f increases when $\lambda > x$ and decreases for $\lambda < x$.
Thus f increases for $x < \lambda = n - .5$. For integer n, $x < n - .5$ for $x \le n - 1$.
Thus the density increases up to n - 1 and decreases thereafter. Therefore the mode is n - 1.

4.41. A. We want the chance of less than 3 claims to be less than .9. For a Poisson with mean λ, the probability of 0, 1 or 2 claims is: $e^{-\lambda}(1+\lambda+\lambda^2/2)$. Over two years we have a Poisson with mean 2λ. Thus we want $e^{-2\lambda}(1+2\lambda+2\lambda^2) < .9$. Trying the endpoints of the given intervals we determine that the smallest such λ must be less than **0.7**.
Comment: In fact the smallest such λ is about 0.56.

4.42. D. If there are more than two losses, we are not concerned about those beyond the first two. Since the sum of two independent Poissons is a Poisson, the portfolio has a Poisson frequency distribution with mean of 0.3. Therefore, the chance of zero claims is $e^{-0.3}$ = 0.7408, one claim is 0.3 $e^{-0.3}$ = 0.2222, and of two or more claims is 1 - 0.7408 - 0.2222 = 0.0370.
0.1/0.3 = 1/3 of the losses are aircraft , and 0.2/0.3 = 2/3 of the losses are marine.
Thus the probability of the first two events, in the case of two or more events, is divided up as (1/3)(1/3) = 1/9, 2(1/3)(2/3) = 4/9, (2/3)(2/3) = 4/9, between 2 aircraft, 1 aircraft and 1 marine, and 2 marine, using the binomial expansion for two events.
⇒ (0.2222)/(1/3) = 0.0741 = probability of one aircraft, (0.2222)(2/3) = probability of one marine, (0.0370)(1/9) = 0.0041 = probability of 2 aircraft, (0.0370)(4/9) = probability one of each type, (0.0370)(4/9) = 0.0164 = probability of 2 marine.
If there are zero claims, the insurer pays nothing. If there is one aircraft loss, the insurer pays nothing. If there is one marine loss, the insurer pays 10 million. If there are two or more events there are three possibilities for the first two events. If the first two events are aircraft, the insurer pays 10 million. If the first two events are one aircraft and one marine, the insurer pays 20 million.
If the first two events are marine, the insurer pays 30 million.

Events (first 2 only)	Probability	Losses from First 2 events ($ million)	Amount Paid by the Insurer ($ million)
None	0.7408	0	0
1 Aircraft	0.0741	10	0
1 Marine	0.1481	20	10
2 Aircraft	0.0041	20	10
1 Aircraft, 1 Marine	0.0164	30	20
2 Marine	0.0164	40	30
	1		**2.345**

Thus the insurer's expected annual payment is **$2.345 million**.
Comment: Beyond what you can expect to be asked on your exam.

4.43. The total number of claims from inexperienced drivers is Poisson with mean: (20)(.15) = 3.
The total number of claims from experienced drivers is Poisson with mean: (40)(.1) = 4.
The total number of claims from all drivers is Poisson with mean: 3 + 4 = 7.
Prob[# claims ≤ 3] = $e^{-7}(1 + 7 + 7^2/2 + 7^3/6)$ = **8.177%**.

4.44. D. f(2) =3 f(4). ⇒ $e^{-\lambda}\lambda^2/2 = 3e^{-\lambda}\lambda^4/24$. ⇒ $\lambda = 2$. Variance = λ = **2**.

4.45. C. The finding of the three different types of coins are independent Poisson processes. Over the course of 60 minutes, Tom expects to find (.6)(.5)(60) = 18 coins worth 1 each and (.2)(.5)(60) = 6 coins worth 10 each. Tom finds 10 coins worth 5 each. The expected worth of the coins he finds is: (18)(1) + (10)(5) + (6)(10) = **128**.

4.46. C. $E[(X-1)_+] = E[X] - E[X \wedge 1] = 1.5 - \{0f(0) + 1(1 - f(0))\} = .5 + f(0) = .5 + e^{-1.5} = .7231$.

Expected Amount Paid is: $10,000E[(X-1)_+]$ = **7231**.

Alternately, Expected Amount Paid is: $10,000\{1f(2) + 2f(3) + 3f(4) + 4f(5) + 5f(6) + \ ...\}$ =

$(10,000)e^{-1.5}\{1.5^2/2 + (2)(1.5^3/6) + (3)(1.5^4/24) + (4)(1.5^5/120) + (5)(1.5^6/720) + ...\}$ =

$2231\{1.125 + 1.125 + .6328 + .2531 + .0791 + .0203 + ...) \cong 7200$.

4.47. B. For this Lognormal Distribution, $S(28,000) = 1 - \Phi[\ln(28000) - 10.12)/0.12]$ =

$1 - \Phi(1) = 1 - 0.8413 = 0.1587$. Acceptable offers arrive via a Poisson Process at rate

$2\ S(28000) = (2)(.01587) = 0.3174$ per month. Thus the number of acceptable offers over the first 3 months is Poisson distributed with mean $(3)(0.3174) = .9522$.

The probability of no acceptable offers over the first 3 months is: $e^{-0.9522}$ = **0.386**.

Alternately, the probability of no acceptable offers in a month is: $e^{-0.3174}$.

Probability of no acceptable offers in 3 months is: $(e^{-0.3174})^3 = e^{-0.9522}$ = **0.386.**

4.48. B. $E[X \wedge 2] = E[\text{Min}[X, 2]] = (0)f(0) + 1f(1) + 2\{1 - f(0) - f(1)\}$

$= 0.6e^{-0.6} + 2\{1 - e^{-0.6} - 0.6e^{-0.6}\} = 0.573$.

$E[(X \wedge 2)^2] = E[\text{Min}[X, 2]^2] = (0)f(0) + 1f(1) + 4\{1 - f(0) - f(1)\}$

$= 0.6e^{-0.6} + 4\{1 - e^{-0.6} - 0.6e^{-0.6}\} = 0.8169$.

$\text{Var}[X \wedge 2] = \text{Var}[\text{Min}[X, 2]] = 0.8169 - 0.573^2 = 0.4886$.

$\text{Var}[1000(X \wedge 2)] = (1000^2)(0.4886) = 488,600$.

$\text{StdDev}[1000(X \wedge 2)] = \sqrt{488,600}$ = **699**.

Comment: The limited expected value, $E[X \wedge x]$, is discussed in “Mahler’s Guide to Loss Dists.”

4.49. D. Trucks arrive at a Poisson rate of: (30%)(20) = 6 per hour.

$f(0) = e^{-6}$. $f(1) = 6e^{-6}$. $f(2) = 6^2e^{-6}/2$. $1 - \{f(0) + f(1) + f(2)\} = 1 - 25e^{-6}$ = **0.938**.

4.50. D. If there is a storm within the first three years, then there is ruin, since the fund would have only 65 + 30 = 95 or less. If there are two or more storms in the first ten years, then the fund is ruined. Thus survival requires no storms during the first three years and at most one storm during the next seven years. Prob[survival through 10 years] =

Prob[0 storms during 3 years] Prob[0 or 1 storm during 7 years] = $(e^{-.3})(e^{-.7} + .7e^{-.7})$ = **0.625**.

4.51. B. Claims of $100,000 or more are Poisson with mean: (5)(1 - 0.9) = 0.5 per year.

The number of large claims during 3 years is Poisson with mean: (3)(0.5) = 1.5.

$f(0) = e^{-1.5}$ = **0.2231.**

4.52. C. For $\lambda = 1$, $E[N \wedge 1] = 0f(0) + 1(1 - f(0)) = 1 - e^{-1} = .6321$.

For $\lambda = 2$, $E[N \wedge 2] = 0f(0) + 1f(1) + 2(1 - f(0) - f(1)) = 2e^{-2} + 2(1 - e^{-2} - 2e^{-2}) = 1.4587$.

Expected number of claims before change: .6321 + 1.4587 = 2.091.

The sum of number of root canals and the number of fillings is Poisson with $\lambda = 3$.

For $\lambda = 3$, $E[N \wedge 3] = 0f(0) + 1f(1) + 2f(2) + 3(1 - f(0) - f(1) - f(2)) =$

$3e^{-3} + (2)(9e^{-3}/2) + 3(1 - e^{-3} - 3e^{-3} - 4.5e^{-3}) = 2.328$. Change is: 2.328 - 2.091 = **0.237**.

Comment: Although it is not stated, we must assume that the number of root canals and the number of fillings are independent.

4.53. D. Poisoned glasses of wine are Poisson with mean: (0.01)(2) = 0.02 per day.

The probability of no poisoned glasses over 30 days is: $e^{-(30)(0.02)} = e^{-0.6} =$ **0.549**.

Comment: Survival corresponds to zero poisoned glasses of wine.
The king can drink any number of non-poisoned glasses of wine.
The poisoned and non-poisoned glasses are independent Poisson Processes.

4.54. B. After implementing a $500 deductible, only losses of size greater than 500 result in a claim payment. Prob[loss > 500] = 1 - F(500) = 1 - .25 = .75.
Via thinning, large losses are Poisson with mean: (.75)(.01) = .0075.

Prob[at least one large loss] = $1 - e^{-.0075} =$ **0.00747**.

4.55. C. By thinning, the number of goals is Poisson with mean (30)(0.1) = 3.

This Poisson has variance 3, and standard deviation: $\sqrt{3} =$ **1.732**.

4.56. B. Large losses are Poisson with mean: (1 - .6)(.3) = 0.12.

Prob[at least one large loss] = $1 - e^{-.12} =$ **11.3%**.

4.57. D. Payouts of size one are Poisson with λ = (1/2)(5) = 2.5 per hour.

Payouts of size one are Poisson with λ = (1/4)(5) = 1.25 per hour.

Payouts of size one are Poisson with λ = (1/8)(5) = 0.625 per hour.

Prob[0 of size 1 over 1/3 of an hour] = $e^{-2.5/3}$.

Prob[0 of size 2 over 1/3 of an hour] = $e^{-1.25/3}$.

Prob[0 of size 3 over 1/3 of an hour] = $e^{-0.625/3}$.

The three Poisson Processes are independent, so we can multiply the above probabilities:
$e^{-2.5/3}e^{-1.25/3}e^{-0.625/3} = e^{-1.458}$ = **0.233**.

Alternately, payouts of sizes one, two, or three are Poisson with

λ = (1/2 + 1/4 + 1/8)(5) = 4.375 per hour.

Prob[0 of sizes 1, 2, or 3, over 1/3 of an hour] = $e^{-4.375/3}$ = **0.233**.

4.58. C. Catalogs are Poisson with mean over two days of: (2)(30%)(10) = 6.
Letters are Poisson with mean over two days of: (2)(20%)(10) = 4.
The Poisson processes are all independent. Therefore, knowing he got 20 applications tells us nothing about the number of catalogs or letters.

Prob[at least 3 letters] = $1 - e^{-4} - e^{-4}\,4 - e^{-4}\,4^2/2$ = 0.7619.

Prob[5 catalogs] = $e^{-6}\,6^5/120$ = 0.1606.

Prob[at least 3 letters and exactly 5 catalogs] = (0.7619)(0.1606) = **12.2%.**

4.59. C. The number of endorsements and cancellations is Poisson with λ = (250)(60%) = 150.

Applying the normal approximation with mean and variance equal to 150:

Prob[more than 156] $\cong 1 - \Phi[(156.5 - 150)/\sqrt{150}] = 1 - \Phi[0.53]$ = **29.8%.**

Comment: Using the continuity correction, 156 is out and 157 is in:

156 156.5 157

$|\!\rightarrow$

4.60. D. $S(50) = e^{-50/15}$ = 0.03567. Large losses are Poisson with mean: 10 S(50) = 0.3567.

Over 30 days, large losses are Poisson with mean: (30)(0.3567) = 10.70.

Prob[exactly 9 large losses in a 30-day period] = $e^{-10.7}\,10.7^9 / 9!$ = **11.4%.**

Section 5, Geometric Distribution

The Geometric Distribution, a special case of the Negative Binomial Distribution, will be discussed first.

Geometric Distribution

Support: $x = 0, 1, 2, 3, ...$ Parameters: $\beta > 0$.

D. f. : $F(x) = 1 - \left(\frac{\beta}{1+\beta}\right)^{x+1}$

P. d. f. : $\mathbf{f(x)} = \frac{\beta^x}{(1+\beta)^{x+1}}$

f(0) = 1/ (1+β). $f(1) = \beta / (1 + \beta)^2$. $f(2) = \beta^2 / (1 + \beta)^3$. $f(3) = \beta^3 / (1 + \beta)^4$.

Mean = β

Variance = β(1+β) **Variance / Mean = 1 + β > 1**.

Coefficient of Variation = $\sqrt{\frac{1 + \beta}{\beta}}$. Skewness = $\frac{1 + 2\beta}{\sqrt{\beta(1 + \beta)}}$.

Kurtosis = $3 + \frac{6\beta^2 + 6\beta + 1}{\beta(1+\beta)}$.

Mode = 0.

Probability Generating Function: $P(z) = \frac{1}{1 - \beta(z-1)}$, $z < 1 + 1/\beta$.

$f(x+1)/f(x) = a + b/(x+1)$, $a = \beta/(1+\beta)$, $b = 0$, $f(0) = 1/(1+\beta)$.

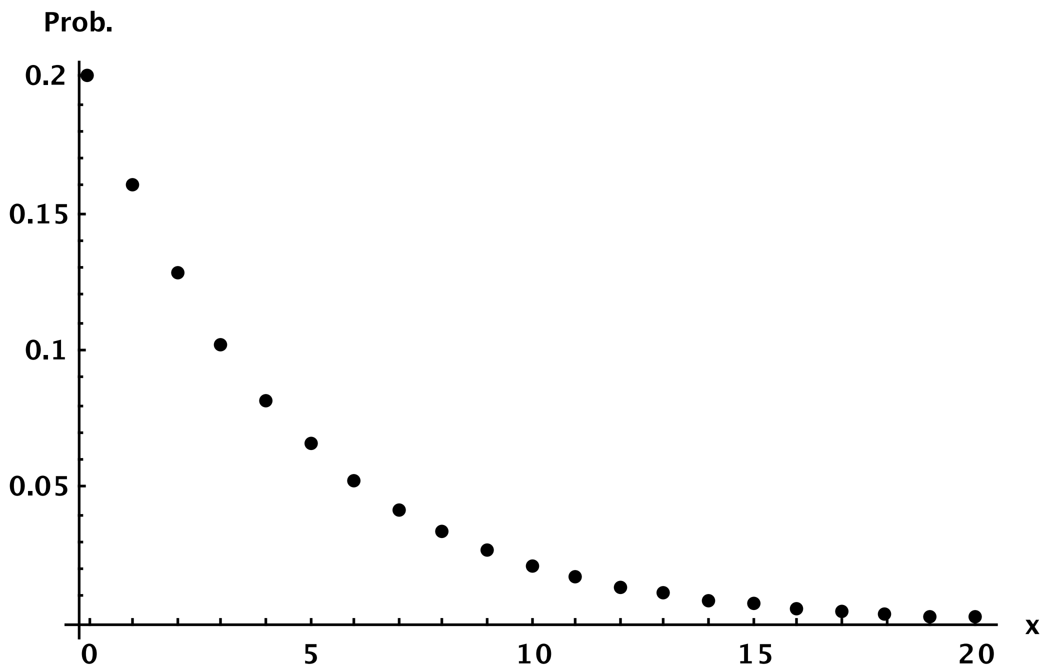
Moment Generating Function: $M(s) = 1/\{1-\beta(e^s-1)\}$, $s < \ln(1+\beta) - \ln(\beta)$.

Using the notation in Loss Models, the Geometric Distribution is:

$$f(x) = \frac{\left(\frac{\beta}{1+\beta}\right)^x}{1+\beta} = \frac{\beta^x}{(1+\beta)^{x+1}},\ x = 0, 1, 2, 3, \ldots$$

For example, for $\beta = 4$, $f(x) = 4^x/5^{x+1}$, x = 0, 1, 2, 3, ...

A Geometric Distribution for $\beta = 4$:



The densities decline geometrically by a factor of $\beta/(1+\beta)$; $f(x+1)/f(x) = \beta/(1+\beta)$.

This is similar to the Exponential Distribution, $f(x+1)/f(x) = e^{-1/\theta}$.

The Geometric Distribution is the discrete analog of the continuous Exponential Distribution.

For q = 0.3 or β = 0.7/0.3 = 2.333, the Geometric distribution is:

Number of Claims	f(x)	F(x)	Number of Claims times f(x)	Square of Number of Claims times f(x)
0	0.30000	0.30000	0.00000	0.00000
1	0.21000	0.51000	0.21000	0.21000
2	0.14700	0.65700	0.29400	0.58800
3	0.10290	0.75990	0.30870	0.92610
4	0.07203	0.83193	0.28812	1.15248
5	0.05042	0.88235	0.25211	1.26052
6	0.03529	0.91765	0.21177	1.27061
7	0.02471	0.94235	0.17294	1.21061
8	0.01729	0.95965	0.13836	1.10684
9	0.01211	0.97175	0.10895	0.98059
10	0.00847	0.98023	0.08474	0.84743
11	0.00593	0.98616	0.06525	0.71777
12	0.00415	0.99031	0.04983	0.59794
13	0.00291	0.99322	0.03779	0.49123
14	0.00203	0.99525	0.02849	0.39880
15	0.00142	0.99668	0.02136	0.32046
16	0.00100	0.99767	0.01595	0.25523
17	0.00070	0.99837	0.01186	0.20169
18	0.00049	0.99886	0.00879	0.15828
19	0.00034	0.99920	0.00650	0.12345
20	0.00024	0.99944	0.00479	0.09575
21	0.00017	0.99961	0.00352	0.07390
22	0.00012	0.99973	0.00258	0.05677
23	0.00008	0.99981	0.00189	0.04343
24	0.00006	0.99987	0.00138	0.03311
25	0.00004	0.99991	0.00101	0.02515
Sum			2.33	13.15

As computed above, the mean is about 2.33. The second moment is about 13.15, so that the variance is about 13.15 - 2.33^2 = 7.72. Since the Geometric has a significant tail, the terms involving the number of claims greater than 25 would have to be taken into account in order to compute a more accurate value of the variance or higher moments. Rather than taking additional terms it is better to have a general formula for the moments.

The mean can be computed as follows:

$$E[X] = \sum_{j=0}^{\infty} \text{Prob}[X > j] = \sum_{j=0}^{\infty} \left(\frac{\beta}{1+\beta}\right)^{j+1} = \frac{\frac{\beta}{1+\beta}}{1 - \frac{\beta}{1+\beta}} = \beta.$$

Thus the mean for the Geometric distribution is β. For the example, β = 2.333 = mean.

The variance of the Geometric is β(1+β), which for β = 2.333 is 7.78.[23]

<u>Survival Function</u>:

Note that there is a small but positive chance of any very large number of claims.
Specifically for the Geometric distribution the chance of x > j is:

$$\sum_{i=j+1}^{\infty} \frac{\beta^i}{(1+\beta)^{i+1}} = \frac{1}{(1+\beta)} \sum_{i=j+1}^{\infty} \left(\frac{\beta}{(1+\beta)}\right)^i = \frac{1}{(1+\beta)} \frac{\{\beta/(1+\beta)\}^{j+1}}{1 - \beta/(1+\beta)} = \{\beta/(1+\beta)\}^{j+1}.$$

1 - F(x) = **S(x) = $\{\beta/(1+\beta)\}^{x+1}$**.

For example, the chance of more than 19 claims is $.7^{20}$ = .00080, so that
F(19) = 1 - 0.00080 = 0.99920, which matches the result above.

Thus for a Geometric Distribution, for n > 0, the chance of at least n claims is $(\beta/(1+\beta))^n$.
The survival function decreases geometrically. The chance of 0 claims from a Geometric is:
1/(1+β) = 1 - β/(1+β) = 1 - geometric factor of decline of the survival function.

Exercise: There is a 0.25 chance of 0 claims, 0.75 chance of at least one claim,
0.75^2 chance of at least 2 claims, 0.75^3 chance of at least 3 claims, etc. What distribution is this?
[Solution: This is a Geometric Distribution with 1/(1+β) = 0.25, β/(1+β) = 0.75, or β = 3.]

For the Geometric, $F(x) = 1 - \{\beta/(1+\beta)\}^{x+1}$. Thus the Geometric distribution is the discrete analog of the continuous Exponential Distribution which has $F(x) = 1 - e^{-x/\theta} = 1 - (\exp[-1/\theta])^x$.
In each case the density function decreases by a constant multiple as x increases.
For the Geometric Distribution: $f(x) = \{\beta/(1+\beta)\}^x / (1+\beta)$,

while for the Exponential Distribution: $f(x) = e^{-x/\theta}/\theta = (\exp[-1/\theta])^x / \theta$.

[23] The variance is shown in Appendix B attached to the exam. One way to get the variance as well as higher moments is via the probability generating function and factorial moments, as will be discussed subsequently.

Memoryless Property:

The geometric shares with the exponential distribution, the "**memoryless property**."[24] "Given that there are at least m claims, the probability distribution of the number of claims in excess of m does not depend on m." In other words, **if one were to truncate and shift a Geometric Distribution, then one obtains the same Geometric Distribution.**

Exercise: Let the number of claims be given by an Geometric Distribution with $\beta = 1.7$.
Eliminate from the data all instances where there are 3 or fewer claims and subtract 4 from each of the remaining data points. (Truncate and shift at 4.)
What is the resulting distribution?
[Solution: Due to the memoryless property, the result is a Geometric Distribution with $\beta = 1.7$.]

Generally, let f(x) be the original Geometric Distribution. Let g(x) be the truncated and shifted distribution. Take as an example, a truncation point of 4 as in the exercise.

Then $g(x) = \dfrac{f(x+4)}{1 - \{f(0) + f(1) + f(2) + f(3)\}} = f(x+4) / S(3) = \dfrac{\beta^{x+4} / (1+\beta)^{x+5}}{\beta^4 / (1+\beta)^4} = \beta^x/(1+\beta)^{x+1}$,

which is again a Geometric Distribution with the same parameter β.

Constant Force of Mortality:

Another application where the Geometric Distribution arises is constant force of mortality, when one only looks at regular time intervals rather than at time continuously.[25]

Exercise: Every year in which Jim starts off alive, he has a 10% chance of dying during that year.
If Jim is currently alive, what is the distribution of his curtate future lifetime?
[Solution: There is a 10% chance he dies during the first year, and has a curtate future lifetime of 0.
If he survives the first year, there is a 10% chance he dies during the second year. Thus there is a (0.9)(0.1) = 0.09 chance he dies during the second year, and has a curtate future lifetime of 1. If he survives the second year, which has probability 0.9^2, there is a 10% chance he dies during the third year. Prob[curtate future lifetime = 2] = $(0.9^2)(0.1)$.
Similarly, Prob[curtate future lifetime = n] = $(0.9^n)(0.1)$.
This is a Geometric Distribution with $\beta/(1+\beta) = 0.9$ or $\beta = 0.9/0.1 = 9$.]

[24] See Section 6.3 of Loss Models. It is due to this memoryless property of the Exponential and Geometric distributions, that they have constant mean residual lives, as discussed subsequently.
[25] When one has a constant force of mortality and looks at time continuously, one gets the Exponential Distribution, the continuous analog of the Geometric Distribution.

In general, if there is a constant probability of death each year q, then the curtate future lifetime,[26] K(x), follows a Geometric Distribution, with β = (1-q)/q =
probability of continuing sequence / probability of ending sequence.
Therefore, for a constant probability of death each year, q, the curtate expectation of life,[27] e_x, is β = (1-q)/q, the mean of this Geometric Distribution. The variance of the curtate future lifetime is:
$\beta(1+\beta) = \{(1-q)/q\}(1/q) = (1-q)/q^2$.

Exercise: Every year in which Jim starts off alive, he has a 10% chance of dying during that year. What is Jim's curtate expectation of life and variance of his curtate future lifetime?
[Solution: Jim's curtate future lifetime is Geometric, with mean = β = (1 - 0.1)/0.1 = 9,
and variance $\beta(1+\beta)$ = (9)(10) = 90.]

Exercise: Jim has a constant force of mortality, μ = 0.10536.
What is the distribution of Jim's future lifetime.
What is Jim's complete expectation of life?
What is the variance of Jim's future lifetime?
[Solution: It is Exponential, with mean $\theta = 1/\mu$ = 9.49, and variance θ^2 = 90.1.
Comment: Jim has a 1 - $e^{-0.10536}$ = 10% chance of dying each year in which he starts off alive. However, here we look at time continuously.]

With a constant force of mortality:
observe continuously $\Leftrightarrow$ Exponential Distribution
observe at discrete intervals $\Leftrightarrow$ Geometric Distribution.

Exercise: Every year in which Jim starts off alive, he has a 10% chance of dying during that year.
Jim's estate will be paid $1 at the end of the year of his death.
At a 5% annual rate of interest, what is the present value of this benefit?

[26] The curtate future lifetime is the number of whole years completed prior to death.
See page 54 of Actuarial Mathematics.
[27] The curtate expectation of life is the expected value of the curtate future lifetime.
See page 69 of Actuarial Mathematics.

[Solution: If Jim's curtate future lifetime is n, there is a payment of 1 at time n + 1.

Present value = $\sum_0^\infty f(n)\, v^{n+1} = \sum_0^\infty (9^n / 10^{n+1})\, 0.9524^{n+1}$ = (0.09524)/(1 - 0.8572) = 0.667.]

In general, if there is a constant probability of death each year q, the present value of $1 paid at the end of the year of death is: $\sum_0^\infty \frac{\beta^n}{(1+\beta)^{n+1}} v^{n+1} = \frac{v}{(1+\beta)} \frac{1}{1 - v\,\beta/(1+\beta)} = \frac{1}{(1+\beta)/v - \beta} =$

$\frac{1}{(1+\beta)(1+i) - \beta} = \frac{1}{(1+i)/q - (1-q)/q}$ = q/(q + i).[28]

Exercise: Every year in which Jim starts off alive, he has a 10% chance of dying during that year. At a 5% annual rate of interest, what is the present value of the benefits from an annuity immediate with annual payment of 1?

[Solution: If Jim's curtate future lifetime is n, there are n payments of 1 each at times: 1, 2, 3,.., n. The present value of these payments is: $v + v^2 + \ldots + v^n = (1 - v^n)/i$.

Present value = $\sum_{n=0}^\infty f(n)(1 - v^n) / i = (1/i)\{\sum f(n) - \sum (9^n/10^{n+1})\, v^n\}$ =

20{1 - (0.1)/(1- 0.9/1.05)} = 6.]

In general, if there is a constant probability of death each year q, the present value of the benefits from an annuity immediate with annual payment of 1:

$$\sum_{n=0}^\infty f(n)(1 - v^n) / i = \{\sum_{n=0}^\infty f(n) - \sum_{n=0}^\infty \frac{\beta^n}{(1+\beta)^{n+1}} (v^n)\}/i =$$

$$(1/i)\{1 - \frac{1/(1+\beta)}{1 - v\beta/(1+\beta)}\} = (1/i)\{1 - 1/(1 + \beta - v\beta)\} = (1/i)\{(\beta - v\beta)/(1 + \beta - v\beta)\} =$$

$$= (1/(1+i))\,\{\beta/(1 + \beta - v\beta)\} = (1/(1+i))\,\{1/(1/\beta + 1 - v)\} = (1/(1+i))\,\{1/(q/(1-q) + 1 - v)\} =$$

$(1/(1+i))\,\{(1-q)/(q + (1 - v)(1-q))\} = (1-q)/(q(1+i) + i(1-q))\} = (1-q)/(q+i)$.[29]

In the previous exercise, the present value of benefits is: (1 - 0.1)/(0.1 + 0.05) = 0.9/0.15 = 6.

For i = 0, (1-q)/(q+i) becomes (1-q)/q, the mean of the Geometric Distribution of curtate future lifetimes. For q = 0, (1-q)/(q+i) becomes 1/i, the present value of a perpetuity, with the first payment one year from now.

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[28] With a constant force of mortality μ, the present value of $1 paid at the time of death is: $\mu/(\mu+\delta)$. See page 99 of Actuarial Mathematics.

[29] With a constant force of mortality μ, the present value of a life annuity paid continuously is: $1/(\mu+\delta)$. See page 136 of Actuarial Mathematics.

Series of Bernoulli Trials:

For a series of Bernoulli trials with chance of success 0.3, the probability that there are no success in the first four trials is: $(1 - 0.3)^4 = 0.24$.

Exercise: What is the probability that there is no success in the first four trials and the fifth trial is a success?
[Solution: $(1 - 0.3)^4(0.3) = 0.072$ = the probability of the first success occurring on the fifth trial.]

In general, the chance of the first success after x failures is: $(1 - 0.3)^x(0.3)$.

More generally, take a series of Bernoulli trials with chance of success q. The probability of the first success on trial x+1 is: $(1-q)^x q$.

$$f(x) = (1-q)^x q, \quad x = 0, 1, 2, 3,...$$

This is the Geometric distribution. It is a special case of the Negative Binomial Distribution.[30]
Loss Models uses the notation β, where $q = 1/(1+\beta)$.
$\beta = (1-q) / q$ = probability of a failure / probability of a success.

For a series of independent identical Bernoulli trials, the chance of the first success following x failures is given by a Geometric Distribution with mean:
β = chance of a failure / chance of a success.
The number of trials = 1 + number of failures = 1 + Geometric.

The Geometric Distribution shows up in many applications, including Markov Chains and Ruin Theory. In many contexts:
β = probability of continuing sequence / probability of ending sequence
= probability of remaining in the loop / probability of leaving the loop.

[30] The Geometric distribution with parameter β is the Negative Binomial Distribution with parameters β and r=1.

Problems:

The following five questions all deal with a Geometric distribution with $\beta = 0.6$.

5.1 (1 point) What is the mean?
(A) 0.4 (B) 0.5 (C) 0.6 (D) 2/3 (E) 1.5

5.2 (1 point) What is the variance?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

5.3 (2 points) What is the chance of having 3 claims?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

5.4 (2 points) What is the mode?
A. 0 B. 1 C. 2 D. 3 E. None of A, B, C, or D.

5.5 (2 points) What is the chance of having 3 claims or more?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

5.6 (1 point) The variable N is generated by the following algorithm:
(1) N = 0.
(2) 25% chance of exiting.
(3) N = N + 1.
(4) Return to step #2.
What is the variance of N?
A. less than 10
B. at least 10 but less than 15
C. at least 15 but less than 20
D. at least 20 but less than 25
E. at least 25

5.7 (2 points) Use the following information:
- Assume a Rating Bureau has been making Workers' Compensation classification rates for a very, very long time.
- Assume every year the rate for the Carpenters class is based on a credibility weighting of the indicated rate based on the latest year of data and the current rate.
- Each year, the indicated rate for the Carpenters class is given 20% credibility.
- Each year, the rate for year Y, was based on the data from year Y-3 and the rate in the year Y-1. Specifically, the rate in the year 2001 is based on the data from 1998 and the rate in the year 2000.

What portion of the rate in the year 2001 is based on the data from the year 1990?
A. less than 1%
B. at least 1% but less than 2%
C. at least 2% but less than 3%
D. at least 3% but less than 4%
E. at least 4%

5.8 (3 points) An insurance company has stopped writing new general liability insurance policies. However, the insurer is still paying claims on previously written policies. Assume for simplicity that payments are made at the end of each quarter of a year. It is estimated that at the end of each quarter of a year the insurer pays 8% of the total amount remaining to be paid. The next payment will be made today.
Let X be the total amount the insurer has remaining to pay.
Let Y be the present value of the total amount the insurer has remaining to pay.
If the <u>annual</u> rate of interest is 5%, what is the Y/X?
A. 0.80 B. 0.82 C. 0.84 D. 0.86 E. 0.88

Use the following information for the next 4 questions:
There is a constant force of mortality of 3%.
There is an annual interest rate of 4%.

5.9 (1 point) What is the curtate expectation of life?
(A) 32.0 (B) 32.2 (C) 32.4 (D) 32.6 (E) 32.8

5.10 (1 point) What is variance of the curtate future lifetime?
(A) 900 (B) 1000 (C) 1100 (D) 1200 (E) 1300

5.11 (2 points) What is the actuarial present value of a life insurance which pays 100,000 at the end of the year of death?
(A) 41,500 (B) 42,000 (C) 42,500 (D) 43,000 (E) 43,500

5.12 (2 points) What is the actuarial present value of an annuity immediate which pays 10,000 per year?
(A) 125,000 (B) 130,000 (C) 135,000 (D) 140,000 (E) 145,000

5.13 (1 point) After each time Mark Orfe eats at a restaurant, there is 95% chance he will eat there again at some time in the future. Mark has eaten today at the Phoenicia Restaurant.
What is the probability that Mark will eat at the Phoenicia Restaurant precisely 7 times in the future?
A. 2.0% B. 2.5% C. 3.0% D. 3.5% E. 4.0%

5.14 (3 points) Use the following information:
- The number of days of work missed by a work related injury to a workers' arm is Geometrically distributed with $\beta = 4$.
- If a worker is disabled for 5 days or less, nothing is paid for his lost wages under workers compensation insurance.
- If he is disabled for more than 5 days due to a work related injury, workers compensation insurance pays him his wages for all of the days he was out of work.

What is the average number of days of wages reimbursed under workers compensation insurance for a work related injury to a workers' arm?
(A) 2.2 (B) 2.4 (C) 2.6 (D) 2.8 (E) 3.0

Use the following information for the next two questions:
The variable X is generated by the following algorithm:
(1) X = 0.
(2) Roll a fair die with six sides and call the result Y.
(3) X = X + Y.
(4) If Y = 6 return to step #2.
(5) Exit.

5.15 (2 points) What is the mean of X?
A. less than 4.0
B. at least 4.0 but less than 4.5
C. at least 4.5 but less than 5.0
D. at least 5.0 but less than 5.5
E. at least 5.5

5.16 (2 points) What is the variance of X?
A. less than 8
B. at least 8 but less than 9
C. at least 9 but less than 10
D. at least 10 but less than 11
E. at least 11

5.17 (1 point) N follows a Geometric Distribution with $\beta = 0.2$. What is Prob[N = 1 | N ≤ 1]?
A. 8% B. 10% C. 12% D. 14% E. 16%

5.18 (1 point) N follows a Geometric Distribution with $\beta = 0.4$. What is Prob[N = 2 | N ≥ 2]?
A. 62% B. 65% C. 68% D. 71% E. 74%

5.19 (2 points) N follows a Geometric Distribution with $\beta = 1.5$. What is E[1/(N+1)]?
Hint: $x + x^2/2 + x^3/3 + x^4/4 + ... = -\ln(1-x)$, for $0 < x < 1$.
A. 0.5 B. 0.6 C. 0.7 D. 0.8 E. 0.9

5.20 (2 points) N follows a Geometric Distribution with $\beta = 0.8$. What is E[N | N > 1]?
A. 2.6 B. 2.7 C. 2.8 D. 2.9 E. 3.0

5.21 (2 points) Use the following information:

- Larry, his brother Darryl, and his other brother Darryl are playing as a three man basketball team at the school yard.
- Larry, Darryl, and Darryl have a 20% chance of winning each game, independent of any other game.
- When a team's turn to play comes, they play the previous winning team.
- Each time a team wins a game it plays again.
- Each time a team loses a game it sits down and waits for its next chance to play.
- It is currently the turn of Larry, Darryl, and Darryl to play again after sitting for a while.

Let X be the number of games Larry, Darryl, and Darryl play until they sit down again.
What is the variance of X?
A. 0.10 B. 0.16 C. 0.20 D. 0.24 E. 0.31

Use the following information for the next two questions:
N follows a Geometric Distribution with $\beta = 1.3$.
Define $(N - j)_+ = n\text{-}j$ if $n \geq j$, and 0 otherwise.

5.22 (2 points) Determine $E[(N - 1)_+]$.
A. 0.73 B. 0.76 C. 0.79 D. 0.82 E. 0.85

5.23 (2 points) Determine $E[(N-2)_+]$.
A. 0.30 B. 0.33 C. 0.36 D. 0.39 E. 0.42

Use the following information for the next three questions:
Ethan is an unemployed worker. Ethan has a 25% probability of finding a job each week.

5.24 (2 points) What is the probability that Ethan is still unemployed after looking for a job for 6 weeks?
A. 12% B. 14% C. 16% D. 18% E. 20%

5.25 (1 point) If Ethan finds a job the first week he looks, count this as being unemployed 0 weeks.
If Ethan finds a job the second week he looks, count this as being unemployed 1 week, etc.
What is the mean number of weeks that Ethan remains unemployed?
A. 2 B. 3 C. 4 D. 5 E. 6

5.26 (1 point) What is the variance of the number of weeks that Ethan remains unemployed?
A. 12 B. 13 C. 14 D. 15 E. 16

5.27 (3 points) For a discrete density p_k, define the entropy as: $-\sum_{k=0}^{\infty} p_k \ln[p_k]$.

Determine the entropy for a Geometric Distribution as per Loss Models.

Hint: Why is the mean of the Geometric Distribution β?

5.28 (2, 5/85, Q.44) (1.5 points) Let X denote the number of independent rolls of a fair die required to obtain the first "3". What is $P[X \geq 6]$?

A. $(1/6)^5(5/6)$ B. $(1/6)^5$ C. $(5/6)^5(1/6)$ D. $(5/6)^6$ E. $(5/6)^5$

5.29 (2, 5/88, Q.22) (1.5 points) Let X be a discrete random variable with probability function $P[X = x] = 2/3^x$ for x = 1, 2, 3, . . . What is the probability that X is even?

A. 1/4 B. 2/7 C. 1/3 D. 2/3 E. 3/4

5.30 (2, 5/90, Q.5) (1.7 points) A fair die is tossed until a 2 is obtained. If X is the number of trials required to obtain the first 2, what is the smallest value of x for which $P[X \leq x] \geq 1/2$?

A. 2 B. 3 C. 4 D. 5 E. 6

5.31 (2, 5/92, Q.35) (1.7 points) Ten percent of all new businesses fail within the first year. The records of new businesses are examined until a business that failed within the first year is found. Let X be the total number of businesses examined which did not fail within the first year, prior to finding a business that failed within the first year. What is the probability function for X?

A. $0.1(0.9^x)$ for x = 0, 1, 2, 3,... B. $0.9x(0.1^x)$ for x = 1, 2, 3,... C. $0.1x(0.9^x)$ for x = 0, 1, 2, 3,...

D. $0.9x(0.1^x)$ for x = 1, 2,3,... E. $0.1(x - 1)(0.9^x)$ for x = 2, 3, 4,...

5.32 (Course 1 Sample Exam, Q. 7) (1.9 points) As part of the underwriting process for insurance, each prospective policyholder is tested for high blood pressure. Let X represent the number of tests completed when the first person with high blood pressure is found.
The expected value of X is 12.5.
Calculate the probability that the sixth person tested is the first one with high blood pressure.

A. 0.000 B. 0.053 C. 0.080 D. 0.316 E. 0.394

5.33 (1, 5/00, Q.36) (1.9 points) In modeling the number of claims filed by an individual under an automobile policy during a three-year period, an actuary makes the simplifying assumption that for all integers $n \geq 0$, $p_{n+1} = p_n/5$, where p_n represents the probability that the policyholder files n claims during the period. Under this assumption, what is the probability that a policyholder files more than one claim during the period?

(A) 0.04 (B) 0.16 (C) 0.20 (D) 0.80 (E) 0.96

5.34 (1, 11/01, Q.33) (1.9 points) An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of failure during that year is 0.4.
What is the expected benefit under this policy?
(A) 2234 (B) 2400 (C) 2500 (D) 2667 (E) 2694

Solutions to Problems:

5.1. C. mean = β = **0.6**.

5.2. A. variance = $\beta(1+\beta)$ = (0.6)(1.6) = **0.96.**

5.3. B. $f(x) = \beta^x / (1+\beta)^{x+1}$. $f(3) = (.6)^3 / (1.6)^{3+1}$ = **3.30%**.

5.4. A. The mode is **0**, since f(0) is larger than any other value.

n	0	1	2	3	4	5	6	7
f(n)	0.6250	0.2344	0.0879	0.0330	0.0124	0.0046	0.0017	0.0007

Comment: Just as with the Exponential Distribution, the Geometric Distribution always has a mode of zero.

5.5. D. 1 - {f(0) + f(1) + f(2)} = 1 - (0.6250 + 0.2344 + 0.0879) = **5.27%**.
Alternately, $S(x) = (\beta/(1+\beta))^{x+1}$. $S(2) = (0.6/1.6)^3$ = **5.27%**.

5.6. B. This is a loop, in which each time through there is a 25% of exiting and a 75% chance of staying in the loop. Therefore, N is Geometric with
β = probability of remaining in the loop / probability of leaving the loop = 0.75/0.25 = 3.
Variance = $\beta(1+\beta)$ = (3)(4) = **12**.

5.7. D. In making the rate for the year 2000, we give 20% weight to the data for the year 1997 and the remaining weight of 80% to the then current rate, that for 1999.
The weight given to the data from year 1997 in the rate for year 2000 is 0.20.
In making the rate for the year 2001, we give 20% weight to the data for the year 1987 and the remaining weight of 80% to the then current rate, that for year 2000.
Therefore, the weight given to the data from year 1997 in the rate for year 2001 is: (1 - 0.2)(0.2).
One could go through and do similar reasoning to determine how much weight the data for 1996 gets in the year 2011 rate.
Similarly, the weight given to the data from year 1996 in the rate for year 2001 is: $(1-0.2)^2(0.2)$.
Given the pattern, we infer that the weight given to the data from year 1990 in the rate for year 2001 is: $(1-0.2)^8(0.2)$ = **3.4%**.
Comment: The weights are from a geometric distribution with $\beta = 1/Z - 1$, and $\beta/(1+\beta) = 1 - Z$.
The weights are: $(1-Z)^nZ$ for n = 0,1,2,... Older years of data get less weight.
This is a simplification of a real world application, as discussed in "An Example of Credibility and Shifting Risk Parameters", by Howard C. Mahler, PCAS 1990.

5.8. E. Let Z be the amount remaining to be paid prior to quarter n. Then the payment in quarter n is 0.08Z. This leaves 0.92Z remaining to be paid prior to quarter n+1. Thus the payment in quarter n+1 is (0.08)(0.92)Z. The payment in quarter n+1 is 0.92 times the payment in quarter n.
The payments each quarter decline by a factor of 0.92.
Therefore, the proportion of the total paid in each quarter is a Geometric Distribution with
$\beta/(1+\beta) = 0.92. \Rightarrow \beta = 0.92/(1-0.92) = 11.5$. The payment at the end of quarter n is:
$X\ f(n) = X\ \beta^n/(1+\beta)^{n+1}$, n = 0, 1, 2, ... (The sum of these payment is X.)
The present value of the payment at the end of quarter n is:
$X\ f(n)/(1.05)^{n/4} = X(.9879^n)\beta^n/(1+\beta)^{n+1}$, n = 0, 1, 2, ...
Y, the total present value is:

$$\sum_{n=0}^{\infty} X\ (0.9879^n)\ \beta^n / (1+\beta)^{n+1} = \{X/(1+\beta)\} \sum_{n=0}^{\infty} (0.9879\ \beta / (1+\beta)\}^n = (X/12.5) \sum_{n=0}^{\infty} 0.9089^n$$

= (X/12.5) / (1-0.9089) = 0.878X. Y/X = **0.878**.
Comment: *In "Measuring the Interest Rate Sensitivity of Loss Reserves," by Richard Gorvett and Stephen D'Arcy, PCAS 2000 , a geometric payment pattern is used in order to estimate Macaulay durations, modified durations, and effective durations.*

5.9. E. If a person is alive at the beginning of the year, the chance they die during the next year is: $1 - e^{-\mu} = 1 - e^{-0.03} = 0.02955$. Therefore, the distribution of curtate future lifetimes is Geometric with mean $\beta = (1-q)/q = 0.97045/0.02955 =$ **32.84** years.
Alternately, the complete expectation of life is: $1/\mu = 1/0.03 = 33.33$.
The curtate future lifetime is on average about 1/2 less; 33.33 - 1/2 = **32.83**.

5.10. C. The variance of the Geometric Distribution is: $\beta(1+\beta)$ = (32.84)(33.84) = **1111**.
Alternately, the future lifetime is Exponentially distributed with mean $\theta = 1/\mu = 1/.03 = 33.33$, and variance $\theta^2 = 33.33^2 = 1111$. Since approximately they differ by a constant, 1/2, the variance of the curtate future lifetimes is approximately that of the future lifetimes, **1111**.

5.11. C. With constant probability of death, q, the present value of the insurance is: q/(q + i).
(100,000)q/(q + i) = (100000)(0.02955) / (0.02955 + 0.04) = **42,487**.
Alternately, the present value of an insurance that pays at the moment of death is:
$\mu/(\mu+\delta) = 0.03 / (0.03 + \ln(1.04)) = 0.03 / (0.03 + 0.03922) = 0.43340$.
(100000)(0.43340) = 43,340.
The insurance paid at the end of the year of death is paid on average about 1/2 year later;
$43340/(1.04^{.5})$ = **42,498**.

5.12. D. With constant probability of death, q, the present value of an annuity immediate is: (1-q)/(q +i).
(10000)(1-q)/(q +i) = (10000)(1 - .02955)/(.02955 + .04) = **139,533**.
Alternately, the present value of an annuity that pays continuously is:
$1/(\mu+\delta) = 1/(.03 + \ln(1.04)) = 1/(.03 + .03922) = 14.4467$. (10000)(14.4467) = 144,467.
Discounting for another half year of interest and mortality, the present value of the annuity immediate is approximately: $144{,}467/((1.04^{.5})(1.03^{.5}))$ = **139,583**.

5.13. D. There is a 95% chance Mark will return. If he returns, there is another 95% chance he will return again, etc. The chance of returning 7 times and then not returning an 8th time is:
$(.95^7)(.05)$ = **3.5%**.
Comment: The number of future visits is a Geometric Distribution with β =
probability of continuing sequence / probability of ending sequence = .95/.05 = 19.
$f(7) = \beta^7/(1+\beta)^8 = 19^7/20^8 = 3.5\%$.

5.14. C. If he is disabled for n days, then he is paid 0 if $n \le 5$, and n days of wages if $n \ge 6$.
Therefore, the mean number of days of wages paid is:

$$\sum_{n=6}^{\infty} n\ f(n) = \sum_{n=0}^{\infty} n\ f(n) - \sum_{n=0}^{5} n\ f(n) = E[N] - \{0f(0) + 1f(1) + 2f(2) + 3f(3) + 4f(4) + 5f(5)\} =$$

$4 - \{(1)(0.2)(0.8) + (2)(0.2)(0.8^2) + (3)(0.2)(0.8^3) + (4)(0.2)(0.8^4) + (5)(0.2)(0.8^5)\}$ = **2.62**.
Alternately, due to the memoryless property of the Geometric Distribution (analogous to its continuous analog the Exponential), truncated and shifted from below at 6, we get the same Geometric Distribution. Thus if only those days beyond 6 were paid for, the average nonzero payment is 4. However, in each case where we have at least 6 days of disability we pay the full length of disability which is 6 days longer, so the average nonzero payment is: 4 + 6 = 10.
The probability of a nonzero payment is: 1 - {f(0) + f(1) + f(2) + f(3) + f(4) + f(5)} =
$1 - \{0.2 + (0.2)(0.8) + (0.2)(0.8^2) + (0.2)(0.8^3) + (0.2)(0.8^4) + (0.2)(0.8^5)\} = 0.262$.
Thus the average payment (including zero payments) is: (0.262)(10 days) = 2.62 days.
Comment: Just an exam type question, not intended as a model of the real world.
Asks for the average payment per injury, including the zeros.
The solution is: $E[X \mid X > 5]\ Prob[X > 5] + 0\ Prob[X \le 5]$.

5.15. B. & 5.16. D. The number of additional dies rolled beyond the first is Geometric with β = probability of remaining in the loop / probability of leaving the loop = (1/6)/(5/6) = 1/5.

Let N be the number of dies rolled, then N - 1 is Geometric with β = 1/5.

X = 6(N - 1) + the result of the last 6-sided die rolled.
The result of the <u>last</u> six sided die to be rolled is equally likely to be a 1, 2, 3, 4 or 5 (it can't be a six or we would have rolled an additional die.)

E[X] = (6)(mean of a Geometric with β = 1/5) + (average of 1,2,3,4,5) = (6)(1/5) + 3 = **4.2**.

Variance of the distribution equally likely to be 1, 2, 3, 4, or 5 is: $(2^2 + 1^2 + 0^2 + 1^2 + 2^2)/5 = 2$.

Var[X] = 6^2(variance of a Geometric with β = 1/5) + 2 = (36)(1/5)(6/5) + 2 = **10.64**.

5.17. D. Prob[N = 1 | N ≤ 1] = Prob[N = 1]/Prob[N ≤ 1] = $\beta/(1+\beta)^2/\{1/(1+\beta)) + \beta/(1+\beta))^2\}$ = $\beta/(1 + 2\beta)$ = .2/1.4 = **0.143**.

5.18. D. Prob[N = 2 | N ≥ 2] = Prob[N = 2]/Prob[N ≥ 2] = $\beta^2/(1+\beta)^3/\{\beta^2/(1+\beta)^2\} = 1/(1+\beta)$.

Alternately, from the memoryless property, Prob[N = 2 | N ≥ 2] = Prob[N = 0] = $1/(1+\beta)$ = **.714**.

5.19. B. $E[1/(N+1)] = \sum_{n=0}^{\infty} f(n)/(n+1) = \sum_{m=1}^{\infty} f(m-1)/m = (1/\beta) \sum_{m=1}^{\infty} \left(\frac{\beta}{1+\beta}\right)^m / m$

$= (1/\beta)\{-\ln(1 - \beta/(1+\beta))\} = \ln(1+\beta)/\beta = \ln(2.5)/1.5$ = **0.611**.

5.20. C. E[N | N > 1]Prob[N > 1] + (1) Prob[N = 1] + (0) Prob[N = 0] = E[N] = β.

E[N | N > 1] = $\{\beta - \beta/(1+\beta)^2\} / \{\beta^2/(1+\beta)^2\} = 2 + \beta$ = **2.8**.

5.21. E. X is 1 + a Geometric Distribution with
β = (chance of remaining in the loop)/(chance of leaving the loop) = .2/.8 = 1/4.

Variance of X is: $\beta(1+\beta)$ = (1/4)(5/4) = **5/16** = **0.3125**.

<u>Comment</u>: Prob[X = 1] = 1 - .2 = .8. Prob[X = 2] = (.2)(.8). Prob[X = 3] = $(.2^2)(.8)$.

Prob[X = 4] = $(.2^3)(.8)$. Prob[X = x] = $(.2^{x-1})(.8)$. While this is a series of Bernoulli trials, it ends when the team has its first failure. X is the number of trials through the first failure.

5.22. A. $E[(N - 1)_+] = E[N] - E[N \wedge 1] = \beta$ - Prob$[N \geq 1] = \beta - \beta/(1+\beta) = \beta^2/(1+\beta)$ = **0.7348**.

Alternately, $E[(N-1)_+] = E[(1-N)_+] + E[N] - 1$ = Prob[N = 0] + β - 1 =

$\beta + 1/(1+\beta) - 1 = 1.3 + 1/2.3 - 1$ = **0.7348**.

Alternately, the memoryless property of the Geometric $\Rightarrow$ $E[(N-1)_+]$/Prob$[N \geq 1] = E[N] = \beta$. $\Rightarrow$

$E[(N-1)_+] = \beta$ Prob$[N \geq 1] = \beta\,\beta/(1+\beta) = \beta^2/(1+\beta)$ = **0.7348**.

5.23. E. $E[(N - 2)_+] = E[N] - E[N \wedge 2] = \beta$ - (Prob[N = 1] + 2 Prob[N ≥ 2]) =

$\beta - \beta/(1+\beta)^2 - 2\beta^2/(1+\beta)^2 = \{\beta(1+\beta)^2 - \beta - 2\beta^2\}/(1+\beta)^2 = \beta^3/(1+\beta)^2 = 1.3^3/2.3^2$ = **0.415**.

Alternately, $E[(N-2)_+] = E[(2-N)_+] + E[N] - 2$ = 2Prob[N = 0] + Prob[N = 1] + β - 2 =

$\beta + 2/(1+\beta) + \beta/(1+\beta)^2 - 2 = 1.3 + 2/2.3 + (1.3)/2.3^2 - 2$ = **0.415**.

Alternately, the memoryless property of the Geometric $\Rightarrow$ $E[(N-2)_+]$/Prob$[N \geq 2] = E[N] = \beta$. $\Rightarrow$

$E[(N-2)_+] = \beta$ Prob$[N \geq 2] = \beta\,\beta^2/(1+\beta)^2 = \beta^3/(1+\beta)^2$ = **0.415**.

Comment: For integral j, for the Geometric, $E[(N - j)_+] = \beta^{j+1}/(1+\beta)^j$.

5.24. D. Probability of finding a job within six weeks is:
$(.25)\{1 + .75 + .75^2 + .75^3 + .75^4 + .75^5\} = .822$. 1 - .822 = **17.8%**.

5.25. B. The number of weeks he remains unemployed is Geometric with
β = (chance of failure) / (chance of success) = 0.75/0.25 = 3. Mean = β = **3**.

5.26. A. Variance of this Geometric is: $\beta(1 + \beta) = (3)(4)$ = **12**.

5.27. The mean of a Geometric Distribution is β. ⇒ $\sum_{k=0}^{\infty} k\ p_k = \beta$.

$p_k = \frac{\beta^k}{(1+\beta)^{k+1}}$. ⇒ $\ln[p_k] = k \ln[\beta] - (k+1) \ln[1 + \beta] = \{\ln[\beta] - \ln[1 + \beta]\}\ k - \ln[1 + \beta]$.

$-\sum_{k=0}^{\infty} p_k \ln[p_k] = \{\ln[1 + \beta] - \ln[\beta]\} \sum_{k=0}^{\infty} k\ p_k + \ln[1 + \beta] \sum_{k=0}^{\infty} p_k = \{\ln[1 + \beta] - \ln[\beta]\}\ \beta + \ln[1 + \beta]\ (1) =$

$(1 + \beta) \ln[1 + \beta] - \beta \ln[\beta]$.

Comment: The Shannon entropy from information theory, except there the log is to the base 2.

5.28. E. Prob[X ≥ 6] = Prob[first 5 rolls each ≠ 3] = **$(5/6)^5$**.

Alternately, the number failures before the first success, X - 1, is Geometric with

β = chance of failure / chance of success = (5/6)/(1/6) = 5.

Prob[X ≥ 6] = Prob[# failures ≥ 5] = 1 - F(4) = $\{\beta/(1+\beta)\}^{4+1}$ = **$(5/6)^5$**.

5.29. A. Prob[X is even] = $2/3^2 + 2/3^4 + 2/3^6 + ...$ = (2/9)/(1 - 1/9) = **1/4**.

Comment: X - 1 follows a Geometric Distribution with β = 2.

5.30. C. X - 1 is Geometric with β = chance of failure / chance of success = (5/6)/(1/6) = 5.

Prob[X - 1 ≥ x - 1] = $\{\beta/(1+\beta)\}^x = (5/6)^x$.

We want to find where the distribution function is at least 1/2.

Thus we want to find the pace where the survival function is 1/2.

Set Prob[X ≥ x] =Prob[X-1 ≥ x-1] = 1/2: 1/2 = $(5/6)^x$.

x = ln(1/2)/ln(5/6) = 3.8. The next greatest integer is 4. P[X ≤ **4**] = 1 - $(5/6)^4$ = .518 ≥ 1/2.

Alternately, Prob[X = x] = Prob[X - 1 tosses ≠ 2]Prob[X = 2] = $(5/6)^{x-1}/6$.

X	Probability	Cumulative
1	0.1667	0.1667
2	0.1389	0.3056
3	0.1157	0.4213
4	0.0965	**0.5177**
5	0.0804	0.5981
6	0.0670	0.6651

5.31. A. X has a Geometric with β = chance of continuing / chance of ending = (.9)/(.1) = 9.

$f(x) = 9^x/10^{x+1}$ = **$(0.1)(0.9^x)$**, for x = 0, 1, 2, 3,...

5.32. B. This is a series of Bernoulli trials, and X - 1 is the number of failures before the first success.
Thus X - 1 is Geometric. β = E[X - 1] = 12.5 - 1 = 11.5.

Prob[X = 6] = Prob[X - 1 = 5] = f(5) = $\beta^5/(1+\beta)^6 = 11.5^5/12.5^6$ = **0.0527**.
Alternately, Prob[person has high blood pressure] = 1/E[X] = 1/12.5 = 8%.
Prob[sixth person is the first one with high blood pressure]
= Prob[first five don't have high blood pressure] Prob[sixth has high blood pressure]
= $(1 - 0.08)^5$ (0.08) = **0.0527**.

5.33. A. The densities are declining geometrically.
Therefore, this is a Geometric Distribution, with $\beta/(1+\beta) = 1/5. \Rightarrow \beta = 1/4$.
Prob[more than one claim] = 1 - f(0) - f(1) = $1 - 1/(1+\beta) - \beta/(1+\beta)^2$ = 1 - 4/5 - 4/25 = **0.04**.

5.34. E. Expected Benefit =
$(4000)(0.4) + (3000)(0.6)(0.4) + (2000)(0.6^2)(0.4) + (1000)(0.6^3)(0.4)$ = **2694**.
Alternately, the benefit is $1000(4 - N)_+$, where N is the number of years before the device fails.
N is Geometric, with $1/(1 + \beta) = .4. \Rightarrow \beta = 1.5$.
$E[N \wedge 4] = 0f(0) + 1f(1) + 2f(2) + 3f(3) + 4\{1 - f(0) - f(1) - f(2) - f(3)\} = 4 - 4f(0) - 3f(1) - 2f(2) - f(3)$.
Expected Benefit = $1000E[(4 - N)_+] = 1000(4 - E[N \wedge 4]) = 1000\{4f(0) + 3f(1) + 2f(2) + f(3)\}$
$= 1000\{4(0.4) + 3(0.4)(0.6) + 2(0.4)(0.6^2) + (0.4)(0.6^3)\}$ = **2694**.

Section 6, Negative Binomial Distribution

The third and final important frequency distribution is the Negative Binomial, which has the Geometric as a special case.

Negative Binomial Distribution

Support: $x = 0, 1, 2, 3, ...$ Parameters: $\beta > 0$, $r \geq 0$. **r = 1 is a Geometric Distribution**

D. f. : *$F(x) = \beta(r, x+1 ; 1/(1+\beta)) = 1 - \beta(x+1, r ; \beta/(1+\beta))$ Incomplete Beta Function*

P. d. f. : $$\mathbf{f(x)} = \frac{\mathbf{r(r+1)...(r+x-1)}}{\mathbf{x!}} \frac{\beta^{\mathbf{x}}}{(\mathbf{1+\beta})^{\mathbf{x+r}}} = \binom{x+r-1}{x} \frac{\beta^x}{(1+\beta)^{x+r}}.$$

$$\mathbf{f(0)} = \frac{1}{(1 + \beta)^r} \qquad f(1) = \frac{r\beta}{(1 + \beta)^{r+1}}$$

$$f(2) = \frac{r(r+1)}{2} \frac{\beta^2}{(1 + \beta)^{r+2}} \qquad f(3) = \frac{r(r+1)(r+2)}{6} \frac{\beta^3}{(1 + \beta)^{r+3}}$$

Mean = $r\beta$ **Variance = $r\beta(1+\beta)$** **Variance / Mean = $1 + \beta > 1$.**

Coefficient of Variation = $\sqrt{\frac{1+\beta}{r\beta}}$ Skewness = $\frac{1 + 2\beta}{\sqrt{r\beta(1+\beta)}}$ = CV(1+2β)/(1+β).

Kurtosis = $3 + \frac{6\beta^2 + 6\beta + 1}{r\beta(1+\beta)}$.

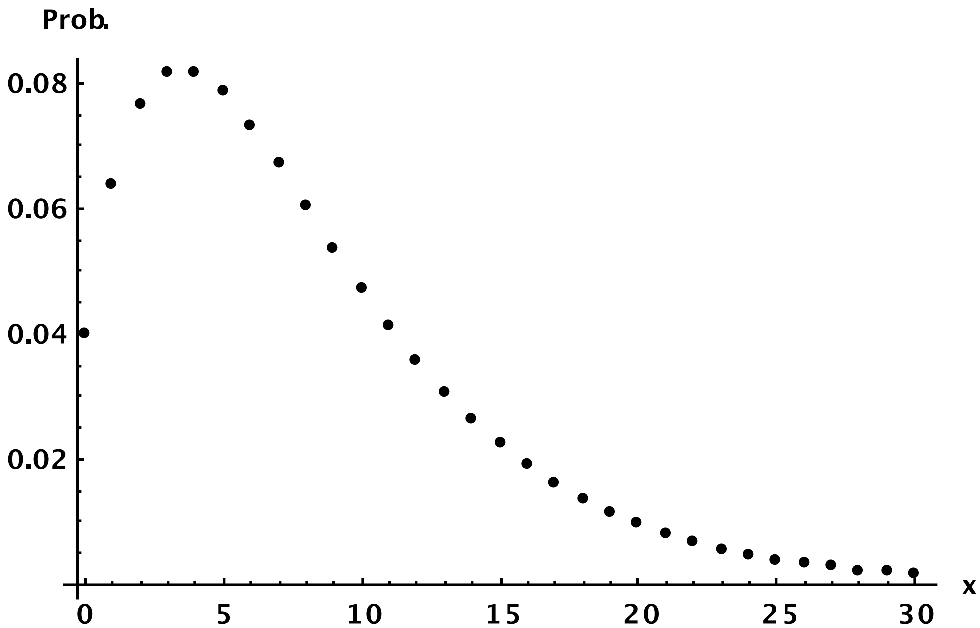
Mode = largest integer in $(r-1)\beta$ (if $(r-1)\beta$ is an integer,
then both $(r-1)\beta$ and $(r-1)\beta - 1$ are modes.)

Probability Generating Function: $P(z) = \{1-\beta(z-1)\}^{-r}$, $z < 1 + 1/\beta$.

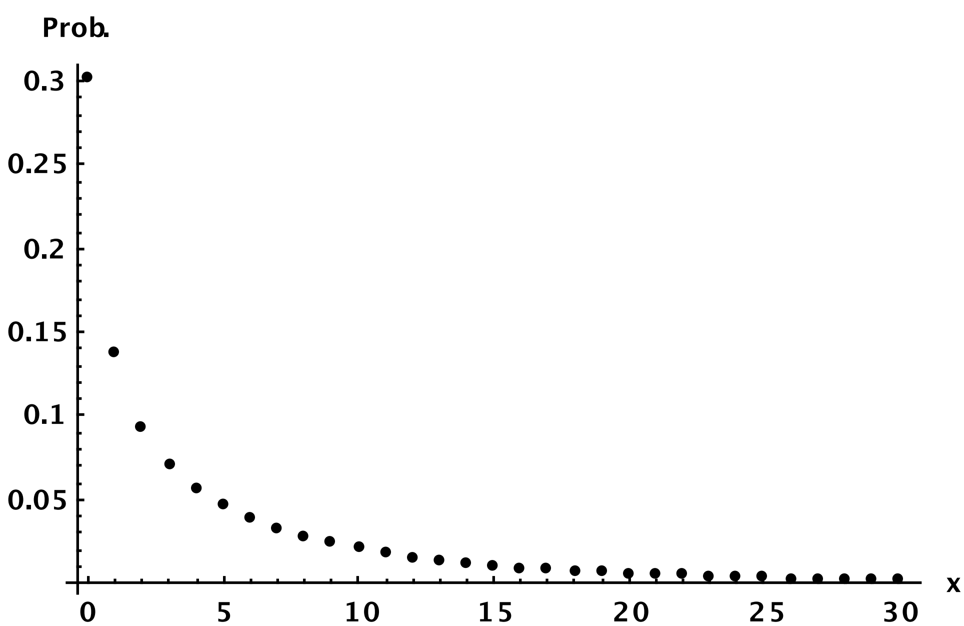
Moment Generating Function: $M(s) = \{1-\beta(e^s-1)\}^{-r}$, $s < \ln(1+\beta) - \ln(\beta)$.

$f(x+1)/f(x) = a + b/(x+1)$, $a = \beta/(1+\beta)$, $b = (r-1)\beta/(1+\beta)$, $f(0) = (1+\beta)^{-r}$.

A Negative Binomial Distribution with r = 2 and β = 4:



A Negative Binomial Distribution with r = 0.5 and β = 10:



Here is a Negative Binomial Distribution with parameters β = 2/3 and r = 8:[31]

Number of Claims	f(x)	F(x)	Number of Claims times f(x)	Square of Number of Claims times f(x)
0	0.0167962	0.01680	0.00000	0.00000
1	0.0537477	0.07054	0.05375	0.05375
2	0.0967459	0.16729	0.19349	0.38698
3	0.1289945	0.29628	0.38698	1.16095
4	0.1418940	0.43818	0.56758	2.27030
5	0.1362182	0.57440	0.68109	3.40546
6	0.1180558	0.69245	0.70833	4.25001
7	0.0944446	0.78690	0.66111	4.62779
8	0.0708335	0.85773	0.56667	4.53334
9	0.0503705	0.90810	0.45333	4.08001
10	0.0342519	0.94235	0.34252	3.42519
11	0.0224194	0.96477	0.24661	2.71275
12	0.0141990	0.97897	0.17039	2.04465
13	0.0087378	0.98771	0.11359	1.47669
14	0.0052427	0.99295	0.07340	1.02757
15	0.0030757	0.99603	0.04614	0.69204
16	0.0017685	0.99780	0.02830	0.45275
17	0.0009987	0.99879	0.01698	0.28863
18	0.0005548	0.99935	0.00999	0.17977
19	0.0003037	0.99965	0.00577	0.10964
20	0.0001640	0.99982	0.00328	0.06560
21	0.0000875	0.99990	0.00184	0.03857
22	0.0000461	0.99995	0.00101	0.02232
23	0.0000241	0.99997	0.00055	0.01273
24	0.0000124	0.99999	0.00030	0.00716
25	0.0000064	0.99999	0.00016	0.00398
26	0.0000032	1.00000	0.00008	0.00218
27	0.0000016	1.00000	0.00004	0.00119
28	0.0000008	1.00000	0.00002	0.00064
29	0.0000004	1.00000	0.00001	0.00034
30	0.0000002	1.00000	0.00001	0.00018
Sum	1.00000		5.33333	37.33314

For example, $f(5) = \{(2/3)^5 / (1+ 2/3)^{8+5}\}(12!) / \{(5!)(7!)\}$
$= (0.000171993)(479{,}001{,}600)/\{(120)(5040)\} = 0.136$.

The mean is: rβ = 8(2/3) = 5.333. The variance is: 8(2/3)(1+2/3) = 8.89.

The variance can also be computed as: (mean)(1+β) = 5.333(5/3) = 8.89.

The variance is indeed = $E[X^2] - E[X]^2 = 37.333 - 5.3333^2 = 8.89$.

According to the formula given previously, the mode should be the largest integer in (r-1)β = (8-1)(2/3) = 4.67, which contains the integer 4. In fact, f(4) = 14.2% is the largest value of the probability density function. Since F(5) = 0.57 ≥ 0.5 and F(4) = 0.44 < 0.5, 5 is the median.

[31] The values for the Negative Binomial probability density function in the table were computed using:
$f(0) = (\beta/(1+\beta))^r$ and $f(x+1) / f(x) = \beta(x+r) / \{(x+1)(1+\beta)\}$.
For example, f(12) = f(11)β(11+r) / {12(1+β)} = (0.02242)(2/3)(19) / 20 = 0.01420.

Mean and Variance of the Negative Binomial Distribution:

The mean of a Geometric distribution is β and its variance is $\beta(1+\beta)$. Since the Negative Binomial is a sum of r Geometric Distributions, it follows that the **mean of the Negative Binomial is $r\beta$ and the variance of the Negative Binomial is $r\beta(1+\beta)$.**
Since $\beta > 0$, $1+\beta > 1$, **for the Negative Binomial Distribution the variance is greater than the mean.**

For the Negative Binomial, the ratio of the variance to the mean is $1+\beta$, while variance/mean = 1 for the Poisson Distribution.
Thus (β)(mean) is the "extra variance" for the Negative Binomial compared to the Poisson.

Non-Integer Values of r:

Note that even if r is not integer, the binomial coefficient in the front of the Negative Binomial Density can be calculated as: $\binom{x+r-1}{x} = \frac{(x+r-1)!}{x!\,(r-1)!} = \frac{(x+r-1)\,(x+r-2)\,...\,(r)}{x!}$.

For example with r = 6.2 if one wanted to compute f(4), then the binomial coefficient in front is:
$\binom{4+6.2-1}{4} = \binom{9.2}{4} = \frac{9.2!}{5.2!\,4!} = \frac{(9.2)\,(8.2)\,(7.2)\,(6.2)}{4!} = 140.32.$

Note that the numerator has 4 factors; in general it will have x factors. These four factors are:
9.2! / (9.2-4)! = 9.2!/5.2!, or if you prefer: $\Gamma(10.2) / \Gamma(6.2)$ = (9.2)(8.2)(7.2)(6.2).

As shown in Loss Models, in general one can rewrite the density of the Negative Binomial as:
$$\mathbf{f(x) = \frac{r(r+1)...(r+x-1)}{x!}\,\frac{\beta^x}{(1+\beta)^{x+r}}},$$ where there are x factors in the product in the numerator.

Exercise: For a Negative Binomial with parameters r = 6.2 and β = 7/3, compute f(4).
[Solution: f(4) = {(9.2)(8.2)(7.2)(6.2)/4!} $(7/3)^4$ / $(1+ 7/3)^{6.2+4}$ = 0.0193.]

<u>Negative Binomial as a Mixture of Poissons</u>:

As discussed subsequently, when Poissons are mixed via a Gamma Distribution, the mixed distribution is always a Negative Binomial Distribution, with $r = \alpha$ = shape parameter of the Gamma and $\beta = \theta$ = scale parameter of the Gamma. The mixture of Poissons via a Gamma distribution produces a Negative Binomial Distribution and increases the variance above the mean.

<u>Series of Bernoulli Trials</u>:

Return to the situation that resulted in the Geometric distribution, involving a series of independent Bernoulli trials each with chance of success $1/(1 + \beta)$, and chance of failure of $\beta/(1 + \beta)$.
What is the probability of two successes and four failures in the first six trials?

It is given by the Binomial Distribution:

$$\binom{6}{2} 1/(1+\beta)^2 \{1 - 1/(1+\beta)\}^4 = \binom{6}{2} \beta^4/(1+\beta)^6.$$

The chance of having the third success on the seventh trial is given by $1/(1 + \beta)$ times the above probability:

$$\binom{6}{2} \beta^4/(1+\beta)^7$$

Similarly the chance of the third success on trial x + 3 is given by $1/(1 + \beta)$ times the probability of 3 - 1 = 2 successes and x failures on the first x + 3 - 1 = x + 2 trials:

$$\binom{x+2}{2} \beta^x/(1+\beta)^{x+3}$$

More generally, the chance of the r^{th} success on trial x+r is given by $1/(1 + \beta)$ times the probability of r-1 success and x failures on the first x+r-1 trials.

$$f(x) = \{1/(1+\beta)\} \binom{x+r-1}{r-1} \beta^x/(1+\beta)^{x+r-1} = \binom{x+r-1}{x} \beta^x/(1+\beta)^{x+r}, \ x = 0, 1, 2, 3...$$

This is the Negative Binomial Distribution. Thus we see that one source of the Negative Binomial is the chance of experiencing failures on a series of independent Bernoulli trials prior to getting a certain number of successes.[32] Note that in the derivation, $1/(1 + \beta)$ is the chance of success on each Bernoulli trial.

Thus, $\beta = \frac{\beta / (1+\beta)}{1/(1+\beta)} = \frac{\text{chance of a failure}}{\text{chance of a success}}$.

For a series of independent identical Bernoulli trials, the chance of success number r following x failures is given by a Negative Binomial Distribution with parameters β = (chance of a failure) / (chance of a success), and r.

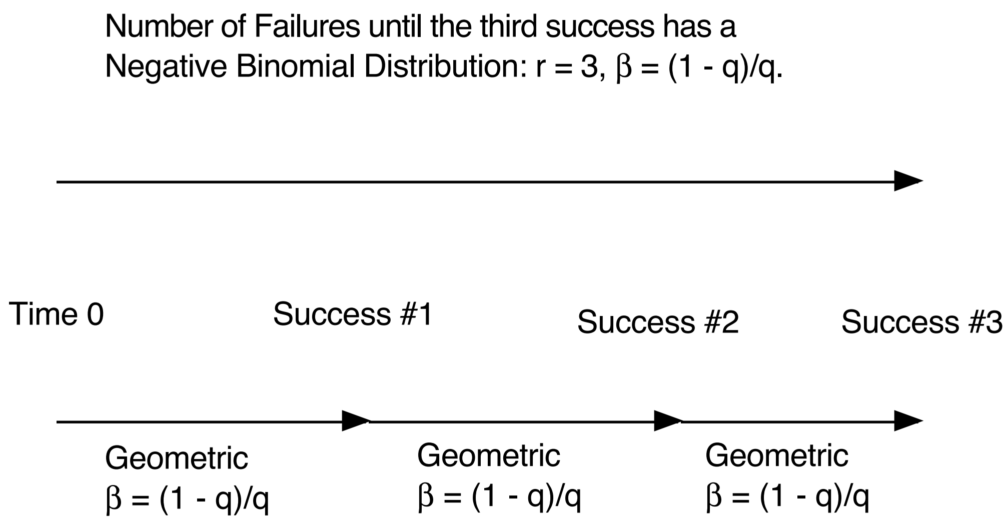
Exercise: One has a series of independent Bernoulli trials, each with chance of success 0.3.
What is the distribution of the number of failures prior to the 5th success?
[Solution: A Negative Binomial Distribution, as per Loss Models, with parameters $\beta = 0.7/0.3 = 7/3$, and $r = 5$.]

While this is one derivation of the Negative Binomial distribution, note that the Negative Binomial Distribution is used to model claim counts in many situations that have no relation to this derivation.

[32] Even though the Negative Binomial Distribution was derived here for integer values of r, as has been discussed, the Negative Binomial Distribution is well defined for r non-integer as well.

Negative Binomial as a Sum of Geometric Distributions:

The number of claims for a Negative Binomial Distribution was modeled as the number of failures prior to getting a total of r successes on a series of independent Bernoulli trials. Instead one can add up the number of failures associated with getting a single success r times independently of each other. As seen before, each of these is given by a Geometric distribution. Therefore, obtaining r successes is the sum of r separate independent variables each involving getting a single success.



Therefore, the Negative Binomial Distribution with parameters β and r, with r integer, can be thought of as the sum of r independent Geometric distributions with parameter β.

The Negative Binomial Distribution for r = 1 is a Geometric Distribution.

Since the Geometric distribution is the discrete analog of the Exponential distribution, the Negative Binomial distribution is the discrete analog of the continuous Gamma Distribution[33].

The parameter r in the Negative Binomial is analogous to the parameter α in the Gamma Distribution.[34]

$(1+\beta)/\beta$ in the Negative Binomial Distribution is analogous to $e^{1/\theta}$ in the Gamma Distribution.

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[33] Recall that the Gamma Distribution is a sum of α independent Exponential Distributions, just as the Negative Binomial is the sum of r independent Geometric Distributions.

[34] Note that the mean and variance of the Negative Binomial and the Gamma are proportional respectively to r and α.

Adding Negative Binomial Distributions:

Since the Negative Binomial is a sum of Geometric Distributions, if one sums independent Negative Binomials with the same β, then one gets another Negative Binomial, with the same β parameter and the sum of their r parameters.[35]

Exercise: X is a Negative Binomial with β = 1.4 and r = 0.8. Y is a Negative Binomial with β = 1.4 and r = 2.2. Z is a Negative Binomial with β = 1.4 and r = 1.7. X, Y, and Z are independent of each other. What form does X + Y + Z have?
[Solution: X + Y + Z is a Negative Binomial with β = 1.4 and r = .8 + 2.2 + 1.7 = 4.7.]

If X is Negative Binomial with parameters β and r_1, and Y is Negative Binomial with parameters β and r_2, X and Y independent, then X + Y is Negative Binomial with parameters β and $r_1 + r_2$.

Specifically, the sum of n independent identically distributed Negative Binomial variables, with the same parameters β and r, is a Negative Binomial with parameters β and nr.

Exercise: X is a Negative Binomial with β = 1.4 and r = 0.8.
What is the form of the sum of 25 independent random draws from X?
[Solution: A random draw from a Negative Binomial with β = 1.4 and r = (25)(.8) = 20.]

Thus if one had 25 exposures, each of which had an independent Negative Binomial frequency process with β = 1.4 and r = 0.8, then the portfolio of 25 exposures has a Negative Binomial frequency process with β = 1.4 and r = 20.

[35] This holds whether or not r is integer. This is analogous to adding independent Gammas with the same θ parameter. One obtains a Gamma, with the same θ parameter, but with the new α parameter equal to the sum of the individual α parameters.

Effect of Exposures:

Assume one has 100 exposures with independent, identically distributed frequency distributions. If each one is Negative Binomial with parameters β and r, then so is the sum, with parameters β and 100r. If we change the number of exposures to for example 150, then the sum is Negative Binomial with parameters β and 150r, or 1.5 times the r parameter in the first case.
In general, as the exposures change, the r parameter changes in proportion.[36]

Exercise: The total number of claims from a portfolio of insureds has a Negative Binomial Distribution with $\beta = 0.2$ and $r = 30$.
If next year the portfolio has 120% of the current exposures, what is its frequency distribution?
[Solution: Negative Binomial with $\beta = 0.2$ and $r = (1.2)(30) = 36$.]

Thinning Negative Binomial Distributions:

Thinning can also be applied to the Negative Binomial Distribution.[37]
The β parameter of the Negative Binomial Distribution is multiplied by the thinning factor.

Exercise: Claim frequency follows a Negative Binomial Distribution with parameters $\beta = 0.20$ and $r = 1.5$. One quarter of all claims involve attorneys. If attorney involvement is independent between different claims, what is the probability of getting two claims involving attorneys in the next year?
[Solution: Claims with attorney involvement are Negative Binomial Distribution with $\beta = (.20)(25\%) = 0.05$ and $r = 1.5$.
Thus $f(2) = r(r+1)\beta^2 / \{2!\ (1+\beta)^{r+2}\} = (1.5)(2.5)(.05)^2 / \{2\ (1.05)^{3.5}\} = 0.395\%$.]

Note that when thinning the parameter β is altered, while when adding the r parameter is affected. As discussed previously, if one adds two independent Negative Binomial Distributions with the same β, then the result is also a Negative Binomial Distribution, with the sum of the r parameters.

[36] See Section 7.4 of Loss Models, not on the syllabus. This same result holds for a Compound Frequency Distribution, to be discussed subsequently, with a primary distribution that is Negative Binomial.
[37] See Table 8.3 in Loss Models. However, unlike the Poisson case, the large and small accidents are not independent processes.

Problems:

The following six questions all deal with a Negative Binomial distribution with parameters $\beta = 0.4$ and $r = 3$.

6.1 (1 point) What is the mean?
A. less than .9
B. at least .9 but less than 1.0
C. at least 1.0 but less than 1.1
D. at least 1.1 but less than 1.2
E. at least 1.2

6.2 (1 point) What is the variance?
A. less than 1.8
B. at least 1.8 but less than 1.9
C. at least 1.9 but less than 2.0
D. at least 2.0 but less than 2.1
E. at least 2.1

6.3 (2 points) What is the chance of having 4 claims?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

6.4 (2 points) What is the mode?
A. 0 B. 1 C. 2 D. 3 E. None of A, B, C, or D.

6.5 (2 points) What is the median?
A. 0 B. 1 C. 2 D. 3 E. None of A, B, C, or D.

6.6 (2 points) What is the chance of having 4 claims or less?
A. 90% B. 92% C. 94% D. 96% E. 98%

6.7 (2 points) Bud and Lou play a series of games. Bud has a 60% chance of winning each game. Lou has a 40% chance of winning each game. The outcome of each game is independent of any other. Let N be the number of games Bud wins prior to Lou winning 5 games.
What is the variance of N?
A. less than 14
B. at least 14 but less than 16
C. at least 16 but less than 18
D. at least 18 but less than 20
E. at least 20

6.8 (1 point) For a Negative Binomial distribution with $\beta = 2/9$ and r = 1.5, what is the chance of having 3 claims?
A. 1% B. 2% C. 3% D. 4% E. 5%

6.9 (2 points) In baseball a team bats in an inning until it makes 3 outs. Assume each batter has a 40% chance of getting on base and a 60% chance of making an out. Then what is the chance of a team sending exactly 8 batters to the plate in an inning? (Assume no double or triple plays. Assume nobody is picked off base, caught stealing or thrown out on the bases. Assume each batter's chance of getting on base is independent of whether another batter got on base.)
A. less than 1%
B. at least 1% but less than 2%
C. at least 2% but less than 3%
D. at least 3% but less than 4%
E. at least 4%

6.10 (1 point) Assume each exposure has a Negative Binomial frequency distribution, as per Loss Models, with $\beta = 0.1$ and r = 0.27. You insure 20,000 independent exposures.
What is the frequency distribution for your portfolio?
A. Negative Binomial with $\beta = 0.1$ and r = 0.27.
B. Negative Binomial with $\beta = 0.1$ and r = 5400.
C. Negative Binomial with $\beta = 2000$ and r = 0.27.
D. Negative Binomial with $\beta = 2000$ and r = 5400.
E. None of the above.

6.11 (3 points) Frequency is given by a Negative Binomial distribution with β = 1.38 and r = 3.

Severity is given by a Weibull Distribution with τ = 0.3 and θ = 1000.

Frequency and severity are independent.

What is chance of two losses each of size greater than $25,000?

A. 1% B. 2% C. 3% D. 4% E. 5%

Use the following information for the next two questions:

Six friends each have their own phone.

The number of calls each friend gets per night from telemarketers is Geometric with β = 0.3.

The number of calls each friend gets is independent of the others.

6.12 (2 points) Tonight, what is the probability that three of the friends get one or more calls from telemarketers, while the other three do not?

A. 11% B. 14% C. 17% D. 20% E. 23%

6.13 (2 points) Tonight, what is the probability that the friends get a total of three calls from telemarketers?

A. 11% B. 14% C. 17% D. 20% E. 23%

6.14 (2 points) The total number of claims from a group of 80 drivers has a Negative Binomial Distribution with β= 0.5 and r = 4.

What is the probability that a group of 40 similar drivers have a total of 2 or more claims?

A. 22% B. 24% C. 26% D. 28% E. 30%

6.15 (2 points) The total number of non-zero payments from a policy with a $1000 deductible follows a Negative Binomial Distribution with β= 0.8 and r = 3.

The ground up losses follow an Exponential Distribution with θ = 2500.

If this policy instead had a $5000 deductible, what would be the probability of having no non-zero payments?

A. 56% B. 58% C. 60% D. 62% E. 64%

6.16 (3 points) The mathematician Stefan Banach smoked a pipe.
In order to light his pipe, he carried a matchbox in each of two pockets.
Each time he needs a match, he is equally likely to take it from either matchbox.
Assume that he starts the month with two matchboxes each containing 20 matches.
Eventually Banach finds that when he tries to get a match from one of his matchboxes it is empty.
What is the probability that when this occurs, the other matchbox has exactly 5 matches in it?
A. less than 6%
B. at least 6% but less than 7%
C. at least 7% but less than 8%
D. at least 8% but less than 9%
E. at least 9%

6.17 (2 points) Total claim counts generated from a portfolio of 400 policies follow a Negative Binomial distribution with parameters r = 3 and β = 0.4. If the portfolio increases to 500 policies, what is the probability of observing exactly 2 claims in total?
A. 21% B. 23% C. 25% D. 27% E. 29%

Use the following information for the next three questions:
Two teams are playing against one another in a seven game series.
The results of each game are independent of the others.
The first team to win 4 games wins the series.

6.18 (3 points) The Flint Tropics have a 45% chance of winning each game.
What is the Flint Tropics chance of winning the series?
A. 33% B. 35% C. 37% D. 39% E. 41%

6.19 (3 points) The Durham Bulls have a 60% chance of winning each game.
What is the Durham Bulls chance of winning the series?
A. 67% B. 69% C. 71% D. 73% E. 75%

6.20 (3 points) The New York Knights have a 40% chance of winning each game.
The Knights lose the first game. The opposing manager offers to split the next two games with the Knights (each team would win one of the next two games.)
Should the Knights accept this offer?

6.21 (3 points) The number of losses follows a Negative Binomial distribution with r = 4 and β = 3. Sizes of loss are uniform from 0 to 15,000.
There is a deductible of 1000, a maximum covered loss of 10,000, and a coinsurance of 90%.
Determine the probability that there are exactly six payments of size greater than 5000.
A. 9.0% B. 9.5% C. 10.0% D. 10.5% E. 11.0%

6.22 (2 points) Define $(N - j)_+ = n - j$ if $n \geq j$, and 0 otherwise.

N follows a Negative Binomial distribution with r = 5 and β = 0.3. Determine $E[(N - 2)_+]$.

A. 0.25 B. 0.30 C. 0.35 D. 0.40 E. 0.45

6.23 (3 points) The number of new claims the State House Insurance Company receives in a day follows a Negative Binomial Distribution r = 5 and β = 0.8. For a claim chosen at random, on average how many other claims were also made on the same day?

A. 4.0 B. 4.2 C. 4.4 D. 4.6 E. 4.8

6.24 (2, 5/83, Q.44) (1.5 points) If a fair coin is tossed repeatedly, what is the probability that the third head occurs on the nth toss?

A. $(n-1)/2^{n+1}$ B. $(n - 1)(n - 2)/2^{n+1}$ C. $(n - 1)(n - 2)/2^n$ D. $(n-1)/2^n$ E. $\binom{n}{3} / 2^n$

6.25 (2, 5/90, Q.45) (1.7 points) A coin is twice as likely to turn up tails as heads. If the coin is tossed independently, what is the probability that the third head occurs on the fifth trial?

A. 8/81 B. 40/243 C. 16/81 D. 80/243 E. 3/5

6.26 (2, 2/96, Q.28) (1.7 points) Let X be the number of independent Bernoulli trials performed until a success occurs. Let Y be the number of independent Bernoulli trials performed until 5 successes occur. A success occurs with probability p and Var(X) = 3/4.

Calculate Var(Y).

A. 3/20 B. $3/(4\sqrt{5})$ C. 3/4 D. 15/4 E. 75/4

6.27 (1, 11/01, Q.11) (1.9 points) A company takes out an insurance policy to cover accidents that occur at its manufacturing plant. The probability that one or more accidents will occur during any given month is 3/5. The number of accidents that occur in any given month is independent of the number of accidents that occur in all other months.

Calculate the probability that there will be at least four months in which no accidents occur before the fourth month in which at least one accident occurs.

(A) 0.01 (B) 0.12 (C) 0.23 (D) 0.29 (E) 0.41

6.28 (1, 11/01, Q.21) (1.9 points) An insurance company determines that N, the number of claims received in a week, is a random variable with $P[N = n] = 1/2^{n+1}$, where $n \geq 0$.

The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly seven claims will be received during a given two-week period.

(A) 1/256 (B) 1/128 (C) 7/512 (D) 1/64 (E) 1/32

6.29 (CAS3, 11/03, Q.18) (2.5 points) A new actuarial student analyzed the claim frequencies of a group of drivers and concluded that they were distributed according to a negative binomial distribution and that the two parameters, r and β, were equal.
An experienced actuary reviewed the analysis and pointed out the following:
"Yes, it is a negative binomial distribution. The r parameter is fine, but the value of the β parameter is wrong. Your parameters indicate that 1/9 of the drivers should be claim-free, but in fact, 4/9 of them are claim-free."
Based on this information, calculate the variance of the corrected negative binomial distribution.
A. 0.50 B. 1.00 C. 1.50 D. 2.00 E. 2.50

6.30 (CAS3, 11/04, Q.21) (2.5 points) The number of auto claims for a group of 1,000 insured drivers has a negative binomial distribution with β = 0.5 and r = 5.
Determine the parameters β and r for the distribution of the number of auto claims for a group of 2,500 such individuals.
A. β = 1.25 and r = 5
B. β = 0.20 and r = 5
C. β = 0.50 and r = 5
D. β = 0.20 and r= 12.5
E. β = 0.50 and r = 12.5

6.31 (CAS3, 5/05, Q.28) (2.5 points)
You are given a negative binomial distribution with r = 2.5 and β = 5.
For what value of k does p_k take on its largest value?
A. Less than 7 B. 7 C. 8 D. 9 E. 10 or more

6.32 (CAS3, 5/06, Q.32) (2.5 points) Total claim counts generated from a portfolio of 1,000 policies follow a Negative Binomial distribution with parameters r = 5 and β = 0.2.
Calculate the variance in total claim counts if the portfolio increases to 2,000 policies.
A. Less than 1.0
B. At least 1.0 but less than 1.5
C. At least 1.5 but less than 2.0
D. At least 2.0 but less than 2.5
E. At least 2.5

6.33 (CAS3, 11/06, Q.23) (2.5 points) An actuary has determined that the number of claims follows a negative binomial distribution with mean 3 and variance 12.
Calculate the probability that the number of claims is at least 3 but less than 6.
A. Less than 0.20
B. At least 0.20, but less than 0.25
C. At least 0.25, but less than 0.30
D. At least 0.30, but less than 0.35
E. At least 0.35

6.34 (CAS3, 11/06, Q.24) (2.5 points) Two independent random variables, X_1 and X_2, follow the negative binomial distribution with parameters (r_1, β_1) and (r_2, β_2), respectively.

Under which of the following circumstances will $X_1 + X_2$ always be negative binomial?

1. $r_1 = r_2$.
2. $\beta_1 = \beta_2$.
3. The coefficients of variation of X_1 and X_2 are equal.

A. 1 only B. 2 only C. 3 only D. 1 and 3 only E. 2 and 3 only

6.35 (CAS3, 11/06, Q.31) (2.5 points)
You are given the following information for a group of policyholders:

- The frequency distribution is negative binomial with $r = 3$ and $\beta = 4$.
- The severity distribution is Pareto with $\alpha = 2$ and $\theta = 2{,}000$.

Calculate the variance of the number of payments if a $500 deductible is introduced.
A. Less than 30
B. At least 30, but less than 40
C. At least 40, but less than 50
D. At least 50, but less than 60
E. At least 60

6.36 (SOA M, 11/06, Q.22 & 2009 Sample Q.283) (2.5 points) The annual number of doctor visits for each individual in a family of 4 has a geometric distribution with mean 1.5.
The annual numbers of visits for the family members are mutually independent.

An insurance pays 100 per doctor visit beginning with the 4^{th} visit per family.

Calculate the expected payments per year for this family.
(A) 320 (B) 323 (C) 326 (D) 329 (E) 332

Solutions to Problems:

6.1. E. mean = $r\beta$ = (3)(.4) = **1.2**.

6.2. A. variance = $r\beta(1+\beta)$ = (3)(.4)(1.4) = **1.68**.

6.3. B. $\binom{x+r-1}{x} \beta^x / (1+\beta)^{x+r} = \binom{6}{4}$ $(0.4)^4 / (1.4)^{4+3}$ = 15 (0.0256)/(10.54) = **0.0364**.

6.4. A. & 6.5. B. The mode is **0**, since f(0) is larger than any other value.

n	0	1	2	3	4
f(n)	0.3644	0.3124	0.1785	0.0850	0.0364
F(n)	0.364	0.677	0.855	0.940	0.977

The median is **1**, since F(0) <.5 and F(1) ≥ .5.

Comment: I've used the formulas: $f(0) = (\beta/(1+\beta))^r$ and $f(x+1) / f(x) = \beta(x+r) / \{(x+1)(1+\beta)\}$.
Just as with the Gamma Distribution, the Negative Binomial can have either a mode of zero or a positive mode. For $r < 1 + 1/\beta$, as is the case here, the mode is zero, and the Negative Binomial looks somewhat similar to an Exponential Distribution.

6.6. E. F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = **97.7%**.

n	0	1	2	3	4
f(n)	0.3644	0.3124	0.1785	0.0850	0.0364
F(n)	0.3644	0.6768	0.8553	0.9403	**0.9767**

Comment: *Using the Incomplete Beta Function:* $F(4) = 1 - \beta(4+1, r; \beta/(1+\beta)) = 1 - \beta(5,3; .4/1.4) = 1 - 0.0233 = 0.9767$.

6.7. D. This is series of Bernoulli trials. Treating Lou's winning as a "success", then chance of success is 40%. N is the number of failures prior to the 5th success.
Therefore N has a Negative Binomial Distribution with r = 5 and
β = chance of failure / chance of success = 60%/40% = 1.5.
Variance is: $r\beta(1+\beta)$ = (5)(1.5)(2.5) = **18.75**.

6.8. A. $f(3) = \{r(r+1)(r+2)\} / 3!\}\ \beta^3 / (1+\beta)^{3+r} = \{(1.5)(2.5)(3.5)/ 6\}\ (2/9)^3\ (11/9)^{-4.5}$ = **0.0097**.

6.9. E. For the defense a batter reaching base is a failure and an out is a success. The number of batters reaching base is the number of failures prior to 3 successes for the defense. The chance of a success for the defense is 0.6. Therefore the number of batters who reach base is given by a Negative Binomial with r = 3 and

β = (chance of failure for the defense)/(chance of success for the defense) = 0.4/0.6 = 2/3.

If exactly 8 batters come to the plate, then 5 reach base and 3 make out. The chance of exactly 5 batters reaching base is f(5) for r = 3 and β = 2/3: $\{(3)(4)(5)(6)(7)/5!\}\ \beta^5 / (1+\beta)^{5+r}$ =

(21)(0.13169)/59.537 = **0.0464**.

Alternately, for there to be exactly 8 batters, the last one has to make an out, and exactly two of the first 7 must make an out. Prob[2 of 7 make out] $\Leftrightarrow$

density at 2 of Binomial Distribution with m = 7 and q = 0.6 $\Leftrightarrow ((7)(6)/2)0.6^2 0.4^5$ = 0.0774.

Prob[8th batter makes an out]Prob[2 of 7 make an out] = (0.6)(0.0774) = **0.0464**.

<u>Comment</u>: Generally, one can use either a Negative Binomial Distribution or some reasoning and a Binomial Distribution in order to answer these type of questions.

For there to be exactly 8 batters, the last one has to make an out, and exactly two of the first 7 must make an out. The team at bat sits down when the third batter makes out.

If instead 6 batters get on base and 2 batters make out, then the ninth batter would get up.

6.10. B. The sum of independent Negative Binomials, each with the same β, is another Negative Binomial, with the sum of the r parameters. In this case we get a Negative Binomial with β = 0.1 and r = (0.27)(20,000) = 5400.

6.11. D. S(25,000) = $\exp(-(25000/1000)^{0.3})$ = 0.0723. The losses greater than $25,000 is another Negative Binomial with r = 3 and β = (1.38)(0.0723) = 0.0998.

For a Negative Binomial, $f(2) = (r(r+1)/2)\beta^2/(1+\beta)^{r+2} = \{(3)(4)/2\}0.0998^2 /(1.0998)^5$= **3.71%**.

<u>Comment</u>: An example of thinning a Negative Binomial.

6.12. A. For the Geometric, f(0) = 1/(1+β) = 1/1.3. 1 - f(0) = .3/1.3.

Prob[3 with 0 and 3 not with 0] = $\{6! / (3!\ 3!)\}\ (1/1.3)^3\ (.3/1.3)^3$ **= 0.112**.

6.13. B. The total number of calls is Negative Binomial with r = 6 and β = .3.

$f(3) = (r(r+1)(r+2)/3!)\beta^3/(1+\beta)^{3+r} = ((6)(7)(8)/3!).3^3/1.3^9$ = **0.143**.

6.14. C. The frequency for the 40 drivers is Negative Binomial Distribution with parameters r = (40/80)(4) = 2 and β= 0.5.

$f(0) = 1/1.5^2$ = 44.44%. $f(1) = 2(.5/1.5^3)$ = 29.63%. 1 - f(0) - f(1) = **25.9%**.

6.15. E. For the Exponential, S(1000) = exp[-1000/2500] = .6703.
S(5000) = exp[-5000/2500] = .1353. Therefore, with the $5000 deductible, the non-zero payments are Negative Binomial Distribution with r = 3 and β = (.1353/.6703)(0.8) = .16.
f(0) = $1/1.16^3$ = **64%**.

6.16. E. Let us assume the righthand matchbox is the one discovered to be empty.
Call a "success" choosing the righthand box and a "failure" choosing the lefthand box.
Then we have a series of Bernoulli trials, with chance of success 1/2.
The number of "failures" prior to the 21st "success" (looking in the righthand matchbox 20 times and getting a match and once more finding no matches are left) is Negative Binomial with r = 21 and
β = (chance of failure)/(chance of success) = (1/2)/(1/2) = 1.
For the lefthand matchbox to then have 5 matches, we must have had 15 "failures".
Density at 15 for this Negative Binomial is: {(21)(22)...(35) / 15!} $1^{15}/(1 + 1)^{15+21}$ = 4.73%.
However, it is equally likely that the lefthand matchbox is the one discovered to be out of matches.
Thus we double this probability: (2)(4.73%) = **9.5%**.
<u>Comment</u>: Difficult. The famous Banach Match problem.

6.17. A. When one changes the number of exposures, the r parameter changes in proportion.
For 500 policies, total claim counts follow a Negative Binomial distribution with parameters
r = 3(500/400) = 3.75 and β = 0.4.
f(2) = $\{r(r+1)/2\}\beta^2/(1+\beta)^{r+2}$ = $(3.75)(4.75)(.5)(.4^2)/(1.4^{5.75})$ = **20.6%**.
<u>Comment</u>: Similar to CAS3, 5/06, Q.32.

6.18. D. Ignoring the fact that once a team wins four games, the final games of the series will not be played, the total number of games won out of seven by the Tropics is Binomial with q = 0.45 and m = 7. We want the sum of the densities of this Binomial from 4 to 7:
$35(0.45^4)(0.55^3) + 21(0.45^5)(0.55^2) + 7(0.45^6)(0.55) + 0.45^7$
= 0.2388 + 0.1172 + 0.0320 + 0.0037 = **0.3917**.
Alternately, the number of failures by the Tropics prior to their 4th success is Negative Binomial with r = 4 and β = .55/.45 = 11/9.
For the Tropics to win the series they have to have 3 or fewer loses prior to their 4th win.
The probability of this is the sum of the densities of the Negative Binomial at 0 to 3:
$1/(20/9)^4 + 4(11/9)/(20/9)^5 + \{(4)(5)(11/9)^2/2!\}/(20/9)^6 + \{(4)(5)(6)(11/9)^3/3!\}/(20/9)^7$
= 0.0410 + 0.0902 + 0.1240 + 0.1364 = **0.3916**.
<u>Comment</u>: The question ignores any effect of home field advantage.

6.19. C. Ignoring the fact that once a team wins four games, the final games of the series will not be played, the total number of games won out of seven by the Bulls is Binomial with q = 0.60 and m = 7. We want the sum of the densities of this Binomial from 4 to 7:

$35(.6^4)(.4^3) + 21(.6^5)(.4^2) + 7(.6^6)(.4) + .6^7$

= 0.2903 + 0.2613 + 0.1306 + 0.0280 = **0.7102**.

Alternately, the number of failures by the Bulls prior to their 4th success is Negative Binomial with r = 4 and β = .4/.6 = 2/3.

For the Bulls to win the series they have to have 3 or fewer loses prior to their 4th win.

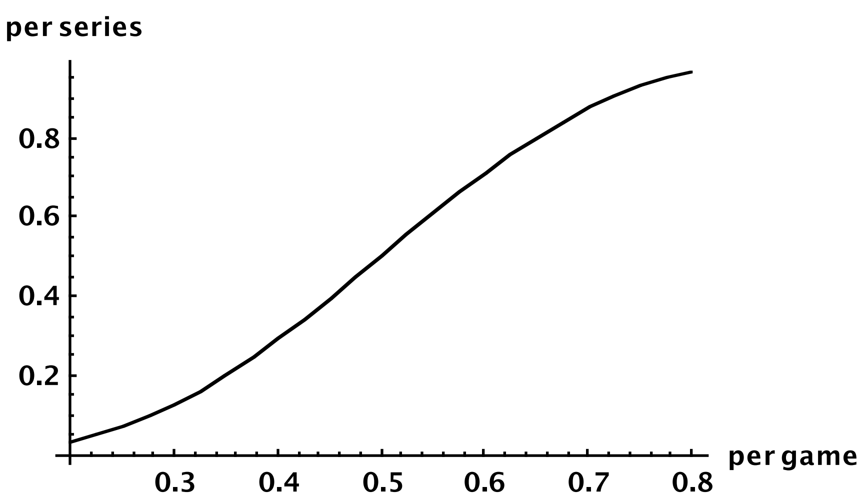
The probability of this is the sum of the densities of the Negative Binomial at 0 to 3:

$1/(5/3)^4 + 4(2/3)/(5/3)^5 + \{(4)(5)(2/3)^2/2!\}/(5/3)^6 + \{(4)(5)(6)(2/3)^3/3!\}/(5/3)^7$

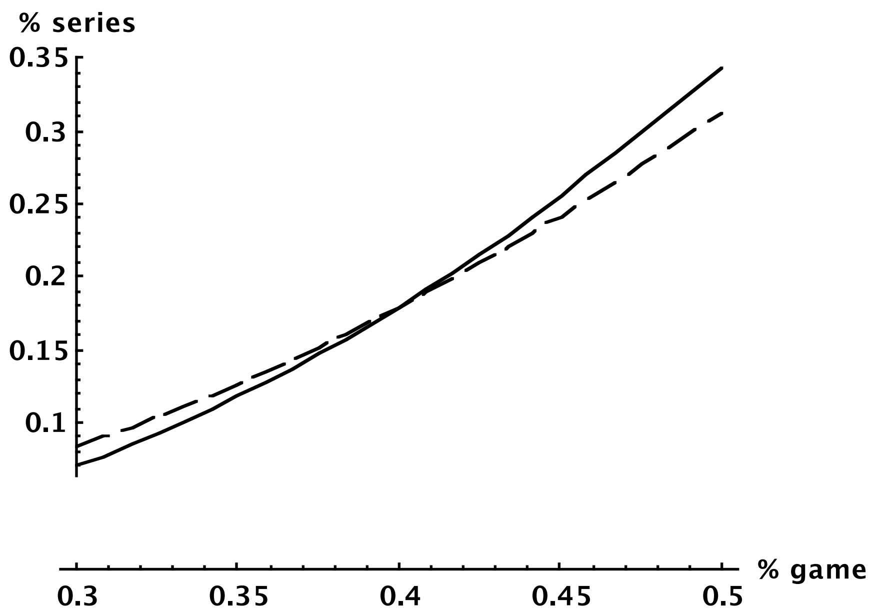
= 0.1296 + 0.2074 + 0.2074 + 0.1659 = **0.7103**.

Comment: According to Bill James, "A useful rule of thumb is that the advantage doubles in a seven-game series. In other words, if one team would win 51% of the games between two opponents, then they would win 52% of the seven-game series. If one team would win 55% of the games, then they would win 60% of the series."

Here is a graph of the chance of winning the seven game series, as a function of the chance of winning each game:



6.20. If the Knights do not accept the offer, then they need to win four of six games.
We want the sum of the densities from 4 to 6 of a Binomial with q = .4 and m = 6:
$15(.4^4)(.6^2) + 6(.4^5)(.6) + .4^6 = 0.1382 + 0.0369 + 0.0041 = 0.1792.$
If the Knights accept the offer, then they need to win three of four games.
We want the sum of the densities from 3 to 4 of a Binomial with q = .4 and m = 4:
$4(.4^3)(.6) + .4^4 = 0.1536 + 0.0256 = 0.1792.$
Thus the **Knights are indifferent between accepting this offer or not**.
Alternately, if the Knights do not accept the offer, then they need to win four of six games.
The number of failures by the Knights prior to their 4th success is Negative Binomial with
r = 4 and β = .6/.4 = 1.5. The Knights win the series if they have 2 or fewer failures:
$1/2.5^4 + 4(1.5)/2.5^5 + \{(4)(5)(1.5)^2/2!\}/2.5^6 = 0.0256 + 0.0614 + 0.0922 = 0.1792.$
If the Knights accept the offer, then they need to win three of four games.
The number of failures by the Knights prior to their 3rd success is Negative Binomial with
r = 3 and β = .6/.4 = 1.5. The Knights win the series if they have 1 or fewer failures:
$1/2.5^3 + 3(1.5)/2.5^4 = 0.0640 + 0.1152 = 0.1792.$
Thus the **Knights are indifferent between accepting this offer or not**.
Comment: A comparison of their chances of winning the series as a function of their chance of winning a game, accepting the offer (dashed) and not accepting the offer (solid):



The Knights should accept the offer if their chance of winning each game is less than 40%.

6.21. C. A payment is of size greater than 5000 if the loss is of size greater than:
5000/.9 + 1000 = 6556. Probability of a loss of size greater than 6556 is: 1 - 6556/15000 = 56.3%. The large losses are Negative Binomial with r = 4 and β = (56.3%)(3) = 1.69.

$f(6) = \{r(r+1)(r+2)(r+3)(r+4)(r+5)/6!\}\beta^6/(1+\beta)^{r+6} = \{(4)(5)(6)(7)(8)(9)/720\}1.69^6/2.69^{10}$ = **9.9%**.

Comment: An example of thinning a Negative Binomial.

6.22. C. $f(0) = 1/1.3^5 = 0.2693$. $f(1) = (5)0.3/1.3^6 = 0.3108$.

$E[N] = 0f(0) + 1f(1) + 2f(2) + 3f(3) + 4f(4) + 5f(5) + ...$ $E[(N - 2)_+] = 1f(3) + 2f(4) + 3f(5) + ...$

$E[N] - E[(N - 2)_+] = f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + ... = f(1) + 2\{1 - f(0) - f(1)\} = 2 - 2f(0) - f(1)$.

$E[(N - 2)_+] = E[N] - \{2 - 2f(0) - f(1)\} = (5)(.3) - \{2 - (2)(.2693) -.3108\}$ = **0.3494**.

Alternately, $E[N \wedge 2] = 0f(0) + 1f(1) + 2\{1 - f(0) - f(1)\} = 1.1506$.
$E[(N - 2)_+] = E[N] - E[N \wedge 2] = (5)(0.3) - 1.1506$ = **0.3494**.

Alternately, $E[(N - 2)_+] = E[(2-N)_+] + E[N] - 2 = 2f(0) + f(1) + (5)(0.3) - 2$ = **0.3494**.

Comment: See the section on Limited Expected Values in "Mahler's Guide to Fitting Loss Distributions."

6.23. E. Let n be the number of claims made on a day.
The probability that the claim picked is on a day of size n is proportional to the product of the number of claims on that day and the proportion of days of that size: n f(n).

Thus, Prob[claim is from a day with n claims] = $n f(n) / \Sigma n f(n) = n f(n) / E[N]$.

For n > 0, the number of other claims on the same day is n - 1.

Average number of other claims is: $\dfrac{\sum_1 n\, f(n)\, (n-1)}{E[N]} = \dfrac{\sum_1 (n^2 - n)\, f(n)}{E[N]} = \dfrac{E[N^2] - E[N]}{E[N]} =$

$\dfrac{E[N^2]}{E[N]} - 1 = \dfrac{Var[N] + E[N]^2}{E[N]} - 1 = \dfrac{Var[N]}{E[N]} + E[N] - 1 = 1 + \beta + r\beta - 1 = (r + 1)\beta = (6)(0.8)$ = **4.8**.

Comment: The average day has four claims; on the average day there are three other claims. However, a claim chosen at random is more likely to be from a day that had a lot of claims.

6.24. B. This is a Negative Binomial with r = 3, β = chance of failure / chance of success = 1, and x = number of failures = n - 3.

$f(x) = \{r(r+1)...(r+x-1)/x!\}\beta^x/(1+\beta)^{r+x} = \{(3)(4)...(x+2)/x!\}/2^{3+x} = (x+1)(x+2)/2^{4+x}$.

$f(n-3) =$ **$(n-2)(n-1)/2^{n+1}$**.

Alternately, for the third head to occur on the nth toss, for n ≥ 3, we have to have had two head out of the first n-1 tosses, which has probability $\binom{n-1}{2} / 2^{n-1} = (n-2)(n-1) / 2^n$, and a head on the nth toss, which has probability 1/2. Thus the total probability is: **$(n-2)(n-1)/2^{n+1}$**.

6.25. A. The number of tails before the third head is Negative Binomial, with r = 3 and β = chance of failure / chance of success = chance of tail / chance of head = 2.

Prob[third head occurs on the fifth trial] = Prob[2 tails when the get 3rd head] = f(2) =

$\{r(r+1)/2\}\beta^2/(1+\beta)^{r+2} = (6)(4)/3^5 =$ **8/81**.

Alternately, need 2 heads and 2 tails out of the first 4 tosses, and then a head on the fifth toss:

$\{4!/(2!2!)\}(1/3)^2(2/3)^2\ (1/3) =$ **8/81**.

6.26. D. X-1 is Geometric with β = chance of failure / chance of success = (1 - p)/p = 1/p - 1.

Therefore, 3/4 = Var(X) = Var(X-1) = β(1 + β) = (1/p - 1){1/p).

$0.75p^2 + p - 1 = 0. \Rightarrow p = \{-1 + \sqrt{1+3}\}/1.5 = 2/3$.

β = 3/2 - 1 = 1/2. Y-5 is Negative Binomial with r = 5 and β = 1/2.

Var[Y - 5] = Var[Y] = (5)(1/2)(3/2) = **15/4**.

Alternately, once one has gotten the first success, the number of additional trials until the second success is independent of and has the same distribution as X, the number of additional trials until the first success. ⇒ Y = X + X + X + X + X. ⇒ Var[Y] = 5Var[X] = (5)(3/4) = **15/4**.

6.27. D. Define a “success” as a month in which at least one accident occurs.
We have a series of independent Bernoulli trials, and we stop upon the fourth success.
The number of failures before the fourth success is Negative Binomial with r = 4 and
β = chance of failure / chance of success = (2/5)/(3/5) = 2/3.

$f(0) = 1/(1 + 2/3)^4 = 0.1296$. $f(1) = 4(2/3)/(5/3)^5 = 0.20736$.

$f(2) = \{(4)(5)/2!\}(2/3)^2/(5/3)^6 = 0.20736$. $f(3) = \{(4)(5)(6)/3!\}(2/3)^3/(5/3)^7 = 0.165888$.

Prob[at least 4 failures] = 1 - (0.1296 + 0.20736 + 0.20736 + 0.165888) = **0.289792.**
Alternately, instead define a “success” as a month in which no accident occurs.
We have a series of independent Bernoulli trials, and we stop upon the fourth success.
The number of failures before the fourth success is Negative Binomial with r = 4 and
β = chance of failure / chance of success = (3/5)/(2/5) = 1.5.

$f(0) = 1/(1 + 1.5)^4 = 0.0256$. $f(1) = (4)1.5/2.5^5 = 0.06144$.

$f(2) = \{(4)(5)/2!\}1.5^2/2.5^6 = 0.09216$. $f(3) = \{(4)(5)(6)/3!\}1.5^3/2.5^7 = 0.110592$.

The event we want will occur if at the time of the fourth success, the fourth month in which no accidents occur, there have been fewer than four failures, in other words fewer than four months in which at least one accident occurs.
Prob[fewer than 4 failures] = 0.0256 + 0.06144 + 0.09216 + 0.110592 = **0.289792.**

6.28. D. The number of claims in a week is Geometric with $\beta/(1+\beta) = 1/2. \Rightarrow \beta = 1$.

The sum of two independent Geometrics is a Negative Binomial with r = 2 and β = 1.

$f(7) = \{(2)(3)(4)(5)(6)(7)(8)/7!\}\beta^7/(1+\beta)^9$ = **1/64**.

6.29. C. For the student’s Negative Binomial, r = β: $f(0) = 1/(1+\beta)^r = 1/(1+r)^r = 1/9. \Rightarrow r = 2$.

For the corrected Negative Binomial, r = 2 and: $f(0) = 1/(1+\beta)^r = 1/(1+\beta)^2 = 4/9. \Rightarrow \beta = .5$.

Variance of the corrected Negative Binomial = $r\beta(1+\beta)$ = (2)(.5)(1.5) = **1.5**.

6.30. E. For a Negative Binomial distribution, as the exposures change we get another Negative Binomial; the r parameter changes in proportion, while β remains the same.

The new r = (2500/1000)(5) = 12.5. **β = 0.5 and r = 12.5**.

6.31. B. For a Negative Binomial, a = $\beta/(1 + \beta)$ = 5/6,

and b = $(r - 1)\beta/(1 + \beta)$ = (1.5)(5/6) = 5/4.

f(x)/f(x-1) = a + b/x = 5/6 + (5/4)/x, x = 1, 2, 3, ...
To find the mode, where the density is largest, find when this ratio is greater than 1.

5/6 + (5/4)/x = 1. $\Rightarrow$ x/6 = 5/4. x = 7.5.

So f(7)/f(6) > 1 while f(8)/f(7) < 1, and **7** is the mode.
Comment: f(6) = .0556878. f(7) = .0563507. f(8) = .0557637.

6.32. D. Doubling the exposures, multiplies r by 2. For 2000 policies, total claim counts follow a Negative Binomial distribution with parameters r = 10 and β = 0.2.

Variance = $r\beta(1+\beta)$ = (10)(0.2)(1.2) = **2.4**.

Alternately, for 1000 policies, the variance of total claim counts is: (5)(0.2)(1.2) = 1.2.

2000 policies. $\Leftrightarrow$ 1000 policies + 1000 policies.

$\Rightarrow$ For 2000 policies, the variance of total claim counts is: 1.2 + 1.2 = **2.4**.

Comment: When one adds independent Negative Binomial Distribution with the same β, one gets another Negative Binomial Distribution with the sum of the r parameters. When one changes the number of exposures, the r parameter changes in proportion.

6.33. B. $r\beta = 3$. $r\beta(1+\beta) = 12$. $\Rightarrow 1 + \beta = 12/3 = 4$. $\Rightarrow \beta = 3$. $\Rightarrow r = 1$.

f(3) + f(4) + f(5) = $3^3/4^4 + 3^4/4^5 + 3^5/4^6$ = **0.244**.
Comment: We have fit via Method of Moments. Since r = 1, this is a Geometric Distribution.

6.34. B. 1. False. 2. True.
3. CV = $\sqrt{r\beta(1+\beta)}\ /\ (r\beta) = \sqrt{(1+\beta)/r}$. False.

Comment: For the Negative Binomial, $P(z) = 1/\{1 - \beta(z-1)\}^r$.
The p.g.f. of the sum of two independent variables is the product of their p.g.f.s:
$1/(\{1 - \beta_1(z-1)\}^{r_1}\ \{1 - \beta_2(z-1)\}^{r_2})$.

This only has the same form as a Negative Binomial if and only if $\beta_1 = \beta_2$.

6.35. A. For the Pareto, S(500) = $(2/2.5)^2$ = 0.64. Thus the number of losses of size greater than 500 is Negative Binomial with r = 3 and β = (0.64)(4) = 2.56.
The variance of the number of large losses is: (3)(2.56)(3.56) = **27.34**.

6.36. D. The total number of visits is the sum of 4 independent, identically distributed Geometric Distributions, which is a Negative Binomial with r = 4 and β = 1.5.

$f(0) = 1/2.5^4 = 0.0256$. $f(1) = (4)1.5/2.5^5 = 0.06144$. $f(2) = \{(4)(5)/2\}1.5^2/2.5^6 = 0.09216$.

$E[N \wedge 3] = 0f(0) + 1f(1) + 2f(2) + 3\{1 - f(0) - f(1) - f(2)\} = 2.708$.

$E[(N-3)_+] = E[N] - E[N \wedge 3] = (4)(1.5) - 2.708 = 3.292$. $100E[(N-3)_+] =$ **329.2**.

Alternately, $E[(N-3)_+] = E[(3-N)_+] + E[N] - 3 = 3f(0) + 2f(1) + f(2) + (4)(1.5) - 3 = 3.292$.

Comment: The exam question is intending to ask the expected amount that the insurer will to pay due to claims from this family. This could have been made clearer.

See the section on Limited Expected Values in “Mahler’s Guide to Fitting Loss Distributions.”

Section 7, Normal Approximation

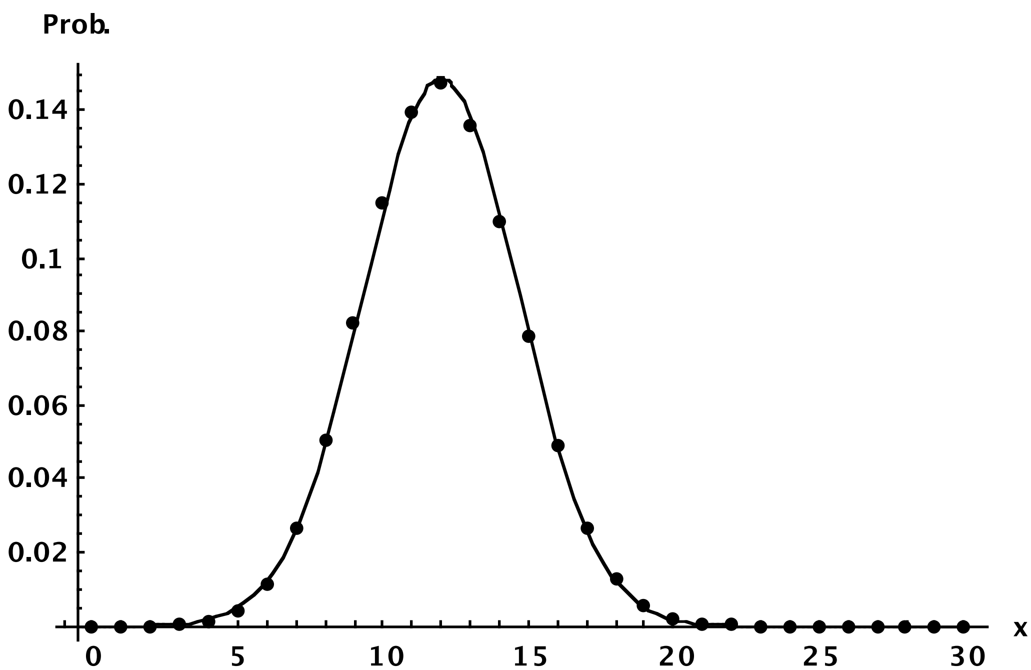
This section will go over important information that Loss Models assumes the reader already knows concerning the Normal Distribution and its use to approximate frequency distributions. These ideas are important for practical applications of frequency distributions.[38]

The Binomial Distribution with parameters q and m is the sum of m independent Bernoulli trials, each with parameter q. The Poisson Distribution with λ integer, is the sum of λ independent Poisson variables each with mean of one. The Negative Binomial Distribution with parameters β and r, with r integer, is the sum of r independent Geometric distributions each with parameter β.

Thus by the Central Limit Theorem, each of these distributions can be approximated by a Normal Distribution with the same mean and variance.

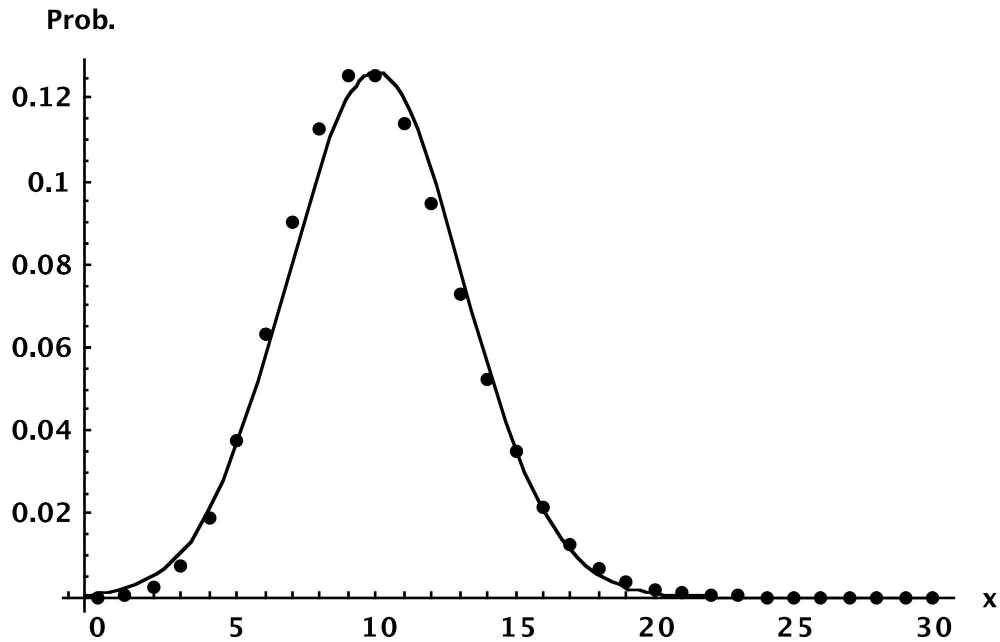
For the Binomial as $m \to \infty$, for the Poisson as $\lambda \to \infty$, and for the Negative Binomial as $r \to \infty$, the distribution approaches a Normal[39]. The approximation is quite good for large values of the relevant parameter, but not very good for extremely small values.

For example, here is the graph of a Binomial Distribution with q = 0.4 and m = 30.
It is has mean (30)(0.4) = 12 and variance = (30)(0.4)(0.6) = 7.2.
Also shown is a Normal Distribution with μ = 12 and $\sigma = \sqrt{7.2}$ = 2.683.

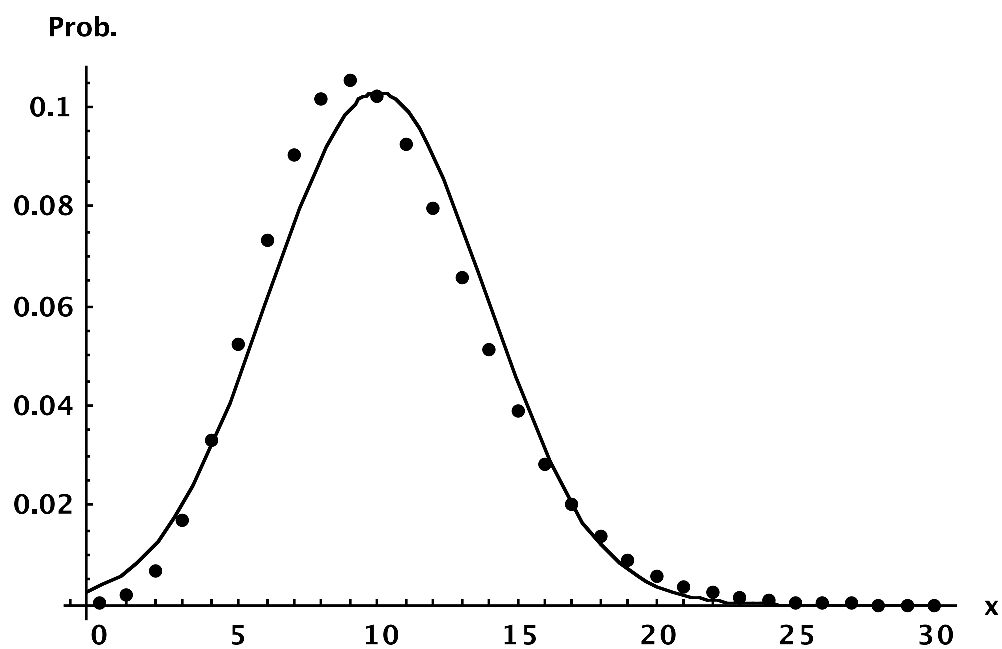


[38] These ideas also underlay Classical Credibility.
[39] In fact as discussed in a subsequent section, the Binomial and the Negative Binomial each approach a Poisson which in turn approaches a Normal.

Here is the graph of a Poisson Distribution with $\lambda = 10$, and the approximating Normal Distribution with $\mu = 10$ and $\sigma = \sqrt{10} = 3.162$:



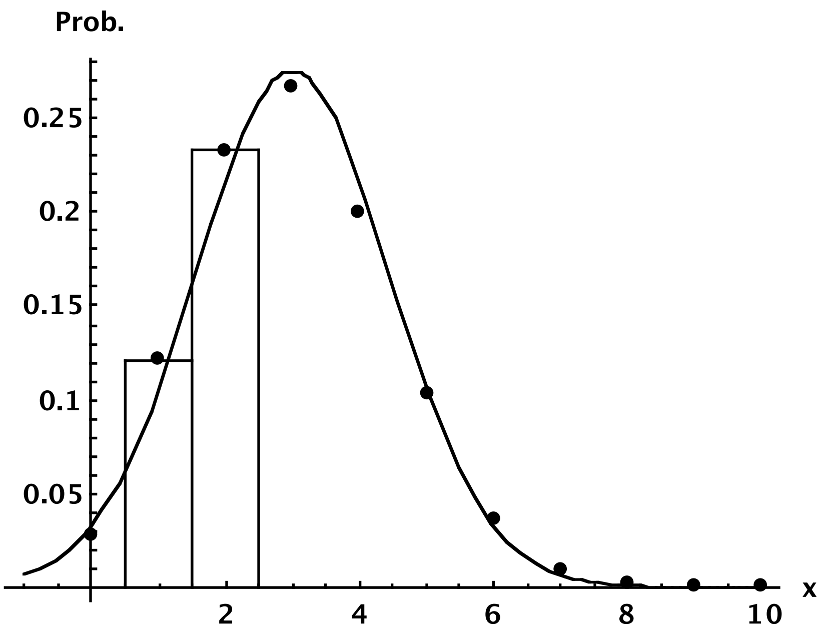
Here is the graph of a Negative Binomial Distribution with $\beta = 0.5$ and $r = 20$, with mean $(20)(0.5) = 10$ and variance $(20)(0.5)(1.5) = 15$, and the approximating Normal Distribution with $\mu = 10$ and $\sigma = \sqrt{15} = 3.873$:



A typical use of the Normal Approximation would be to find the probability of observing a certain range of claims. For example, given a certain distribution, what is the probability of at least 10 and no more than 20 claims.

Exercise: Given a Binomial with parameters q = 0.3 and m = 10, what is the chance of observing 1 or 2 claims?
[Solution: $10(0.3^1)(0.7^9) + 45(0.3^2)(0.7^8) = 0.1211 + 0.2335 = 0.3546$.]

In this case one could compute the exact answer as the sum of only two terms.
Nevertheless, let us illustrate how the Normal Approximation could be used in this case.
The Binomial distribution with q = 0.3 and m = 10 has a mean of: (0.3)(10) = 3, and a variance of: (10)(0.3)(0.7) = 2.1. This Binomial Distribution can be approximated by a Normal Distribution with mean of 3 and variance of 2.1, as shown below:

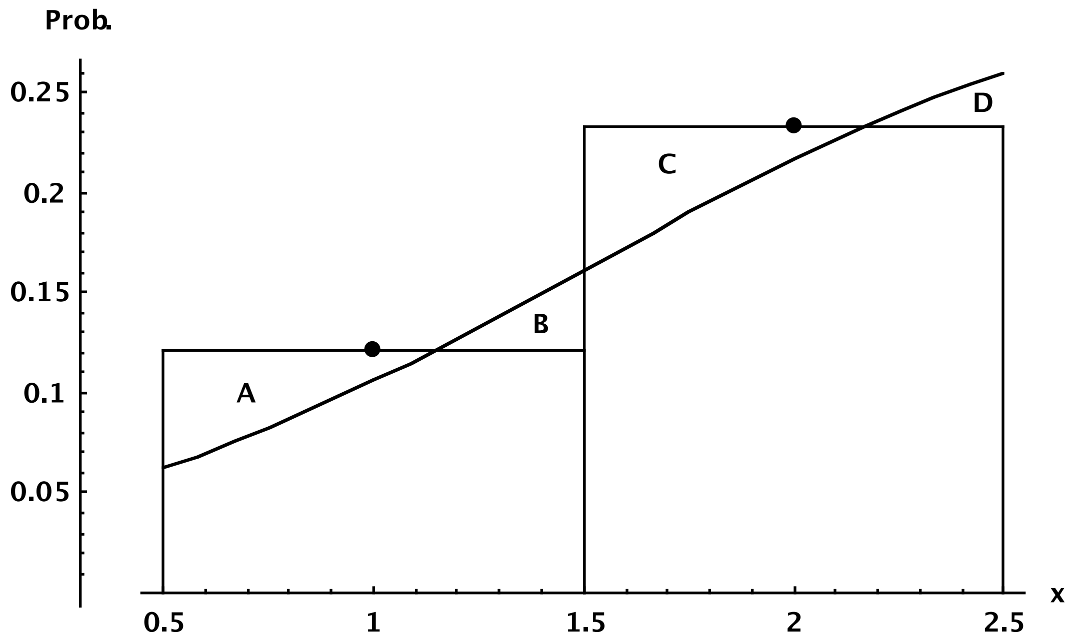


Prob[1 claim] = the area of a rectangle of width one and height f(1) = 0.1211.
Prob[2 claims] = the area of a rectangle of width one and height f(2) = 0.2335.
The chance of either one or two claims is the sum of these two rectangles; this is approximated by the area under this Normal Distribution, with mean 3 and variance 2.1, from 1 - .5 = .5 to 2 + .5 = 2.5.

Prob[1 or 2 claims] $\cong \Phi[(2.5-3)/\sqrt{2.1}] - \Phi[(.5-3)/\sqrt{2.1}] = \Phi[-0.345] - \Phi[-1.725]$
$= 0.365 - 0.042 = 0.323$.

Note that in order to get the probability for two values on the discrete Binomial Distribution, one has to cover an interval of length two on the real line for the continuous Normal Distribution. We subtracted 1/2 from the lower end of 1 and added 1/2 to the upper end of 2.
This is called the “**continuity correction**”.

Below, I have zoomed in on the relevant of part of the previous diagram:



It should make it clear why the continuity correction is needed. In this case the chance of having 1 or 2 claims is equal to the area under the two rectangles, which is not close to the area under the Normal from 1 to 2, but is approximated by the area under the Normal from 0.5 to 2.5.

In order to use the Normal Approximation, one must translate to the so called "Standard" Normal Distribution[40]. In this case, we therefore need to standardize the variables by subtracting the mean of 3 and dividing by the standard deviation of $\sqrt{2.1}$ = 1.449. In this case,

0.5 $\leftrightarrow$ (0.5 - 3) / 1.449 = -1.725, while 2.5 $\leftrightarrow$ (2.5 - 3) / 1.449 = -0.345. Thus, the chance of observing either 1 or 2 claims is approximately: Φ[-0.345] - Φ[-1.725] = 0.365 - 0.042 = 0.323.

This compares to the exact result of .3546 calculated above. The diagram above shows why the approximation was too small in this particular case[41]. Area A is within the first rectangle, but not under the Normal Distribution. Area B is not within the first rectangle, but is under the Normal Distribution. Area C is within the second rectangle, but not under the Normal Distribution. Area D is not within the second rectangle, but is under the Normal Distribution.
Normal Approximation minus Exact Result = (Area B - Area A) + (Area D - Area C).

While there was no advantage to using the Normal approximation in this example, it saves a lot of time when trying to deal with many terms.

[40] Attached to the exam and shown below.
[41] The approximation gets better as the mean of the Binomial gets larger. The error can be either positive or negative.

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately: $\Phi[\frac{(j + 0.5) - \mu}{\sigma}] - \Phi[\frac{(i - 0.5) - \mu}{\sigma}]$.

Exercise: Use the Normal Approximation in order to estimate the probability of observing at least 10 claims but no more than 18 claims from a Negative Binomial Distribution with parameters β = 2/3 and r = 20.

[Solution: Mean = rβ = 13.33 and variance = rβ(1+β) = 22.22.

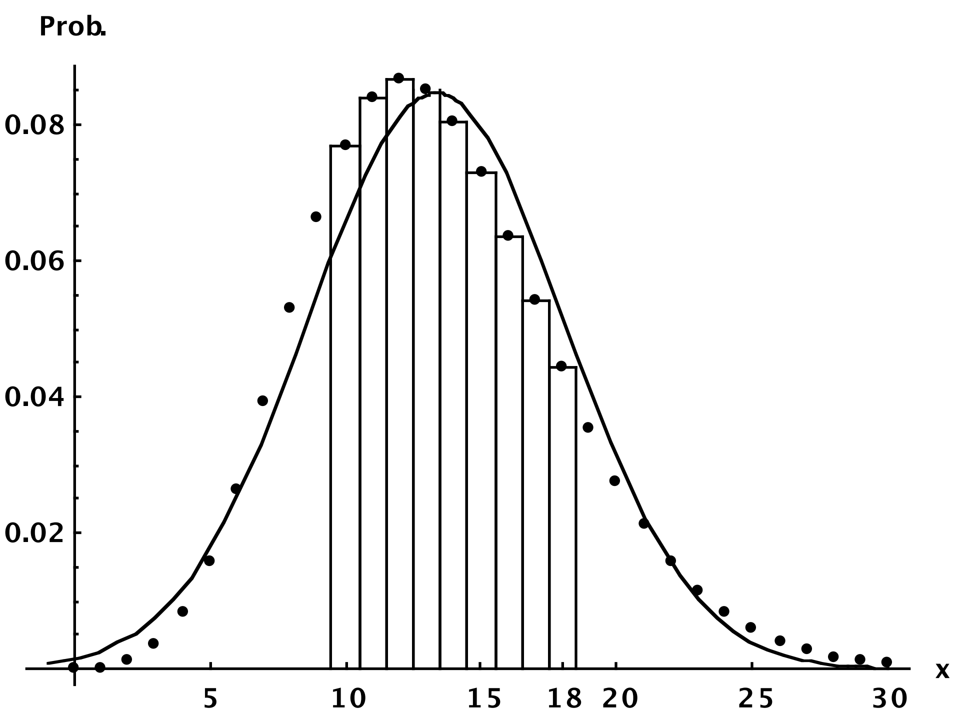
Prob[at least 10 claims but no more than 18 claims] ≅

$\Phi[(18.5 - 13.33) / \sqrt{22.22}] - \Phi[(9.5-13.33) / \sqrt{22.22}] = \Phi[1.097] - \Phi[-0.813] =$

0.864 - 0.208 = 0.656.

Comment: The exact answer is 0.648.]

Here is a graph of the Normal Approximation used in this exercise:



The continuity correction in this case: at least 10 claims but no more than 18 claims ↔ 10 - 1/2 = 9.5 to 18 + 1/2 = 18.5 on the Normal Distribution.

Note that Prob[10 ≤ # claims ≤ 18] = Prob[9 < # claims < 19]. Thus one must be careful to carefully check the wording, to distinguish between open and closed intervals.

Prob[9 < # claims < 19] = Prob[10 ≤ # claims ≤ 18] ≅ $\Phi[\{18.5 - \mu\} / \sigma] - \Phi[\{9.5 - \mu\} / \sigma]$.

One should **use the continuity correction whenever one is using the Normal Distribution in order to approximate the probability associated with a discrete distribution**.

Do not use the continuity correction when one is using the Normal Distribution in order to approximate continuous distributions, such as aggregate distributions[42] or the Gamma Distribution.

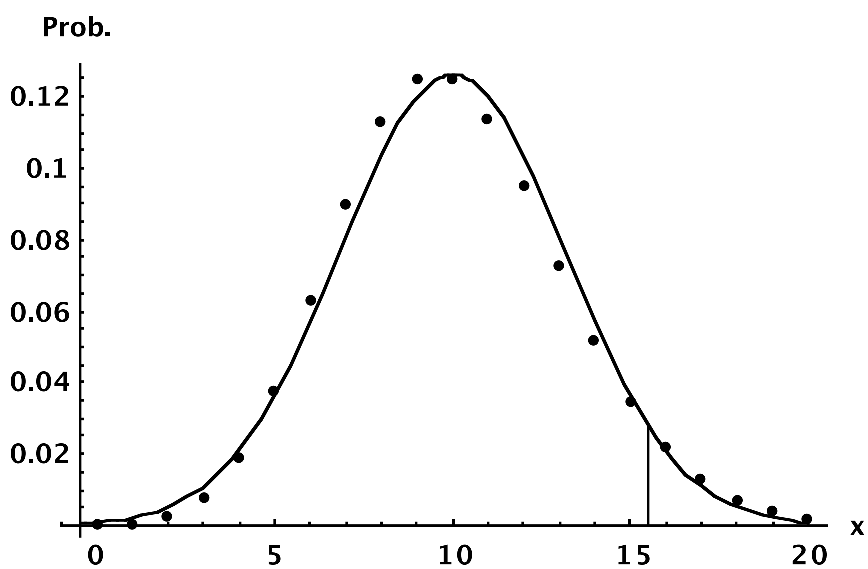
Exercise: Use the Normal Approximation in order to estimate the probability of observing more than 15 claims from a Poisson Distribution with $\lambda = 10$.

[Solution: Mean = variance = 10. Prob[# claims > 15] = 1 - Prob[# claims ≤ 15] ≅

$1 - \Phi[(15.5-10)/\sqrt{10}] = 1 - \Phi[1.739] = 1 - 0.9590 = 4.10\%$.

Comment: The exact answer is 4.87%.]

The area under the Normal Distribution and to the right of the vertical line at 15.5 is the approximation used in this exercise:



[42] See "Mahler's Guide to Aggregate Distributions."

<u>Diagrams</u>:

Some of you will find the following simple diagrams useful when applying the Normal Approximation to discrete distributions.

More than 15 claims ⇔ At least 16 claims ⇔ 16 claims or more

15 15.5 16

|⟶

Prob[More than 15 claims] ≅ 1 - Φ[(15.5 - μ)/σ].

Exercise: For a frequency distribution with mean 14 and standard deviation 2, using the Normal Approximation, what is the probability of at least 16 claims?

[Solution: Prob[At least 16 claims] = Prob[More than 15 claims] ≅ 1 - Φ[(15.5 - μ)/σ] =

1 - Φ[(15.5 - 14)/2] = 1 - Φ[0.75] = 1 - 0.07734 = 22.66%.]

Less than 12 claims ⇔ At most 11 claims ⇔ 11 claims or less

11 11.5 12

⟵|

Prob[Less than 12 claims] ≅ Φ[(11.5 - μ)/σ].

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of at most 11 claims?

[Solution: Prob[At most 11 claims] = Prob[Less than 12 claims] ≅ Φ[(11.5 - μ)/σ] =

Φ[(11.5 - 10)/4] = Φ[0.375] = 64.6%.]

At least 10 claims and at most 13 claims ⇔ More than 9 claims and less than 14 claims

9 9.5 10 11 12 13 13.5 14

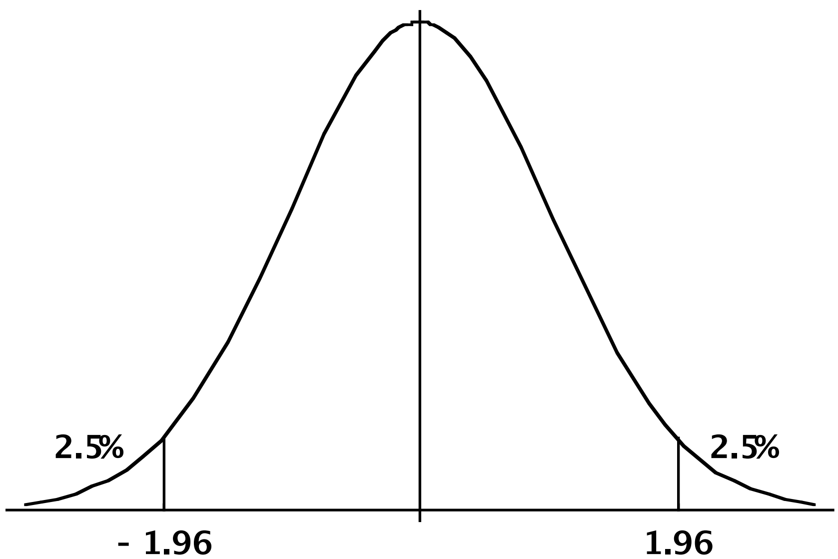
Prob[At least 10 claims and at most 13 claims] $\cong \Phi[(13.5 - \mu)/\sigma] - \Phi[(9.5 - \mu)/\sigma]$.

Exercise: For a frequency distribution with mean 10 and standard deviation 4, using the Normal Approximation, what is the probability of more than 9 claims and less than 14 claims?
[Solution: Prob[more than 9 claims and less than 14 claims] =
Prob[At least 10 claims and at most 13 claims] $\cong \Phi[(13.5 - \mu)/\sigma] - \Phi[(9.5 - \mu)/\sigma]$ =
$\Phi[(13.5 - 10)/4] - \Phi[(9.5 - 10)/4] = \Phi[0.875] - \Phi[-0.125] = 0.809 - 0.450 = 35.9\%$.]

<u>Confidence Intervals</u>:

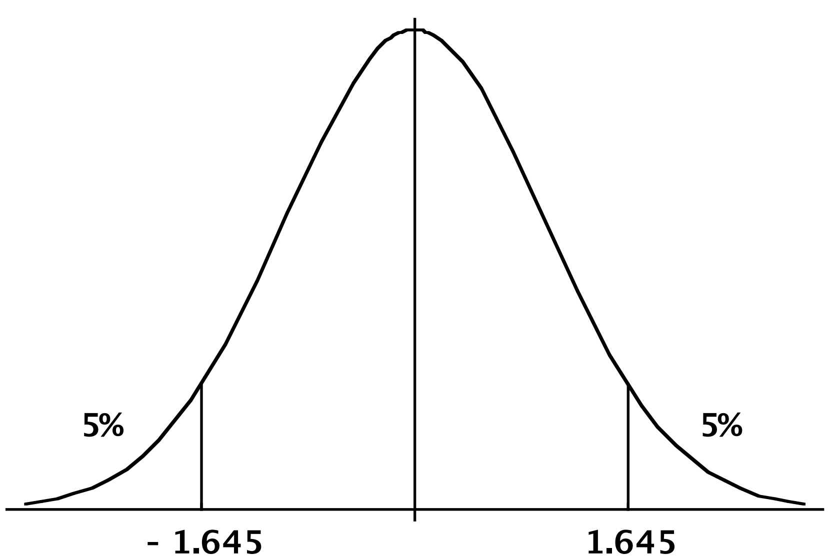
One can use the lower portion of the Normal Distribution table in order to get confidence intervals.

For example, in order to get a 95% confidence interval, one allows 2.5% probability on either tail.
$\Phi(1.96) = (1 + 95\%)/2 = 97.5\%$.
Thus 95% of the probability on the Standard Normal Distribution is between -1.96 and 1.96:



Thus a 95% confidence interval for a Normal would be: mean ± 1.960 standard deviations.

Similarly, since $\Phi(1.645) = (1 + 90\%)/2 = 95\%$, a 90% confidence interval is:
mean ± 1.645 standard deviations.

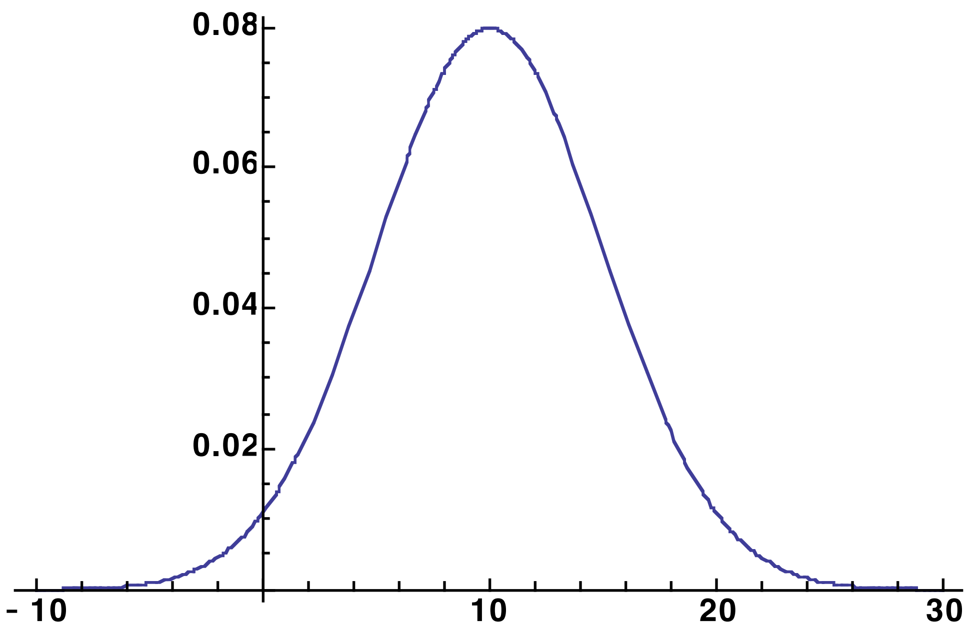


Normal Distribution:

The Normal Distribution is a bell-shaped symmetric distribution. Its two parameters are its mean μ and its standard deviation σ. $f(x) = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$, $-\infty < x < \infty$.

The sum of two independent Normal Distributions is also a Normal Distribution, with the sum of the means and variances. **If X is normally distributed, then so is aX + b**, but with mean $a\mu$+b and standard deviation $a\sigma$. If one standardizes a normally distributed variable by subtracting μ and dividing by σ, then one obtains a Standard Normal with mean 0 and standard deviation of 1.

A Normal Distribution with μ = 10 and σ = 5:



The density of the Standard Normal is denoted by $\phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}$, $-\infty < x < \infty$.[43]

The corresponding distribution function is denoted by $\Phi(x)$.

$\Phi(x) \cong 1 - \phi(x)\{.4361836t - .1201676t^2 + .9372980t^3\}$, *where* $t = 1/(1+.33267x)$.[44]

[43] As shown near the bottom of the first page of the Tables for Exam C.
[44] See pages 103-104 of Simulation by Ross or 26.2.16 in Handbook of Mathematical Functions.

Normal Distribution

Support: $\infty > x > -\infty$ Parameters: $\infty > \mu > -\infty$ (location parameter)

$\sigma > 0$ (scale parameter)

D. f. : **F(x) =** $\Phi[(x-\mu)/\sigma]$

P. d. f. : $f(x) = \phi\left[\frac{x - \mu}{\sigma}\right] / \sigma = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}$. $\phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}$.

Central Moments: $E[(X-\mu)^n] = \sigma^n\, n! / \{2^{n/2}\,(n/2)!\}$ *n even,* $n \geq 2$

$E[(X-\mu)^n] = 0$ *n odd,* $n \geq 1$

Mean = μ **Variance =** σ^2

Coefficient of Variation = Standard Deviation / Mean = σ/μ

Skewness = 0 (distribution is symmetric) Kurtosis = 3

Mode = μ Median = μ

Limited Expected Value Function:

$E[X \wedge x] = \mu\Phi[(x-\mu)/\sigma] - \sigma\exp[-(x-\mu)^2/(2\sigma^2)]/\sqrt{2\pi} + x\{1 - \Phi[(x-\mu)/\sigma]\}$

Excess Ratio: $R(x) = \{1 - x/\mu\}\{1-\Phi((x-\mu)/\sigma)\} + (\sigma/\mu)\exp(-[(x-\mu)^2]/[2\sigma^2])/\sqrt{2\pi}$

Mean Residual Life: $e(x) = \mu - x + \sigma\exp(-[(x-\mu)^2]/[2\sigma^2])/\{\{1 - \Phi((x-\mu)/\sigma)\sqrt{2\pi}\}\}$

Derivatives of d.f. : $\partial F(x)/\partial\mu = -\phi((x-\mu)/\sigma)$ $\partial F(x)/\partial\sigma = -\phi((x-\mu)/\sigma)/\sigma^2$

Method of Moments: $\mu = \mu_1'$, $\sigma = (\mu_2' - \mu_1'^2)^{0.5}$

Percentile Matching: Set $g_i = \Phi^{-1}(p_i)$, *then* $\sigma = (x_1 - x_2)/(g_1 - g_2)$, $\mu = x_1 - \sigma g_1$

Method of Maximum Likelihood: Same as Method of Moments.

<u>Using the Normal Table</u>:

When using the normal distribution, choose the nearest z-value to find the probability, or if the probability is given, chose the nearest z-value. No interpolation should be used.
Example: If the given z-value is 0.759, and you need to find Pr(Z < 0.759) from the normal distribution table, then choose the probability value for z-value = 0.76; Pr(Z < 0.76) = 0.7764.
When using the Normal Approximation to a discrete distribution, use the continuity correction.[45]

When using the top portion of the table, use the symmetry of the Standard Normal Distribution around zero: $\Phi[-x] = 1 - \Phi[x]$.

For example, $\Phi[-0.4] = 1 - \Phi[0.4] = 1 - 0.6554 = 0.3446$.

The bottom portion of the table can be used to get confidence intervals.
To cover a confidence interval of probability P, find y such that $\Phi[y] = (1 + P)/2$.

For example, in order to get a 95% confidence interval, find y such that $\Phi[y] = 97.5\%$.
Thus, y = 1.960.
[-1.960, 1.960] covers 95% probability on a Standard Normal Distribution.

[45] The instructions for Exam C from the SOA.

Normal Distribution Table

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z < z). The value of z to the first decimal place is given in the left column. The second decimal is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Values of z for selected values of Pr(Z < z)						
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z < z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995

Problems:

7.1 (2 points) You roll 1000 6-sided dice.
What is the chance of observing exactly 167 sixes?
(Use the Normal Approximation.)
A. less than 2.5%
B. at least 2.5% but less than 3.0%
C. at least 3.0% but less than 3.5%
D. at least 3.5% but less than 4.0%
E. at least 4.0%

7.2 (2 points) You roll 1000 6-sided dice.
What is the chance of observing 150 or more sixes but less than or equal to 180 sixes?
(Use the Normal Approximation.)
A. less than 78%
B. at least 78% but less than 79%
C. at least 79% but less than 80%
D. at least 80% but less than 81%
E. at least 81%

7.3 (2 points) You conduct 100 independent Bernoulli Trials, each with chance of success 1/4.
What is the chance of observing a total of at least 16 but not more than 20 successes?
(Use the Normal Approximation.)
A. less than 11%
B. at least 11% but less than 12%
C. at least 12% but less than 13%
D. at least 13% but less than 14%
E. at least 14%

7.4 (2 points) One observes 10,000 independent lives, each of which has a 2% chance of death over the coming year. What is the chance of observing 205 or more deaths?
(Use the Normal Approximation.)
A. less than 36%
B. at least 36% but less than 37%
C. at least 37% but less than 38%
D. at least 38% but less than 39%
E. at least 39%

7.5 (2 points) The number of claims in a year is given by a Poisson distribution with parameter $\lambda = 400$. What is the probability of observing at least 420 but no more than 440 claims over the next year? (Use the Normal Approximation.)

A. less than 11%
B. at least 11% but less than 12%
C. at least 12% but less than 13%
D. at least 13% but less than 14%
E. at least 14%

Use the following information in the next three questions:
The Few States Insurance Company writes insurance in the states of Taxachusetts, Florgia and Calizonia. Claims frequency for Few States Insurance in each state is Poisson, with expected claims per year of 400 in Taxachusetts, 500 in Florgia and 1000 in Calizonia. The claim frequencies in the three states are independent.

7.6 (2 points) What is the chance of Few States Insurance having a total of more than 1950 claims next year? (Use the Normal Approximation.)

A. less than 10%
B. at least 10% but less than 11%
C. at least 11% but less than 12%
D. at least 12% but less than 13%
E. at least 13%

7.7 (3 points) What is the chance that Few States Insurance has more claims next year from Taxachusetts and Florgia combined than from Calizonia?
(Use the Normal Approximation.)

A. less than 1.0%
B. at least 1.0% but less than 1.2%
C. at least 1.2% but less than 1.4%
D. at least 1.4% but less than 1.6%
E. at least 1.6%

7.8 (3 points) Define a large claim as one larger than $10,000. Assume that 30% of claims are large in Taxachusetts, 25% in Florgia and 20% in Calizonia. Which of the following is an approximate 90% confidence interval for the number of large claims observed by Few States Insurance over the next year? Frequency and severity are independent.
(Use the Normal Approximation.)

A. [390, 500] B. [395, 495] C. [400, 490] D. [405, 485] E. [410, 480]

7.9 (2 points) A six-sided die is rolled five times. Using the Central Limit Theorem, what is the estimated probability of obtaining a total of 20 on the five rolls?
A. less than 9.0%
B. at least 9% but less than 9.5%
C. at least 9.5% but less than 10%
D. at least 10% but less than 10.5%
E. at least 10.5%

7.10 (2 points) The number of claims in a year is given by the negative binomial distribution:

$P[X=x] = \binom{9999+x}{x} 0.6^{10000}\ 0.4^x$, x = 0,1,2,3...

Using the Central Limit Theorem, what is the estimated probability of having 6800 or more claims in a year?
A. less than 10.5%
B. at least 10.5% but less than 11%
C. at least 11% but less than 11.5%
D. at least 11.5% but less than 12%
E. at least 12%

7.11 (2 points) In order to estimate 1 - $\Phi(4)$, use the formula:

$\Phi(x) \cong 1 - \phi(x)\{.4361836t - .1201676t^2 + .9372980t^3\}$, where $t = 1/(1 + .33267x)$,

A. less than .0020%
B. at least .0020% but less than .0020%
C. at least .0025% but less than .0030%
D. at least .0030% but less than .0035%
E. at least .0035%

7.12 (2 points) You are given the following:

- The New York Yankees baseball team plays 162 games.
- Assume the Yankees have an a priori chance of winning each game of 65%.
- Assume the results of the games are independent of each other.

What is the chance of the Yankees winning 114 or more games?
(Use the Normal Approximation.)
A. less than 6%
B. at least 6% but less than 7%
C. at least 7% but less than 8%
D. at least 8% but less than 9%
E. at least 9%

7.13 (2 points) You are given the following:

- Sue takes an actuarial exam with 40 multiple choice questions, each of equal value.
- Sue knows the answers to 13 questions and answers them correctly.
- Sue guesses at random on the remaining 27 questions, with a 1/5 chance of getting each such question correct, with each question independent of the others.

If 22 correct answers are needed to pass the exam, what is the probability that Sue passed her exam?
Use the Normal Approximation.
A. 4% B. 5% C. 6% D. 7% E. 8%

7.14 (3 points) You are given the following:

- The New York Yankees baseball team plays 162 games, 81 at home and 81 on the road.
- The Yankees have an a priori chance of winning each home game of 80%.
- The Yankees have an a priori chance of winning each road game of 50%.
- Assume the results of the games are independent of each other.

What is the chance of the Yankees winning 114 or more games?
(Use the Normal Approximation.)
A. less than 6%
B. at least 6% but less than 7%
C. at least 7% but less than 8%
D. at least 8% but less than 9%
E. at least 9%

7.15 (2 points) You are given the following:

- Lucky Tom takes an actuarial exam with 40 multiple choice questions, each of equal value.
- Lucky Tom knows absolutely nothing about the material being tested.
- Lucky Tom guesses at random on each question, with a 40% chance of getting each question correct, independent of the others.

If 24 correct answers are needed to pass the exam, what is the probability that Lucky Tom passed his exam? Use the Normal Approximation.
A. 0.4% B. 0.5% C. 0.6% D. 0.7% E. 0.8%

7.16 (4 points) X has a Normal Distribution with mean μ and standard deviation σ.
Determine the expected value of $|x|$.

7.17 (4, 5/86, Q.48) (2 points) Assume an insurer has 400 claims drawn independently from a distribution with mean 500 and variance 10,000.
Assuming that the Central Limit Theorem applies, find M such that the probability of the sum of these claims being less than or equal to M is approximately 99%.
In which of the following intervals is M?
A. Less than 202,000
B. At least 202,000, but less than 203,000
C. At least 203,000, but less than 204,000
D. At least 204,000, but less than 205,000
E. 205,000 or more

7.18 (4, 5/86, Q.51) (1 point) Suppose X has a Poisson distribution with mean q.
Let Φ be the (Cumulative) Standard Normal Distribution.
Which of the following is an approximation for Prob($1 \le x \le 4$) for sufficiently large q?
A. $\Phi[(4 - q) / \sqrt{q}] - \Phi[(1 - q) / \sqrt{q}]$
B. $\Phi[(4.5 - q) / \sqrt{q}] - \Phi[(.5 - q) / \sqrt{q}]$
C. $\Phi[(1.5 - q) / \sqrt{q}] - \Phi[(3.5 - q) / \sqrt{q}]$
D. $\Phi[(3.5 - q) / \sqrt{q}] - \Phi[(1.5 - q) / \sqrt{q}]$
E. $\Phi[(4 - q) / q] - \Phi[(1 - q) / q]$

7.19 (4, 5/87, Q.51) (2 points) Suppose that the number of claims for an individual policy during a year has a Poisson distribution with mean 0.01. What is the probability that there will be 5, 6, or 7 claims from 400 identical policies in one year, assuming a normal approximation?
A. Less than 0.30
B. At least 0.30, but less than 0.35
C. At least 0.35, but less than 0.40
D. At least 0.40, but less than 0.45
E. 0.45 or more.

7.20 (4, 5/88, Q.46) (1 point) A random variable X is normally distributed with mean 4.8 and variance 4. The probability that X lies between 3.6 and 7.2 is Φ(b) - Φ(a) where Φ is the distribution function of the unit normal variable. What are a and b, respectively?
A. 0.6, 1.2 B. 0.6, -0.3 C. -0.3, 0.6 D. -0.6, 1.2 E. None A, B, C, or D.

7.21 (4, 5/88, Q.49) (1 point) An unbiased coin is tossed 20 times. Using the normal approximation, what is the probability of obtaining at least 8 heads?
The cumulative unit normal distribution is denoted by Φ(x).
A. $\Phi(-1.118)$ B. $\Phi(-0.671)$ C. $1 - \Phi(-0.447)$ D. $\Phi(0.671)$ E. $\Phi(1.118)$

7.22 (4, 5/90, Q.25) (1 point) Suppose the distribution of claim amounts is normal with a mean of \$1,500. If the probability that a claim exceeds \$5,000 is .015, in what range is the standard deviation, σ, of the distribution?

A. $\sigma < 1{,}600$

B. $1{,}600 \le \sigma < 1{,}625$

C. $1{,}625 \le \sigma < 1{,}650$

D. $1{,}650 \le \sigma < 1{,}675$

E. $\sigma \ge 1{,}675$

7.23 (4, 5/90, Q.36) (2 points) The number of claims for each insured written by the Homogeneous Insurance Company follows a Poisson process with a mean of .16.
The company has 100 independent insureds.
Let p be the probability that the company has more than 12 claims and less than 20 claims.
In what range does p fall? You may use the normal approximation.
A. $p < 0.61$
B. $0.61 \le p < 0.63$
C. $0.63 \le p < 0.65$
D. $0.65 \le p < 0.67$
E. $0.67 \le p$

7.24 (4, 5/91, Q.29) (2 points) A sample of 1,000 policies yields an estimated claim frequency of 0.210. Assuming the number of claims for each policy has a Poisson distribution, use the Normal Approximation to find a 95% confidence interval for this estimate.
A. (0.198, 0.225) B. (0.191, 0.232) C. (0.183, 0.240)
D. (0.173, 0.251) E. (0.161, 0.264)

7.25 (4B, 5/92, Q.5) (2 points) You are given the following information:
- Number of large claims follows a Poisson distribution.
- Exposures are constant and there are no inflationary effects.
- In the past 5 years, the following number of large claims has occurred: 12, 15, 19, 11, 18

Estimate the probability that more than 25 large claims occur in one year.
(The Poisson distribution should be approximated by the normal distribution.)
A. Less than .002
B. At least .002 but less than .003
C. At least .003 but less than .004
D. At least .004 but less than .005
E. At least .005

7.26 (4B, 11/92, Q.13) (2 points) You are given the following information:

- The occurrence of hurricanes in a given year has a Poisson distribution.
- For the last 10 years, the following number of hurricanes has occurred:
  2, 4, 3, 8, 2, 7, 6, 3, 5, 2

Using the normal approximation to the Poisson, determine the probability of more than 10 hurricanes occurring in a single year.

A. Less than 0.0005
B. At least 0.0005 but less than 0.0025
C. At least 0.0025 but less than 0.0045
D. At least 0.0045 but less than 0.0065
E. At least 0.0065

7.27 (4B, 5/94, Q.20) (2 points) The occurrence of tornadoes in a given year is assumed to follow a binomial distribution with parameters m = 50 and q = 0.60.
Using the Normal approximation to the binomial, determine the probability that at least 25 and at most 40 tornadoes occur in a given year.

A. Less than 0.80
B. At least 0.80, but less than 0.85
C. At least 0.85, but less than 0.90
D. At least 0.90, but less than 0.95
E. At least 0.95

7.28 (5A, 11/94, Q.35) (1.5 points)
An insurance contract was priced with the following assumptions:
Claim frequency is Poisson with mean 0.01.
All claims are of size $5000.
Premiums are 110% of expected losses.
The company requires a 99% probability of not having losses exceed premiums.
(3/4 point) a. What is the minimum number of policies that the company must write given the above surplus requirement?
(3/4 point) b. After the rate has been established, it was discovered that the claim severity assumption was incorrect and that the claim severity should be 5% greater than originally assumed. Now, what is the minimum number of policies that the company must write given the above surplus requirement?

7.29 (4B, 11/96, Q.31) (2 points) You are given the following:
- A portfolio consists of 1,600 independent risks.
- For each risk the probability of at least one claim is 0.5.

Using the Central Limit Theorem, determine the approximate probability that the number of risks in the portfolio with at least one claim will be greater than 850.
A. Less than 0.01
B. At least 0.01, but less than 0.05
C. At least 0.05, but less than 0.10
D. At least 0.10, but less than 0.20
E. At least 0.20

7.30 (4B, 11/97, Q.1) (2 points) You are given the following:
- A portfolio consists of 10,000 identical and independent risks.
- The number of claims per year for each risk follows a Poisson distribution with mean λ.
- During the latest year, 1000 claims have been observed for the entire portfolio.

Determine the lower bound of a symmetric 95% confidence interval for λ.
A. Less than 0.0825
B. At least 0.0825, but less than 0.0875
C. At least 0.0875, but less than 0.0925
D. At least 0.0925, but less than 0.0975
E. At least 0.0975

7.31 (IOA 101, 9/00, Q.3) (1.5 points) The number of claims arising in a period of one month from a group of policies can be modeled by a Poisson distribution with mean 24.
Using the Normal Approximation, determine the probability that fewer than 20 claims arise in a particular month.

7.32 (IOA 101, 4/01, Q.4) (1.5 points) For a certain type of policy the probability that a policyholder will make a claim in a year is 0.001. If a random sample of 10,000 policyholders is selected, using the Normal Approximation, calculate an approximate value for the probability that not more than 5 will make a claim next year.

Solutions to Problems:

7.1. C. This is Binomial with q = 1/6 and m =1000. Mean = 1000 /6 = 166.66.

Standard Deviation = $\sqrt{(1000)(5/6)(1/6)}$ = 11.785.

$\Phi[\frac{167.5 - 166.66}{11.785}] - \Phi[\frac{166.5 - 166.66}{11.785}]$ = Φ[0.07] - Φ[-0.01] = 0.5279 - 0.4960 = **0.0319**.

7.2. D. This is Binomial with q = 1/6 and m =1000. Mean = 1000 /6 = 166.66.

Standard Deviation = $\sqrt{(1000)(5/6)(1/6)}$ = 11.785. The interval from 150 to 180 corresponds on the Standard Normal to the interval from {(149.5-166.66)/11.785}
to {(180.5-166.66)/11.785}. Therefore the desired probability is:

Φ((180.5-166.66)/11.785) - Φ((149.5-166.66)/11.785) = Φ(1.17) - Φ(-1.46) =

.8790 - .0721 = **0.8069**.

Comment: The exact answer is 0.8080, so the Normal Approximation is quite good.

7.3. D. This is the Binomial Distribution with q =.25 and m = 100. Therefore the mean is (100)(.25) = 25. The Variance is: (100)(.25)(.75) = 18.75 and the Standard Deviation is: $\sqrt{18.75}$ = 4.330.
Therefore the desired probability is:

Φ[(20.5-25)/4.330) - Φ((15.5-25)/4.330] = Φ(-1.04) - Φ(-2.19) = .1492 - .0143 = **0.1349**.

Comment: The exact answer is .1377, so the Normal Approximation is okay.

7.4. C. Binomial Distribution with mean = 200 and variance = (10,000)(.02)(1-.02) = 196.

Standard deviation = 14. Chance of 205 or more claims = 1 - chance of 204 claims or less ≅

1 - Φ((204.5-200)/14) = 1 - Φ(.32) =1 - .6255 = **0.3745**.

7.5. E. Mean = 400 = variance. Standard deviation = 20.

Φ((440.5-400)/20) - Φ((419.5-400)/20) = Φ(2.03) - Φ(0.98) =.9788 - .8365 = **0.1423**.

7.6. D. The total claims follow a Poisson Distribution with mean 400 + 500 + 1000 = 1900, since independent Poisson variables add. This has a variance equal to the mean of 1900 and therefore a standard deviation of $\sqrt{1900}$ = 43.59.

Prob[more than 1950 claims] ≅ 1 - Φ((1950.5-1900)/43.59) = 1 - Φ(1.16) = 1 - 0.8770 = **0.123**.

7.7. B. The number of claims in Taxachusetts and Florgia is given by a Poisson with mean 400 + 500 = 900. (Since the sum of independent Poisson variables is a Poisson.) This is approximated by a Normal distribution with a mean of 900 and variance of 900. The number of claims in Calizonia is approximated by the Normal distribution with mean 1000 and variance of 1000. The difference between the number of claims in Calizonia and the sum of the claims in Taxachusetts and Florgia is therefore approximately a Normal Distribution with mean = 1000 - 900 = 100 and variance = 1000 + 900 = 1900.

More claims next year from Taxachusetts and Florgia combined than from Calizonia $\Leftrightarrow$

(# in Calizonia) - (# in Taxachusetts + # in Florgia) < 0 $\Leftrightarrow$

(# in Calizonia) - (# in Taxachusetts + # in Florgia) $\leq$ -1.

The probability of this is approximately: $\Phi(\{(0 - .5) - 100\} / \sqrt{1900}) = \Phi(-2.31) =$ **.0104**.

Comment: The sum of independent Normal variables is a Normal.
If X is Normal, then so is -X, so the difference of Normal variables is also Normal.
Also E[X - Y] = E[X] - E[Y]. For X and Y independent variables Var[X - Y] = Var[X] + Var[Y].

7.8. E. The number of large claims in Taxachusetts is Poisson with mean (30%)(400) = 120. (This is the concept of “thinning” a Poisson.) Similarly the number of large claims in Florgia and Calizonia are Poisson with means of 125 and 200 respectively. Thus the large claims from all three states is Poisson with mean = 120 + 125 + 200 = 445. (The sum of independent Poisson variables is a Poisson.) This Poisson is approximated by a Normal with a mean of 445 and a variance of 445. The standard deviation is $\sqrt{445}$ = 21.10. $\Phi(1.645)$ = .95 and thus a 90% confidence interval $\cong$ the mean ± 1.645 standard deviations, which in this case is about 445 ± (1.645)(21.10) = 410.3 to 479.7. Thus **[410, 480]** covers a little more than 90% of the probability.

7.9. A. For a six-sided die the mean is 3.5 and the variance is 35/12. For 5 such dice the mean is: (5)(3.5) = 17.5 and the variance is: (5)(35/12) = 175/12. The standard deviation = 3.819. Thus the interval from 19.5 to 20.5 corresponds to (19.5-17.5)/3.819 = .524 to (20.5-17.5)/3.819 = .786 on the standard unit normal distribution. Using the Standard Normal Table, this has a probability of $\Phi(.79) - \Phi(.52)$ = .7852 - .6985 = **0.0867**.

7.10. A. A Negative Binomial distribution with β = 2/3 and r =10000.

Mean = $r\beta$= (10000)(2/3) = 6666.66. Variance = mean(1+ β) = 11111.11

Standard Deviation = 105.4. 1 - $\Phi((6799.5-6666.66)/105.4)$ = 1 - $\Phi(1.26)$ = 1- .8962 = **10.38%**

Comment: You have to recognize that this is an alternate way of writing the Negative Binomial Distribution. In the tables attached to the exam, $f(x) = \{r(r+1)..(r+x-1)/x!\} \beta^x / (1+\beta)^{x+r}$. The factor $\beta/(1 + \beta)$ is taken to the power x. Thus for the form of the distribution in this question, $\beta/(1 + \beta)$ = 0.4, and solving β = 2/3. Then line up with the formula in Appendix B and note that r = 10,000.

7.11. D. t = 1/(1+(.33267)(4)) = .42906.

$\phi(4) = \exp(-4^2/2) / \sqrt{2\pi}$ = 0.00033546/2.5066 = 0.00013383.

1- $\Phi(4) \cong$ (.0013383){.4361836(.42906) -.1201676$(.42906)^2$ +.9372980$(.42906)^3$} =

(.00013383)(.23906) = **0.00003199**.

Comment: The exact answer is 1- $\Phi(4)$ = .000031672.

7.12. D. The number of games won is a binomial with m =162 and q = .65.
The mean is: (162)(.65) = 105.3. The variance is: (162)(.65)(1-.65) = 36.855.

The standard deviation is $\sqrt{36.855}$ = 6.07. Thus the chance of 114 or more wins is about:

1 - Φ((113.5-105.3)/6.07) = 1 - Φ(1.35) = 1 - .9115 = **8.85%**.

Comment: The exact probability is 8.72%.

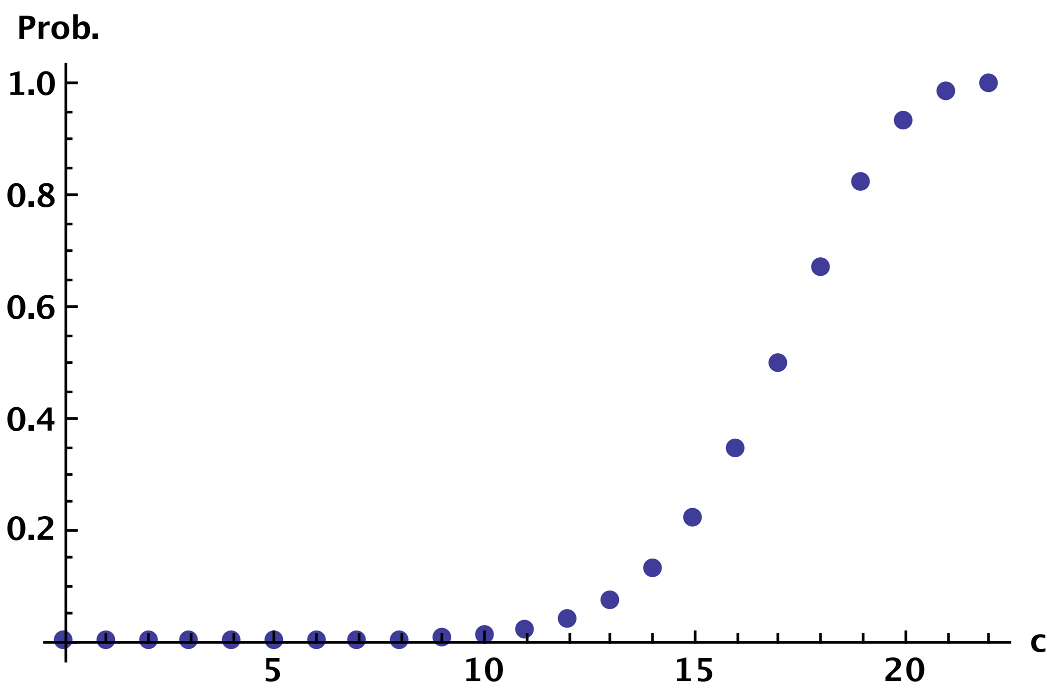
7.13. D. The number of correct guesses is Binomial with parameters m = 27 and
q = 1/5, with mean: (1/5)(27) = 5.4 and variance: (1/5)(4/5)(27) = 4.32.

Therefore, Prob(# correct guesses ≥ 9) ≅ 1 - $\Phi[(8.5-5.4)/\sqrt{4.32}]$ = 1 - Φ(1.49) = **6.81%.**

Comment: Any resemblance between the situation in this question and actual exams is coincidental.

The exact answer is in terms of an incomplete Beta Function: 1 - β(19, 9, 0.8) = 7.4%.

If Sue knows c questions, c ≤ 22, then her chance of passing is: 1 - β(19, 22-c, 0.8),
as displayed below:



7.14. C. The number of home games won is a binomial with m = 81 and q = .8. The mean is: (81)(.8) = 64.8 and variance is: (81)(.8)(1-.8) = 12.96. The number of road games won is a binomial with m = 81 and q = .5. The mean is: (81)(.5) = 40.5 and variance is: (81)(.5)(1-.5) = 20.25. The number of home and road wins are independent random variables, thus the variance of their sum is the sum of their variances: 12.96 + 20.25 = 33.21.

The standard deviation is $\sqrt{33.21}$ = 5.76. The mean number of wins is: 64.8 + 40.5 = 105.3.

Thus the chance of 114 or more wins is about:

1 - Φ((113.5-105.3)/5.76) = 1 - Φ(1.42) = 1 - .9228 = **7.78%**.

Comment: The exact probability is 7.62%, obtained by convoluting two binomial distributions.

7.15. E. Number of questions Lucky Tom guesses correctly is Binomial with mean (0.4)(40) = 16, and variance (40)(0.4)(0.6) = 9.6. The probability he guesses 24 or more correctly is approximately: 1 - Φ[(23.5 - 16)/$\sqrt{9.6}$] = 1 - Φ(2.42) = 1 - .9922 = **0.78%**.

Comment: The exact answer is 0.834177%. An ordinary person would only have a 20% chance of randomly guessing correctly on each question. Therefore, their chance of passing would be approximately: 1 - Φ[(23.5 - 8)/$\sqrt{6.4}$] = 1 - Φ(6.13) = 4.5 x 10^{-10}.

7.16. Let y = (x - μ)/σ. Then y follows a Standard Normal Distribution with mean 0 and standard deviation 1. f(y) = exp[-y^2/2]/$\sqrt{2\pi}$. x = σy + μ.

Expected value of |x| = Expected value of |σy + μ| =

$$\int_{-\infty}^{\infty} \frac{|\sigma y + \mu| \exp[-y^2/2]}{\sqrt{2\pi}} dy = -\int_{-\infty}^{-\mu/\sigma} \frac{(\sigma y + \mu) \exp[-y^2/2]}{\sqrt{2\pi}} dy + \int_{-\mu/\sigma}^{\infty} \frac{(\sigma y + \mu) \exp[-y^2/2]}{\sqrt{2\pi}} dy$$

$$= -\sigma \int_{-\infty}^{-\mu/\sigma} \frac{y \exp[-y^2/2]}{\sqrt{2\pi}} dy - \mu\Phi(-\mu/\sigma) + \sigma \int_{-\mu/\sigma}^{\infty} \frac{y \exp[-y^2/2]}{\sqrt{2\pi}} dy + \mu\{1 - \Phi(-\mu/\sigma)\} =$$

$$\mu\{1 - 2\Phi(-\mu/s)\} + \frac{\sigma}{\sqrt{2\pi}} \exp[-y^2/2]\Big]_{y = -\mu/\sigma}^{y = \infty} - \frac{\sigma}{\sqrt{2\pi}} \exp[-y^2/2]\Big]_{y = -\infty}^{y = -\mu/\sigma} =$$

$$\mathbf{\mu\{1 - 2\Phi[-\mu/\sigma]\} + \sigma \sqrt{\frac{2}{\pi}} \exp[-\frac{\mu^2}{2\sigma^2}]}.$$

Comment: For a Standard Normal, with μ = 0 and σ = 1, E[|X|] = $\sqrt{\frac{2}{\pi}}$.

7.17. D. The sum of 400 claims has a mean of (400)(500) = 200,000 and a variance of (400)(10000). Thus the standard deviation of the sum is (20)(100) = 2000.
In order to standardize the variables one subtracts the mean and divides by the standard deviation, thus standardizing M gives: (M - 200,000)/2000.
We wish the probability of the sum of the claims being less than or equal to M to be 99%.
For the standard Normal Distribution, $\Phi(2.327) = 0.99$.
Setting 2.327 = (M - 200,000)/2000, we get M = 200000 + (2.327)(2000) = **204,654**.

7.18. B. $\Phi[(4.5 - q) / \sqrt{q}] - \Phi[(0.5 - q) / \sqrt{q}]$.

7.19. C. The portfolio has a mean of (400)(.01) = 4. Since each policy has a variance of .01 and they should be assumed to be independent, then the variance of the portfolio is (400)(.01) = 4.
Thus the probability of 5, 6 or 7 claims is approximately:
$\Phi[(7.5-4)/2] - \Phi[4.5-4)/2] = \Phi[1.75] - \Phi[.25] = 0.9599 - 0.5987 =$ **0.3612**.

7.20. D. The standard deviation of the Normal is $\sqrt{4} = 2$.
Thus 3.6 corresponds to (3.6-4.8)/2 = **-0.6**, while 7.2 corresponds to (7.2-4.8)/2 = **1.2**.

7.21. E. The distribution is Binomial with q =.5 and m = 20. That has mean (20)(.5) = 10 and variance (20)(.5)(1-.5) = 5. The chance of obtaining 8 or more heads is approximately:
$1 - \Phi[(7.5-10)/\sqrt{5}] = 1 - \Phi(-1.118) = 1 - \{1 - \Phi(1.118)\} = \Phi(1.118)$.

7.22. B. The chance that a claim exceeds 5000 is $1 - \Phi((5000-1500)/\sigma) = .015$.
Thus $\Phi(3500/\sigma) = .985$. Consulting the Standard Normal Distribution, $\Phi(2.17) = .985$,
therefore $3500 / \sigma = 2.17$. $\sigma = 3500 / 2.17 =$ **1613**.

7.23. B. The sum of independent Poisson variables is a Poison. The mean number of claims is (100)(.16) = 16. Since for a Poisson the mean and variance are equal, the variance is also 16.
The standard deviation is 4. The probability is:
$\Phi((19.5-16) / 4) - \Phi((12.5-16) / 4) = \Phi(0.87) - \Phi(-0.88) = 0.8078 - 0.1894 =$ **0.6184**.
Comment: More than 12 claims (greater than or equal to 13 claims) corresponds to 12.5 due to the "continuity correction".

7.24. C. The observed number of claims is (1000)(.210) = 210. Since for the Poisson Distribution the variance is equal to the mean, the estimated variance for the sum is also 210. The standard deviation is $\sqrt{210}$ = 14.49. Using the Normal Approximation, an approximate 95% confidence interval is ± 1.96 standard deviations. $\Phi(1.96) = 0.975$. Therefore a 95% confidence interval for the number of claims from 1000 policies is 210 ± (1.96)(14.49) =
210 ± 28.4. A 95% confidence interval for the claim frequency is: 0.210 ± 0.028.
Alternately, the standard deviation for the estimated frequency declines as the square root of the number of policies used to estimate it: $\sqrt{0.210} / \sqrt{1000}$ = .458 / 31.62 = 0.01449. Thus a 95% confidence interval for the claim frequency is: 0.210 ± (1.96)(0.01449) = 0.210 ± 0.028.
Alternately, one can be a little more "precise" and let λ be the Poisson frequency. Then the standard deviation is: $\sqrt{\lambda} / \sqrt{1000}$ and the 95% confidence interval has λ within 1.96 standard deviations of 0.210: $-1.96\sqrt{\lambda} / \sqrt{1000} \le (0.210 - \lambda) \le 1.96\sqrt{\lambda} / \sqrt{1000}$. We can solve for the boundaries of this interval: $1.96^2 \lambda / 1000 = (0.210 - \lambda)^2$. $\Rightarrow \lambda^2 - 0.4238416\lambda + 0.0441 = 0$.

$\lambda = \{0.4238416 \pm \sqrt{0.4238416^2 - (4)(1)(0.0441)}\} / \{(2)(1)\} = 0.2119 \pm 0.0285$.

Thus the boundaries are 0.2119 - 0.0285 = **0.183** and 0.2119 + 0.0285 = **0.240**.
Comment: One needs to assume that the policies have independent claim frequencies.
The sum of independent Poisson variables is again a Poisson.

7.25. C. The average number of large claims observed per year is: (12+15+19+11+18)/5 = 15.
Thus we estimate that the Poisson has a mean of 15 and thus a variance of 15.
Thus Prob(N > 25) ≅ $1 - \Phi[(25.5 - 15)/\sqrt{15}] = 1 - \Phi(2.71)$ ≅ 1 - 0.9966 = **0.0034.**

7.26. B. The observed mean is 42 / 10 = 4.2. Assume a Poisson with mean of 4.2 and therefore variance of 4.2. Using the "continuity correction", more than 10 on the discrete Poisson, (11, 12, 13, ...) will correspond to more than 10.5 on the continuous Normal Distribution.
With a mean of 4.2 and a standard deviation of $\sqrt{4.2}$ = 2.05, 10.5 is "standardized" to:
(10.5 - 4.2) / 2.05 = 3.07. Thus P(N > 10) ≅ $1 - \Phi(3.07)$ = 1 - .9989 = **0.0011**.

7.27. D. The mean of the Binomial is mq = (50)(.6) = 30.
The variance of the Binomial is mq(1-q) = (50)(.6)(1-.6) = 12.
Thus the standard deviation is $\sqrt{12}$ = 3.464.
$\Phi[(40.5 - 30) / 3.464] - \Phi[(24.5 - 30) / 3.464] = \Phi(3.03) - \Phi(-1.59)$ = 0.9988 - 0.0559 = **0.9429**.

7.28. (a) With N policies, the mean aggregate loss = N(.01)(5000) and the variance of aggregate losses = N(.01)(5000^2). Thus premiums are 1.1N(.01)(5000).
The 99th percentile of the Standard Normal Distribution is 2.326. Thus we want:
Premiums - Expected Losses = 2.326(standard deviation of aggregate losses).
$0.1N(0.01)(5000) = 2.326\sqrt{N(0.01)(5000^2)}$. Therefore, $N = (2.326/0.1)^2/0.01 =$ **54,103**.
(b) With a severity of (1.05)(5000) = 5250 and N policies, the mean aggregate loss = N(.01)(5250), and the variance of aggregate losses = N(.01)(5250^2).
Premiums are still: 1.1N(0.01)(5000). Therefore, we want:
$1.1N(0.01)(5000) - N(0.01)(5250) = 2.326\sqrt{N(0.01)(5250^2)}$.
$2.5N = 1221.15\sqrt{N}$. N = **238,593**.

7.29. A. The mean is 800 while the variance is: (0.5)(1 - 0.5)(1600) = 400.
Thus the standard deviation is 20.
Using the continuity correction, more than 850 corresponds on the continuous Normal Distribution to:
$1 - \Phi[(850.5\text{-}800)/20] = 1 - \Phi(2.53) =$ **0.0057**.

7.30. D. The observed frequency is 1000/10000 = .1, which is the point estimate for λ.
Since for a Poisson the variance is equal to the mean, the estimated variance for a single insured of its observed frequency is .1. For the sum of 10,000 identical insureds, the variance is divided by 10000; thus the variance of the observed frequency is: 0.1/10,000 = 1/100,000. The standard deviation is $\sqrt{1/100{,}000}$ = 0.00316. Using the Normal Approximation, ±1.96 standard deviations would produce an approximate 95% confidence interval:
0.1 ± (1.96)(0.00316) = 0.1 ± 0.0062 = [**0.0938**, 0.1062].

7.31. $\text{Prob}[N < 20] \cong \Phi[(19.5 - 24)/\sqrt{24}] = \Phi[-0.92] =$ **17.88%**.

7.32. The Binomial has mean: (0.001)(10,000) = 10, and variance: (0.001)(.999)(10,000) = 9.99.
$\text{Prob}[N \le 5] \cong \Phi[(5.5 - 10)/\sqrt{9.99}] = \Phi[-1.42] =$ **7.78%**.
Comment: The exact answer is 0.06699.

Section 8, Skewness

Skewness is one measure of the shape of a distribution.[46]

For example, take the following frequency distribution:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	0.1	0	0	0.0
1	0.2	0.2	0.2	0.2
2	0	0	0	0.0
3	0.1	0.3	0.9	2.7
4	0	0	0	0.0
5	0	0	0	0.0
6	0.1	0.6	3.6	21.6
7	0	0	0	0.0
8	0	0	0	0.0
9	0.1	0.9	8.1	72.9
10	0.3	3	30	300.0
11	0.1	1.1	12.1	133.1
Sum	1	6.1	54.9	530.5

E[X] = 1st moment about the origin = 6.1

$E[X^2]$ = 2nd moment about the origin = 54.9

$E[X^3]$ = 3rd moment about the origin = 530.5

Variance ≡ 2nd Central Moment = $E[X^2] - E[X]^2 = 54.9 - 6.1^2 = 17.69$.

Standard Deviation = $\sqrt{17.69}$ = 4.206.

3rd Central Moment ≡ $E[(X - E[X])^3] = E[X^3 - 3X^2E[X] + 3XE[X]^2 - E[X]^3]$

$= E[X^3] - 3E[X]E[X^2] + 2E[X]^3 = 530.5 - (3)(6.1)(54.9) + (2)(6.1^3) = -20.2$.

(Coefficient of) **Skewness** ≡ Third Central Moment / $STDDEV^3$ = $-20.2/4.206^3 = -0.27$.

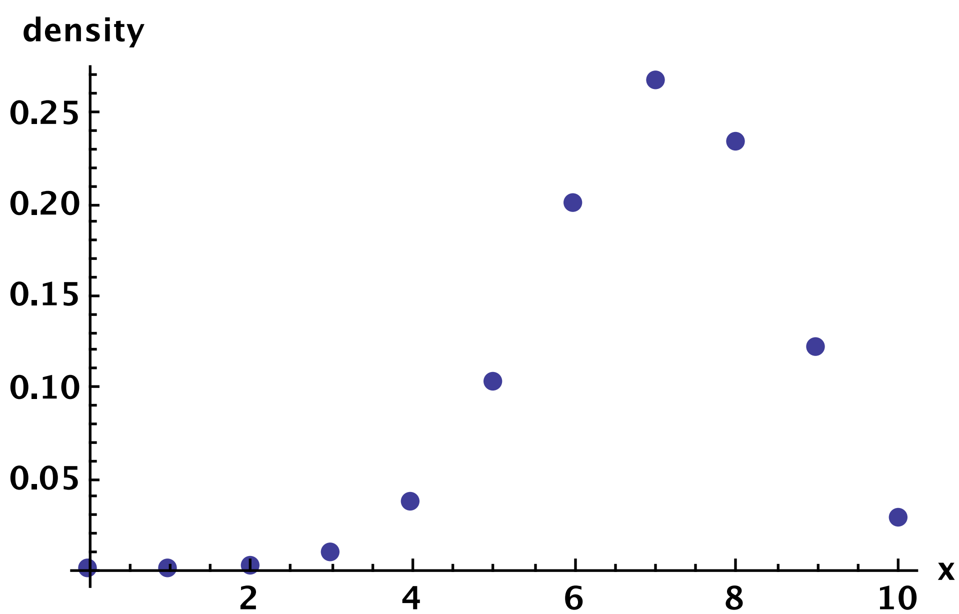
The third central moment: $\mu_3 \equiv$ **$E[(X - E[X])^3]$** $= E[X^3] - 3E[X]\,E[X^2] + 2E[X]^3$.

The (coefficient of) skewness is defined as the 3rd central moment divided by the cube of the standard deviation = $E[(X - E[X])^3] / STDDEV^3$.

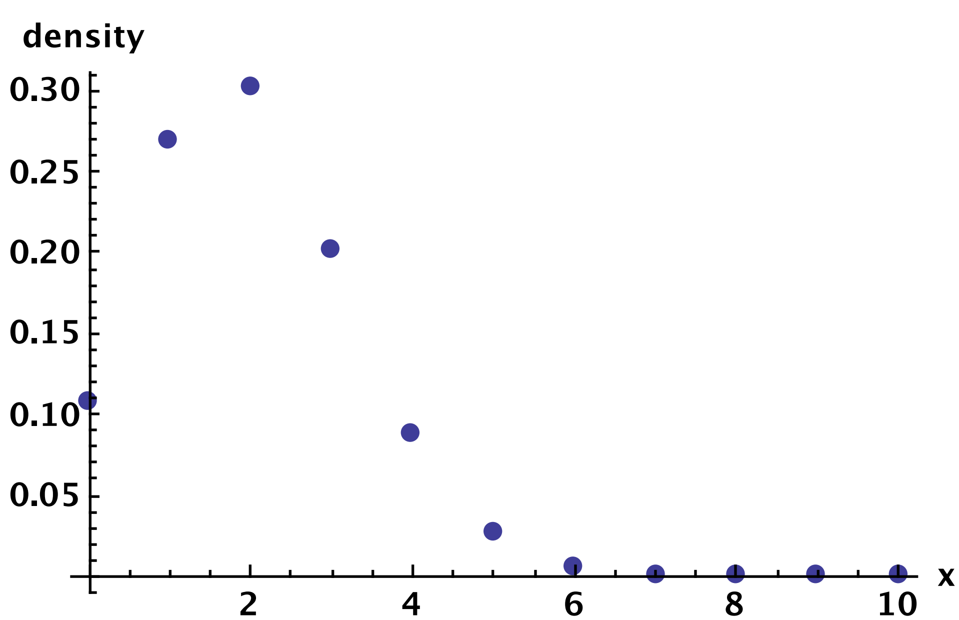
[46] The coefficient of variation and kurtosis are others. See “Mahler’s Guide to Loss Distributions.”

In the above example, the skewness is -0.27.
A negative skewness indicates a curve skewed to the left.

The Binomial Distribution for $q > 0.5$ is skewed to the left.
For example, here is a Binomial Distribution with m = 10 and q = 0.7:

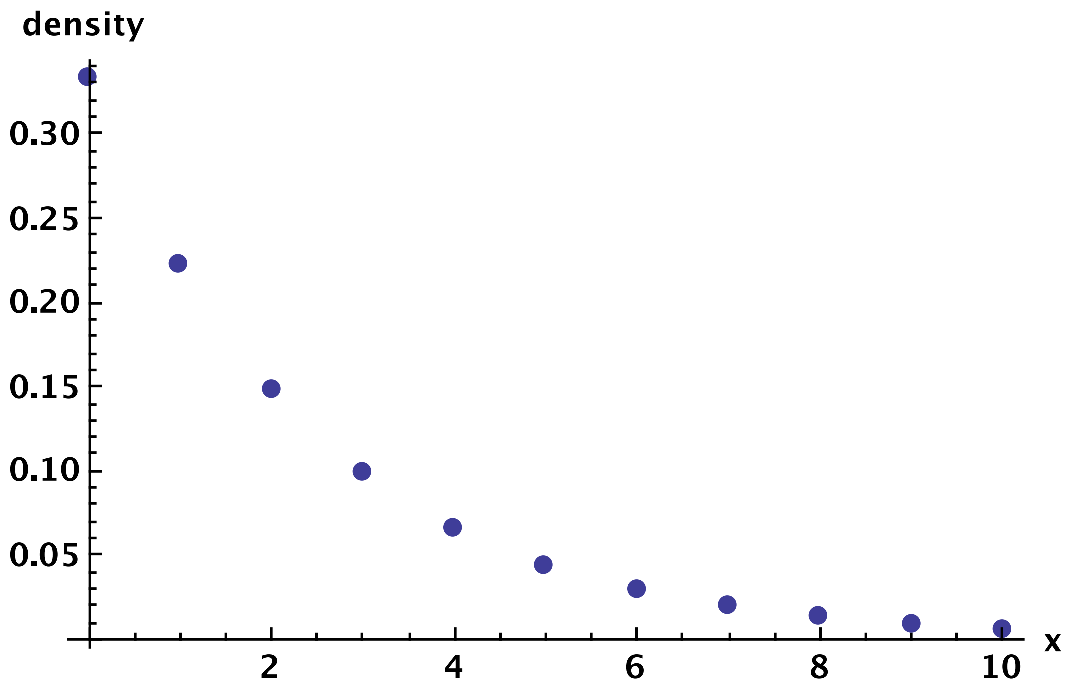


In contrast, the Binomial Distribution for $q < 0.5$ has positive skewness and is skewed to the right.
For example, here is a Binomial Distribution with m = 10 and q = 0.2:



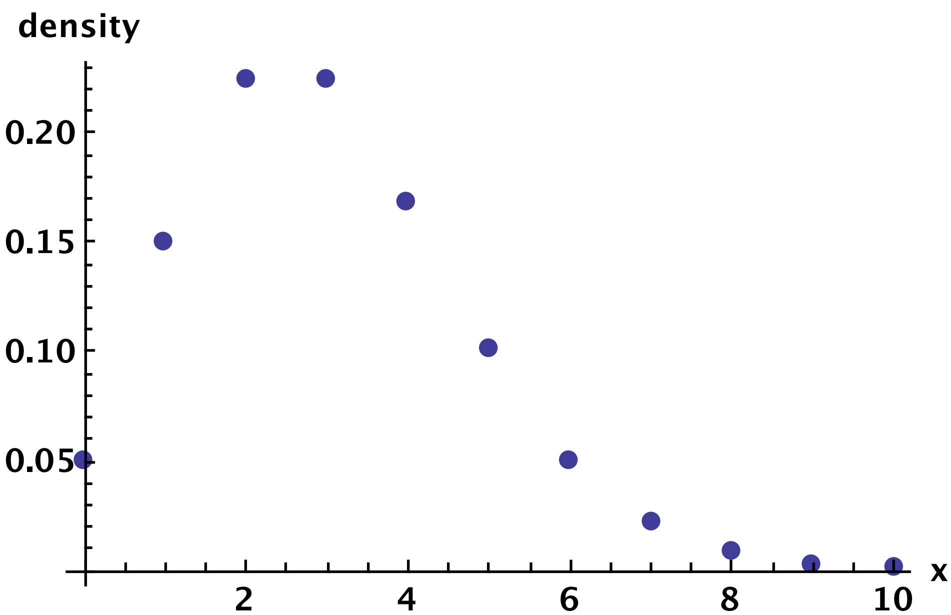
The Poisson Distribution, the Negative Binomial Distribution (including the special case of the Geometric Distribution), as well as most size of loss distributions, are skewed to the right; they have a small but significant probability of very large values.

For example, here is a Geometric Distribution with $\beta = 2$:[47]



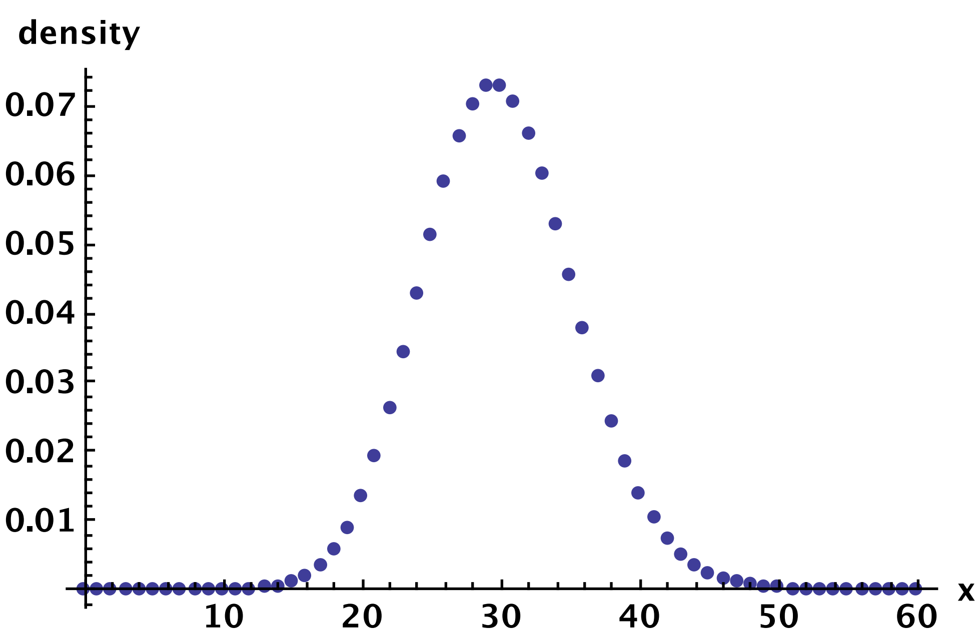
[47] Even through only densities up to 10 are shown, the Geometric Distribution has support from zero to infinity.

As another example of a distribution skewed to the right, here is a Poisson Distribution with $\lambda = 3$:[48]



For the Poisson distribution the skewness is positive and therefore the distribution is skewed to the right. However, as λ gets very large, the skewness of a Poisson approaches zero; in fact the Poisson approaches a Normal Distribution.[49]

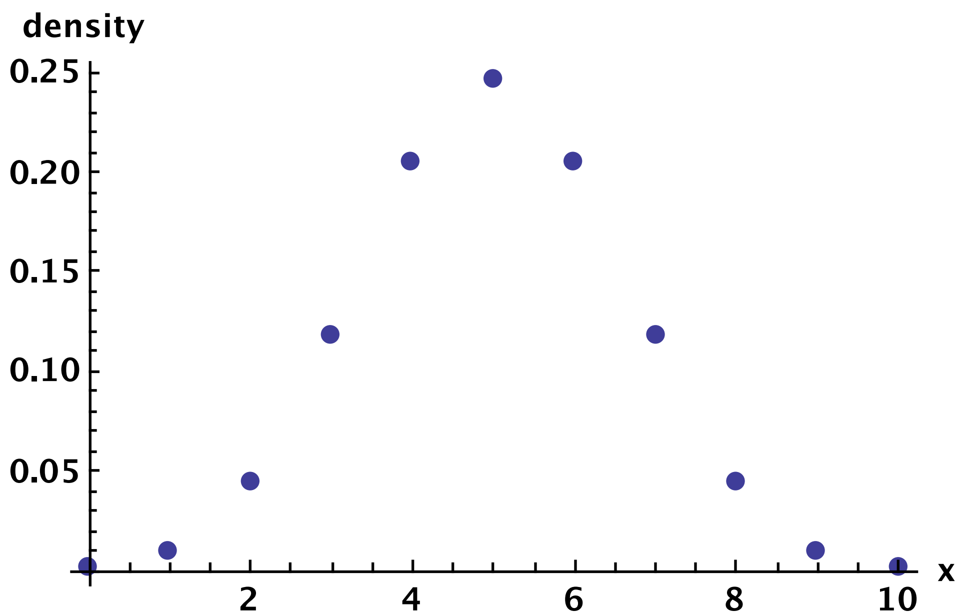
For example, here is a Poisson Distribution with $\lambda = 30$:



[48] Even through only densities up to 10 are shown, the Poisson Distribution has support from zero to infinity.

[49] This follows from the Central Limit Theorem and the fact that for integral N, a Poisson with parameter N is the sum of N independent variables each with a Poisson distribution with a parameter of unity. The Normal Distribution is symmetric and therefore has zero skewness.

A symmetric distribution has zero skewness.
Therefore, the Binomial Distribution for q = 0.5 and the Normal Distribution each have zero skewness. For example, here is a Binomial Distribution with m = 10 and q = 0.5:



<u>Binomial Distribution</u>:

For a Binomial Distribution with m = 5 and q = 0.1:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	59.049%	0.00000	0.00000	0.00000
1	32.805%	0.32805	0.32805	0.32805
2	7.290%	0.14580	0.29160	0.58320
3	0.810%	0.02430	0.07290	0.21870
4	0.045%	0.00180	0.00720	0.02880
5	0.001%	0.00005	0.00025	0.00125
Sum	1	0.50000	0.70000	1.16000

The mean is: 0.5 = (5)(0.1) = mq.

The variance is: $E[X^2] - E[X]^2 = 0.7 - 0.5^2 = 0.45 = (5)(0.1)(0.9) = mq(1-q)$.

The skewness is: $$\frac{E[X^3] - 3\,E[X]\,E[X^2] + 2\,E[X]^3}{\sigma^3} = \frac{1.16 - (3)(0.7)(0.5) + 2\,(0.5^3)}{0.45^{3/2}}$$

$$= 1.1925 = \frac{0.8}{\sqrt{0.45}} = \frac{1 - 2q}{\sqrt{m\,q\,(1 - q)}}.$$

For a Binomial Distribution, the skewness is: $\dfrac{1 - 2q}{\sqrt{m\,q\,(1 - q)}}$.

Binomial Distribution with q < 1/2 ⇔ positive skewness ⇔ skewed to the right.

Binomial Distribution q = 1/2 ⇔ symmetric ⇒ zero skewness.

Binomial Distribution q > 1/2 ⇔ negative skewness ⇔ skewed to the left.

Poisson Distribution:

For a Poisson distribution with $\lambda = 2.5$:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims	Distribution Function
0	0.08208500	0.00000000	0.00000000	0.00000000	0.08208500
1	0.20521250	0.20521250	0.20521250	0.20521250	0.28729750
2	0.25651562	0.51303124	1.02606248	2.05212497	0.54381312
3	0.21376302	0.64128905	1.92386716	5.77160147	0.75757613
4	0.13360189	0.53440754	2.13763017	8.55052069	0.89117802
5	0.06680094	0.33400471	1.67002357	8.35011786	0.95797896
6	0.02783373	0.16700236	1.00201414	6.01208486	0.98581269
7	0.00994062	0.06958432	0.48709021	3.40963146	0.99575330
8	0.00310644	0.02485154	0.19881233	1.59049864	0.99885975
9	0.00086290	0.00776611	0.06989496	0.62905464	0.99972265
10	0.00021573	0.00215725	0.02157252	0.21572518	0.99993837
11	0.00004903	0.00053931	0.00593244	0.06525687	0.99998740
12	0.00001021	0.00012257	0.00147085	0.01765024	0.99999762
13	0.00000196	0.00002554	0.00033196	0.00431553	0.99999958
14	0.00000035	0.00000491	0.00006875	0.00096250	0.99999993
15	0.00000006	0.00000088	0.00001315	0.00019731	0.99999999
16	0.00000001	0.00000015	0.00000234	0.00003741	1.00000000
17	0.00000000	0.00000002	0.00000039	0.00000660	1.00000000
18	0.00000000	0.00000000	0.00000006	0.00000109	1.00000000
19	0.00000000	0.00000000	0.00000001	0.00000017	1.00000000
20	0.00000000	0.00000000	0.00000000	0.00000002	1.00000000
Sum	1.00000000	2.50000000	8.75000000	36.87500000	

The mean is: $2.5 = \lambda$. The variance is: $E[X^2] - E[X]^2 = 8.75 - 2.5^2 = 2.5 = \lambda$.

The coefficient of variation = $\dfrac{\sqrt{\text{variance}}}{\text{mean}} = \dfrac{\sqrt{\lambda}}{\lambda} = \dfrac{1}{\sqrt{\lambda}}$.

The skewness is: $\dfrac{E[X^3] - 3\,E[X]\,E[X^2] + 2\,E[X]^3}{\sigma^3}$ =

$$\frac{36.875 - (3)(2.5)(8.75) + 2\,(2.5^3)}{2.5^{3/2}} = \frac{1}{\sqrt{2.5}} = \frac{1}{\sqrt{\lambda}}.$$

For the Poisson Distribution, the skewness is: $\dfrac{1}{\sqrt{\lambda}}$. *For the Poisson, Skewness = CV.*

Poisson Distribution ⇔ positive skewness ⇔ skewed to the right.

Negative Binomial Distribution:

For a Negative Binomial distribution with r = 3 and β = 0.4:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims	Distribution Function
0	0.36443149	0.00000000	0.00000000	0.00000000	0.36443149
1	0.31236985	0.31236985	0.31236985	0.31236985	0.67680133
2	0.17849705	0.35699411	0.71398822	1.42797644	0.85529839
3	0.08499860	0.25499579	0.76498738	2.29496213	0.94029699
4	0.03642797	0.14571188	0.58284753	2.33139010	0.97672496
5	0.01457119	0.07285594	0.36427970	1.82139852	0.99129614
6	0.00555093	0.03330557	0.19983344	1.19900062	0.99684707
7	0.00203912	0.01427382	0.09991672	0.69941703	0.99888619
8	0.00072826	0.00582605	0.04660838	0.37286706	0.99961445
9	0.00025431	0.00228880	0.02059924	0.18539316	0.99986876
10	0.00008719	0.00087193	0.00871926	0.08719255	0.99995595
11	0.00002944	0.00032386	0.00356244	0.03918682	0.99998539
12	0.00000981	0.00011777	0.00141320	0.01695839	0.99999520
13	0.00000324	0.00004206	0.00054677	0.00710805	0.99999844
14	0.00000106	0.00001479	0.00020706	0.00289887	0.99999950
15	0.00000034	0.00000513	0.00007697	0.00115454	0.99999984
16	0.00000011	0.00000176	0.00002815	0.00045038	0.99999995
17	0.00000004	0.00000060	0.00001015	0.00017251	0.99999998
18	0.00000001	0.00000020	0.00000361	0.00006501	0.99999999
19	0.00000000	0.00000007	0.00000127	0.00002414	1.00000000
20	0.00000000	0.00000002	0.00000044	0.00000885	1.00000000
Sum	1.00000000	1.19999999	3.11999977	10.79999502	

The mean is: 1.2 = (3)(0.4) = $r\beta$.
The variance is: $E[X^2] - E[X]^2 = 3.12 - 1.2^2 = 1.68 = (3)(.4)(1.4) = r\beta(1+\beta)$.

The third central moment is: $E[X^3] - 3\,E[X]\,E[X^2] + 2\,E[X]^3 = 10.8 - (3)(1.2)(3.12) + 2\,(1.2^3) = 3.024 = (1.8)(3)(.4)(1.4) = (1 + 2\beta)r\beta(1 + \beta)$.[50]

The skewness is: $3.024 / 1.68^{3/2} = 1.389 = \frac{1.8}{\sqrt{(3)(0.4)(1.4)}} = \frac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$

For the Negative Binomial Distribution, the skewness is: $\frac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$.

Negative Binomial Distribution ⇔ positive skewness ⇔ skewed to the right.

[50] For the Negative Binomial Distribution, $3\sigma^2 - 2\mu + 2(\sigma^2 - \mu)^2/\mu = 3r\beta(1+\beta) - 2r\beta + 2\{r\beta(1+\beta) - r\beta\}^2/(r\beta) = r\beta + 3r\beta^2 + 2r\beta^3 = r\beta(1+\beta)(1+2\beta)$ = third central moment.
This property of the Negative Binomial is discussed in Section 7.2 of Loss Models, not on the syllabus.

Problems:

8.1 (3 points) What is the skewness of the following frequency distribution?

Number of Claims	Probability
0	0.02
1	0.04
2	0.14
3	0.31
4	0.36
5	0.13

A. less than -1.0
B. at least -1.0 but less than -0.5
C. at least -0.5 but less than 0
D. at least 0 but less than 0.5
E. at least 0.5

8.2 (2 points) A distribution has first moment = m, second moment about the origin = $m + m^2$, and third moment about the origin = $m + 3m^2 + m^3$.
Which of the following is the skewness of this distribution?
A. m B. $m^{0.5}$ C. 1 D. $m^{-0.5}$ E. m^{-1}

8.3 (3 points) The number of claims filed by a commercial auto insured as the result of at-fault accidents caused by its drivers is shown below:

Year	Claims
2002	7
2001	3
2000	5
1999	10
1998	5

Calculate the skewness of the empirical distribution of the number of claims per year.
A. Less than 0.50
B. At least 0.50, but less than 0.75
C. At least 0.75, but less than 1.00
D. At least 1.00, but less than 1.25
E. At least 1.25

8.4 (4 points) You are given the following distribution of the number of claims on 100,000 motor vehicle comprehensive policies:

Number of claims	Observed number of policies
0	88,585
1	10,577
2	779
3	54
4	4
5	1
6 or more	0

Calculate the skewness of this distribution.

A. 1.0 B. 1.5 C. 2.0 D. 2.5 E. 3.0

8.5 (4, 5/87, Q.33) (1 point) There are 1000 insurance policies in force for one year.
The results are as follows:

Number of Claims	Policies
0	800
1	130
2	50
3	20
	1000

Which of the following statements are true?

1. The mean of this distribution is 0.29.
2. The variance of this distribution is at least 0.45.
3. The skewness of this distribution is negative.

A. 1 B. 1, 2 C. 1, 3 D. 2, 3 E. 1, 2, 3

8.6 (CAS3, 5/04, Q.28) (2.5 points) A pizza delivery company has purchased an automobile liability policy for its delivery drivers from the same insurance company for the past five years.
The number of claims filed by the pizza delivery company as the result of at-fault accidents caused by its drivers is shown below:

Year	Claims
2002	4
2001	1
2000	3
1999	2
1998	15

Calculate the skewness of the empirical distribution of the number of claims per year.
A. Less than 0.50
B. At least 0.50, but less than 0.75
C. At least 0.75, but less than 1.00
D. At least 1.00, but less than 1.25
E. At least 1.25

Solutions to Problems:

8.1. B. Variance = $E[X^2] - E[X]^2 = 12.4 - 3.34^2 = 1.244$.

Standard Deviation = $\sqrt{1.244}$ = **1.116**.

skewness = $\{E[X^3] - (3\ E[X]\ E[X^2]) + (2\ E[X]^3)\} / \text{STDDEV}^3$ =

$\{48.82 - (3)(3.34)(12.4) + (2)\ (3.34^3)\} / (1.116^3)$ = **-0.65**.

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	2%	0	0	0.0
1	4%	0.04	0.04	0.0
2	14%	0.28	0.56	1.1
3	31%	0.93	2.79	8.4
4	36%	1.44	5.76	23.0
5	13%	0.65	3.25	16.2
Sum	1	3.34	12.4	48.82

8.2. D. $\sigma^2 = \mu_2' - \mu_1'^2 = (m + m^2) - m^2 = m$.

skewness = $\{\mu_3' - (3\ \mu_1'\ \mu_2') + (2\ \mu_1'^3)\} / \sigma^3$ =

$\{(m + 3m^2 + m^3) - 3(m + m^2)m + 2\ m^3\} / m^{3/2}$ = $\mathbf{m^{-0.5}}$.

Comment: The moments are those of the Poisson Distribution with mean m.

8.3. B. $E[X] = (7 + 3 + 5 + 10 + 5)/5 = 6$. $E[X^2] = (7^2 + 3^2 + 5^2 + 10^2 + 5^2)/5 = 41.6$.

$\text{Var}[X] = 41.6 - 6^2 = 5.6$. $E[X^3] = (7^3 + 3^3 + 5^3 + 10^3 + 5^3)/5 = 324$.

Skewness = $\{E[X^3] - 3\ E[X^2]E[X] + 2E[X]^3\} / \text{Var}[X]^{1.5}$

= $\{324 - (3)(41.6)(6) + (2)(6^3)\}/5.6^{1.5} = 7.2/13.25$ = **0.54.**

Comment: Similar to CAS3, 5/04, Q.28. $E[(X-\overline{X})^3] = (1^3 + (-3)^3 + (-1)^3 + 4^3 + (-1)^3)/5 = 7.2$.

8.4. E. E[X] = 12318/100000 = 0.12318. E[X^2] = 14268/100000 = 0.14268.

Number of Claims	Number of Policies	Contribution to First Moment	Contribution to Second Moment	Contribution to Third Moment
0	88,585	0	0	0
1	10,577	10577	10577	10577
2	779	1558	3116	6232
3	54	162	486	1458
4	4	16	64	256
5	1	5	25	125
Total	100,000	12,318	14,268	18,648

Var[X] = 0.14268 - 0.12318^2 = 0.12751.

E[X^3] = 18648/100000 = 0.18648.

Third Central Moment = E[X^3] - 3 E[X]E[X^2] + 2E[X]^3

= 0.18648 - (3)(0.12318)(0.14268) + (2)(0.12318^3) = 0.13749.

Skewness = (Third Central Moment) / Var[X]$^{1.5}$ = 0.13749 / $0.12751^{1.5}$ = **3.02**.

Comment: Data taken from Table 5.9.1 in Introductory Statistics with Applications in General Insurance by Hossack, Pollard and Zehnwirth.

8.5. A. 1. True. The mean is {(0)(800) + (1)(130) + (2)(50) + (3)(20)} / 1000 = 0.290.

2. False. The second moment is {(0^2)(800) + (1^2)(130) + (2^2)(50) + (3^2)(20)} / 1000 = 0.510.

Thus the variance = 0.510 - 0.29^2 = 0.4259.

3. False. The distribution is skewed to the right and thus of positive skewness. The third moment is:

{(0^3)(800) + (1^3)(130) + (2^3)(50) + (3^3)(20)} / 1000 = 1.070.

Therefore, skewness = $\{\mu_3' - (3\,\mu_1'\,\mu_2') + (2\,\mu_1'^3)\}$ / STDDEV^3 =

{1.070 - (3)(.29)(.51) +(2)($.29^3$)} / .278 = 2.4 > 0.

8.6. E. E[X] = (4 + 1 + 3 + 2 + 15)/5 = 5. E[X^2] = (4^2 + 1^2 + 3^2 + 2^2 + 15^2)/5 = 51.

Var[X] = 51 - 5^2 = 26. E[X^3] = (4^3 + 1^3 + 3^3 + 2^3 + 15^3)/5 = 695.

Skewness = {E[X^3] - 3 E[X^2]E[X] + 2E[X]^3} / Var[X]$^{1.5}$

= {695 - (3)(51)(5) + (2)(5^3)}/$26^{1.5}$ = 180/133.425 = **1.358**.

Alternately, the third central moment is:

{$(4 - 5)^3$ + $(1 - 5)^3$ + $(3 - 5)^3$ + $(2 - 5)^3$ + $(15 - 5)^3$}/5 = 180. Skewness = 180/$26^{1.5}$ = **1.358**.

Section 9, Probability Generating Functions[51]

The **Probability Generating Function**, **p.g.f.**, is useful for working with frequency distributions.[52]

$$\mathbf{P(z) = \text{Expected Value of } z^n = E[z^n] = \sum_{n=0}^{\infty} f(n)\, z^n.}$$

Note that as with other generating functions, there is a dummy variable, in this case z.

Exercise: Assume a distribution with 1-q chance of 0 claims and q chance of 1 claim. (This a Bernoulli distribution with parameter q.) What is the probability generating function?
[Solution: $P(z) = E[z^n] = (1-q)(z^0) + q(z^1) = 1 + q(z-1)$.]

The Probability Generating Function of the sum of independent frequencies is the product of the individual Probability Generating Functions.

Specifically, if X and Y are independent random variables, then
$P_{X+Y}(z) = E[z^{x+y}] = E[z^x z^y] = E[z^x]E[z^y] = P_X(z)P_Y(z)$.

Exercise: What is the probability generating function of the sum of two independent Bernoulli distributions each with parameter q?
[Solution: It is the product of the probability generating functions of each Bernoulli:
$\{1 + q(z-1)\}^2$. Alternately, one can compute that for the sum of the two Bernoulli there is:
$(1-q)^2$ chance of zero claims, $2q(1-q)$ chance of 1 claims and q^2 chance of 2 claims.
Thus $P(z) = (1-q)^2z^0 + 2q(1-q)z^1 + q^2z^2 = 1 - 2q + q^2 + 2qz + 2q^2z + q^2z^2 =$
$1 + 2q(z-1) + (z^2 - 2z + 1)q^2 = \{1 + q(z-1)\}^2$.]

As discussed, a Binomial distribution with parameters q and m is the sum of m independent Bernoulli distributions each with parameter q. Therefore the probability generating function of a Binomial distribution is that of the Bernoulli, to the power m: $\{1 + q(z-1)\}^m$.

The probability generating functions, as well as much other useful information on each frequency distribution, are given in the tables attached to the exam.

[51] See Definition 3.8 in Loss Models.

[52] The Probability Generating Function is similar to the Moment Generating Function: $M(z) = E[e^{zn}]$.
See "Mahler's Guide to Aggregate Distributions." They are related via $P(z) = M(\ln(z))$. They share many properties.
Loss Models uses the Probability Generating Function when dealing with frequency distributions.

Densities:

The distribution determines the probability generating function and vice versa.

Given a p.g.f., one can obtain the probabilities by repeated differentiation as follows:

$$f(n) = \frac{\left(\frac{d^n P(z)}{dz^n}\right)_{z=0}}{n!}.$$

f(0) = P(0).[53] f(1) = P'(0). f(2) = P''(0)/2. f(3) = P'''(0)/6. f(4) = P''''(0)/24.

Exercise: Given the probability generating function: $e^{\lambda(z-1)}$, what is the probability of three claims?

[Solution: $P(z) = e^{\lambda(z-1)} = e^{\lambda z}e^{-\lambda}$. $P'(z) = \lambda e^{\lambda z}e^{-\lambda}$. $P''(z) = \lambda^2 e^{z\lambda}e^{-\lambda}$. $P'''(z) = \lambda^3 e^{z\lambda}e^{-\lambda}$.

$f(3) = (d^3 P(z) / dz^3)_{z=0} / 3! = (d^3 e^{\lambda(z-1)} / dz^3)_{z=0} / 3! = \lambda^3 e^{0\lambda}e^{-\lambda} / 3! = \lambda^3 e^{-\lambda} / 3!$.

Note that this is the p.g.f. of a Poisson Distribution with parameter lambda, and this is indeed the probability of 3 claims for a Poisson.]

Alternately, the probability of n claims is the coefficient of z^n in the p.g.f.

So for example, given the probability generating function: $e^{\lambda(z-1)} = e^{\lambda z}e^{-\lambda} =$

$e^{-\lambda}\sum_{i=0}^{\infty}\frac{(\lambda z)^i}{i!} = \sum_{i=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}z^i$. Thus for this p.g.f., $f(i) = e^{-\lambda}\lambda^i / i!$,

which is the density function of a Poisson distribution.

Mean:

$$P(z) = \sum_{n=0}^{\infty} f(n)z^n. \Rightarrow P(1) = \sum_{n=0}^{\infty} f(n) = 1.$$

$$P'(z) = \sum_{n=1}^{\infty} f(n)nz^{n-1}. \Rightarrow P'(1) = \sum_{n=1}^{\infty} f(n)n = \sum_{n=0}^{\infty} f(n)n = \text{Mean}.$$

P'(1) = Mean.[54]

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[53] As $z \longrightarrow 0$, $z^n \longrightarrow 0$ for $n > 0$. Therefore, $P(z) = \Sigma f(n) z^n \longrightarrow f(0)$ as $z \longrightarrow 0$.

[54] This is a special case of a result discussed subsequently in the section on factorial moments.

Proof of Results for Adding Distributions:

One can use the probability generating function, in order to determine the results of adding Poisson, Binomial, or Negative Binomial Distributions.

Assume one has two independent Poisson Distributions with means λ_1 and λ_2.

The p.g.f. of a Poisson is $P(z) = e^{\lambda(z-1)}$.
The p.g.f. of the sum of these two Poisson Distributions is the product of the p.g.f.s of the two Poisson Distributions: $\exp[\lambda_1(z-1)]\exp[\lambda_2(z-1)] = \exp[(\lambda_1 + \lambda_2)(z-1)]$.

This is the p.g.f. of a Poisson Distribution with mean $\lambda_1 + \lambda_2$.

In general, the sum of two independent Poisson Distributions is also Poisson with mean equal to the sum of the means.

Similarly, assume we are summing two independent Binomial Distributions with parameters m_1 and q, and m_2 and q. The p.g.f. of a Binomial is $P(z) = \{1 + q(z-1)\}^m$.

The p.g.f. of the sum is: $\{1 + q(z-1)\}^{m1} \{1 + q(z-1)\}^{m2} = \{1 + q(z-1)\}^{m1 + m2}$.
This is the p.g.f. of a Binomial Distribution with parameters $m_1 + m_2$ and q.
In general, the sum of two independent Binomial Distributions with the same q parameter is also Binomial with parameters $m_1 + m_2$ and q.

Assume we are summing two independent Negative Binomial Distributions with parameters r_1 and β, and r_2 and β. The p.g.f. of the Negative Binomial is $P(z) = \{1 - \beta(z-1)\}^{-r}$.

The p.g.f. of the sum is: $\{1 - \beta(z-1)\}^{-r1} \{1 - \beta(z-1)\}^{-r2} = \{1 - \beta(z-1)\}^{-(r1 + r2)}$.

This is the p.g.f. of a Negative Binomial Distribution with parameters $r_1 + r_2$ and β.

In general, the sum of two independent Negative Binomial Distributions with the same β parameter is also Negative Binomial with parameters $r_1 + r_2$ and β.

Infinite Divisibility:[55]

If a distribution is infinitely divisible, then if one takes the probability generating function to any positive power, one gets the probability generating function of another member of the same family of distributions.[56]

For example, for the Poisson $P(z) = e^{\lambda(z-1)}$. If we take this p.g.f. to the power $\rho > 0$,

$P(z)^\rho = e^{\rho\lambda(z-1)}$, which is the p.g.f. of a Poisson with mean $\rho\lambda$.

The p.g.f. of a sum of r independent identically distributed variables, is the individual p.g.f. to the power r. Since for the Geometric $P(z) = 1/\{1- \beta(z-1)\}$, for the Negative Binomial distribution:
$P(z) = \{1-\beta(z-1)\}^{-r}$, for $r > 0$, $\beta > 0$.

Exercise: $P(z) = \{1-\beta(z-1)\}^{-r}$, for $r > 0$, $\beta > 0$.
Is the corresponding distribution infinitely divisible?
[Solution: $P(z)^\rho = \{1-\beta(z-1)\}^{-\rho r}$. Which is of the same form, but with ρr rather than r. Thus the corresponding Negative Binomial Distribution is infinitely divisible.]

Infinitely divisible distributions include: Poisson, Negative Binomial, Compound Poisson, Compound Negative Binomial, Normal, Gamma, and Inverse Gaussian.[57]

Exercise: $P(z) = \{1 + q(z-1)\}^m$, for m a positive integer and $0 < q < 1$.
Is the corresponding distribution infinitely divisible?
[Solution: $P(z)^\rho = \{1 + q(z-1)\}^{\rho m}$. Which is of the same form, but with ρm rather than m. However, unless ρ is integral, ρm is not. Thus the corresponding distribution is not infinitely divisible. This is a Binomial Distribution. While Binomials can be added up, they can not be divided into pieces smaller than a Bernoulli Distribution.]

If a distribution is infinitely divisible, and one adds up independent identically distributed random variables, then one gets a member of the same family. As has been discussed this is the case for the Poisson and for the Negative Binomial.

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[55] See Definition 7.6 of Loss Models not on the syllabus, and in Section 9.2 of Loss Models on the syllabus.
[56] One can work with either the probability generating function, the moment generating function, or the characteristic function.
[57] Compound Distributions will be discussed in a subsequent section.

In particular infinitely divisible distributions are preserved under a change of exposure.[58] One can find a distribution of the same type such that when one adds up independent identical copies they add up to the original distribution.

Exercise: Find a Poisson Distribution, such that the sum of 5 independent identical copies will be a Poisson Distribution with $\lambda = 3.5$.

[Solution: A Poisson Distribution with $\lambda = 3.5/5 = 0.7$.]

Exercise: Find a Negative Binomial Distribution, such that the sum of 8 independent identical copies will be a Negative Binomial Distribution with $\beta = .37$ and $r = 1.2$.

[Solution: A Negative Binomial Distribution with $\beta = .37$ and $r = 1.2/8 = .15$.]

Distribution	Probability Generating Function, P(z)[59]	Infinitely Divisible
Binomial	$\left(\frac{1}{1 + q(z-1)}\right)^m$	No[60]
Poisson	$e^{\lambda(z-1)}$	Yes
Negative Binomial	$\left(\frac{1}{1 - \beta(z-1)}\right)^r$, $z < 1 + 1/\beta$	Yes

[58] See Section 7.4 of Loss Models, not on the syllabus.
[59] As shown in Appendix B, attached to the exam.
[60] Since m is an integer.

<u>*A(z)*</u>:[61]

Let a_n = Prob[x > n].

Define $A(z) = \sum_{n=0}^{\infty} a_n z^n$.

$$(1 - z) A(z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^{n+1} = a_0 - \sum_{n=1}^{\infty} (a_{n-1} - a_n) z^n = 1 - p_0 - \sum_{n=1}^{\infty} p_n z^n = 1 - P(z).$$

Thus $A(z) = \frac{1 - P(z)}{1 - z}$.

P(1) = 1. Therefore, as $z \to 1$, $A(z) = \frac{P(z) - 1}{z - 1} \to P'(1) = E[X]$. Thus $\sum_{n=0}^{\infty} a_n = A(1) = E[X]$.

This is analogous to the result that the mean is the integral of the survival function from 0 to ∞.

For example, for a Geometric Distribution, $P(z) = \frac{1}{1 - \beta(z-1)}$.

Thus $A(z) = \frac{1 - P(z)}{1 - z} = \frac{-\beta(z-1)}{1 - \beta(z-1)} \frac{1}{1 - z} = \frac{\beta}{1 + \beta - \beta z}$. ⇒ A(1) = β = mean.

Now in general, $1 / (1 - x/c) = 1 + (x/c) + (x/c)^2 + (x/c)^3 + (x/c)^4 + ...$

Thus $A(z) = \frac{\beta}{1 + \beta - \beta z} = \frac{\beta}{1 + \beta} \frac{1}{1 - z\beta / (1+\beta)} =$

$$\frac{\beta}{1 + \beta} \{1 + \{z\beta/(1+\beta)\} + \{z\beta/(1+\beta)\}^2 + \{z\beta/(1+\beta)\}^3 + \{z\beta/(1+\beta)\}^4 + ...\} =$$

$$\frac{\beta}{1 + \beta} + z \left(\frac{\beta}{1 + \beta}\right)^2 + z^2 \left(\frac{\beta}{1 + \beta}\right)^3 + z^3 \left(\frac{\beta}{1 + \beta}\right)^4 + z^4 \left(\frac{\beta}{1 + \beta}\right)^5 + ...$$

Thus matching up coefficients of z^n, we have: $a_n = \left(\frac{\beta}{1 + \beta}\right)^{n+1}$.

Thus for the Geometric, Prob[x > n] = $\left(\frac{\beta}{1 + \beta}\right)^{n+1}$, a result that has been discussed previously.

[61] See Exercise 6.34 in the <u>third</u> Edition of <u>Loss Models</u>, <u>not</u> on the syllabus.

Problems:

9.1 (2 points) The number of claims, N, made on an insurance portfolio follows the following distribution:

n	Pr(N=n)
0	0.35
1	0.25
2	0.20
3	0.15
4	0.05

What is the Probability Generating Function, P(z)?

A. $1 + 0.65z + 0.4z^2 + 0.2z^3 + 0.05z^4$

B. $0.35 + 0.6z + 0.8z^2 + 0.95z^3 + z^4$

C. $0.35 + 0.25z + 0.2z^2 + 0.15z^3 + 0.05z^4$

D. $0.65 + 0.75z + 0.8z^2 + 0.85z^3 + 0.95z^4$

E. None of A, B, C, or D.

9.2 (1 point) For a Poisson Distribution with $\lambda = 0.3$,
what is the Probability Generating Function at 5?
A. less than 3
B. at least 3 but less than 4
C. at least 4 but less than 5
D. at least 5 but less than 6
E. at least 6

9.3 *(1 point) Which of the following distributions is not infinitely divisible?*
A. Binomial *B. Poisson* *C. Negative Binomial*
D. Normal *E. Gamma*

9.4 (3 points) Given the Probability Generating Function, $P(z) = \frac{e^{0.4z} - 1}{e^{0.4} - 1}$, what is the density at 3 for the corresponding frequency distribution?
A. 1/2% B. 1% C. 2% D. 3% E. 4%

9.5 (5 points) You are given the following data on the number of runs scored during half innings of major league baseball games from 1980 to 1998:

Runs	Number of Occurrences
0	518,228
1	105,070
2	47,936
3	21,673
4	9736
5	4033
6	1689
7	639
8	274
9	107
10	36
11	25
12	5
13	7
14	1
15	0
16	1
Total	709,460

WIth the aid of computer, from z = -2.5 to z = 2.5, graph P(z) the probability generating function of the empirical model corresponding to this data.

9.6 (2 points) A variable B has probability generating function $P(z) = 0.8z^2 + 0.2z^4$.

A variable C has probability generating function $P(z) = 0.7z + 0.3z^5$.

B and C are independent.

What is the probability generating function of B + C.

A. $1.5z^3 + 0.9z^5 + 1.1z^7 + 0.5z^9$

B. $0.25z^3 + 0.25z^5 + 0.25z^7 + 0.25z^9$

C. $0.06z^3 + 0.24z^5 + 0.14z^7 + 0.56z^9$

D. $0.56z^3 + 0.14z^5 + 0.24z^7 + 0.06z^9$

E. None of A, B, C, or D.

9.7 (1 point) Given the Probability Generating Function, $P(z) = 0.5z + 0.3z^2 + 0.2z^4$, what is the density at 2 for the corresponding frequency distribution?

A. 0.2 B. 0.3 C. 0.4 D. 0.5 E. 0.6

9.8 (1 point) For a Binomial Distribution with m = 4 and q = 0.7, what is the Probability Generating Function at 10?
A. less than 1000
B. at least 1000 but less than 1500
C. at least 1500 but less than 2000
D. at least 2000 but less than 2500
E. at least 2500

9.9 (1 point) N follows a Poisson Distribution with $\lambda = 5.6$. Determine $E[3^N]$.
A. 10,000 B. 25,000 C. 50,000 D. 75,000 E. 100,000

9.10 (7 points) A frequency distribution has $P(z) = 1 - (1-z)^{-r}$,
where r is a parameter between 0 and -1.
(a) (3 points) Determine the density at 0, 1, 2, 3, etc.
(b) (1 point) Determine the mean.
(c) (2 points) Let a_n = Prob[x > n]. Define $A(z) = \sum_{n=0}^{\infty} a_n z^n$.

Show that in general, $A(z) = \frac{1 - P(z)}{1 - z}$.

(d) (2 points) Using the result in part (c), show that for this distribution,

$$a_n = \binom{n+r}{n} = (r+1)\ (r+2)\ ....\ (r+n)\ /\ n!.$$

9.11 (3 points) For a Binomial Distribution with m = 2 and q = 0.3:
(a) From the definition, determine the form of the probability generating function, P(z).
(b) Confirm that the result in (a) matches the form given in Appendix A of Loss Models.
(c) Using P(z), recover the densities.

9.12 (2 points) X_1, X_2, and X_3 are independent, identically distributed variables.
$X_1 + X_2 + X_3$ is Poisson. Prove that X_1, X_2, and X_3 are each Poisson.

9.13 (3 points) Given the Probability Generating Function, $P(z) = \left(\frac{0.25}{1 - 0.75z}\right)^5$,

what is the density at 4 for the corresponding frequency distribution?
A. 1/2% B. 1% C. 2% D. 3% E. 4%

9.14 (3 points) Given the Probability Generating Function, $P(z) = \frac{2z^3 / 9}{(1 - z/3)(1 - 2z/3)}$,

what is the mean of the corresponding frequency distribution?

A. 5.5 B. 6.0 C. 6.5 D. 7.0 E. 7.5

9.15 (2, 2/96, Q.15) (1.7 points) Let $X_1,..., X_n$ be independent Poisson random variables with expectations $\lambda_1, . . . , \lambda_n$, respectively. Let $Y = \sum_{i=1}^{n} cX_i$, where c is a constant.

Determine the probability generating function of Y.

A. $\exp[(zc + z^2c^2/2) \sum_{i=1}^{n} \lambda_i]$

B. $\exp[(zc - 1) \sum_{i=1}^{n} \lambda_i]$

C. $\exp[zc \sum_{i=1}^{n} \lambda_i + (z^2c^2/2) \sum_{i=1}^{n} \lambda_i^2]$

D. $\exp[(z^c - 1) \sum_{i=1}^{n} \lambda_i]$

E. $(z^c - 1)^n \prod_{i=1}^{n} \lambda_i$

9.16 (IOA 101, 4/00, Q.10) *(4.5 points) Under a particular model for the evolution of the size of a population over time, the probability generating function of X_t, the size at time t, is given by:*

$P(z) = \frac{z + \lambda t(1 - z)}{1 + \lambda t(1 - z)}$, $\lambda > 0$.

If the population dies out, it remains in this extinct state for ever.

(i) (2.25 points) Determine the expected size of the population at time t.

(ii) (1.5 points) Determine the probability that the population has become extinct by time t.

(iii) (0.75 points) Comment briefly on the future prospects for the population.

9.17 (IOA 101, 9/01, Q.2) (1.5 points) Let X_1 and X_2 be independent Poisson random variables with respective means μ_1 and μ_2. Determine the probability generating function of $X_1 + X_2$ and hence state the distribution of $X_1 + X_2$.

Solutions to Problems:

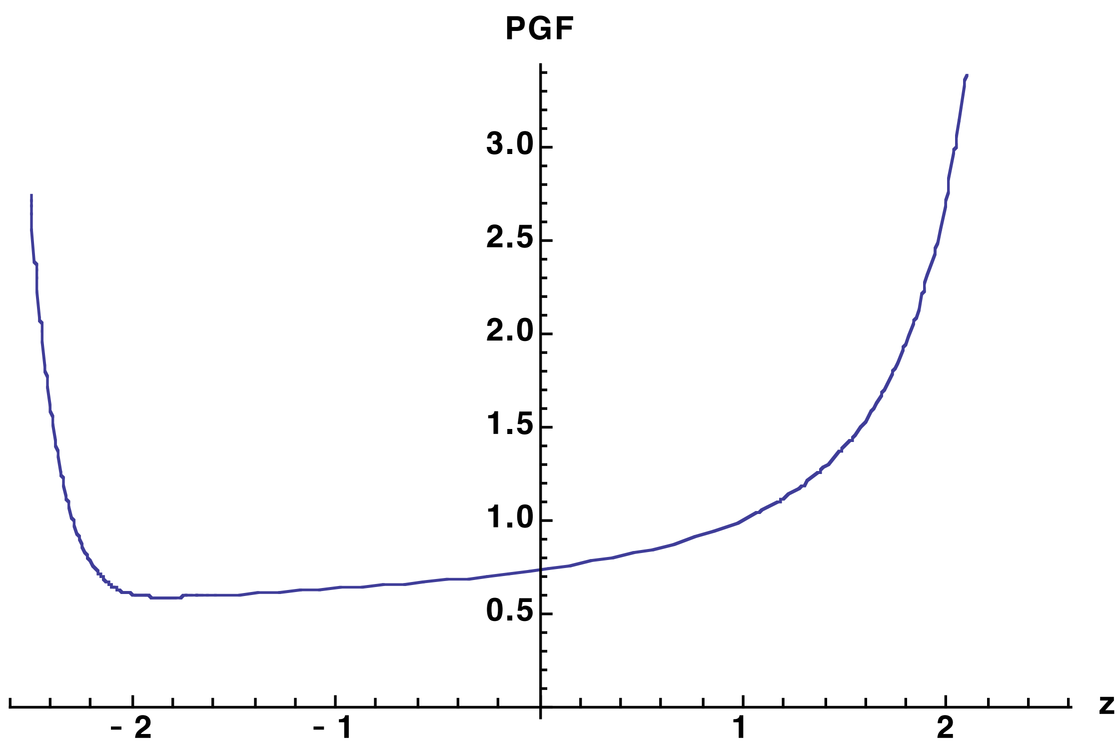
9.1. C. $P(z) = E[z^n] = (0.35)(z^0) + (0.25)(z^1) + (0.20)(z^2) + (0.15)(z^3) + (0.05)(z^4) =$
$\mathbf{0.35 + 0.25z + 0.2z^2 + 0.15z^3 + 0.05z^4}$.

9.2. B. As shown in the Appendix B attached to the exam, for a Poisson Distribution
$P(z) = e^{\lambda(z-1)}$. $P(5) = e^{4\lambda} = e^{1.2} =$ **3.32**.

9.3. A. The **Binomial** is not infinitely divisible.
Comment: In the Binomial m is an integer. For m =1 one has a Bernoulli. One can not divide a Bernoulli into smaller pieces.

9.4. C. $P(z) = (e^{0.4z} - 1)/(e^{0.4} - 1)$. $P'(z) = 0.4e^{0.4z}/(e^{0.4} - 1)$. $P''(z) = 0.16e^{0.4z}/(e^{0.4} - 1)$.
$P'''(z) = 0.064e^{0.4z}/(e^{0.4} - 1)$. $f(3) = (d^3 P(z) / dz^3)_{z=0} / 3! = (0.064/(e^{0.4} - 1))/6 =$ **2.17%**.
Comment: This is a zero-truncated Poisson Distribution with $\lambda = 0.4$.

9.5. $P(z) = \{518,228 + 105,070 z + 47,936 z^2 + ... + z^{16}\} / 709,460$.
Here is a graph of P(z):



Comment: For example, P(-2) = 0.599825, and P(2) = 2.73582.

9.6. D. The probability generating function of a sum of independent variables is the product of the probability generating functions.

$P_{B+C}(z) = P_B(z)P_C(z) = (0.8z^2 + 0.2z^4)(0.7z + 0.3z^5) = \mathbf{0.56z^3 + 0.14z^5 + 0.24z^7 + 0.06z^9}$.

Alternately, B has 80% probability of being 2 and 20% probability of being 4.
C has 70% probability of being 1 and 30% probability of being 5.
Therefore, B + C has: (80%)(70%) = 56% chance of being 1+ 2 = 3,
(20%)(70%) = 14% chance of being 4 + 1 = 5, (80%)(30%) = 24% chance of being 2 + 5 = 7,
and (20%)(30%) = 6% chance of being 4 + 5 = 9.

$\Rightarrow P_{B+C}(z) = (0.8z^2 + 0.2z^4)(0.7z + 0.3z^5) = \mathbf{0.56z^3 + 0.14z^5 + 0.24z^7 + 0.06z^9}$.

Comment: An example of a convolution.

9.7. B. P(z) = Expected Value of $z^n = \sum f(n)\, z^n$. Thus f(2) = **0.3**.

Alternately, $P(z) = 0.5z + 0.3z^2 + 0.2z^4$. $P'(z) = 0.5 + 0.6z + 0.8z^3$. $P''(z) = 0.6 + 2.4z^3$.

$f(2) = (d^2 P(z) / dz^2)_{z=0} / 2! = 0.6/2 = \mathbf{0.3}$.

9.8. E. As shown in the Appendix B attached to the exam, for a Binomial Distribution

$P(z) = \{1 + q(z-1)\}^m = \{1 + (.7)(z-1)\}^4$. $P(10) = \{1 + (.7)(9)\}^4 = \mathbf{2840}$.

9.9. D. The p.g.f. of the Poisson Distribution is: $P(z) = e^{\lambda(z-1)} = e^{5.6(z-1)}$.

$E[3^N] = P(3) = e^{5.6(3-1)} = e^{11.2} = \mathbf{73,130}$.

9.10. (a) f(0) = P(0) = 0.

P'(z) = -r $(1-z)^{-(r+1)}$. f(1) = P'(0) = -r. 0

P''(z) = -r (r+1)$(1-z)^{-(r+1)}$. f(2) = P''(0)/2 = -r(r+1)/2.

P'''(z) = -r (r+1)(r+2)$(1-z)^{-(r+1)}$. f(3) = P'''(0)/3! = -r(r+1)(r+2)/6.

f(x) = -r(r+1) ... (r+x - 1)/x! = $-\frac{\Gamma[x+r]}{\Gamma[x+1]\,\Gamma[r]}$, x = 1, 2, 3, ...

(b) Mean = P'(1) = infinity. The densities go to zero too slowly; thus there is no finite mean.

(c) $(1 - z)\,A(z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^{n+1} = a_0 - \sum_{n=1}^{\infty} (a_{n-1} - a_n) z^n = 1 - p_0 - \sum_{n=1}^{\infty} p_n z^n = 1 - P(z)$.

Thus $A(z) = \frac{1 - P(z)}{1 - z}$.

(d) Thus for this distribution $A(z) = (1-z)^{-r} / (1-z) = (1-z)^{-(r+1)} = \sum_{n=0}^{\infty} \binom{n+r}{n} z^n$, from the Taylor series.

Thus since $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n = \binom{n+r}{n}$ = (r+1) (r+2) .... (r+n) / n!.

Comment: *This is called a Sibuya frequency distribution.*

It is the limit of an Extended Zero-Truncated Negative Binomial Distribution, as $\beta \to \infty$.

See Exercises 6.7 and 8.32 in Loss Models.

For r = -0.7, the densities at 1 through 10: 0.7, 0.105, 0.0455, 0.0261625, 0.0172672, 0.0123749, 0.00936954, 0.00737851, 0.00598479, 0.00496738.

For r = -0.7, Prob[n > 10] = (0.3)(1.3) ... (9.3) / 10! = 0.065995.

P(1) = 1. Therefore, as $z \to 0$, $A(z) = \frac{1 - P(z)}{1 - z} \to P'(1) = E[X]$. Thus $\sum_{n=0}^{\infty} a_n = A(1) = E[X]$.

This is analogous to the result that the mean is the integral of the survival function from 0 to ∞.

For this distribution, Mean = A(1) = $(1 - 1)^{-(r+1)} = \infty$.

9.11. (a) f(0) = 0.7^2 = 0.49. f(1) = (2)(0.3)(0.7) = 0.42. f(2) = 0.3^2 = 0.09.

P(z) = E[z^n] = f(0) z^0 + f(1) z^1 + f(2) z^2 = **0.49 + 0.42z + 0.09z²**.

(b) P(z) = $\{1 + q(z-1)\}^m = \{1 + (0.3)(z-1)\}^2 = (0.7 + 0.3z)^2 = 0.49 + 0.42z + 0.09z^2$.

(c) $f(n) = \frac{\left(\frac{d^n P(z)}{dz^n}\right)_{z=0}}{n!}$.

f(0) = P(0) = 0.49.
P'(z) = 0.42 + 0.18 z. f(1) = P'(0) = 0.42.
P"(z) = 0.18. f(2) = P"(0)/2 = 0.09.
P"(z) = 0. f(3) = P'"(0)/6 = 0.
<u>Comment</u>: Since the Binomial has finite support, P(z) has a finite number of terms.

9.12. Since X_1, X_2, and X_3 are identically distributed variables, they have the same p.g.f.: $P_X(z)$.
Let Y = $X_1 + X_2 + X_3$.

Then since X_1, X_2, and X_3 are independent, $P_Y(z) = P_X(z)\, P_X(z)\, P_X(z) = P_X(z)^3$.

The probability generating function of Y is that of a Poisson: $e^{\lambda(z-1)} = P_X(z)^3$.

$\Rightarrow P_X(z) = e^{(\lambda/3)(z-1)}$. ⇒ X is Poisson with mean 1/3 of that of Y.

9.13. C. For the Negative Binomial, $P(z) = \left(\frac{1}{1 - \beta(z-1)}\right)^r = \left(\frac{1/(1+\beta)}{1 - z\beta/(1+\beta)}\right)^r$.

Matching this to the given p.g.f. of $\left(\frac{0.25}{1 - 0.75z}\right)^5$,

this is a Negative Binomial Distribution with r = 5 and β = 3.

Thus $f(4) = \frac{(5)(6)(7)(8)}{4!} \frac{3^4}{(1+3)^{4+5}}$ = **0.02163**.

Alternately, $P'(z) = \frac{0.25^5}{(1 - 0.75z)^6}$ (-5)(-0.75) = 0.0036621 / $(1 - 0.75z)^6$.

P"(z) = (-6)(-0.75) (0.0036621) / $(1 - 0.75z)^7$ = 0.0164795 / $(1 - 0.75z)^7$.

P'"(z) = (-7)(-0.75) (0.0164795) / $(1 - 0.75z)^8$ = 0.0865173 / $(1 - 0.75z)^8$.

P""(z) = (-8)(-0.75) (0.0865173) / $(1 - 0.75z)^9$ = 0.5191040 / $(1 - 0.75z)^9$.
f(4) = P""(0) / 4! = 0.5191040 / 24 = **0.02163**.

9.14. A. $P'(z) = (2/9) \frac{3z^2 (1 - z/3)(1 - 2z/3) - z^3 (-1/3)(1 - 2z/3) - z^3 (-2/3)(1 - z/3)}{\{(1 - z/3)(1 - 2z/3)\}^2}$.

mean = $P'(1) = (2/9) \frac{3 (1 - 1/3)(1 - 2/3) - (-1/3)(1 - 2/3) - (-2/3)(1 - 1/3)}{\{(1 - 1/3)(1 - 2/3)\}^2}$ = **5.5**.

Comment: Assume we have three different coupons equally likely to be on a given bag of chips. Then this is the probability generating function for the number of bags one has to buy in order to get a complete collection of coupons.
This p.g.f. is the product of the probability generating functions for three zero-truncated Geometric Distributions, with $\beta = 0$, $\beta = 1/2$, and $\beta = 2$.
The mean of the sum of these three independent zero-truncated Geometrics is:
(1 + 0) + (1 + 1/2) + (1 + 2) = 5.5.

9.15. D. For each Poisson, the probability generating function is: $P(z) = \exp[\lambda_i(z-1)]$.

The definition of the probability generating function is $P_N(z) = E[z^n]$. Here we take n = cx.

Multiplying a variable by a constant c: $P_{cX}[z] = E[z^{cx}] = E[(z^c)^x] = P_X[z^c]$.

Thus for each Poisson times c, the p.g.f. is: $\exp[\lambda_i(z^c - 1)]$.

The p.g.f. of the sum of variables is a product of the p.g.f.s: $P_Y(z) = \exp[(z^c - 1) \sum_{i=1}^{n} \lambda_i]$.

Comment: Multiplying a Poisson variable by a constant does not result in another Poisson; *rather it results in what is called an Over-Dispersed Poisson Distribution.*
Since Var[cX]/E[cX] = cVar[X]/E[X], for a constant > 1, the Over-Dispersed Poisson Distribution has a variance greater than it mean. See for example "A Primer on the Exponential Family of Distributions", by David R. Clark and Charles Thayer, CAS 2004 Discussion Paper Program.

9.16. (i) $P'(z) = (\{1-\lambda t\}\{1 + \lambda t(1-z)\} + \lambda t\{z + \lambda t(1-z)\}) / \{1 + \lambda t(1-z)\}^2 = 1 / \{1 + \lambda t(1- z)\}^2$.
E[X] = P'(1) = 1. The expected size of the population is 1 regardless of time.
(ii) $f(0) = P(0) = \lambda t/(1 + \lambda t)$. This is the probability of extinction by time t.

The probability of survival to time t is: $1 - \lambda t/(1 + \lambda t) = 1/(1 + \lambda t) = (1/\lambda)/\{(1/\lambda) + t)$,

the survival function of a Pareto Distribution with $\alpha = 1$ and $\theta = 1/\lambda$.
(iii) As t approaches infinity, the probability of survival approaches zero.
Comment: $P''(z) = 2\lambda t/\{1 + \lambda t(1- z)\}^3$. $E[X(X -1)] = P''(1) = 2\lambda t$.

$\Rightarrow E[X^2] = E[X] + 2\lambda t = 1 + 2\lambda t. \Rightarrow Var[X] = 1 + 2\lambda t - 1^2 = 2\lambda t$.

9.17. $P_1(z) = \exp[\mu_1(z-1)]$. $P_2(z) = \exp[\mu_2(z-1)]$.

Since X_1 and X_2 are independent, the probability generating function of $X_1 + X_2$ is:

$P_1(z)P_2(z) = \exp[\mu_1(z-1) + \mu_2(z-1)] = \exp[(\mu_1 + \mu_2)(z-1)]$.

This is the probability generating function of a Poisson with mean $\mu_1 + \mu_2$, which must therefore be the distribution of $X_1 + X_2$.

Section 10, Factorial Moments

When working with frequency distributions, in addition to moments around the origin and central moments, one sometimes uses factorial moments. The nth factorial moment is the expected value of the product of the n factors: X(X-1) .. (X+1-n).

$$\mu_{(n)} = E[X(X-1) \ ... \ (X+1-n)].$$ [62]

So for example, $\mu_{(1)} = E[X]$, $\mu_{(2)} = E[X(X-1)]$, $\mu_{(3)} = E[X(X-1)(X-2)]$.

Exercise: What is the second factorial moment of a Binomial Distribution with parameters m = 4 and q = 0.3?
[Solution: The density function is:
$f(0) = 0.7^4$, $f(1) = (4)(0.7^3)(0.3)$, $f(2) = (6)(0.7^2)(0.3^2)$, $f(3) = (4)(0.7)(0.3)^3$, $f(4) = 0.3^4$.
$E[X(X-1)] = (0)(-1)f(0) + (1)(0)f(1) + (2)(1)f(2) + (3)(2)f(3) + (4)(3)f(4) =$
$(12)(0.7^2)\ (0.3^2) + (24)(0.7)(0.3^3) + 12(0.3^4) = 12(0.3^2)\{(0.7^2) + (2)(0.7)(0.3) + (0.3^2)\} =$
$(12)(0.3^2)(0.7 + 0.3)^2 = (12)(0.3^2) = 1.08$.]

The factorial moments are related to the moments about the origin as follows:[63]

$\mu_{(1)} = \mu_1' = \mu$

$\mu_{(2)} = \mu_2' - \mu_1'$

$\mu_{(3)} = \mu_3' - 3\mu_2' + 2\mu_1'$

$\mu_{(4)} = \mu_4' - 6\mu_3' + 11\mu_2' - 6\mu_1'$

The moments about the origin are related to the factorial moments as follows:

$\mu_1' = \mu_{(1)} = \mu$

$\mu_2' = \mu_{(2)} + \mu_{(1)}$

$\mu_3' = \mu_{(3)} + 3\mu_{(2)} + \mu_{(1)}$

$\mu_4' = \mu_{(4)} + 6\,\mu_{(3)} + 7\mu_{(2)} + \mu_{(1)}$

Note that one, can use the factorial moments to compute the variance, etc.

[62] See the first page of Appendix B of Loss Models.
[63] Moments about the origin are sometimes referred to as "raw moments."

For example for a Binomial Distribution with m = 4 and q = 0.3, the mean is mq = 1.2, while the second factorial moment was computed to be 1.08. Thus the second moment around the origin is

$\mu_2' = \mu_{(2)} + \mu_{(1)} = \mu_{(2)} + \mu$ = 1.08 + 1.2 = 2.28. Thus the variance is 2.28 - 1.2^2 = 0.84.

This in fact equals mq(1-q) = (4)(0.3)(0.7) = 0.84.

In general the variance (the second central moment) is related to the factorial moments as follows:

variance = $\mu_2 = \mu_2' - \mu_1'^2 = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

<u>Using the Probability Generating Function to Get Factorial Moments</u>:

One can use the Probability Generating Function to get the factorial moments.

To get the n^{th} factorial moment, one differentiates the p.g.f. n times and sets z =1:

$$\mu_{(n)} = \left(\frac{d^n P(z)}{dz^n}\right)_{z=1} = P^n(1).$$

So for example, $\mu_{(1)}$ = **E[X] = P'(1)**, and $\mu_{(2)}$ = **E[X(X-1)] = P''(1)**.[64]

Exercise: Given that the p.g.f. of a Poisson Distribution is $e^{\lambda(z-1)}$, determine its first four factorial moments.

[Solution: $P(z) = e^{\lambda(z-1)} = e^{\lambda z}e^{-\lambda}$. $P'(z) = \lambda e^{\lambda z}e^{-\lambda}$. $P''(z) = \lambda^2 e^{z\lambda}e^{-\lambda}$. $P'''(z) = \lambda^3 e^{z\lambda}e^{-\lambda}$

$P''''(z) = \lambda^4 e^{z\lambda}e^{-\lambda}$. $\mu_{(1)} = P'(1) = \lambda e^{1\lambda}e^{-\lambda} = \lambda$. $\mu_{(2)} = P''(1) = \lambda^2 e^{\lambda}e^{-\lambda} = \lambda^2$.

$\mu_{(3)} = P'''(1) = \lambda^3 e^{1\lambda}e^{-\lambda} = \lambda^3$. $\mu_{(4)} = P''''(1) = \lambda^4$.

<u>Comment</u>: For the Poisson Distribution, $\mu_{(n)} = \lambda^n$.]

Exercise: Using the first four factorial moments of a Poisson Distribution, determine the first four moments of a Poisson Distribution.

[Solution: $\mu_1' = \mu_{(1)} = \lambda$.

$\mu_2' = \mu_{(2)} + \mu_{(1)} = \lambda^2 + \lambda$.

$\mu_3' = \mu_{(3)} + 3\mu_{(2)} + \mu_{(1)} = \lambda^3 + 3\lambda^2 + \lambda$.

$\mu_4' = \mu_{(4)} + 6\mu_{(3)} + 7\mu_{(2)} + \mu_{(1)} = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$.]

[64] Exercise 6.1 in <u>Loss Models</u>.

Exercise: Using the first four moments of a Poisson Distribution, determine its coefficient of variation, skewness, and kurtosis.

[Solution: variance = $\mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Coefficient of variation = standard deviation / mean = $\sqrt{\lambda}/\lambda = 1/\sqrt{\lambda}$.

third central moment = $\mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = \lambda^3 + 3\lambda^2 + \lambda - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$.

skewness = third central moment / variance$^{1.5}$ = $\lambda/\lambda^{1.5} = 1/\sqrt{\lambda}$.

fourth central moment = $\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 =$

$\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4 = 3\lambda^2 + \lambda$.

kurtosis = fourth central moment / variance^2 = $(3\lambda^2 + \lambda)/\lambda^2 = 3 + 1/\lambda$.

<u>Comment</u>: While there is a possibility you might use the skewness of the Poisson Distribution, you are extremely unlikely to ever use the kurtosis of the Poisson Distribution!
Kurtosis is discussed in "Mahler's Guide to Loss Distributions."
As lambda approaches infinity, the kurtosis of a Poisson approaches 3, that of a Normal Distribution.
As lambda approaches infinity, the Poisson approaches a Normal Distribution.]

Exercise: Derive the p.g.f. of the Geometric Distribution and use it to determine the variance.

[Solution: P(z) = Expected Value of $z^n = \sum_{n=0}^{\infty} f(n)\, z^n = \sum_{n=0}^{\infty} \frac{\beta^n}{(1+\beta)^{n+1}} z^n = \frac{1}{1+\beta} \sum_{n=0}^{\infty} \left(\frac{\beta z}{1+\beta}\right)^n$

$= \frac{1}{1+\beta} \frac{1}{1 - z\beta/(1+\beta)} = \frac{1}{1 - \beta(z-1)}$, $z < 1 + 1/\beta$.

$P'(z) = \frac{\beta}{\{1 - \beta(z-1)\}^2}$. $P''(z) = \frac{2\beta^2}{\{1 - \beta(z-1)\}^3}$. $\mu_{(1)} = P'(1) = \beta$. $\mu_{(2)} = P''(1) = 2\beta^2$.

Thus the variance of the Geometric distribution is: $\mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2 = 2\beta^2 + \beta - \beta^2 = \beta(1+\beta)$.]

Formulas for the (a, b, 0) class:[65]

One can use iteration to calculate the factorial moments of a member of the (a,b,0) class.[66]

$\mu_{(1)} = (a + b)/(1-a)$ $\mu_{(n)} = (an + b)\mu_{(n-1)}/(1-a)$

Exercise: Use the above formulas to compute the first three factorial moments about the origin of a Negative Binomial Distribution.

[Solution: For a Negative Binomial Distribution: $a = \beta/(1+\beta)$ and $b = (r-1)\beta/(1+\beta)$.

$\mu_{(1)} = (a + b)/(1 - a) = r\beta/(1+\beta) /\{1/(1+\beta)\} = r\beta$.

$\mu_{(2)} = (2a + b)\mu_{(1)}/(1 - a) = \{(r+1)\beta/(1+\beta)\}\ r\beta/\{1/(1+\beta)\} = r(r+1)\beta^2$.

$\mu_{(3)} = (3a + b)\mu_{(2)}/(1 - a) = \{(r+2)\beta/(1+\beta)\}\ r(r+1)\beta^2/\{1/(1+\beta)\} = r(r+1)(r+2)\beta^3$.]

In general, the nth factorial moment of a Negative Binomial Distribution is:

$\mu_{(n)} = r(r+1)...(r+n-1)\beta^n$.

Exercise: Use the first three factorial moments to compute the first three moments about the origin of a Negative Binomial Distribution.

[Solution: $\mu_1' = \mu_{(1)} = r\beta$. $\mu_2' = \mu_{(2)} + \mu_{(1)} = r(r+1)\beta^2 + r\beta$.

$\mu_3' = \mu_{(3)} + 3\mu_{(2)} + \mu_{(1)} = r(r+1)(r+2)\beta^3 + 3r(r+1)\beta^2 + r\beta$.]

Exercise: Use the first two moments about the origin of a Negative Binomial Distribution to compute its variance.

[Solution: The variance of the Negative Binomial is $\mu_2' - \mu_1'^2 = r(r+1)\beta^2 + r\beta - (r\beta)^2 = r\beta(1+\beta)$.]

Exercise: Use the first three moments about the origin of a Negative Binomial Distribution to compute its skewness.

[Solution: Third central moment = $\mu_3' - (3\,\mu_1'\,\mu_2') + (2\,\mu_1'^3)$ =

$r(r+1)(r+2)\beta^3 + 3r(r+1)\beta^2 + r\beta - (3)(r\beta)(r(r+1)\beta^2 + r\beta) + 2(r\beta)^3$

$= 2r\beta^3 + 3r\beta^2 + r\beta$. Variance = $r\beta(1+\beta)$.

Therefore, skewness = $\dfrac{2r\beta^3 + 3r\beta^2 + r\beta}{\{r\beta(1+\beta)\}^{1.5}} = \dfrac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$.]

[65] The (a, b, 0) class will be discussed subsequently.

[66] See Appendix B.2 of Loss Models.

One can derive that for any member of the (a,b,0) class, the variance = $(a+b) / (1-a)^2$.[67]

For example for the Negative Binomial Distribution, $a = \beta/(1+\beta)$ and $b = (r-1)\beta/(1+\beta)$
variance = $(a+b)/(1-a)^2 = \{r\beta/(1+\beta)\}/\{1/(1+\beta)^2\} = r\beta(1+\beta)$.

The derivation is as follows:
$\mu_{(1)} = (a+b)/(1-a)$. $\mu_{(2)} = (2a+b)\mu_{(1)}/(1-a) = (2a+b)(a+b)/(1-a)^2$.

$\mu_2' = \mu_{(2)} + \mu_{(1)} = (2a+b)(a+b)/(1-a)^2 + (a+b)/(1-a) = (a+b+1)(a+b)/(1-a)^2$

variance = $\mu_2' - \mu_1'^2 = (a+b+1)(a+b)/(1-a)^2 - \{(a+b)/(1-a)\}^2 = (a+b)/(1-a)^2$.

Exercise: Use the above formula for the variance of a member of the (a,b,0) class to compute the variance of a Binomial Distribution.
[Solution: For the Binomial, $a = -q/(1-q)$ and $b = (m+1)q/(1-q)$.
variance = $(a+b)/(1-a)^2 = mq/(1-q) / \{1/(1-q)\}^2 = mq(1-q)$.]

Distribution	nth Factorial Moment
Binomial	$m(m-1)...(m+1-n)q^n$ for $n \le m$, 0 for $n > m$
Poisson	λ^n
Negative Binomial	$r(r+1)...(r+n-1)\beta^n$

[67] See Appendix B.2 of Loss Models.

Problems:

10.1 (2 points) The number of claims, N, made on an insurance portfolio follows the following distribution:

n	Pr(N=n)
0	0.3
1	0.3
2	0.2
3	0.1
4	0.1

What is the second factorial moment of N?

A. 1.6 B. 1.8 C. 2.0 D. 2.2 E. 2.4

10.2 (3 points) Determine the third moment of a Poisson Distribution with $\lambda = 5$.

A. less than 140
B. at least 140 but less than 160
C. at least 160 but less than 180
D. at least 180 but less than 200
E. at least 200

10.3 (2 points) The random variable X has a Binomial distribution with parameters q and m = 8. Determine the expected value of X(X -1)(X - 2).

A. 512 B. 512q(q-1)(q-2) C. q(q-1)(q-2) D. q^3 E. None of A, B, C, or D

10.4 (2 points) You are given the following information about the probability generating function for a discrete distribution:

- P'(1) = 10
- P"(1) = 98

Calculate the variance of the distribution.

A. 8 B. 10 C. 12 D. 14 E. 16

10.5 (3 points) The random variable X has a Negative Binomial distribution with parameters $\beta = 7/3$ and r = 9. Determine the expected value of X(X -1)(X - 2)(X-3).

A. less than 200,000
B. at least 200,000 but less than 300,000
C. at least 300,000 but less than 400,000
D. at least 400,000 but less than 500,000
E. at least 500,000

10.6 (3 points) Determine the third moment of a Binomial Distribution with m = 10 and q = 0.3.
A. less than 40
B. at least 40 but less than 50
C. at least 50 but less than 60
D. at least 60 but less than 70
E. at least 70

10.7 (3 points) Determine the third moment of a Negative Binomial Distribution with r = 10 and $\beta = 3$.

A. less than 36,000
B. at least 36,000 but less than 38,000
C. at least 38,000 but less than 40,000
D. at least 40,000 but less than 42,000
E. at least 42,000

10.8 (4B, 11/97, Q.21) (2 points) The random variable X has a Poisson distribution with mean λ. Determine the expected value of X(X -1)...(X - 9).

A. 1 B. λ C. $\lambda(\lambda-1)...(\lambda-9)$ D. λ^{10} E. $\lambda(\lambda+1)...(\lambda+9)$

10.9 (CAS3, 11/06, Q.25) (2.5 points) You are given the following information about the probability generating function for a discrete distribution:

- P'(1) = 2
- P''(1) = 6

Calculate the variance of the distribution.
A. Less than 1.5
B. At least 1.5, but less than 2.5
C. At least 2.5, but less than 3.5
D. At least 3.5, but less than 4.5
E. At least 4.5

Solutions to Problems:

10.1. **D.** The 2nd factorial moment is:
E[N(N-1)] = (.3)(0)(-1) + (.3)(1)(0) + (.2)(2)(1) + (.1)(3)(2) + (.1)(4)(3) = **2.2**.

10.2. E. The factorial moments for a Poisson are: λ^n. mean = first factorial moment = λ = 5.

Second factorial moment = 5^2 = 25 = E[X(X-1)] = E[X^2] - E[X]. ⇒ E[X^2] = 25 + 5 = 30.

Third factorial moment = 5^3 = 125 = E[X(X-1)(X-2)] = E[X^3] - 3E[X^2] + 2E[X].

⇒ E[X^3] = 125 + (3)(30) - (2)(5) = **205**.

Alternately, for the Poisson P(z) = $e^{\lambda(z-1)}$.

P(z) = $e^{\lambda(z-1)}$. P'(z) = $\lambda e^{\lambda(z-1)}$. P''(z) = $\lambda^2 e^{\lambda(z-1)}$. P'''(z) = $\lambda^3 e^{\lambda(z-1)}$.

mean = first factorial moment = P'(1) = λ. Second factorial moment = P''(1) = λ^2.

Third factorial moment = P'''(1) = λ^3. Proceed as before.

Alternately, the skewness of a Poisson is $1/\sqrt{\lambda}$.

Since the variance is λ, the third central moment is: $\lambda^{1.5}/\sqrt{\lambda} = \lambda$.

$\lambda = E[(X - \lambda)^3] = E[X^3] - 3\lambda E[X^2] + 3\lambda^2 E[X] - \lambda^3$.

⇒ $E[X^3] = \lambda + 3\lambda E[X^2] - 3\lambda^2 E[X] + \lambda^3 = \lambda + 3\lambda(\lambda + \lambda^2) - 3\lambda^2\lambda + \lambda^3 = \lambda^3 + 3\lambda^2 + \lambda$

= 125 + 75 + 5 = **205**.

Comment: The third moment of a Poisson Distribution is: $\lambda^3 + 3\lambda^2 + \lambda$.

One could compute enough of the densities and then calculate the third moment:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	0.674%	0.00000	0.00000	0.00000
1	3.369%	0.03369	0.03369	0.03369
2	8.422%	0.16845	0.33690	0.67379
3	14.037%	0.42112	1.26337	3.79010
4	17.547%	0.70187	2.80748	11.22991
5	17.547%	0.87734	4.38668	21.93342
6	14.622%	0.87734	5.26402	31.58413
7	10.444%	0.73111	5.11780	35.82459
8	6.528%	0.52222	4.17779	33.42236
9	3.627%	0.32639	2.93751	26.43761
10	1.813%	0.18133	1.81328	18.13279
11	0.824%	0.09066	0.99730	10.97034
12	0.343%	0.04121	0.49453	5.93437
13	0.132%	0.01717	0.22323	2.90193
14	0.047%	0.00660	0.09246	1.29444
15	0.016%	0.00236	0.03538	0.53070
16	0.005%	0.00079	0.01258	0.20127
17	0.001%	0.00025	0.00418	0.07101
Sum	0.99999458366	4.99990	29.99818	204.96644

10.3. E. $f(x) = \frac{8!}{x!\,(8-x)!} q^x (1-q)^{8-x}$, for x = 0 to 8.

$$E[X(X-1)(X-2)] = \sum_{x=3}^{x=8} x(x-1)(x-2)f(x) = \sum_{x=3}^{x=8} x(x-1)(x-2)\,\frac{8!}{x!\,(8-x)!} q^x (1-q)^{8-x} =$$

$$(8)(7)(6)q^3 \sum_{x=3}^{x=8} \frac{5!}{(x-3)!\,(8-x)!} q^{x-3} (1-q)^{8-x} = 336q^3 \sum_{y=0}^{y=5} \frac{5!}{y!\,(5-y)!} q^y (1-q)^{5-y} = \mathbf{336q^3}.$$

Alternately, the 3rd factorial moment is the 3rd derivative of the p.g.f. at z = 1.

For the Binomial: $P(z) = \{1+ q(z-1)\}^m$. $dP/dz = mq\{1+ q(z-1)\}^{(m-1)}$.

$P''(z) = m(m-1)q^2\{1+ q(z-1)\}^{(m-2)}$. $P'''(z) = m(m-1)(m-2)q^3\{1+ q(z-1)\}^{(m-3)}$.

$P'''(1) = m(m-1)(m-2)q^3 = (8)(7)(6)q^3 = \mathbf{336q^3}$.

Comment: Note that the product x(x -1)(x-2) is zero for x = 0,1 and 2, so only terms for $x \geq 3$ contribute to the sum. Then a change of variables is made: y = x-3. Then the resulting sum is the sum of Binomial terms from y = 0 to 5, which sum is one, since the Binomial is a Distribution, with a support in this case 0 to 5. The expected value of: X(X -1)(X - 2), is an example of what is referred to as a factorial moment.

In the case of the Binomial, the kth factorial moment for $k \leq m$ is:

$p^k(m!)/(m-k)! = p^k(m)(m-1)...(m-(k-1))$. In our case we have the 3rd factorial moment (involving the product of 3 terms) equal to: $q^3(m)(m-1)(m-2)$.

10.4. A. 10 = P'(1) = E[N]. 98 = P"(1) = E[N(N-1)] = $E[N^2]$ - E[N]. $E[N^2]$ = 98 + 10 = 108.

$Var[N] = E[N^2] - E[N]^2 = 108 - 10^2 =$ **8**.

Comment: Similar to CAS3, 11/06, Q.25.

10.5. C. $f(x) = \frac{(x+8)!}{x!\ 8!} \frac{(7/3)^x}{(1+\ 7/3)^{x+9}}$. E[X(X-1)(X-2)(X-3)] =

$$\sum_{x=0}^{x=\infty} x(x-1)(x-2)(x-3)f(x) = \sum_{x=4}^{x=\infty} x(x-1)(x-2)(x-3) \frac{(x+8)!}{x!\ 8!} \frac{(7/3)^x}{(1+\ 7/3)^{x+9}} =$$

$$(12)(11)(10)(9)(7/3)^4 \sum_{x=4}^{x=\infty} \frac{(x+8)!}{(x-4)!\ 12!} \frac{(7/3)^{x-4}}{(1+\ 7/3)^{x+9}} =$$

$$352{,}147 \sum_{y=0}^{y=\infty} \frac{(y+12)!}{y!\ 12!} \frac{(7/3)^y}{(1+\ 7/3)^{y+13}} = \mathbf{352{,}147.}$$

Alternately, the 4th factorial moment is the 4th derivative of the p.g.f. at z = 1.
For the Negative Binomial: $P(z) = \{1-\beta(z-1)\}^{-r}$. $dP/dz = r\beta\{1-\beta(z-1)\}^{-(r+1)}$.

$P''(z) = r(r+1)\beta^2\{1-\beta(z-1)\}^{-(r+2)}$. $P'''(z) = r(r+1)(r+2)\beta^3\{1-\beta(z-1)\}^{-(r+3)}$.

$P''''(z) = r(r+1)(r+2)(r+3)\beta^4\{1-\beta(z-1)\}^{-(r+4)}$.

$P''''(1) = r(r+1)(r+2)(r+3)\beta^4 = (9)(10)(11)(12)(7/3)^4 =$ **352,147.**

Comments: Note that the product x(x -1)(x-2)(x-3) is zero for x = 0,1,2 and 3, so only terms for x ≥4 contribute to the sum. Then a change of variables is made: y = x-4. Then the resulting sum is the sum of Negative Binomial terms, with β = 7/3 and r = 13, from y = 0 to infinity, which sum is one, since the Negative Binomial is a Distribution with support from 0 to ∞.
The expected value of X(X -1)(X - 2)(X-3), is an example of a factorial moment.
In the case of the Negative Binomial, the mth factorial moment is: $\beta^m (r)(r+1)...(r+m-1)$.

In our case we have the 4^{th} factorial moment (involving the product of 4 terms) equal to:

$\beta^4 (r)(r+1)(r+2)(r+3)$, with β = 7/3 and r = 9.

10.6. B. $P(z) = \{1 + q(z-1)\}^m = \{1 + 0.3(z-1)\}^{10} = \{0.7 + 0.3z\}^{10}$.

$P'(z) = (10)(0.3)\{0.7 + 0.3z\}^9$. $P''(z) = (3)(2.7)\{0.7 + 0.3z\}^8$. $P'''(z) = (3)(2.7)(2.4)\{0.7 + 0.3z\}^7$.
mean = first factorial moment = P'(1) = 3.
Second factorial moment = P''(1) = (3)(2.7) = 8.1.
Third factorial moment = P'''(1) = (3)(2.7)(2.4) = 19.44.

Second factorial moment = 8.1 = E[X(X-1)] = $E[X^2]$ - E[X]. $\Rightarrow E[X^2] = 8.1 + 3 = 11.1$.

Third factorial moment = 19.44 = E[X(X-1)(X-2)] = $E[X^3] - 3E[X^2] + 2E[X]$.

$\Rightarrow E[X^3] = 19.44 + (3)(11.1) - (2)(3) =$ **46.74**.

Comment: $E[X^2]$ = variance + mean^2 = $2.1 + 3^2 = 11.1$.

One could compute all of the densities and then calculate the third moment:

Number of Claims	Probability Density Function	Probability x # of Claims	Probability x Square of # of Claims	Probability x Cube of # of Claims
0	2.825%	0.00000	0.00000	0.00000
1	12.106%	0.12106	0.12106	0.12106
2	23.347%	0.46695	0.93390	1.86780
3	26.683%	0.80048	2.40145	7.20435
4	20.012%	0.80048	3.20194	12.80774
5	10.292%	0.51460	2.57298	12.86492
6	3.676%	0.22054	1.32325	7.93949
7	0.900%	0.06301	0.44108	3.08758
8	0.145%	0.01157	0.09259	0.74071
9	0.014%	0.00124	0.01116	0.10044
10	0.001%	0.00006	0.00059	0.00590
Sum	1	3.00000	11.10000	46.74000

10.7. C. $P(z) = \{1 - \beta(z-1)\}^{-r} = \{1 - 3(z-1)\}^{-10} = (4 - 3z)^{-10}$.

$P'(z) = (-10)(-3)(4 - 3z)^{-11}$. $P''(z) = (30)(33)(4 - 3z)^{-12}$. $P'''(z) = (30)(33)(36)(4 - 3z)^{-13}$.
mean = first factorial moment = P'(1) = 30.
Second factorial moment = P''(1) = (30)(33) = 990.
Third factorial moment = P'''(1) = (30)(33)(36) = 35,640.

Second factorial moment = 990 = E[X(X-1)] = $E[X^2]$ - E[X]. $\Rightarrow E[X^2] = 990 + 30 = 1020$.

Third factorial moment = 35,640 = E[X(X-1)(X-2)] = $E[X^3] - 3E[X^2] + 2E[X]$.

$\Rightarrow E[X^3] = 35,640 + (3)(1020) - (2)(30) =$ **38,640**.

Comment: $E[X^2]$ = variance + mean^2 = $(10)(3)(4) + 30^2 = 1020$.

10.8. D. For a discrete distribution, the expected value of a quantity is determined by taking the sum of its product with the probability density function. In this case, the density of the Poisson is:
$e^{-\lambda}\lambda^x / x!$, x = 0, 1, 2... Thus E[X(X -1)...(X - 9)] =

$$\sum_{x=0}^{x=\infty} x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9) \frac{e^{-\lambda} \lambda^x}{x!} =$$

$$e^{-\lambda}\lambda^{10} \sum_{x=10}^{x=\infty} \frac{\lambda^{x-10}}{(x-10)!} = e^{-\lambda}\lambda^{10} \sum_{y=0}^{y=\infty} \frac{\lambda^y}{y!} = e^{-\lambda}\lambda^{10} e^{\lambda} = \boldsymbol{\lambda^{10}}.$$

Alternately, the 10th factorial moment is the 10th derivative of the p.g.f. at z = 1.
For the Poisson: P(z) = exp(λ(z-1)). dP/dz = λ exp(λ(z-1)). P''(z) = λ^2 exp(λ(z-1)).

P'''(z) = λ^3 exp(λ(z-1)). $P^{10}(z) = \lambda^{10}$ exp(λ(z-1)). $P^{10}(1) = \boldsymbol{\lambda^{10}}$.
Comment: Note that the product x(x -1)...(x - 9) is zero for x = 0,1...,9, so only terms for x ≥10 contribute to the sum. The expected value of X(X -1)...(X - 9), is an example of a factorial moment. In the case of the Poisson, the nth factorial moment is λ to the nth power. In this case we have the 10^{th} factorial moment (involving the product of 10 terms) equal to λ^{10}.

10.9. D. 2 = P'(1) = E[N]. 6 = P''(1) = E[N(N-1)] = E[N^2] - E[N]. E[N^2] = 6 + 2 = 8.
Var[N] = E[N^2] - E[N]^2 = 8 - 2^2 = **4**.
Comment: P(z) = E[z^n] = Σ f(n)z^n. P'(z) = Σ nf(n)z^{n-1}. P'(1) = Σ nf(n) = E[N].
P''(z) = Σ n(n-1)f(n)z^{n-2}. P''(1) = Σ n(n-1)f(n) = E[N(N-1)].

Section 11, (a, b, 0) Class of Distributions

The "(a,b,0) class of frequency distributions" consists of the three common distributions: Binomial, Poisson, and Negative Binomial.

Distribution	Mean	Variance	Variance / Mean	
Binomial	mq	$mq(1-q)$	$1 - q < 1$	**Variance < Mean**
Poisson	λ	λ	1	**Variance = Mean**
Negative Binomial	$r\beta$	$r\beta(1+\beta)$	$1+\beta > 1$	**Variance > Mean**

Distribution	Skewness	
Binomial	$\frac{1 - 2q}{\sqrt{m q (1 - q)}}$	If $q < 0.5$ skewed right, if $q > 0.5$ skewed left
Poisson	$1 / \sqrt{\lambda}$	Skewed to the right
Negative Binomial	$\frac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$	Skewed to the right

Distribution	f(x)	f(x+1)	f(x+1) / f(x)
Binomial	$\frac{m!\, q^x (1-q)^{m-x}}{x!\, (m-x)!}$	$\frac{m!\, q^{x+1} (1-q)^{m-x-1}}{(x+1)!\, (m-x-1)!}$	$\frac{q}{1-q} \frac{m-x}{x+1}$
Poisson	$\lambda^x e^{-\lambda} / x!$	$\lambda^{x+1} e^{-\lambda} / (x+1)!$	$\lambda / (x+1)$
Negative Binomial	$\frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}$	$\frac{r(r+1)...(r+x)}{(x+1)!} \frac{\beta^{x+1}}{(1+\beta)^{x+r+1}}$	$\frac{\beta}{1+\beta} \frac{x+r}{x+1}$

(a,b,0) relationship:

For each of these three frequency distributions: $\mathbf{f(x+1) / f(x) = a + \frac{b}{x+1}}$, x = 0, 1, 2, ...

where a and b depend on the parameters of the distribution:[68]

Distribution	**a**	**b**	**f(0)**
Binomial	**-q/(1-q)**	**(m+1)q/(1-q)**	$(1-q)^m$
Poisson	**0**	λ	$e^{-\lambda}$
Negative Binomial	$\beta/(1+\beta)$	$(r-1)\beta/(1+\beta)$	$1/(1+\beta)^r$

Loss Models writes this recursion formula equivalently as: $p_k/p_{k-1} = a + b/k$, k = 1, 2, 3 ... [69]

This relationship defines the **(a,b,0) class of frequency distributions**.[70] The (a, b ,0) class of frequency distributions consists of the three common distributions: Binomial, Poisson, and Negative Binomial.[71] Therefore, it also includes the Bernoulli, which is a special case of the Binomial, and the Geometric, which is a special case of the Negative Binomial.

Note that a is positive for the Negative Binomial, zero for the Poisson, and negative for the Binomial.

These formula can be useful when programming these frequency distributions into spreadsheets. One calculates f(0) and then one gets additional values of the density function via iteration: f(x+1) = f(x){a + b / (x+1)}.

f(1) = f(0) (a + b). f(2) = f(1) (a + b/2). f(3) = f(2) (a + b/3). f(4) = f(3) (a + b/4), etc.

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[68] These a and b values are shown in the tables attached to the exam. This relationship is used in the Panjer Algorithm (recursive formula), a manner of computing either the aggregate loss distribution or a compound frequency distribution. For a member of the (a,b,0) class, the values of a and b determine everything about the distribution. *Given the density at zero, all of the densities would follow; however, the sum of all of the densities must be one.*

[69] See Definition 6.4 in Loss Models.

[70] See Table 6.1 and Appendix B.2 in Loss Models. The (a, b, 0) class is distinguished from the (a, b, 1) class, to be discussed in a subsequent section, by the fact that the relationship holds starting with the density at zero, rather than possibly only starting with the density at one.

[71] As stated in Loss Models, these are the only members of the (a, b, 0) class. This is proved in Lemma 6.6.1 of Insurance Risk Models, by Panjer and Willmot. Only certain combinations of a and b are acceptable. Each of the densities must be nonnegative and they must sum to one, a finite amount.

Thinning and Adding:

Distribution	Thinning by factor of t	Adding n independent, identical copies
Binomial	$q \to tq$	$m \to nm$
Poisson	$\lambda \to t\lambda$	$\lambda \to n\lambda$
Negative Binomial	$\beta \to t\beta$	$r \to nr$

If for example, we assume 1/4 of all claims are large:

If All Claims	Then Large Claims
Binomial m = 5, q = 0.04	Binomial m = 5, q = 0.01
Poisson λ = 0.20	Poisson λ = 0.05
Negative Binomial r = 2, β = 0.10	Negative Binomial r = 2, β = 0.025

In the Poisson case, small and large claims are independent Poisson Distributions.[72]

For X and Y independent:

X	**Y**	**X + Y**
Binomial(m_1, q)	**Binomial(m_2, q)**	**Binomial($m_1 + m_2$, q)**
Poisson(λ_1)	**Poisson(λ_2)**	**Poisson($\lambda_1 + \lambda_2$)**
Negative Binomial(r_1, β)	**Negative Bin.(r_2, β)**	**Negative Bin.($r_1 + r_2$, β)**

If Claims each Year	Then Claims for 6 Independent Years
Binomial m = 5, q = 0.04	Binomial m = 30, q = 0.04
Poisson λ = 0.20	Poisson λ = 1.20
Negative Binomial r = 2, β = 0.10	Negative Binomial r = 12, β = 0.10

[72] As discussed in the section on the Gamma-Poisson Frequency Process, in the Negative Binomial case, the number of large and small claims are positively correlated. In the Binomial case, the number of large and small claims are negatively correlated.

Probability Generating Functions:

Recall that the probability generating function for a given distribution is $P(z) = E[z^N]$.

Distribution	Probability Generating Function
Binomial	$P(z) = \{1 + q(z-1)\}^m$
Poisson	$P(z) = e^{\lambda(z-1)}$
Negative Binomial	$P(z) = \{1 - \beta(z-1)\}^{-r}$, $z < 1 + 1/\beta$

Parametric Models:

Some advantages of parametric models:
1. They summarize the information in terms of the form of the distribution and the parameter values.
2. They serve to smooth the empirical data.
3. They greatly reduce the dimensionality of the information.

In addition one can use parametric models to extrapolate beyond the largest observation.
As will be discussed in a subsequent section, the behavior in the righthand tail is an important feature of any frequency distribution.

Some advantages of working with separate distributions of frequency and severity:[73]

1. Can obtain a deeper understanding of a variety of issues surrounding insurance.
2. Allows one to address issues of modification of an insurance contract (for example, deductibles.)
3. Frequency distributions are easy to obtain and do a good job of modeling the empirical situations.

[73] See Section 6.1 of Loss Models.

Limits:

Since, the probability generating function determines the distribution, one can take limits of a distribution by instead taking limits of the Probability Generating Function.

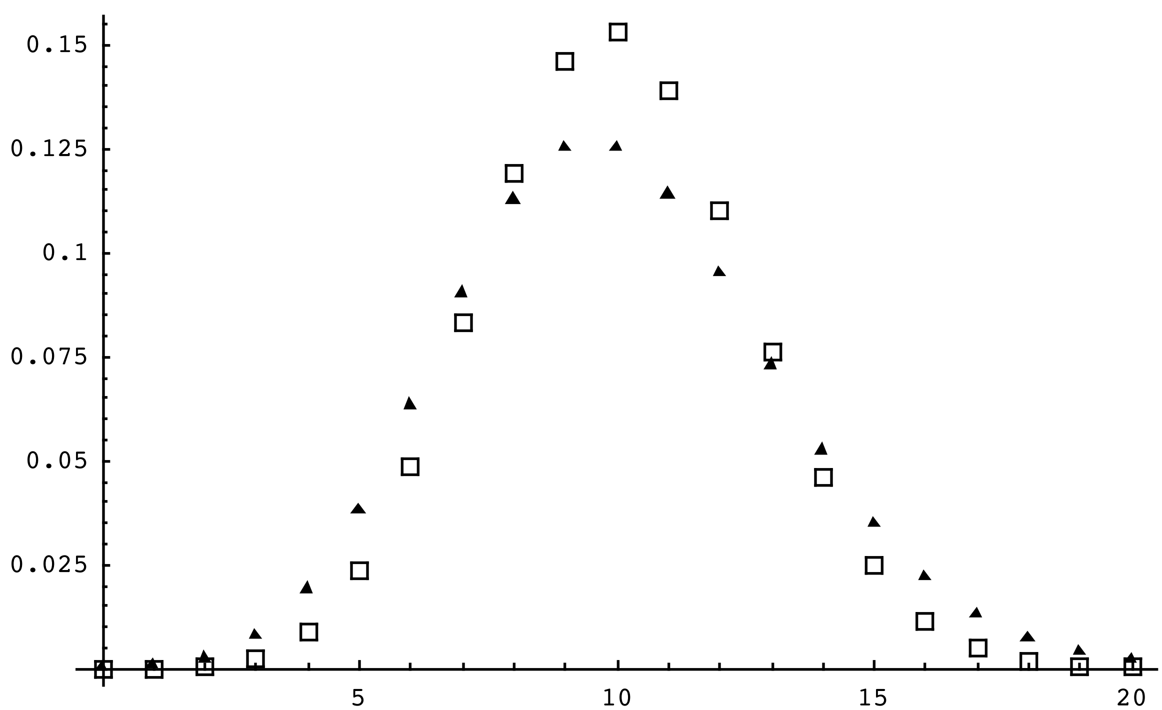
Assume one takes a limit of the probability generating function of a Binomial distribution for qm = λ as $m \to \infty$ and $q \to 0$:

$P(z) = \{1 + q(z-1)\}^m = \{1 + q(z-1)\}^{\lambda/q} = [\{1 + q(z-1)\}^{1/q}]^{\lambda} \to \{e^{(z-1)}\}^{\lambda} = e^{\lambda(z-1)}$.

Where we have used the fact that as $x \to 0$, $(1+ax)^{1/x} \to e^a$.

Thus the limit of the Binomial Probability Generating Function is the Poisson Probability Generating Function. Therefore, as we let q get very small in a Binomial but keep the mean constant, in the limit one approaches a Poisson with the same mean.[74]

For example, a Poisson (triangles) with mean 10 is compared to a Binomial (squares) with q = 1/3 and m = 30 (mean = 10, variance = 20/3):



While the Binomial is shorter-tailed than the Poisson, they are not that dissimilar.

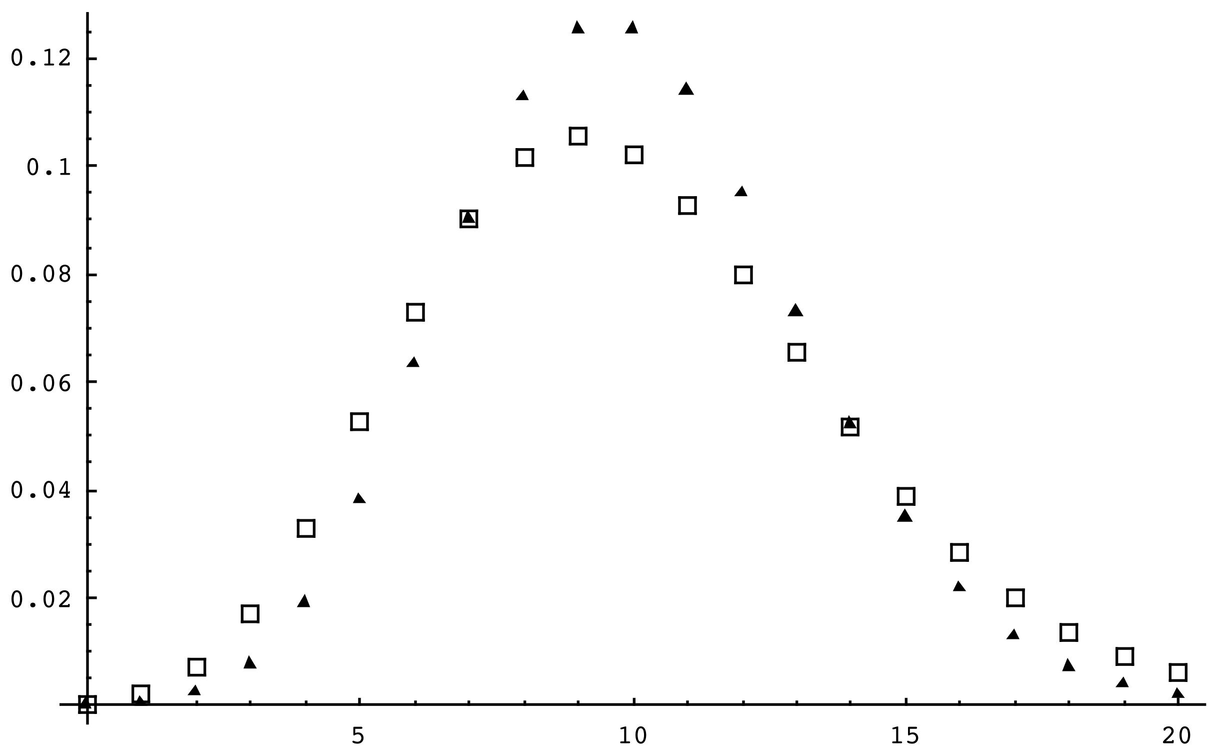
[74] The limit of the probability generating function is the probability generating function of the limit of the distributions if it exists.

Assume one takes a limit of the probability generating function of a Negative Binomial distribution for $r\beta = \lambda$ as $r \to \infty$ and $\beta \to 0$:

$P(z) = \{1-\beta(z-1)\}^{-r} = \{1-\beta(z-1)\}^{-\lambda/\beta} = [\{1-\beta(z-1)\}^{1/\beta}]^{-\lambda} \to \{e^{-(z-1)}\}^{-\lambda} = e^{\lambda(z-1)}$.

Thus the limit of the Negative Binomial Probability Generating Function is the Poisson Probability Generating Function. Therefore, as we let β get very close to zero in a Negative Binomial but keep the mean constant, in the limit one approaches a Poisson with the same mean.[75]

A Poisson (triangles) with mean 10 is compared to a Negative Binomial Distribution (squares) with r = 20 and β = 0.5 (mean = 10, variance = 15):



For the three distributions graphed here and previously, while the means are the same, the variances are significantly different; thus the Binomial is more concentrated around the mean while the Negative Binomial is more dispersed from the mean. Nevertheless, one can see how the three distributions are starting to resemble each other.[76]

[75] The limit of the probability generating function is the probability generating function of the limit of the distributions if it exists.
[76] They are each approximated by a Normal Distribution. While these three Normal Distributions have the same mean, they have different variances.

If the Binomial q were smaller and m larger such that the mean remained 10, for example q = 1/30 and m = 300, then the Binomial would have been much closer to the Poisson. Similarly, if on the Negative Binomial one had β closer to zero with r larger such that the mean remained 10, for example β = 1/9 and r = 90, then the Negative Binomial would have been much closer to the Poisson.

Thus the Poisson is the limit of either a series of Binomial or Negative Binomial Distributions as they "come from different sides."[77] The Binomial has q go to zero; one adds up very many Bernoulli Trials each with a very small chance of success. This approaches a constant chance of success per very small unit of time, which is a Poisson. Note that for each Binomial the mean is greater than the variance, but as q goes to zero the variance approaches the mean.

For the Negative Binomial one lets β goes to zero; one adds up very many Geometric distributions each with very small chance of a claim.[78] Again this limit is a Poisson, but in this case for each Negative Binomial the variance is greater than the mean. As β goes to zero, the variance approaches the mean.

As mentioned previously the Distribution Function of the Binomial Distribution is a form of the Incomplete Beta Function, while that of the Poisson is in the form of an Incomplete Gamma Function. As q → 0 and the Binomial approaches a Poisson, the Distribution Function of the Binomial approaches that of the Poisson. An Incomplete Gamma Function can thus be obtained as a limit of Incomplete Beta Distributions. Similarly, the Distribution Function of the Negative Binomial is a somewhat different form of the Incomplete Beta Distribution.

As β → 0 and the Negative Binomial approaches a Poisson, the Distribution Function of the Negative Binomial approaches that of the Poisson. Again, an Incomplete Gamma Function can be obtained as a limit of Incomplete Beta Distributions.

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[77] One can also show this via the use of Sterling's formula to directly calculate the limits rather than via the use of Probability Generating Functions.

[78] The mean of a Geometric is β, thus as β →0, the chance of a claim becomes very small.
For the Negative Binomial, r = mean/β, so that as β →0 for a fixed mean, r →∞.

Modes:

The mode, where the density is largest, can be located by observing where f(x+1)/f(x) switches from being greater than 1 to being less than 1.[79]

Exercise: For a member of the (a, b, 0) frequency class, when is f(x+1)/f(x) greater than one, equal to one, and less than one?
[Solution: f(x+1)/f(x) = 1 when a + b/(x+1) = 1. This occurs when x = b/(1-a) - 1.
For x < b/(1-a) - 1, f(x+1)/f(x) > 1. For x > b/(1-a) - 1, f(x+1)/f(x) < 1.]

For example, for a Binomial Distribution with m = 10 and q = .23, a = - q/(1-q) = -.2987 and b = (m+1)q/(1-q) = 3.2857. For x > b/(1-a) - 1 = 1.53, f(x+1)/f(x) < 1.
Thus f(3) < f(2). For x < 1.53, f(x+1)/f(x) > 1. Thus f(2) > f(1). Therefore, the mode is 2.

In general, since for x < b/(1-a) - 1, f(x+1) > f(x), if c is the largest integer in b/(1-a), f(c) > f(c-1).
Since for x > b/(1-a) - 1, f(x+1) < f(x), f(c+1) > f(c). Thus c is the mode.

For a member of the (a, b, 0) class, the mode is the largest integer in b/(1-a).
If b/(1-a) is an integer, then f(b/(1-a) - 1) = f(b/(1-a)), and there are two modes.

For the Binomial Distribution, a = - q/(1-q) and b = (m+1)q/(1-q), so b/(1-a) = (m+1)q.
Thus the mode is the largest integer in (m+1)q.
If (m+1)q is an integer, there are two modes at: (m+1)q and (m+1)q - 1.

For the Poisson Distribution, a = 0 and b = λ, so b/(1-a) = λ. Thus the mode is the largest integer in λ. If λ is an integer, there are two modes at: λ and λ - 1.

For the Negative Binomial Distribution, a = $\beta/(1+\beta)$ and b = $(r-1)\beta/(1+\beta)$, so b/(1-a) = $(r-1)\beta$. Thus the mode is the largest integer in $(r-1)\beta$.
If $(r-1)\beta$ is an integer, there are two modes at: $(r-1)\beta$ and $(r-1)\beta$ - 1.

Note that in each case the mode is close to the mean.[80]
So one could usefully start a numerical search for the mode at the mean.

[79] In general this is only a local maximum, but members of the (a,b, 0) class do not have local maxima other than the mode.
[80] The means are mq, λ, and $r\beta$.

Moments:

Formulas for the Factorial Moments of the (a, b, 0) class have been discussed in a previous section.

It can be derived from those formulas that for a member of the (a, b, 0) class:

Mean	$(a + b)/(1 - a)$
Second Moment	$(a + b)(a + b + 1)/(1 - a)^2$
Variance	$(a + b)/(1 - a)^2$
Third Moment	$(a + b)\{(a + b + 1)(a + b + 2) + a - 1\}/(1 - a)^3$
Skewness	$(a + 1)/\sqrt{a + b}$

A Generalization of the (a, b, 0) Class:

The (a, b, 0) relationship is: $f(x+1) / f(x) = a + \{b / (x+1)\}$, x = 0, 1, 2, ...
or equivalently: $p_k / p_{k-1} = a + b/k$, k = 1, 2, 3, ...

A more general relationship is: $p_k / p_{k-1} = (ak + b) / (k + c)$, k = 1, 2, 3, ...

If c = 0, then this would reduce to the (a, b, 0) relationship.

Contagion Model:[81]

Assume one has a claim intensity of λ. Then the chance of having a claim over an extremely small period of time Δt is approximately $\lambda(\Delta t)$.[82] We assume there is (virtually) no chance of having more than one claim over extremely small time period Δt.

As mentioned previously, if the claim intensity is a constant over time, then the number of claims observed over a period time t is given by a Poisson Distribution, with mean λt. If the claim intensity depends on the number of claims that have occurred so far then the frequency distribution is other than Poisson.

Given one has had k-1 claims so far, let $\lambda_k \Delta t$ be the chance of having the k^{th} claim in small time period Δt. Then the times between claims are independent Exponentials; the mean time between claim k -1 and k is $1/\lambda_k$. Assume that this claims intensity is linear in k: $\lambda_k = c + d\,k$, $c > 0$.

Then for $d > 0$, it turns out that one gets a Negative Binomial Distribution. As one observes more claims the chance of observing another claim goes up. This is referred to as positive contagion; examples might be claims due to a contagious disease or from a very large fire. Over time period (0, t), the parameters of the Negative Binomial are: $r = c/d$, and $\beta = e^{dt} - 1$.

For $d < 0$, it turns out that one gets a Binomial distribution. As one observes more claims, the chance of future claims goes down. This is referred to as negative contagion. Over time period (0, t), the parameters of the Binomial are: $m = -c/d$, and $q = 1 - e^{dt}$.

For $d = 0$ one gets the Poisson. There is no contagion and the claim intensity is constant. Thus the contagion model is another mathematical connection between these three common frequency distributions. We expect as $d \to 0$ in either the Binomial or Negative Binomial that we approach a Poisson. This is indeed the case as discussed previously.

As discussed in "Mahler's Guide to Simulation," in Section 20.2.3 of Loss Models this model is used to simulate members of the (a, b, 0) class.
There we simulate the number of claims from time 0 to time 1. Thus $\beta = e^d - 1$, and $q = 1 - e^d$.
Thus for the Negative Binomial: $d = \ln[1 + \beta]$, and $c = r \ln[1 + \beta]$.
For the Binomial: $d = \ln[1 - q]$, and $c = -m \ln[1 - q]$.

[81] Not on the syllabus of your exam. See pages 52-53 of Mathematical Methods of Risk Theory by Buhlmann.
[82] The claim intensity is analogous to the force of mortality in Life Contingencies.

As used in the Heckman-Meyers algorithm to calculate aggregate losses, the frequency distributions are parameterized in a related but somewhat different manner via their mean λ and a "contagion parameter" c:[83]

	λ	c
Binomial	mq	-1/m
Poisson	λ	0
Negative Binomial	$r\beta$	1/r

HyperGeometric Distribution:

For the HyperGeometric Distribution with parameters m, n, and N, the density is:[84]

$$f(x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{m}}, \; x = 0, 1, ..., n.$$

$$\frac{f(x+1)}{f(x)} = \frac{\binom{m}{x+1}\binom{N-m}{n-x-1}}{\binom{m}{x}\binom{N-m}{n-x}} = \frac{(m-x)!\; x!}{(m-x-1)!\; (x+1)!} \; \frac{(N-m-n+x)!\; (n-x)!}{(N-m-n+x+1)!\; (n-x-1)!}.$$

$$= \frac{m-x}{x+1} \; \frac{n-x}{N-m-n+x+1}.$$

Thus the HyperGeometric Distribution is <u>not</u> a member of the (a, b, 0) family.

Mean = $\frac{nm}{N}$. Variance = $\frac{nm\,(N-m)\,(N-n)}{(N-1)\,N^2}$.

[83] <u>Not</u> on the syllabus of your exam. See PCAS 1983 p. 35-36, "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," by Phil Heckman and Glenn Meyers.

[84] <u>Not</u> on the syllabus of your exam. See for example, <u>A First Course in Probability</u>, by Sheldon Ross.
If we had an urn with N balls,of which m were white, and we took a sample of size n, then f(x) is the probability that x of the balls in the sample were white.
For example, tests with 35 questions will be selected at random from a bank of 500 questions.
Treat the 35 questions on the first randomly selected test as white balls.
Then the number of white balls in a sample of size n from the 500 balls is HyperGeometric with m = 35 and N = 500.
Thus the number of questions a second test of 35 questions has in common with the first test is HyperGeometric with m = 35, n = 35, and N = 500. The densities from 0 to 10 are: 0.0717862, 0.204033, 0.272988, 0.228856, 0.134993, 0.0596454, 0.0205202, 0.00564155, 0.00126226, 0.000232901, 0.000035782.

Problems:

11.1 (1 point) Which of the following statements are true?

1. The variance of the Negative Binomial Distribution is less than the mean.

2. The variance of the Poisson Distribution only exists for $\lambda > 2$.

3. The variance of the Binomial Distribution is greater than the mean.

A. 1 B. 2 C. 3 D. 1, 2, and 3 E. None of A, B, C, or D

11.2 (1 point) A member of the (a, b, 0) class of frequency distributions has a = -2.
Which of the following types of Distributions is it?
A. Binomial B. Poisson C. Negative Binomial D. Logarithmic E. None A, B, C, or D.

11.3 (1 point) A member of the (a, b, 0) class of frequency distributions has a = 0.4 and b = 2.
Given f(4) = 0.1505, what is f(7)?
A. Less than 0.06
B. At least 0.06, but less than 0.07
C. At least 0.07, but less than 0.08
D. At least 0.08, but less than 0.09
E. At least 0.09

11.4 (2 points) X is a discrete random variable with a probability function which is a member of the (a,b,0) class of distributions. P(X = 1) = 0.0064. P(X = 2) = 0.0512. P(X = 3) = 0.2048.
Calculate P(X = 4).
(A) 0.37 (B) 0.38 (C) 0.39 (D) 0.40 (E) 0.41

11.5 (2 points) For a discrete probability distribution, you are given the recursion relation:

$$f(x+1) = \left\{\frac{1}{3} + \frac{0.6}{x+1}\right\} f(x),\ x = 0, 1, 2, \ldots$$

Determine f(3).
(A) 0.09 (B) 0.10 (C) 0.11 (D) 0.12 (E) 0.13

11.6 (2 points) A member of the (a, b, 0) class of frequency distributions has a = 0.4, and b = 2.8. What is the mode?
A. 0 or 1 B. 2 C. 3 D. 4 E. 5 or more

11.7 (2 points) For a discrete probability distribution, you are given the recursion relation:

$$p(x) = (-2/3 + 4/x)p(x-1),\ x = 1, 2, \ldots$$

Determine p(3).
(A) 0.19 (B) 0.20 (C) 0.21 (D) 0.22 (E) 0.23

11.8 (3 points) X is a discrete random variable with a probability function which is a member of the (a, b, 0) class of distributions.
P(X = 100) = 0.0350252. P(X = 101) = 0.0329445. P(X = 102) = 0.0306836.
Calculate P(X = 105).
(A) .022 (B) .023 (C) .024 (D) .025 (E) .026

11.9 (2 points) X is a discrete random variable with a probability function which is a member of the (a, b, 0) class of distributions.
P(X = 10) = 0.1074. P(X = 11) = 0.
Calculate P(X = 6).
(A) 6% (B) 7% (C) 8% (D) 9% (E) 10%

11.10 (3 points) Show that the (a, b, 0) relationship with a = -2 and b = 6 leads to a legitimate distribution while a = -2 and b = 5 does not.

11.11 (2 points) A discrete probability distribution has the following properties:
(i) $p_k = c(-1 + 4/k)p_{k-1}$ for k = 1, 2,…
(ii) $p_0 = 0.7$.
Calculate c.
(A) 0.06 (B) 0.13 (C) 0.29 (D) 0.35 (E) 0.40

***11.12** (3 points) Show that the (a, b, 0) relationship with a = 1 and b = -1/2 does not lead to a legitimate distribution.*

11.13 (3 points) A member of the (a, b, 0) class of frequency distributions has been fit via maximum likelihood to the number of claims observed on 10,000 policies.

Number of claims	Number of Policies	Fitted Model
0	6587	6590.79
1	2598	2586.27
2	647	656.41
3	136	136.28
4	25	25.14
5	7	4.29
6 or more	0	0.80

Determine what type of distribution has been fit and the value of the fitted parameters.

11.14 (4, 5/86, Q.50) (1 point) Which of the following statements are true?
1. For a Poisson distribution the mean and variance are equal.
2. For a binomial distribution the mean is less than the variance.
3. The negative binomial distribution is a useful model of the distribution of claim frequencies of a heterogeneous group of risks.

A. 1 B. 1, 2 C. 1, 3 D. 2, 3 E. 1, 2, 3

11.15 (4B, 11/92, Q.21) (1 point) A portfolio of 10,000 risks yields the following:

Number of Claims	Number of Insureds
0	6,070
1	3,022
2	764
3	126
4	18

Based on the portfolio's sample moments, which of the following distributions provides the best fit to the portfolio's number of claims?

A. Binomial B. Poisson C. Negative Binomial D. Lognormal E. Pareto

11.16 (5A, 11/94, Q.24) (1 point) Let X and Y be random variables representing the number of claims for two separate portfolios of insurance risks. You are asked to model the distributions of the number of claims using either the Poisson or Negative Binomial distributions. Given the following information about the moments of X and Y, which distribution would be the best choice for each?

$E[X] = 2.40$ $E[Y] = 3.50$

$E[X^2] = 8.16$ $E[Y^2] = 20.25$

A. X is Poisson and Y is Negative Binomial
B. X is Poisson and Y is Poisson
C. X is Negative Binomial and Y is Negative Binomial
D. X is Negative Binomial and Y is Poisson
E. Neither distribution is appropriate for modeling numbers of claims.

11.17 (5A, 11/99, Q.39) (2 points) You are given the following information concerning the distribution, S, of the aggregate claims of a particular line of business:

E[S] = \$500,000 and Var[S] = 7.5×10^9.

The claim severity follows a Normal Distribution with both mean and standard deviation equal to \$5,000.

What conclusion can be drawn regarding the individual claim propensity of the insureds in this line of business?

11.18 (3, 5/01, Q.25 & 2009 Sample Q.108) (2.5 points) For a discrete probability distribution, you are given the recursion relation

$p(k) = (2/k)\, p(k-1)$, $k = 1, 2, \ldots$.

Determine p(4).

(A) 0.07 (B) 0.08 (C) 0.09 (D) 0.10 (E) 0.11

11.19 (3, 11/02, Q.28 & 2009 Sample Q.94) (2.5 points) X is a discrete random variable with a probability function which is a member of the (a,b,0) class of distributions.

You are given:

(i) $P(X = 0) = P(X = 1) = 0.25$

(ii) $P(X = 2) = 0.1875$

Calculate $P(X = 3)$.

(A) 0.120 (B) 0.125 (C) 0.130 (D) 0.135 (E) 0.140

11.20 (CAS3, 5/04, Q.32) (2.5 points) Which of the following statements are true about the sums of discrete, independent random variables?

1. The sum of two Poisson random variables is always a Poisson random variable.
2. The sum of two negative binomial random variables with parameters (r, β) and (r', β') is a negative binomial random variable if $r = r'$.
3. The sum of two binomial random variables with parameters (m, q) and (m', q') is a binomial random variable if $q = q'$.

A. None of 1, 2, or 3 is true. B. 1 and 2 only C. 1 and 3 only D. 2 and 3 only E. 1, 2, and 3

11.21 (CAS3, 5/05, Q.16) (2.5 points)

Which of the following are true regarding sums of random variables?

1. The sum of two independent negative binomial distributions with parameters (r_1, β_1) and (r_2, β_2) is negative binomial if and only if $r_1 = r_2$.
2. The sum of two independent binomial distributions with parameters (q_1, m_1) and (q_2, m_2) is binomial if and only if $m_1 = m_2$.
3. The sum of two independent Poison distributions with parameters λ_1 and λ_2 is Poisson if and only if $\lambda_1 = \lambda_2$.

A. None are true B. 1 only C. 2 only D. 3 only E. 1 and 3 only

11.22 (SOA M, 5/05, Q.19 & 2009 Sample Q.166) (2.5 points)
A discrete probability distribution has the following properties:
(i) $p_k = c\,(1 + 1/k)\,p_{k-1}$ for k = 1, 2,…
(ii) $p_0 = 0.5$.
Calculate c.
(A) 0.06 (B) 0.13 (C) 0.29 (D) 0.35 (E) 0.40

11.23 (CAS3, 5/06, Q.31) (2.5 points)
N is a discrete random variable from the (a, b, 0) class of distributions.
The following information is known about the distribution:

- Pr(N = 0) = 0.327680
- Pr(N = 1) = 0.327680
- Pr(N = 2) = 0.196608
- E(N) = 1.25

Based on this information, which of the following are true statements?
I. Pr(N = 3) = 0.107965
II. N is from a Binomial distribution.
III. N is from a Negative Binomial distribution.
A. I only B. II only C. III only D. I and II E. I and III

Solutions to Problems:

11.1. E. 1. The variance of the Negative Binomial Distribution is greater than the mean. Thus Statement #1 is false. 2. The variance of the Poisson always exists (and is equal to the mean.) Thus Statement #2 is false.
3. The variance of the Binomial Distribution is less than the mean. Thus Statement #3 is false.

11.2. A. For $a < 0$, one has a Binomial Distribution.
Comment: Since $a = -q/(1-q)$, $q = a/(a-1) = -2/(-3) = 2/3$.
$a = 0$ is a Poisson, $1 > a > 0$ is a Negative Binomial. The Logarithmic Distribution is not a member of the (a,b,0) class. The Logarithmic Distribution is a member of the (a,b,1) class.

11.3. B. $f(x+1) = f(x)\ \{a + b/(x+1)\} = f(x)\{.4 + 2/(x+1)\} = f(x)(.4)(x+6)/(x+1)$.
Then proceed iteratively. For example $f(5) = f(4)(.4)(10)/5 = (.1505)(.8) = .1204$.

n	0	1	2	3	4	5	6	7
f(n)	0.0467	0.1120	0.1568	0.1672	0.1505	0.1204	0.0883	**0.0605**

Comment: Since $0 < a < 1$ we have a Negative Binomial Distribution. $r = 1 + b/a = 1 + (2/.4) = 6$.
$\beta = a/(1-a) = .4/.6 = 2/3$. Thus once a and b are given in fact f(4) is determined. Normally one would compute $f(0) = (1+\beta)^{-r} = .6^6 = .0467$, and proceed iteratively from there.

11.4. E. For a member of the (a,b,0) class of distributions, $f(x+1) / f(x) = a + \{b / (x+1)\}$.
$f(2)/f(1) = a + b/2. \Rightarrow 0.0512/0.0064 = 8 = a + b/2$.
$f(3)/f(2) = a + b/3. \Rightarrow 0.2048/0.0512 = 4 = a + b/3$.
Therefore, $a = -4$ and $b = 24$. $f(4) = f(3)(a + b/4) = (.2048)(-4 + 24/4) =$ **0.4096**.
Alternately, once one solves for a and b, $a < 0 \Rightarrow$ a Binomial Distribution.
$-4 = a = -q/(1-q) \Rightarrow q = 0.8$. $24 = b = (m+1)q/(1-q). \Rightarrow m + 1 = 6. \Rightarrow m = 5$.
$f(4) = (5)(0.8^4)(0.2) =$ **0.4096**.
Comment: Similar to 3, 11/02, Q.28.

11.5. B. This is a member of the (a, b, 0) class of frequency distributions with $a = 1/3$ and $b = 0.6$. Since $a > 0$, this is a Negative Binomial, with $a = \beta/(1+\beta) = 1/3$, and
$b = (r - 1)\beta/(1+\beta) = 0.6$. Therefore, $r - 1 = 0.6/(1/3) = 1.8. \Rightarrow r = 2.8$. $\beta = 0.5$.
$f(3) = \{(2.8)(3.8)(4.8)/3!\}\ 0.5^3/(1.5^{2.8+3}) =$ **0.1013**.
Comment: Similar to 3, 5/01, Q.25. $f(x+1) = f(x)\ \{a + b/(x+1)\}$, $x = 0, 1, 2, \ldots$

11.6. D. For a member of the (a, b, 0) class, the mode is the largest integer in b/(1-a) = 2.8/(1-.4) = 4.667. Therefore, the mode is **4**.
Alternately, f(x+1)/f(x) = a + b/(x+1) = .4 + 2.8/(x+1).

x	0	1	2	3	4	5	6
f(x+1)/f(x)	3.200	1.800	1.333	1.100	0.960	0.867	0.800

Therefore, f(4) = 1.1 f(3) > f(3), but f(5) = .96f(4) < f(4). Therefore, the mode is **4**.

Alternately, since a > 0, this a Negative Binomial Distribution with a = β/(1+β) and

b = (r-1)β/(1+β). Therefore, β = a/(1-a) = 0.4/0.6 = 2/3 and r = b/a + 1 = 2.8/0.4 + 1 = 8.

The mode of a Negative Binomial is the largest integer in: (r-1)β = (7)(2/3) = 4.6667.

Therefore, the mode is **4**.

11.7. E. This is a member of the (a, b, 0) class of frequency distributions with a = -2/3 and b = 4. Since a < 0, this is a Binomial, with a = -q/(1-q) = -2/3, and b = (m+1)q/(1-q) = 4.

Therefore, m + 1 = 4/(2/3) = 6; m = 5. q = .4. f(3) = {(5!)/((3!)(2!))}$0.4^3 0.6^2$ = **0.2304**.

Comment: Similar to 3, 5/01, Q.25. f(x) = f(x-1) {a + b/x}, x = 1, 2, 3, ...

11.8. B. For a member of the (a,b,0) class of distributions, f(x+1) / f(x) = a + {b / (x+1)}.

f(101)/f(100) = a + b/101. ⇒ 0.0329445/0.0350252 = .940594 = a + b/101.

f(102)/f(101) = a + b/102. ⇒ 0.0306836 /0.0329445 = .931372 = a + b/102.

Therefore, a = 0 and b = 95.0. f(105) = f(102)(a + b/103)(a + b/104)(a + b/105) = (0.0306836)(95/103)(95/104)(95/105) = **0.0233893**.

Comment: Alternately, once one solves for a and b, a = 0 ⇒ a Poisson Distribution.

λ = b = 95. f(105) = $e^{-95}(95^{105})$(105!) = .0233893, difficult to calculate using most calculators.

11.9. D. P(X = 11) = 0. ⇒ finite support.

The only member of the (a, b, 0) class with finite support is the Binomial Distribution.

P(X = 11) = 0 and P(X = 10) > 0 ⇒ m = 10. 0.1074 = P(X = 10) = q^{10}. ⇒ q = .800.

P(X = 6) = $\frac{10!}{6!\ 4!}(1-q)^4 q^6 = (210)(0.2^4)(0.8^6)$ = **0.088**.

11.10. f(1) = f(0) (a + b). f(2) = f(1) (a + b/2). f(3) = f(2) (a + b/3). f(4) = f(3) (a + b/4), etc.
For a = -2 and b = 6:
f(1) = f(0) (-2 + 6) = 4 f(0). f(2) = f(1) (-2 + 6/2) = f(1). f(3) = f(2) (-2 + 6/3) = 0. f(4) = 0, etc.
This is a Binomial with m = 2 and q = a/(a-1) = 2/3.
f(0) = 1/9. f(1) = 4/9. f(2) = 4/9.
For a = -2 and b = 5:
f(1) = f(0) (-2 + 5) = 3 f(0). f(2) = f(1) (-2 + 5/2) = 1.5f(1). f(3) = f(2) (-2 + 5/3) < 0. No good!
Comment: Similar to Exercise 6.3 in Loss Models.
For a < 0, we require that b/a be a negative integer.

11.11. B. This is the (a, b, 0) relationship, with a = -c and b = 4c.
For the Binomial, a < 0. For the Poisson a = 0. For the Negative Binomial, a > 0.
c must be positive, since the densities are positive, therefore, a < 0 and this is a
Binomial. For the Binomial, a = -q/(1-q) and b = (m+1)q/(1-q).

b = -4a. ⇒ m + 1 = 4. ⇒ m = 3.

$0.7 = p_0 = (1 - q)^m = (1 - q)^3$. ⇒ q = .1121.

c = -a = q/(1-q) = .1121/.8879 = **0.126.**
Comment: Similar to SOA M, 5/05, Q.19.

11.12. f(1) = f(0) (1 - 1/2) = (1/2) f(0). f(2) = f(1) (1 - 1/4) = (3/4)f(1).
f(3) = f(2) (1 - 1/6) = (5/6)f(2). f(4) = f(3) (1 - 1/8) = (7/8)f(3). f(5) = f(4) (1 - 1/10) = (9/10)f(4).
The sum of these densities is:
f(0){1 + 1/2 + (3/4)(1/2) + (5/6)(3/4)(1/2) + (7/8)(5/6)(3/4)(1/2) + (9/10)(7/8)(5/6)(3/4)(1/2) + ...}
f(0){1 + 1/2 + 3/8 + 5/16 + 35/128 + 315/1280 + ...} > f(0){1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + ...}.
However, the sum 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + ..., diverges.
Therefore, these densities would sum to infinity.

Comment: We require that a < 1. a is positive for a Negative Binomial; a = β/(1 + β) < 1.

11.13. For a member of the (a, b, 0) class f(1)/f(0) = a + b, and f(2)/f(1) = a + b/2.
a + b = 2586.27/6590.79 = 0.39241.
a + b/2 = 656.41/2586.27 = 0.25381.

$\Rightarrow$ b = 0.27720. $\Rightarrow$ a = 0.11521.

Looking in Appendix B in the tables attached to the exam, a is positive for the Negative Binomial. Therefore, we have a Negative Binomial.

0.11521 = a = $\beta/(1+\beta)$.

$\Rightarrow$ $1/\beta$ = 1/0.11521 - 1 = 7.6798. $\Rightarrow$ **β = 0.1302**.

0.27720 = b = (r-1) $\beta/(1+\beta)$.

$\Rightarrow$ r - 1 = 0.27720/0.11521 = 2.4060. $\Rightarrow$ **r = 3.406**.

Comment: Similar to Exercise 16.21b in Loss Models.

11.14. C. 1. True.
2. False. The variance = nq(1-q) is less than the mean = nq, since q < 1.
3. True. Statement 3 is referring to the mixture of Poissons via a Gamma, which results in a Negative Binomial frequency distribution for the entire portfolio.

11.15. B. The mean frequency is .5 and the variance is: 0.75 - 0.5^2 = 0.5.

	Number of Insureds	Number of Claims	Square of Number of Claims
	6070	0	0
	3022	1	1
	764	2	4
	126	3	9
	18	4	16
Average		0.5000	0.7500

Since estimated mean = estimated variance, we expect the Poisson to provide the best fit.
Comment: If the estimated mean is approximately equal to the estimated variance, then the Poisson is likely to provide a good fit. The Pareto and the LogNormal are continuous distributions not used to fit discrete frequency distributions.

11.16. A. Var[X] = E[X^2] - E[X]^2 = 8.16 - 2.40^2 = 2.4 = E[X], so a Poisson Distribution is a good choice for X. Var[Y] = E[Y^2] - E[Y]^2 = 20.25 - 3.50^2 = 8 > 3.5 = E[Y], so a Negative Binomial Distribution is a good choice for Y.

11.17. Mean frequency = \$500,000/\$5000 = 100. Assuming frequency and severity are independent: Var[S] = $7.5 \times 10^9 = (100)(5000^2) + (5000^2)$ (Variance of the frequency).
Variance of the frequency = 200. Thus if each insured has the same frequency distribution, then it has variance > mean, so it might be a Negative Binomial. Alternately, each insured could have a Poisson frequency, but with the means varying across the portfolio. In that case, the mean of mixing distribution = 100. When mixing Poisons, Variance of the mixed distribution
= Mean of mixing Distribution + Variance of the mixing distribution,
so the variance of the mixing distribution = 200 - 100 = 100.
Comment: There are many possible other answers.

11.18. C. f(x+1)/f(x) = 2/(x+1), x = 0, 1, 2,...
This is a member of the (a, b , 0) class of frequency distributions:
with f(x+1)/f(x) = a + b/(x+1), for a = 0 and b = 2.
Since a = 0, this is a Poisson with λ = b = 2. f(4) = $e^{-2}\, 2^4/4!$ = **0.090**.
Alternately, let f(0) = c. Then f(1) = 2c, f(2) = $2^2c/2!$, f(3) = $2^3c/3!$, f(4) = $2^4c/4!$, ...
$1 = \Sigma$ f(i) = $\Sigma 2^i c/i! = c\Sigma 2^i/i! = c\, e^2$. Therefore, c = e^{-2}. f(4) = $e^{-2}\, 2^4/4!$ = **0.090**.

11.19. B. For a member of the (a, b, 0) class of distributions, f(x+1) / f(x) = a + {b / (x+1)}.
f(1)/f(0) = a + b. $\Rightarrow$ 0.25/0.25 = 1 = a + b.
f(2)/f(1) = a + b/2. $\Rightarrow$ 0.1875/0.25 = 0.75 = a + b/2.
Therefore, a = 0.5 and b = 0.5.
f(3) = f(2)(a + b/3) = (0.1875)(0.5 + .5/3) = **0.125**.
Alternately, once one solves for a and b, a > 0 $\Rightarrow$ a Negative Binomial Distribution.
1/2 = a = $\beta/(1 + \beta)$. $\Rightarrow \beta = 1$. 1/2 = b = $(r-1)\beta/(1 + \beta)$. $\Rightarrow$ r - 1 = 1. $\Rightarrow$ r = 2.
f(3) = $r(r + 1)(r + 2)\, \beta^3/\{(1 + \beta)^{r+3}\, 3!\}$ = $(2)(3)(4)/\{(2^5)(6)\}$ = **0.125**.

11.20. C. 1. True. 2. False. Would be true if $\beta = \beta'$, in which case the sum would have the sum of the r parameters. 3. True. The sum would have the sum of the m parameters.
Comment: Note the requirement that the variables be independent.

11.21. A. The sum of two independent negative binomial distributions with parameters (r_1, β_1) and (r_2, β_2) is negative binomial if and only if $\beta_1 = \beta_2$. Statement 1 is false.

The sum of two independent binomial distributions with parameters (q_1, m_1) and (q_2, m_2) is binomial if and only if $q_1 = q_2$. Statement 2 is false.

The sum of two independent Poison distributions with parameters λ_1 and λ_2 is Poisson, <u>regardless</u> of the values of lambda. Statement 3 is false.

11.22. C. This is the (a, b, 0) relationship, with a = c and b = c.
For the Binomial, $a < 0$. For the Poisson $a = 0$. For the Negative Binomial, $a > 0$.
c must be positive, since the densities are positive, therefore, $a > 0$ and this is a Negative Binomial. For the Negative Binomial, $a = \beta/(1+\beta)$ and $b = (r-1)\beta/(1+\beta)$.

$a = b. \Rightarrow r - 1 = 1. \Rightarrow r = 2.$

$0.5 = p_0 = 1/(1+\beta)^r = 1/(1+\beta)^2. \Rightarrow (1+\beta)^2 = 2. \Rightarrow \beta = \sqrt{2} - 1 = 0.4142.$

$c = a = \beta/(1+\beta) = 0.4142/1.4142 =$ **0.293.**

11.23. C. For a member of the (a, b, 0) class, $f(1)/f(0) = a + b$, and $f(2)/f(1) = a + b/2$.

Therefore, $a + b = 1$, and $a + b/2 = 0.196608/0.327680 = 0.6. \Rightarrow a = 0.2$ and $b = 0.8$.

Since a is positive, we have a Negative Binomial Distribution. Statement III is true.
$f(3) = f(2)(a + b/3) = (0.196608)(0.2 + 0.8/3) = 0.0917504$. Statement I is false.

<u>Comment</u>: $0.2 = a = \beta/(1+\beta)$ and $0.8 = b = (r-1)\beta/(1+\beta). \Rightarrow r = 5$ and $\beta = 0.25$.

$E[N] = r\beta = (5)(0.25) = 1.25$, as given.

$f(3) = \{r(r+1)(r+2)/3!\}\beta^3/(1+\beta)^{3+r} = \{(5)(6)(7)/6\}.25^3/1.25^8 = 0.0917504.$

Section 12, Accident Profiles[85]

Constructing an "Accident Profile" is a technique in Loss Models that can be used to decide whether data was generated by a member of the (a, b, 0) class of frequency distributions and if so which member.

As discussed previously, the (a,b,0) class of frequency distributions consists of the three common distributions: Binomial, Poisson, and Negative Binomial. Therefore, it also includes the Bernoulli, which is a special case of the Binomial, and the Geometric, which is a special case of the Negative Binomial. As discussed previously, for members of the (a, b, 0) class:
$f(x+1) / f(x) = a + b / (x+1)$,
where a and b depend on the parameters of the distribution:[86]

Distribution	a	b	f(0)
Binomial	$-q/(1-q)$	$(m+1)q/(1-q)$	$(1-q)^m$
Poisson	0	λ	$e^{-\lambda}$
Negative Binomial	$\beta/(1+\beta)$	$(r-1)\beta/(1+\beta)$	$1/(1+\beta)^r$

Note that $a < 0$ is a Binomial, $a = 0$ is a Poisson, and $1 > a > 0$ is a Negative Binomial.

For the Binomial: $q = a/(a-1) = |a| / (|a| +1)$.

For the Negative Binomial: $\beta = a/(1-a)$.

For the Binomial: $m = -(a+b)/a = (a + b)/ |a|$.[87] The Bernoulli has $m =1$ and $b = -2a$.

For the Poisson: $\lambda = b$.

For the Negative Binomial: $r = 1 + b/a$. The Geometric has $r =1$ and $b = 0$.

Thus given values of a and b, one can determine which member of the (a,b,0) class one has and its parameters.

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[85] See Example 6.2 in Loss Models.
[86] See Appendix B of Loss Models.
[87] Since for the Binomial m is an integer, we require that b/ |a| to be an integer.

Accident Profile:

Also note that for a member of the (a, b, 0) class, (x+1)f(x+1)/f(x) = (x+1)a + b, so that (x+1)f(x+1)/f(x) is linear in x. It is a straight line with slope a and intercept a + b.

Thus graphing (x+1)f(x+1)/f(x) can be a useful method of determining whether one of these three distributions fits the given data.[88] If a straight line does seem to fit this “accident profile”, then one should use a member of the (a, b, 0) class.

The slope determines which of the three distributions is likely to fit: if the slope is close to zero then a Poisson, if significantly negative then a Binomial, and if significantly positive then a Negative Binomial.

For example, here is the accident profile for some data:

Number of Claims	Observed	Observed Density Function	(x+1)f(x+1)/f(x)
0	17,649	0.73932	0.27361
1	4,829	0.20229	0.45807
2	1,106	0.04633	0.62116
3	229	0.00959	0.76856
4	44	0.00184	1.02273
5	9	0.00038	2.66667
6	4	0.00017	1.75000
7	1	0.00004	8.00000
8	1	0.00004	
9 & +	0		

Prior to the tail where the data thins out,(x+1)f(x+1)/f(x) approximately follows a straight line with a positive slope of about 0.2, which indicates a Negative Binomial with $\beta/(1+\beta) \cong 0.2$.[89] [90]

The intercept is $r\beta/(1+\beta)$, so that $r \cong 0.27 / 0.2 \cong 1.4$.[91]

In general, an accident profile is used to see whether data is likely to have come from a member of the (a, b, 0) class. One would do this test prior to attempting to fit a Negative Binomial, Poisson, or Binomial Distribution to the data. One starts with the hypothesis that the data was drawn from a member of the (a, b, 0) class, without specifying which one. If this hypothesis is true the accident profile should be approximately linear.[92]

[88] This computation is performed using the empirical densities.
[89] One should not significantly rely on those ratios involving few observations.
[90] Slope is: $a = \beta/(1+\beta)$.
[91] Intercept is: $a + b = \beta/(1+\beta)+(r-1)\beta/(1+\beta)= r\beta/(1+\beta)$.
[92] Approximate, because any finite sample of data is subject to random fluctuations.

If the accident profile is "approximately" linear, then we do not reject the hypothesis and decide which member of the (a, b, 0) to fit based on the slope of this line.[93]

Comparing the Mean to the Variance:

Another way to decide which of the members of the (a,b,0) class is most likely to fit a given set of data is to compare the sample mean and sample variance.

Binomial	**Mean > Variance**
Poisson	**Mean = Variance**[94]
Negative Binomial	**Mean < Variance**

[93] There is not a numerical statistical test to perform, such as with the Chi-Square Test.

[94] For data from a Poisson Distribution, the sample mean and sample variance will be approximately equal rather than equal, because any finite sample of data is subject to random fluctuations.

Problems:

12.1 (2 points) You are given the following accident data:

Number of accidents	Number of policies
0	91,304
1	7,586
2	955
3	133
4	18
5	3
6	1
7+	0
Total	100,000

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial (B) Poisson (C) Negative Binomial, r ≤ 1

(D) Negative Binomial, r > 1 (E) None of A, B, C, or D

12.2 (3 points) You are given the following accident data:

Number of accidents	Number of policies
0	860
1	2057
2	2506
3	2231
4	1279
5	643
6	276
7	101
8	41
9	4
10	2
11&+	0
Total	10,000

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial

(B) Poisson

(C) Negative Binomial, r ≤ 1

(D) Negative Binomial, r > 1

(E) None of the above

12.3 (3 points) You are given the following data on the number of runs scored during half innings of major league baseball games from 1980 to 1998:

Runs	Number of Occurrences
0	518,288
1	105,070
2	47,936
3	21,673
4	9736
5	4033
6	1689
7	639
8	274
9	107
10	36
11	25
12	5
13	7
14	1
15	0
16	1
Total	709,460

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial
(B) Poisson
(C) Negative Binomial, $r \le 1$
(D) Negative Binomial, $r > 1$
(E) None of the above

12.4 (3 points) You are given the following accident data:

Number of accidents	Number of policies
0	820
1	1375
2	2231
3	1919
4	1397
5	1002
6	681
7	330
8	172
9	56
10	14
11	3
12&+	0
Total	10,000

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial
(B) Poisson
(C) Negative Binomial, $r \le 1$
(D) Negative Binomial, $r > 1$
(E) None of the above

12.5 (2 points) You are given the following distribution of the number of claims per policy during a one-year period for 20,000 policies.

Number of claims per policy	Number of Policies
0	6503
1	8199
2	4094
3	1073
4	128
5	3
6+	0

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial
(B) Poisson
(C) Negative Binomial, $r \le 1$
(D) Negative Binomial, $r > 1$
(E) None of the above

12.6 (2 points) You are given the following distribution of the number of claims on motor vehicle policies:

Number of claims in a year	Observed frequency
0	565,664
1	68,714
2	5,177
3	365
4	24
5	6
6	0

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial
(B) Poisson
(C) Negative Binomial, $r \le 1$
(D) Negative Binomial, $r > 1$
(E) None of the above

12.7 (4, 5/00, Q.40) (2.5 points)
You are given the following accident data from 1000 insurance policies:

Number of accidents	Number of policies
0	100
1	267
2	311
3	208
4	87
5	23
6	4
7+	0
Total	1000

Which of the following distributions would be the most appropriate model for this data?

(A) Binomial
(B) Poisson
(C) Negative Binomial
(D) Normal
(E) Gamma

12.8 (4, 11/03, Q.32 & 2009 Sample Q.25) (2.5 points)
The distribution of accidents for 84 randomly selected policies is as follows:

Number of Accidents	Number of Policies
0	32
1	26
2	12
3	7
4	4
5	2
6	1
Total	84

Which of the following models best represents these data?
(A) Negative binomial
(B) Discrete uniform
(C) Poisson
(D) Binomial
(E) Either Poisson or Binomial

Solutions to Problems:

12.1. C. Calculate (x+1)f(x+1)/f(x). Since it is approximately linear, we seem to have a member of the (a, b, 0) class. f(x+1)/f(x) = a + b/(x+1), so (x+1)f(x+1)/f(x) = a(x+1) + b = ax + a + b. The slope is positive, so a > 0 and we have a **Negative Binomial**.

The slope, a ≅ 0.17. The intercept is about 0.08. Thus a + b ≅ 0.08.

Therefore, b ≅ 0.08 - 0.17 = -0.09 < 0.

For the Negative Binomial b = (r-1)β/(1+β). Thus b < 0, implies **r < 1**.

Number of Accident	Observed	Observed Density Function	(x+1)f(x+1)/f(x)	Differences
0	91,304	0.91304	0.083	
1	7,586	0.07586	0.252	0.169
2	955	0.00955	0.418	0.166
3	133	0.00133	0.541	0.124
4	18	0.00018	0.833	0.292
5	3	0.00003	2.000	
6	1	0.00001		
7+	0	0.00000		

Comment: Similar to 4, 5/00, Q.40. Do not put much weight on the values of (x+1)f(x+1)/f(x) in the righthand tail, which can be greatly affected by random fluctuation.
The first moment is 0.09988, and the second moment is 0.13002.

The variance is: 0.13002 - 0.09988^2 = 0.12004, significantly greater than the mean.

Thus, if we have a member of the (a, b, 0) class, it is a Negative Binomial.
Fitting regressions is not on the syllabus of this exam; thus you should not be asked to do so on your exam. If one fit a regression to all of the points:
{0, 0.083}, {1, 0.252}, {2, 0.418}, {3, 0.541}, {4, 0.833}, {5, 2.000},
one gets a slope of 0.32 and an intercept of -0.13.
If one fits a regression to all but the last point, which is based on very little data:
{0, 0.083}, {1, 0.252}, {2, 0.418}, {3, 0.541}, {4, 0.833},
one gets a slope of 0.179 and an intercept of -0.068.
This is similar to what I discuss in my solution.

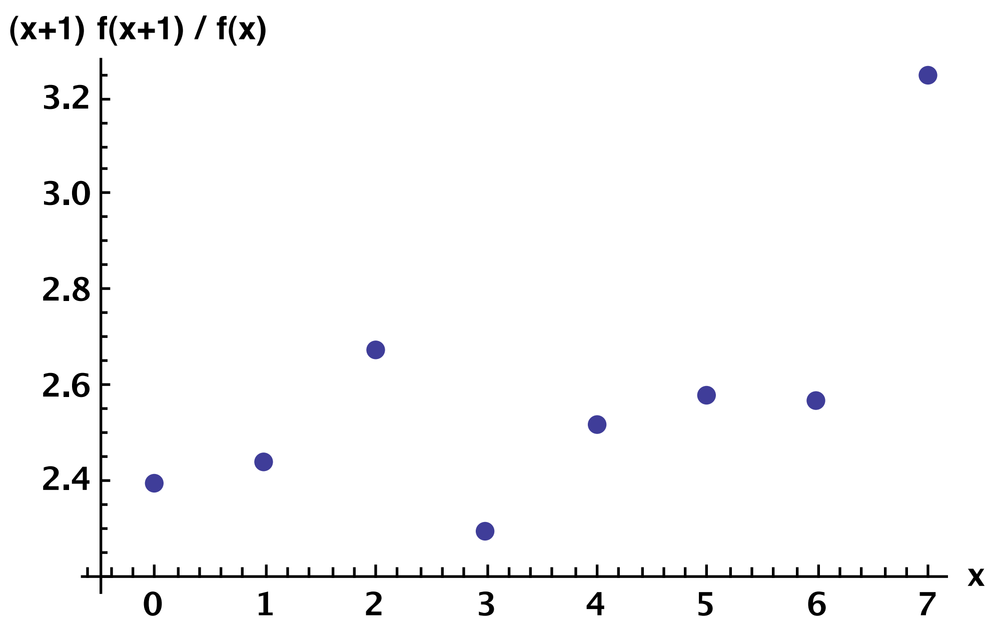
12.2. B. Calculate (x+1)f(x+1)/f(x). Since it is approximately linear, we seem to have a member of the (a, b, 0) class. f(x+1)/f(x) = a + b/(x+1), so (x+1)f(x+1)/f(x) = a(x+1) + b =
ax + a + b. The slope seems close to zero, until the data starts to get thin, so a ≅ 0 and therefore we assume this data probably came from a **Poisson**.

Number of Accident	Observed	Observed Density Function	(x+1)f(x+1)/f(x)
0	860	0.0860	2.392
1	2,057	0.2057	2.437
2	2,506	0.2506	2.671
3	2,231	0.2231	2.293
4	1,279	0.1279	2.514
5	643	0.0643	2.575
6	276	0.0276	2.562
7	101	0.0101	3.248
8	41	0.0041	0.878
9	4	0.0004	5.000
10	2	0.0002	

Comment: Any actual data set is subject to random fluctuation, and therefore the observed slope of the accident profile will never be exactly zero. One can never distinguish between the possibility that the model was a Binomial with q small, a Poisson, or a Negative Binomial with β small.

This data was simulated as 10,000 independent random draws from a Poisson with λ = 2.5.

On the exam they should give you something that is either obviously linear or obviously not linear. On the exam you will not be required to fit a straight line or perform a statistical test to see how good a linear fit is. In contrast this question is a realistic situation. One either graphs or just eyeballs the numbers in the last column not worrying too much about the last few numbers which are based on very little data. Here, the values look approximately linear, in fact they seem approximately flat.



12.3. E. Calculate (x+1)f(x+1)/f(x).
Note that f(x+1)/f(x) = (number with x + 1)/(number with x).
Since (x+1)f(x+1)/f(x) is not linear, we do not have a member of the (a, b, 0) class.

Number of runs	Observed	(x+1)f(x+1)/f(x)	Differences
0	518,228	0.203	
1	105,070	0.912	0.710
2	47,936	1.356	0.444
3	21,673	1.797	0.441
4	9,736	2.071	0.274
5	4,033	2.513	0.442
6	1,689	2.648	0.136
7	639	3.430	0.782
8	274	3.515	0.084
9	107	3.364	-0.150
10	36	7.639	4.274
11	25		

Comment: At high numbers of runs, where the data starts to thin out, one would not put much reliance on the values of (x+1)f(x+1)/f(x). The data is taken from "An Analytic Model for Per-inning Scoring Distributions," by Keith Woolner.

12.4. E. Calculate (x+1)f(x+1)/f(x). Since it is does not appear to be linear, we **do not seem to have a member of the (a, b, 0) class**.

Number of Accident	Observed	Observed Density Function	(x+1)f(x+1)/f(x)
0	820	0.0820	1.677
1	1,375	0.1375	3.245
2	2,231	0.2232	2.580
3	1,919	0.1920	2.912
4	1,397	0.1397	3.586
5	1,002	0.1002	4.078
6	681	0.0681	3.392
7	330	0.0330	4.170
8	172	0.0172	2.930
9	56	0.0056	2.500
10	14	0.0014	2.357
11	3	0.0003	

12.5. A. Calculate (x+1)f(x+1)/f(x) = (x+1)(number with x + 1)/(number with x).

Number of claims	Observed	(x+1)f(x+1)/f(x)	Differences
0	6,503	1.261	
1	8,199	0.999	-0.262
2	4,094	0.786	-0.212
3	1,073	0.477	-0.309
4	128	0.117	-0.360
5	3		

Since (x+1)f(x+1)/f(x) is approximately linear, we probably have a member of the (a, b, 0) class.

a = slope < 0. ⇒ Binomial Distribution.

Comment: The data was simulated from a Binomial Distribution with m = 5 and q = 0.2.
The sample mean is: 20,133/20,000 = 1.00665.
The second moment is: 36,355/20,000 = 1.81775.
The sample variance is: (20,000/19,999) (1.81755 - 1.00665^2) = 0.804.
Since the sample mean is greater than the sample variance by a significant amount, if this is a member of the (a, b, 0) class then it is a Binomial Distribution.

12.6. E. Calculate (x+1)f(x+1)/f(x).
Note that f(x+1)/f(x) = (number with x + 1)/(number with x).

Number of claims	Observed	(x+1)f(x+1)/f(x)	Differences
0	565,664	0.121	
1	68,714	0.151	0.029
2	5,177	0.212	0.061
3	365	0.263	0.052
4	24	1.250	0.987
5	6		

Even ignoring the final value, (x+1)f(x+1)/f(x) is not linear.
Therefore, we do not have a member of the (a, b, 0) class.
Comment: Data taken from Table 6.6.2 in Introductory Statistics with Applications in General Insurance by Hossack, Pollard and Zehnwirth. See also Table 7.1 in Loss Models.

12.7. A. Calculate (x+1)f(x+1)/f(x). Since it seems to be decreasing linearly, we seem to have a member of the (a, b, 0) class, with a < 0, which is a **Binomial Distribution**.

Number of Accident	Observed	Observed Density Function	(x+1)f(x+1)/f(x)
0	100	0.10000	2.67
1	267	0.26700	2.33
2	311	0.31100	2.01
3	208	0.20800	1.67
4	87	0.08700	1.32
5	23	0.02300	1.04
6	4	0.00400	
7+	0	0.00000	

Alternately, the mean is 2, and the second moment is 5.494. Therefore, the sample variance is $(1000/999)(5.494 - 2^2) = 1.495$. Since the variance is significantly less than the mean, this indicates a **Binomial Distribution**.

Comment: One would not use a continuous distribution such as the Normal or the Gamma to model a frequency distribution. (x+1)f(x+1)/f(x) = a(x+1) + b. In this case, a ≅ -0.33.

For the Binomial, a = -q/ (1-q), so q ≅ 0.25. In this case, b ≅ 2.67+ 0.33 = 3.00.

For the Binomial, b = (m+1)q/ (1-q), so m ≅ (3/.33) -1 = 8.

12.8. A. Calculate (x+1)f(x+1)/f(x). For example, (3)(7/84)/(12/84) = (3)(7)/12 = 1.75.

Number of Accident	Observed	(x+1)f(x+1)/f(x)
0	32	0.81
1	26	0.92
2	12	1.75
3	7	2.29
4	4	2.50
5	2	3.00
6	1	

Since this quantity seems to be increasing roughly linearly, we seem to have a member of the (a, b, 0) class, with a = slope > 0, which is a **Negative Binomial Distribution**.
Alternately, the mean is: 103/84 = 1.226, and the second moment is: 287/84 = 3.417.
The sample variance is: $(84/83)(3.417 - 1.226^2) = 1.937$. Since the sample variance is significantly more than the sample mean, this indicates a **Negative Binomial**.

Comment: If (x+1)f(x+1)/f(x) had been approximately linear with a slope that was close to zero, then one could not distinguish between the possibility that the model was a Binomial with q small, a Poisson, or a Negative Binomial with β small. If the correct model were the discrete uniform, then we would expect the observed number of policies to be similar for each number of accidents.

Section 13, Zero-Truncated Distributions[95]

Frequency distributions can be constructed that have support on the positive integers or equivalently have a density at zero of 0.

For example, let $f(x) = (e^{-3}\, 3^x / x!) / (1 - e^{-3})$, for x = 1, 2, 3, ...

x	1	2	3	4	5	6	7
f(x)	15.719%	23.578%	23.578%	17.684%	10.610%	5.305%	2.274%
F(x)	15.719%	39.297%	62.875%	80.558%	91.169%	96.4736%	98.74718%

Exercise: Verify that the sum of $f(x) = (e^{-3}\, 3^x / x!) / (1 - e^{-3})$ from x =1 to ∞ is unity.

[Solution: The sum of the Poisson Distribution from 0 to ∞ is 1. $\sum_{i=0}^{\infty} e^{-3}\, 3^x / x! = 1$.

Therefore, $\sum_{i=1}^{\infty} e^{-3}\, 3^x / x! = 1 - e^{-3}$. $\Rightarrow \sum_{i=1}^{\infty} f(x) = 1$.]

This is an example of a Poisson Distribution Truncated from Below at Zero, with λ = 3.

In general, **if f is a distribution on 0, 1, 2, 3,...,**

then $p_k^T = \frac{f(k)}{1 - f(0)}$ **is a distribution on 1, 2, 3, ...**

This is a special case of truncation from below. The general concept of truncation of a distribution is covered in a "Mahler's Guide to Loss Distributions."

We have the following three examples, shown in Appendix B.3.1 of Loss Models:

Distribution	Density of the Zero-Truncated Distribution	
Binomial	$\dfrac{\frac{m!\, q^x\, (1-q)^{m-x}}{x!\, (m-x)!}}{1 - (1-q)^m}$	x = 1, 2, 3,..., m
Poisson	$\dfrac{e^{-\lambda}\, \lambda^x / x!}{1 - e^{-\lambda}}$	x = 1, 2, 3,...
Negative Binomial	$\dfrac{\frac{r(r+1)...(r+x-1)}{x!}\, \frac{\beta^x}{(1+\beta)^{x+r}}}{1 - 1/(1+\beta)^r}$	x = 1, 2, 3,...

[95] See Section 6.6 in Loss Models.

Moments:

Exercise: For a Zero-Truncated Poisson with $\lambda = 3$, what is the mean?

[Solution: $p_k^T = \frac{f(k)}{1 - f(0)}$, k = 1, 2, 3,. The mean of the zero-truncated distribution is:

$$\sum_{k=1}^{\infty} k\, p_k^T = \sum_{k=1}^{\infty} k \frac{f(k)}{1 - f(0)} = \frac{\sum_{k=0}^{\infty} k\, f(k)}{1 - f(0)} = \frac{\text{mean of f}}{1 - f(0)} = \frac{\lambda}{1 - e^{-\lambda}} = \frac{3}{1 - e^{-3}} = 3.157.]$$

In general, the moments of a zero-truncated distribution are given in terms of those of the corresponding untruncated distribution, f, by: $\mathbf{E^{Truncated}[X^n] = \frac{E_f[X^n]}{1 - f(0)}}$.

For example for the Zero-Truncated Poisson the mean is: $\lambda / (1 - e^{-\lambda})$,

while the second moment is: $(\lambda + \lambda^2) / (1 - e^{-\lambda})$.

Exercise: For a Zero-Truncated Poisson with $\lambda = 3$ what is the second moment?

[Solution: The second moment of untruncated Poisson is its variance plus the square of its mean: $\lambda + \lambda^2$. The second moment of the zero-truncated Poisson is:

(the second moment of f) / {1 - f(0)} = $(\lambda + \lambda^2) / (1 - e^{-\lambda}) = (3 + 3^2) / (1 - e^{-3})$ =12.629.]

Thus a Zero-Truncated Poisson with $\lambda = 3$ has a variance of $12.629 - 3.157^2 = 2.66$.

This matches the result of using the formula in Appendix B of Loss Models:

$\lambda\{1 - (\lambda+1)e^{-\lambda}\} / (1 - e^{-\lambda})^2 = (3)\{1 - 4e^{-3}\} / (1 - e^{-3})^2 = (3)(0.8009)/(0.9502)^2 = 2.66$.

It turns out that for the Zero-Truncated Negative Binomial, the parameter r can take on values between -1 and 0, as well as the usual positive values, $r > 0$.

This is sometimes referred to as the Extended Zero-Truncated Negative Binomial.

However, provided $r \neq 0$, all of the same formulas apply.

As r approaches zero, the Zero-Truncated Negative Binomial approaches the Logarithmic Distribution, to be discussed next.

<u>Logarithmic Distribution</u>:[96]

The Logarithmic Distribution with parameter β has support equal to the positive integers:

$$f(x) = \frac{\left(\frac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)}, \text{ for } x = 1, 2, 3,...$$

with mean: $\frac{\beta}{\ln(1+\beta)}$, and variance: $\beta \frac{1 + \beta - \frac{\beta}{\ln(1+\beta)}}{\ln(1+\beta)}$.

$a = \beta / (1 + \beta)$. $b = -\beta / (1 + \beta)$ $P(z) = 1 - \frac{\ln[1 - \beta(z-1)]}{\ln[1+\beta]}$, $z < 1 + 1/\beta$.

Exercise: Assume the number of vehicles involved in each automobile accident is given by $f(x) = 0.2^x / \{x \ln(1.25)\}$, for x = 1, 2, 3,...
Then what is the mean number of vehicles involved per automobile accident?
[Solution: This is a Logarithmic Distribution with β = 0.25. Mean $\beta/\ln(1+\beta) = 0.25/\ln(1.25) = 1.12$.
<u>Comment</u>: $\beta / (1 + \beta) = 0.25/1.25 = 0.2$.]

The density function of this Logarithmic Distribution with β = 0.25 is as follows:

x	1	2	3	4	5	6	7
f(x)	89.6284%	8.9628%	1.1950%	0.1793%	0.0287%	0.0048%	0.0008%
F(x)	89.628%	98.591%	99.786%	99.966%	99.994%	99.9990%	99.9998%

Exercise: Show that the densities of a Logarithmic Distribution sum to one.

Hint: $\ln[1/(1 - y)] = \sum_{k=1}^{\infty} \frac{y^k}{k}$, for $|y| < 1$.[97]

[Solution: $\sum_{k=1}^{\infty} f(k) = \frac{1}{\ln(1+\beta)} \sum_{k=1}^{\infty} \left(\frac{\beta}{1+\beta}\right)^k / k$.

Let $y = \beta/(1+\beta)$. Then, $1/(1 - y) = 1 + \beta$.

Thus $\sum_{k=1}^{\infty} f(k) = \frac{1}{\ln(1+\beta)} \sum_{k=1}^{\infty} \frac{y^k}{k} = \frac{\ln[1/(1 - y)]}{\ln(1+\beta)} = \frac{\ln(1+\beta)}{\ln(1+\beta)} = 1$.]

[96] *Sometimes called instead a Log Series Distribution.*
[97] <u>Not</u> something you need to know for your exam. This result can be derived as a Taylor series.

(a,b,1) Class:[98]

The **(a,b,1) class of frequency distributions** in Loss Models is a generalization of the (a,b,0) class. As with the (a,b,0) class, the recursion formula applies: $\frac{\text{density at } x+1}{\text{density at } x} = a + \frac{b}{x+1}$.

However, this relationship need only apply now for $x \geq 1$, rather than $x \geq 0$.

Members of the (a,b,1) family include: all the members of the (a,b,0) family,[99] the zero-truncated versions of those distributions: Zero-Truncated Binomial, Zero-Truncated Poisson, and Extended Truncated Negative Binomial,[100] and the Logarithmic Distribution.
In addition the (a,b,1) class includes the zero-modified distributions corresponding to these, to be discussed in the next section.

Loss Models Notation:

p_k the density function of the untruncated frequency distribution at k.

p_k^T the density function of the zero-truncated frequency distribution at k.

p_k^M the density function of the zero-modified frequency distribution at k.[101]

Exercise: Give a verbal description of the following terms: p_7, p_4^M, and p_6^T.

[Solution: p_7 is the density of the frequency at 7, f(7).

p_4^M is the density of the zero-modified frequency at 4, $f_M(4)$.

p_6^T is the density of the zero-truncated frequency at 6, $f_T(6)$.]

[98] See Table 6.4 and Appendix B.3 in Loss Models.
[99] Binomial, Poisson, and the Negative Binomial.
[100] The Zero-Truncated Negative Binomial where in addition to $r > 0$, $-1 < r < 0$ is also allowed.
[101] Zero-modified distributions will be discussed in the next section.

Probability Generating Functions:

The Probability Generating Function, $P(z) = E[z^N]$, for a zero-truncated distribution can be obtained from that for the untruncated distribution.

$$P^T(z) = \frac{P(z) - f(0)}{1 - f(0)},$$

where P(z) is the p.g.f. for the untruncated distribution and $P^T(z)$ is the p.g.f. for the zero-truncated distribution, and f(0) is the probability at zero for the untruncated distribution.

Exercise: What is the Probability Generating Function for a Zero-Truncated Poisson Distribution?
[Solution: For the untruncated Poisson $P(z) = e^{\lambda(z-1)}$. $f(0) = e^{-\lambda}$.
$P^T(z) = \{P(z) - f(0)\} / \{1 - f(0)\} = \{e^{\lambda(z-1)} - e^{-\lambda}\} / \{1- e^{-\lambda}\} = \{e^{\lambda z} - 1\} / \{e^{\lambda} - 1\}$.]

One can derive this relationship as follows:

$$P^T(z) = \sum_{n=1}^{\infty} z^n p_n^T = \frac{\sum_{n=1}^{\infty} z^n f(n)}{1 - f(0)} = \frac{\sum_{n=0}^{\infty} z^n f(n) - f(0)}{1 - f(0)} = \frac{P(z) - f(0)}{1 - f(0)}.$$

In any case, Appendix B of <u>Loss Models</u> displays the Probability Generating Functions for all of the Zero-Truncated Distributions.

For example, for the zero-truncated Geometric Distribution, in Appendix B it is shown that:

$$P^T(z) = \frac{\{1 - \beta(z-1)\}^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}}.$$

This can be simplified:

$$\frac{\{1 - \beta(z-1)\}^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}} = \frac{(1+\beta) / (1 + \beta - \beta z) - 1}{(1+\beta) - 1} = \frac{(1+\beta) - (1 + \beta - \beta z)}{\beta (1 + \beta - \beta z)} = \frac{z}{1 + \beta - \beta z}.$$

Logarithmic Distribution as a Limit of Zero-Truncated Negative Binomial Distributions:

Exercise: Show that the limit as $r \to 0$ of Zero-Truncated Negative Binomial Distributions with the other parameter β fixed, is a Logarithmic Distribution with parameter β.

[Solution: For the Zero-Truncated Negative Binomial Distribution:

$$p_k^T = \frac{\dfrac{r(r+1)...(r+k-1)}{k!} \dfrac{\beta^k}{(1+\beta)^{k+r}}}{1 - 1/(1+\beta)^r} = \frac{r(r+1)...(r+k-1)}{k!} \frac{\{\beta/(1+\beta)\}^k}{(1+\beta)^r - 1}.$$

$$\lim_{r \to 0} p_k^T = \frac{\{\beta/(1+\beta)\}^k}{k!} \lim_{r \to 0} (r+1)...(r+k-1) \frac{r}{(1+\beta)^r - 1} = \frac{\{\beta/(1+\beta)\}^k}{k!} (k-1)! \lim_{r \to 0} \frac{r}{(1+\beta)^r - 1}.$$

Using L'Hospital's Rule, $\lim_{r \to 0} \frac{r}{(1+\beta)^r - 1} = \lim_{r \to 0} \frac{1}{\ln[1+\beta] (1+\beta)^r} = \frac{1}{\ln[1+\beta]}$.

Thus, $\lim_{r \to 0} p_k^T = \frac{\{\beta/(1+\beta)\}^k}{k} \frac{1}{\ln[1+\beta]} = \frac{\left(\frac{\beta}{1+\beta}\right)^k}{k \ln(1+\beta)}$.

This is the density of a Logarithmic Distribution.

Alternately, as shown in Appendix B of <u>Loss Models</u>,
the p.g.f. of a Zero-Truncated Negative Binomial Distribution is:

$$P^T(z) = \frac{\{1 - \beta(z-1)\}^{-r} - (1+\beta)^{-r}}{1 - (1+\beta)^{-r}}.$$

$$\lim_{r \to 0} P^T(z) = \lim_{r \to 0} \frac{\{1 - \beta(z-1)\}^{-r} - (1+\beta)^{-r}}{1 - (1+\beta)^{-r}}.$$

Using L'Hospital's Rule, $\lim_{r \to 0} P^T(z) = \lim_{r \to 0} \frac{-\ln[1-\beta(z-1)] \{1 - \beta(z-1)\}^{-r} + \ln[1+\beta] (1 + \beta)^{-r}}{\ln[1+\beta] (1 + \beta)^{-r}}$

$$= \frac{\ln[1+\beta] - \ln[1-\beta(z-1)]}{\ln[1+\beta]} = 1 - \frac{\ln[1-\beta(z-1)]}{\ln[1+\beta]}.$$

As shown in Appendix B of <u>Loss Models</u>, this is the p.g.f. of a Logarithmic Distribution.]

Problems:

13.1 (1 point) The number of persons injured in an accident is assumed to follow a Zero -Truncated Poisson Distribution with parameter $\lambda = 0.3$.
Given an accident, what is the probability that exactly 3 persons were injured in it?
A. Less than 1.0%
B. At least 1.0% but less than 1.5%
C. At least 1.5% but less than 2.0%
D. At least 2.0% but less than 2.5%
E. At least 2.5%

Use the following information for the next four questions:

The number of vehicles involved in an automobile accident is given by a Zero-Truncated Binomial Distribution with parameters q = 0.3 and m = 5.

13.2 (1 point) What is the mean number of vehicles involved in an accident?
A. less than 1.8
B. at least 1.8 but less than 1.9
C. at least 1.9 but less than 2.0
D. at least 2.0 but less than 2.1
E. at least 2.1

13.3 (2 points) What is the variance of the number of vehicles involved in an accident?
A. less than 0.5
B. at least 0.5 but less than 0.6
C. at least 0.6 but less than 0.7
D. at least 0.7 but less than 0.8
E. at least 0.8

13.4 (1 point) What is the chance of observing exactly 3 vehicles involved in an accident?
A. less than 11%
B. at least 11% but less than 13%
C. at least 13% but less than 15%
D. at least 15% but less than 17%
E. at least 17%

13.5 (2 points) What is the median number of vehicles involved in an accident??
A. 1 B. 2 C. 3 D. 4 E. 5

Use the following information for the next five questions:
The number of family members is given by a Zero-Truncated Negative Binomial Distribution with parameters $r = 4$ and $\beta = 0.5$.

13.6 (1 point) What is the mean number of family members?
A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

13.7 (2 points) What is the variance of the number of family members?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

13.8 (2 points) What is the chance of a family having 7 members?
A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

13.9 (3 points) What is the probability of a family having more than 5 members?
A. less than 1%
B. at least 1%, but less than 3%
C. at least 3%, but less than 5%
D. at least 5%, but less than 7%
E. at least 7%

13.10 (1 point) What is the probability generating function?

Use the following information for the next three questions:

A Logarithmic Distribution with parameter $\beta = 2$.

13.11 (1 point) What is the mean?

A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

13.12 (2 points) What is the variance?

A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

13.13 (1 point) What is the density function at 6?

A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

13.14 (1 point) For a Zero-Truncated Negative Binomial Distribution with parameters $r = -0.6$ and $\beta = 3$, what is the density function at 5?

A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

Use the following information for the next four questions:
Shoeless Joe is a baseball player.
The number of games until Joe goes hitless is a zero-truncated Geometric Distribution with parameter $\beta = 4$.

13.15 (1 point) What is the mean number of games until Joe goes hitless?
A. 3 B. 4 C. 5 D. 6 E. 7

13.16 (1 point) What is the variance of the number of games until Joe goes hitless?
A. 8 B. 12 C. 16 D. 20 E. 24

13.17 (1 point) What is the probability that it is exactly 6 games until Joe goes hitless?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

13.18 (2 points) What is the probability that it is more than 6 games until Joe goes hitless?
A. 26% B. 27% C. 28% D. 29% E. 30%

13.19 (3 points) You are given the following information from a survey of the number of persons per household in the United States:

Number of Persons	Number of Households
1	29,181
2	35,569
3	17,314
4	15,828
5	7,003
6	2,552
7 or more	1,425

Determine whether or not this data seems to be drawn from a member of the (a, b, 1) class.

13.20 (3 points) If N follows a zero-truncated Poisson Distribution, demonstrate that:

$$E\left[\frac{1}{N+1}\right] = \frac{1}{\lambda} - \frac{1}{e^{\lambda} - 1}.$$

Use the following information for the next five questions:
The number of days per hospital stay is given by a Zero-Truncated Poisson Distribution with parameter $\lambda = 2.5$.

13.21 (1 point) What is the mean number of days per hospital stay?
A. less than 2.5
B. at least 2.5 but less than 2.6
C. at least 2.6 but less than 2.7
D. at least 2.7 but less than 2.8
E. at least 2.8

13.22 (2 points) What is the variance of the number of days per hospital stay?
A. less than 2.2
B. at least 2.2 but less than 2.3
C. at least 2.3 but less than 2.4
D. at least 2.4 but less than 2.5
E. at least 2.5

13.23 (1 point) What is the chance that a hospital stay is 6 days?
A. less than 3%
B. at least 3% but less than 4%
C. at least 4% but less than 5%
D. at least 5% but less than 6%
E. at least 6%

13.24 (2 points) What is the chance that a hospital stay is fewer than 4 days?
A. less than 50%
B. at least 50% but less than 60%
C. at least 60% but less than 70%
D. at least 70% but less than 80%
E. at least 80%

13.25 (2 points) What is the mode of this frequency distribution?
A. 1 B. 2 C. 3 D. 4 E. 5

13.26 (4 points) Let X follow an Exponential with mean θ.
Let Y be the minimum of a random sample from X of size k.
However, K in turn follows a Logarithmic Distribution with parameter β.
What is the distribution function of Y?

Use the following information for the next 2 questions:

- Harvey Wallbanker, the Automatic Teller Machine, works 24 hours a day, seven days a week, without a vacation or even an occasional day off.
- Harvey services on average one customer every 10 minutes.
- 60% of Harvey's customers are male and 40% are female.
- The gender of a customer is independent of the gender of the previous customers.
- Harvey's hobby is to observe patterns of customers. For example, FMF denotes a female customer, followed by a male customer, followed by a female customer.

Harvey starts looking at customers who arrive after serving Pat, his most recent customer.
How long does it take on average until he sees the following patterns?

13.27 (2 points) How long on average until Harvey sees "M"?

13.28 (2 points) How long on average until Harvey sees "F"?

<u>13.29</u> (1 point) X and Y are independently, identically distributed
Zero-Truncated Poisson Distributions, each with $\lambda = 3$.
What is the probability generating function of their sum?

13.30 (3 points) Let X follow an Exponential with mean θ.
Let Y be the minimum of a random sample from X of size k.
However, K in turn follows a Zero-Truncated Geometric Distribution with parameter β.
What is the mean of Y?
Hint: The densities of a Logarithmic Distribution sum to one.

A. $\theta / (1 + \beta)$ B. $(\theta/\beta) \ln[1 + \beta]$ C. $\theta / (1 + \ln[1 + \beta])$ D. $\theta (1 + \beta)$

E. None of A, B, C, or D.

13.31 (5 points) At the Hyperion Hotel, the number of days a guest stays is distributed via a zero-truncated Poisson with $\lambda = 4$.
On the day they check out, each guest leaves a tip for the maid equal to \$3 per day of their stay.
The guest in room 666 is checking out today. What is the expected value of the tip?

13.32 (5 points) The Krusty Burger Restaurant has started a new sales promotion.
With the purchase of each meal they give the customer a coupon.
There are ten different coupons, each with the face of a different famous resident of Springfield.
A customer is equally likely to get each type of coupon, independent of the other coupons he has gotten in the past.
Once you get one coupon of each type, you can turn your 10 different coupons for a free meal.
(a) Assuming a customer saves his coupons, and does not trade with anyone else, what is the mean number of meals he must buy until he gets a free meal?
(b) What is the variance of the number of meals until he gets a free meal?

13.33 (2 points) For a member of the (a, b, 1) class, you are given:
p_{21} = 0.04532.
p_{22} = 0.02987.
p_{23} = 0.01818.
Determine p_{24}.
A. 0.9% B. 1.0% C. 1.1% D. 1.2% E. 1.3%

13.34 (3 points) You are given the following information from a survey of the number of rooms per year-round housing units in the United States:

Number of Rooms	Number of Housing Units
1	556
2	1,292
3	10,319
4	21,599
5	27,687
6	24,810
7 or more	34,269

Determine whether or not this data seems to be drawn from a member of the (a, b, 1) class.

13.35 (Course 151 Sample Exam #1, Q.12) (1.7 points)
A new business has initial capital 700 and will have annual net earnings of 1000.
It faces the risk of a one time loss with the following characteristics:
- The loss occurs at the end of the year.
- The year of the loss is one plus a Geometric distribution with β = 0.538.
  (So the loss may either occur at the end of the first year, second year, etc.)
- The size of the loss is uniformly distributed on the ten integers:
  500,1000,1500, ..., 5000.

Determine the probability of ruin.
(A) 0.00 (B) 0.41 (C) 0.46 (D) 0.60 (E) 0.65

Solutions to Problems:

13.1. B. Let f(x) be the density of a Poisson Distribution, then the distribution truncated from below at zero is: g(x) = f(x) / {1-f(0)}. Thus for θ = 0.3, g(x) = $\{.3^x e^{-.3} / x!\} / \{1-e^{-.3}\}$.

g(3) = $\{.3^3 e^{-.3} / 3!\} / \{1-e^{-.3}\}$ = 0.00333 / 0.259 = **1.3%**.

13.2. B. Mean is that of the non-truncated binomial, divided by 1 - f(0): (.3)(5) / $(1-.7^5)$ = **1.803.**

13.3. D. The second moment is that of the non-truncated binomial, divided by 1 - f(0):
$(1.05 + 1.5^2) / (1 - 0.7^5)$ = 3.967. Variance = 3.967 - 1.803^2 = **0.716**.
Comment: Using the formula in Appendix B of Loss Models:
Variance = $mq\{(1-q) - (1 - q + mq)(1-q)^m\} / \{1-(1-q)^m\}^2$
= $(5)(0.3)\{(0.7 - (0.7 + 1.5)(0.7)^5\} / \{1-(0.7)^5\}^2$ = $(1.5)(0.3303)/0.8319^2$ = 0.716.

13.4. D. For a non-truncated binomial, f(3) = $5!/\{(3!)(2!)\}\ .3^3\ .7^2$ = 0.1323. For the zero-truncated distribution one gets the density by dividing by 1 - f(0): (0.1323) / $(1 - 0.7^5)$ = **15.9%.**

13.5. B. For a discrete distribution such as we have here, employ the convention that the median is the first value at which the distribution function is greater than or equal to 0.5.
F(1) = 0.433 < 50%, F(2) = .804 > 50%, and therefore the median is **2**.

Number of Vehicles	Untruncated Binomial	Zero-Truncated Binomial	Cumulative Zero-Truncated Binomial
0	16.81%		
1	36.02%	43.29%	43.29%
2	30.87%	37.11%	80.40%
3	13.23%	15.90%	96.30%
4	2.83%	3.41%	99.71%
5	0.24%	0.29%	100.00%

13.6. E. Mean is that of the non-truncated negative binomial, divided by 1-f(0):
(4)(0.5) / $(1 - 1.5^{-4})$ = 2 / 0.8025 = **2.49**

13.7. D. The second moment is that of the non-truncated negative binomial, divided by 1-f(0):
$(3 + 2^2) / (1 - 1.5^{-4}) = 8.723$. Variance = $8.723 - 2.492^2 = \mathbf{2.51}$.
Comment: Using the formula in Appendix B of Loss Models:
Variance = $r\beta\{(1+\beta) - (1 + \beta + r\beta)(1+\beta)^{-r}\} / \{1-(1+\beta)^{-r}\}^2$
$= (4)(0.5)\{(1.5 - (1 + 0.5 + 2)(1.5^{-4})\} / (1 - 1.5^{-4})^2 = (2)(0.8086)/0.8025^2 = 2.51$.
The non-truncated negative binomial has mean = $r\beta = 2$, and variance = $r\beta(1+\beta) = 3$,
and thus a second moment of: $3 + 2^2 = 7$.

13.8. C. For the non-truncated negative binomial,
$f(7) = (4)(5)(6)(7)(8)(9)(10)\ .5^7 /((7!)(1.5)^{11}) = 1.08\%$. For the zero-truncated distribution one gets the density by dividing by 1-f(0): $(1.08\%) / (1 - 1.5^{-4})$ = **1.35%**.

13.9. D. The chance of more than 5 is: 1 - .9471 = **5.29%**.

Number of Members	Untruncated Neg. Binomial	Zero-Truncated Neg. Binomial	Cumulative Zero-Truncated Neg. Binomial
0	19.75%		
1	26.34%	32.82%	32.82%
2	21.95%	27.35%	60.17%
3	14.63%	18.23%	78.40%
4	8.54%	10.64%	89.04%
5	4.55%	5.67%	94.71%
6	2.28%	2.84%	97.55%
7	1.08%	1.35%	98.90%
8	0.50%	0.62%	99.52%
9	0.22%	0.28%	99.79%

13.10. As shown in Appendix B of Loss Models,

$$P(z) = \frac{\{1 - \beta(z-1)\}^{-r} - (1+\beta)^{-r}}{1 - (1+\beta)^{-r}} = \frac{(1.5 - 0.5z)^{-4} - 1/1.5^4}{1 - 1/1.5^4} = \frac{1.5^4 / (1.5 - 0.5z)^4 - 1}{1.5^4 - 1}.$$

Alternately, for the Negative Binomial, $f(0) = 1/(1+\beta)^r = 1/1.5^4$,
and $P(z) = \{1 - \beta(z-1)\}^{-r} = \{1 - (0.5)(z - 1)\}^{-4} = (1.5 - 0.5z)^{-4}$.

$$P^T(z) = \frac{P(z) - f(0)}{1 - f(0)} = \frac{(1.5 - 0.5z)^{-4} - 1/1.5^4}{1 - 1/1.5^4} = \frac{1.5^4 / (1.5 - 0.5z)^4 - 1}{1.5^4 - 1}.$$

Comment: This probability generating function only exists for $z < 1 + 1/\beta = 1 + 1/0.5 = 3$.

13.11. A. Mean of the logarithmic distribution is: $\beta/\ln(1+\beta) = 2 / \ln(3) = \mathbf{1.82}$.

13.12. B. Variance of the logarithmic distribution is: $\beta\{1+\beta-\beta/\ln(1+\beta)\}/\ln(1+\beta)$ =
2{3 - 1.82}/ ln(3) = **2.15.**

13.13. C. For the logarithmic distribution, $f(x) = \{\beta/(1+\beta)\}^x / \{x \ln(1+\beta)\}$
$f(6) = (2/3)^6 / \{6 \ln(3)\}$ = **1.33%**.

13.14. A. For the zero-truncated Negative Binomial Distribution,
$f(5) = r(r+1)(r+2)(r+3)(r+4)\ (\beta/(1+\beta))^x / \{(5!)((1+\beta)^r - 1)\}$ =
$(-.6)(.4)(1.4)(2.4)(3.4)(3/4)^5 / \{(120)(4^{-.6} - 1) = (-2.742)(.2373) / (120)(-.5647)$ = .96%.
Comment: Note this is an extended zero-truncated negative binomial distribution, with $0 > r > -1$. The same formulas apply as when $r > 0$. (As r approaches zero one gets a logarithmic distribution.) For the untruncated negative binomial distribution we must have $r > 0$. So in this case there is no corresponding untruncated distribution.

13.15. C. mean = $1 + \beta$ = **5**.

13.16. D. variance = $\beta(1 + \beta)$ = (4)(5) = **20**.

13.17. E. $p_6^T = \beta^5 / (1+\beta)^6 = 4^5 / 5^6$ = **6.55%**.
Comment: Due to the memoryless property of the Geometric, if one were to truncate and shift at 1, in other words get rid of the the zeros and subtract one from the number of claims, one gets the same Geometric. Therefore, if N follows a the zero-truncated Geometric with parameter β, then N -1 follows a Geometric Distribution with mean β.
In this case, the number of games in a row in which Joe gets a hit is Geometric with $\beta = 4$.
Prob[6 games until Joe goes hitless] = Prob[Joe hits in exactly 5 games in a row] =
density at 5 of a Geometric with β equals 4 = $4^5 / 5^6$ = 6.55%.

13.18. A. $p_7^T + p_8^T + p_9^T + ... = 4^6 / 5^7 + 4^7 / 5^8 + 4^8 / 5^9 + ... = (4^6 / 5^7) / (1 - 4/5)$ = **26.2%**.
Alternately, due to the memoryless property of the Geometric, the number of games in a row in which Joe gets a hit is Geometric with $\beta = 4$.
The probability that Joe gets a hit in at least the next 6 games is:
$S(5) = \{\beta/(1+\beta)\}^6 = (4/5)^6$ = **26.2%**.

13.19. The result of the accident profile is definitely not linear.

Number of Persons	Number of Households Observed	Observed Density Function	(x+1)f(x+1)/f(x)
1	29,181	0.26803	2.438
2	35,569	0.32670	1.460
3	17,314	0.15903	3.657
4	15,828	0.14538	2.212
5	7,003	0.06432	2.186
6	2,552	0.02344	
7+	1,425	0.01309	

Thus I conclude that this data is <u>not</u> drawn from a member of the (a, b, 1) class.
<u>Comment</u>: Data is for 2005, taken from Table A-59 of "32 Years of Housing Data" prepared for U.S. Department of Housing and Urban Development Office of Policy Development and Research, by Frederick J. Eggers and Alexander Thackeray of Econometrica, Inc.

13.20. $E[\frac{1}{N+1}] = \sum_{n=1}^{\infty} f(n)/(n+1) = \sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n/n!}{1 - e^{-\lambda}} \frac{1}{n+1} = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n+1}}{(n+1)!}$

$$= \frac{1}{e^{\lambda} - 1} \frac{1}{\lambda} \sum_{j=2}^{\infty} \frac{\lambda^j}{j!} = \frac{1}{e^{\lambda} - 1} \frac{1}{\lambda} \{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} - 1 - \lambda\} = \frac{1}{e^{\lambda} - 1} \frac{1}{\lambda} (e^{\lambda} - 1 - \lambda) = \frac{1}{\lambda} - \frac{1}{e^{\lambda} - 1}.$$

13.21. D. Mean is that of the non-truncated Poisson, divided by 1- f(0):
$(2.5) / (1 - e^{-2.5}) = 2.5/0.9179 =$ **2.724.**
<u>Comment</u>: Note that since the probability at zero has been distributed over the positive integers, the mean is larger for the zero-truncated distribution than for the corresponding untruncated distribution.

13.22. A. The second moment is that of the non-truncated Poisson, divided by 1 - f(0):
$(2.5 + 2.5^2) / (1 - e^{-2.5}) = 9.533$. Variance $= 9.533 - 2.724^2 =$ **2.11**.
<u>Comment</u>: Using the formula in Appendix B of <u>Loss Models</u>:
Variance $= \lambda\{1 - (\lambda+1)e^{-\lambda}\} /(1-e^{-\lambda})^2 = (2.5)\{1 - 3.5e^{-2.5}\}/(1 - e^{-2.5})^2 = (2.5)(.7127)/.9179^2 = 2.11$.

13.23. B. For a untruncated Poisson, $f(6) = (2.5^6)e^{-2.5}/6! = 0.0278$. For the zero-truncated distribution one gets the density by dividing by 1-f(0): $(0.0278) / (1-e^{-2.5}) =$ **3.03%.**

13.24. D. One adds up the chances of 1, 2 and 3 days, and gets **73.59%.**

Number of Days	Untruncated Poisson	Zero-Truncated Poisson	Cumulative Zero-Truncated Poisson
0	8.21%		
1	20.52%	22.36%	22.36%
2	25.65%	27.95%	50.30%
3	21.38%	23.29%	**73.59%**
4	13.36%	14.55%	88.14%
5	6.68%	7.28%	95.42%
6	2.78%	3.03%	98.45%
7	0.99%	1.08%	99.54%
8	0.31%	0.34%	99.88%

Comment: By definition, there is no probability of zero items for a zero-truncated distribution.

13.25. B. The mode is where the density function is greatest, **2.**

Number of Days	Untruncated Poisson	Zero-Truncated Poisson
0	8.21%	
1	20.52%	22.36%
2	25.65%	27.95%
3	21.38%	23.29%
4	13.36%	14.55%

Comment: Unless the mode of the untruncated distribution is 0, the mode of the zero-truncated distribution is the same as that of the untruncated distribution. For example, in this case all the densities on the positive integers are increased by the same factor 1/(1 - 0.0821). Thus since the density at 2 was largest prior to truncation, it remains the largest after truncation at zero.

13.26. Assuming a sample of size k, then

Prob[Min > y | k] = Prob[all elements of the sample > y] = $(e^{-y/\theta})^k = e^{-yk/\theta}$.

Let p_k be the Logarithmic density.

$$\text{Prob[Min > y]} = \sum_{k=1}^{\infty} \text{Prob[Min > y | k]}\ p_k = \sum_{k=1}^{\infty} (e^{-y/\theta})^k\ p_k = E_K[(e^{-y/\theta})^k].$$

However, the P.G.F. of a frequency distribution is defined as $E[z^k]$.

For the Logarithmic Distribution, $P(z) = 1 - \frac{\ln[1 - \beta(z-1)]}{\ln(1+\beta)}$.

Therefore, taking $z = e^{-y/\theta}$,

$$\text{Prob[Min > y]} = 1 - \frac{\ln[1 - \beta(e^{-y/\theta} - 1)]}{\ln(1+\beta)}.$$

Thus Prob[Min ≤ y], in other words the distribution function is:

$$F(y) = \frac{\ln[1 - \beta(e^{-y/\theta} - 1)]}{\ln(1+\beta)},\ y > 0.$$

Comment: *The distribution of Y is called an Exponential-Logarithmic Distribution.*

If one lets $p = 1/(1+\beta)$, then one can show that $F(y) = 1 - \ln[1 - (1-p)\, e^{-y/\theta}] / \ln(p)$.

As β approaches 0, in other words as p approaches 1, the distribution of Y approaches an Exponential Distribution.

The Exponential-Logarithmic Distribution has a declining hazard rate.

In general, if S(x) is the survival function of severity, Y is the minimum of a random sample from X of size k, and K in turn follows a frequency distribution with support k ≥ 1 and Probability Generating Function P(z), then F(y) = 1 - P(S(y)).

13.27. The number of customers he has to wait is a Zero-Truncated Geometric Distribution with β = chance of failure / chance of success = (1 - 0.6)/0.6 = 1/0.6 - 1.

So the mean number of customers is 1/0.6 = 1.67. ⇒ **16.7 minutes** on average.

Comment: The mean of the Zero-Truncated Geometric Distribution is: $\frac{\beta}{1 - 1/(1+\beta)} = 1 + \beta$.

13.28. The number of customers he has to wait is a Zero-Truncated Geometric Distribution with β = chance of failure / chance of success = (1 - 0.4)/0.4 = 1/0.4 - 1.

So the mean number of customers is 1/0.4 = 2.5. ⇒ **25 minutes** on average.

Comment: Longer patterns can be handled via Markov Chain ideas not on the syllabus.
See Example 4.20 in Probability Models by Ross.

13.29. As shown in Appendix B of Loss Models, for the zero-truncated Poisson:

$$P(z) = \frac{e^{\lambda z} - 1}{e^{\lambda} - 1} = \frac{e^{3z} - 1}{e^{3} - 1}.$$

The p.g.f. for the sum of two independently, identically distributed variables is: P(z) P(z) = $P(z)^2$:

$$\boldsymbol{\left(\frac{e^{3z} - 1}{e^{3} - 1}\right)^{2}}.$$

Comment: The sum of two zero-truncated distributions has a minimum of two events.
Therefore, the sum of two zero-truncated Poissons is not a zero-truncated Poisson.

13.30. B. Prob[Min > y | k] = Prob[all elements of the sample > y] = $(e^{-y/\theta})^k = e^{-yk/\theta}$.

Thus the minimum from a sample of size k, follows an Exponential Distribution with mean θ/k.

Therefore, E[Y] = E[E[Y|k]] = E[θ/k] = θ E[1/k].

For a Zero-Truncated Geometric, $p_k = \beta^{k-1} / (1+\beta)^k$, for k = 1, 2, 3,...

Thus $E[1/k] = (1/\beta) \sum_{k=1}^{\infty} \left(\frac{\beta}{1+\beta}\right)^k / k$.

However, for the Logarithmic: $p_k = \frac{\left(\frac{\beta}{1+\beta}\right)^k}{k \ln(1+\beta)}$, for k = 1, 2, 3,...

Therefore, since these Logarithmic densities sum to one: $\sum_{k=1}^{\infty} \left(\frac{\beta}{1+\beta}\right)^k / k = \ln(1 + \beta)$.

Thus E[1/k] = (1/β) ln[1 +β]. Thus E[Y] = θ E[1/k] = **(θ/β) ln[1 + β]**.

13.31. The probability of a stay of length k is p_k^T.

If a stay is of length k, the probability that today is the last day is 1/k.
Therefore, for an occupied room picked at random, the probability that its guest is checking out today is: $\sum_{k=1}^{\infty} p_k^T \ / \ k$.

The tip for a stay of length k is 3k.
Thus, the expected tip left by the guest checking out of room 666 is:

$$\frac{\sum_{k=1}^{\infty} 3k \ p_k^T \ / \ k}{\sum_{k=1}^{\infty} p_k^T \ / \ k} = \frac{3 \sum_{k=1}^{\infty} p_k^T}{\sum_{k=1}^{\infty} p_k^T \ / \ k} = \frac{3}{\sum_{k=1}^{\infty} p_k^T \ / \ k}.$$

For the zero-truncated Poisson, $\sum_{k=1}^{\infty} p_k^T \ / \ k =$

$$\frac{e^{-\lambda}}{1 - e^{-\lambda}} (\lambda + \lambda^2/4 + \lambda^3/18 + \lambda^4/96 + \lambda^5/600 + \lambda^6/4320 + \lambda^7/35{,}280 + \lambda^8/322{,}560$$

$$+ \lambda^9/3{,}265{,}920 + \lambda^{10}/36{,}288{,}000 + \lambda^{11}/439{,}084{,}800 + ...) = 0.330.$$

Thus, the expected tip left by the guest checking out of room 666 is: 3 / 0.330 = **9.09**.
Alternately, the (average) tip per day is 3.
3 = (0)(probability not last day) + (average tip if last day)(probability last day).
3 = (average tip if last day)(0.333).
Therefore, the average tip if it is the last day is: 3 / 0.330 = **9.09**.

13.32. (a) After the customer gets his first coupon, there is 9/10 probability that his next coupon is different. Therefore, the number of meals it takes him to get his next unique coupon after his first is a zero-truncated Geometric Distribution,

with β = (probability of failure) / (probability of success) = (1/10)/(9/10) = 1/9.

(Alternately, it is one plus a Geometric Distribution with β = 1/9.)

Thus the mean number of meals from the first to the second unique coupon is: 1 + 1/9 = 10/9.
After the customer gets his second unique coupon, there is 8/10 probability his next coupon is different than those he already has. Therefore, the number of meals it takes him to get his third unique coupon after his second is a zero-truncated Geometric Distribution,

with β = (probability of failure) / (probability of success) = (2/10)/(8/10) = 2/8.

Thus the mean number of meals from the second to the third unique coupon is: 1 + 2/8 = 10/8.
Similarly, the number of meals it takes him to get his fourth unique coupon after his third is a zero-truncated Geometric Distribution, with β = 3/7, and mean 10/7.

Proceeding in a similar manner, the means to get the remaining coupons are: 10/6 + ... + 10/1.
Including one meal to get the first coupon, the mean total number of meals is:
(10) (1/10 + 1/9 + 1/8 + 1/7 + 1/6 + 1/5 + 1/4 + 1/3 + 1/2 + 1/1) = **29.29**.
(b) It takes one meal to get the first coupon; variance is zero.
The number of additional meals to get the second unique coupon is a zero-truncated Geometric Distribution, with β = 1/9 and variance: (1/9)(10/9).

Similarly, the variance of the number of meals from the second to the third unique coupon is: (2/8)(10/8).
The number of meals in intervals between unique coupons are independent, so their variances add.
Thus, the variance of the total number of meals is:

(10) $(1/9^2 + 2/8^2 + 3/7^2 + 4/6^2 + 5/5^2 + 6/4^2 + 7/3^2 + 8/2^2 + 9/1^2)$ = **125.69**.

Comment: The coupon collector's problem.

13.33. B. $p_{22} / p_{21} = a + b/22$. ⇒ 0.02987 / 0.04532 = 0.65909 = a + b/22.

$p_{23} / p_{22} = a + b/23$. ⇒ 0.01818 / 0.02987 = 0.60864 = a + b/23.

⇒ 0.05045 = b(1/22 - 1/23). ⇒ b = 25.53. ⇒ a = -0.5014.

$p_{24} = (a + b/24)\ p_{23}$ = (-0.5014 + 25.53/24) (0.01818) = **0.01022**.

13.34. The result of the accident profile is definitely not linear.

Number of Rooms	Number of Housing Units Observed	Observed Density Function	(x+1)f(x+1)/f(x)
1	556	0.00461	4.647
2	1,292	0.01072	23.961
3	10,319	0.08561	8.373
4	21,599	0.17920	6.409
5	27,687	0.22971	5.377
6	24,810	0.20584	
7+	34,269	0.28431	

Thus I conclude that this data is not drawn from a member of the (a, b, 1) class.
Comment: Data is for 2005, taken from Table A-19 of “32 Years of Housing Data” prepared for U.S. Department of Housing and Urban Development Office of Policy Development and Research, by Frederick J. Eggers and Alexander Thackeray of Econometrica, Inc.

13.35. D. At the end of year one the business has 1700. Thus, if the loss occurs at the end of year one, there is ruin if the size of loss is > 1700, a 70% chance. Similarly, at the end of year 2, if the loss did not occur in year 1, the business has 2700. Thus, if the loss occurs at the end of year two there is ruin if the size of loss is > 2700, a 50% chance.
If the loss occurs at the end of year three there is ruin if the size of loss is > 3700, a 30% chance.
If the loss occurs at the end of year four there is ruin if the size of loss is > 4700, a 10% chance.
If the loss occurs in year 5 or later there is no chance of ruin.
The probability of the loss being in year n is: $(1/(1+\beta))(\beta/(1+\beta))^{n-1} = .65(.35^{n-1})$.

A	B	C	D
Year	Probability of Loss in this year	Probability of Ruin if Loss Occurs in this year	Column B times Column C
1	0.6500	0.7	0.4550
2	0.2275	0.5	0.1138
3	0.0796	0.3	0.0239
4	0.0279	0.1	0.0028
5	0.0098	0	0.0000
			0.5954

Alternately, if the loss is of size 500, 1000, or 1500 there is not ruin. If the loss is of size 2000 or 2500, then there is ruin if the loss occurs in year 1. If the loss is of size 3000 or 3500, then there is ruin if the loss occurs by year 2. If the loss is of size 4000 or 4500, then there is ruin if the loss occurs by year 3. If the loss is of size 5000, then there is ruin if the loss occurs by year 4.

A	B	C	D	E
Size of Loss	Probability of a Loss of this Size	Year by which loss occurs for Ruin	Probability that Loss Occurs by this year	Column B times Column D
500, 1000, 1500	0.3	none	0.000	0.000
2000 or 2500	0.2	1	0.650	0.130
3000 or 3500	0.2	2	0.877	0.175
4000 or 4500	0.2	3	0.957	0.191
5000	0.1	4	0.985	0.099
				0.595

Section 14, Zero-Modified Distributions[102]

Frequency distributions can be constructed whose densities on the positive integers are proportional to those of a well-known distribution, but with f(0) having any value between zero and one.

For example, let $g(x) = \frac{e^{-3}\, 3^x}{x!} \frac{1 - 0.25}{1 - e^{-3}}$, for x = 1, 2, 3, ..., and g(0) = 0.25.

Exercise: Verify that the sum of this density is in fact is unity.

[Solution: The sum of the Poisson Distribution from 0 to ∞ is 1. $\sum_{i=0}^{\infty} e^{-3}\, 3^x / x! = 1$.

Therefore, $\sum_{i=1}^{\infty} e^{-3}\, 3^x / x! = 1 - e^{-3}$. $\Rightarrow \sum_{i=1}^{\infty} g(x)\ (1 - 0.25)$. $\Rightarrow \sum_{i=0}^{\infty} g(x) = 1 - 0.25 + 0.25 = 1$.]

This is just an example of a Poisson Distribution Modified at Zero, with λ = 3 and 25% probability placed at zero.

For a Zero-Modified distribution, an arbitrary amount of probability has been placed at zero. In the example above it is 25%. Loss Models uses p_0^M to denote this probability at zero.
The remaining probability is spread out proportional to some well-known distribution such as the Poisson. In general if f is a distribution on 0, 1, 2, 3,..., and $0 < p_0^M < 1$,

then probability at zero is p_0^M, $p_k^M = f(k)\ \frac{1 - p_0^M}{1 - f(0)}$, k = 1, 2, 3,... is a distribution on 0, 1, 2, 3, ...

Exercise: For a Poisson Distribution Modified at Zero, with λ = 3 and 25% probability placed at zero, what are the densities at 0, 1, 2, 3, and 4?

[Solution: For example the density at 4 is: $\frac{3^4\, e^{-3}}{4!} \frac{1 - 0.25}{1 - e^{-3}} = 0.133$.

x	0	1	2	3	4
f(x)	0.250	0.118	0.177	0.177	0.133

In the case of a Zero-Modified Distribution, there is no relationship assumed between the density at zero and the other densities, other than the fact that all of the densities sum to one.

[102] See Section 6.6 and Appendix B.3.2 in Loss Models.

We have the following four cases:

Distribution	Zero-Modified Distribution, density at zero p_0^M
Binomial	$(1-p_0^M) \dfrac{\dfrac{m!\, q^x (1-q)^{m-x}}{x!\,(m-x)!}}{1 - (1-q)^m}$ $x = 1, 2, 3, \ldots, m$
Poisson	$(1-p_0^M) \dfrac{e^{-\lambda} \lambda^x / x!}{1 - e^{-\lambda}}$ $x = 1, 2, 3, \ldots$
Negative Binomial[103]	$(1-p_0^M) \dfrac{\dfrac{r(r+1)\ldots(r+x-1)}{x!} \dfrac{\beta^x}{(1+\beta)^{x+r}}}{1 - 1/(1+\beta)^r}$ $x = 1, 2, 3, \ldots$
Logarithmic	$(1-p_0^M) \dfrac{\left(\dfrac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)}$ $x = 1, 2, 3, \ldots$

These four zero-modified distributions complete the (a, b, 1) class of frequency distributions.[104]

They each follow the formula: $\dfrac{\text{density at } x+1}{\text{density at } x} = a + \dfrac{b}{x+1}$, for $x \geq 1$.

Note that if $p_0^M = 0$, the zero-modified distribution reduces to a zero-truncated distribution.

However, even though it might be useful to think of the zero-truncated distributions as a special case of the zero-modified distributions, Loss Models restricts the term zero-modified to those cases where the density at zero is positive.

Moments:

The moments of a zero-modified distribution are given in terms of those of unmodified f by

$E^{\text{Modified}}[X^n] = (1 - p_0^M) \dfrac{E_f[X^n]}{1 - f(0)} = (1 - p_0^M)\, E^{\text{Truncated}}[X^n]$.

For example for the Zero-Truncated Poisson the mean is:

$(1 - p_0^M) \dfrac{\lambda}{1 - e^{-\lambda}}$, while the second moment is: $(1 - p_0^M) \dfrac{\lambda + \lambda^2}{1 - e^{-\lambda}}$.

[103] The zero-modified version of the Negative Binomial is referred to by Loss Models as the Zero-Modified Extended Truncated Negative Binomial.

[104] See Table 6.4 and Appendix B.3 in Loss Models.

Exercise: For a Zero-modified Poisson with $\lambda = 3$ and 25% chance of zero claims, what is the mean?

[Solution: $E^{Modified}[X] = (1 - p_0^M) \frac{E_f[X]}{1 - f(0)} = (1 - p_0^M) \frac{\lambda}{1 - e^{-\lambda}} = (1 - 0.25) \frac{3}{1 - e^{-3}} = 2.3679$.

Alternately, mean of zero-truncated Poisson is: $\frac{\lambda}{1 - e^{-\lambda}} = \frac{3}{1 - e^{-3}} = 3.1572$.

mean of zero-modified Poisson is: $(1 - p_0^M)$(mean of zero-truncated) = (1- 0.25)(3.1572) = 2.3679.

Alternately, we can do the calculation from first principles.

Let f(x) be the untruncated Poisson, and p_k^M be the zero-modified distribution.

Then $p_k^M = \frac{0.75 \, f(x)}{1 - f(0)}$, $x > 0$. The mean of the zero-modified distribution is:

$$\sum_{k=0}^{\infty} k \, p_k^M = \sum_{k=1}^{\infty} k \, p_k^M = \sum_{x=1}^{\infty} \frac{0.75 \, x \, f(x)}{1 - f(0)} = 0.75 \frac{\sum_{x=0}^{\infty} x \, f(x)}{1 - f(0)} = 0.75 \frac{\text{mean of f}}{1 - f(0)} = 0.75 \frac{\lambda}{1 - e^{-\lambda}} =$$

$$(0.75) \frac{3}{1 - e^{-3}} = (0.75)(3.1572) = 2.3679.$$

<u>Comment</u>: In the summation, the term involving k = 0 would contribute nothing to the mean.]

Exercise: For a Zero-modified Poisson with $\lambda = 3$ and 25% chance of zero claims, what is the variance?

[Solution: The second moment of the zero-modified Poisson is:

$(1 - p_0^M)$ (second moment of Poisson) / {1 - f(0)} = $(1 - 0.25)\,(3 + 3^2) / (1 - e^{-3}) = 9.4716$.

Thus the variance of the zero-modified Poisson is: $9.4716 - 2.3679^2 = 3.8646$.

Alternately, for the zero-truncated Poisson:

mean = $\lambda / (1 - e^{-\lambda}) = 3 / (1 - e^{-3}) = 3.1572$, and

$$\text{variance} = \frac{\lambda \{1 - (\lambda + 1) e^{-\lambda}\}}{(1 - e^{-\lambda})^2} = \frac{(3)\,(1 - 4e^{-3})}{(1 - e^{-3})^2} = 2.6609.$$

Then for the zero-modified Poisson, as shown in the Tables attached to the exam:

variance = $(1 - p_0^M)$ (variance of zero truncated) + $p_0^M (1 - p_0^M)$ (mean of zero truncated)^2 =

$(1 - 0.25)(2.6609) + (0.25)(1 - 0.25)(3.1572^2) = 3.8647$.]

Exercise: For a Negative Binomial with r = 0.7 and $\beta = 3$ what is the second moment?
[Solution: The mean is (0.7)(3) = 2.1, the variance is (0.7)(3)(1+3) = 8.4, so the second moment is: $8.4 + 2.1^2 = 12.81$.]

Exercise: For a Zero-Truncated Negative Binomial with r = 0.7 and $\beta = 3$ what is the second moment?
[Solution: For a Negative Binomial with r = 0.7 and $\beta = 3$,
the density at zero is: $1/(1+\beta)^r = 4^{-0.7} = 0.3789$, and the second moment is 12.81.
Thus the second moment of the zero-truncated distribution is: 12.81 / (1 - 0.3789) = 20.625.]

Exercise: For a Zero-Modified Negative Binomial with r = 0.7 and $\beta = 3$, with a 15% chance of zero claims, what is the second moment?
[Solution: For a Zero-Truncated Negative Binomial with r = 0.7 and $\beta = 3$, the second moment is 20.625. Thus the second moment of the zero-modified distribution with a 15% chance of zero claims is: (20.625)(1 - 0.15) = 17.531.]

Exercise: For a Zero-Modified Negative Binomial with r = 0.7 and $\beta = 3$, with a 15% chance of zero claims, what is the variance?
[Solution: mean of the zero-truncated Negative Binomial is:
(mean of Negative Binomial) / {1 - f(0)} = (0.7)(3) / (1 - 0.3789) = 3.3811.
mean of the zero-modified Negative Binomial is:
$(1 - p_0^M)$(mean of zero-truncated) = (1 - 0.15)(3.3811) = 2.8739.

Thus, the variance of the zero-modified Negative Binomial is: $17.531 - 2.8739^2 = 9.272$.
Alternately, for the zero-truncated Negative Binomial:

$$\text{mean} = \frac{r\beta}{1 - (1+\beta)^{-r}} = \frac{(0.7)(3)}{1 - 4^{-0.7}} = 3.3813, \text{ and}$$

$$\text{variance} = \frac{r\beta\,\{(1+\beta) - (1 + \beta + r\beta)(1+\beta)^{-r}\}}{\{1 - (1+\beta)^{-r}\}^2} = \frac{(0.7)(3)\,\{4 - (4 + 2.1)(4^{-0.7})\}}{\{1 - 4^{-0.7}\}^2} = 9.1928.$$

Then for the zero-modified Negative Binomial, as shown in the Tables attached to the exam:
variance = $(1-p_0^M)$ (variance of zero truncated) + $p_0^M(1-p_0^M)$ (mean of zero truncated)^2 =
$(1 - 0.15)(9.1928) + (0.15)(1-0.15)(3.3813^2) = 9.272$.]

Probability Generating Functions:

The zero-modified distribution, can be thought of a mixture of a point mass of probability at zero and a zero-truncated distribution. The probability generating function of a mixture is the mixture of the probability generating functions. A point mass of probability at zero, has a probability generating function $E[z^n] = E[z^0] = 1$. Therefore, the Probability generating function, $P(z) = E[z^N]$, for a zero-modified distribution can be obtained from that for zero-truncated distribution:[105]

$$P^M(z) = p_0^M + (1 - p_0^M)\, P^T(z).$$

where $P^M(z)$ is the p.g.f. for the zero-modified distribution and $P^T(z)$ is the p.g.f. for the zero-truncated distribution, and p_0^M is the probability at zero for the zero-modified distribution.

Exercise: What is the Probability Generating Function for a Zero-Modified Poisson Distribution, with 30% probability placed at zero?
[Solution: For the zero-truncated Poisson. $P^T(z) = (e^{\lambda z} - 1) / (e^{\lambda} - 1)$.
$P^M(z) = p_0^M + (1 - p_0^M)\, P^T(z) = 0.3 + 0.7 \dfrac{e^{\lambda z} - 1}{e^{\lambda} - 1}$.]

One can derive this relationship as follows:

$p_k^M = p_k^T\, (1 - p_0^M)$ for $k > 0$.

$$P^M(z) = \sum_{n=0}^{\infty} z^n p_n^M = p_0^M + \sum_{n=1}^{\infty} z^n (1 - p_0^M)\, p_n^T = p_0^M + (1 - p_0^M) \sum_{n=1}^{\infty} z^n p_n^T = p_0^M + (1 - p_0^M)\, P^T(z).$$

[105] The probability generating functions of the zero-modified distributions are shown in Appendix B.

Thinning:[106]

If we take at random a fraction of the events, then we get a distribution of the same family.
One parameter is altered by the thinning as per the non-zero-modified case.
In addition, the probability at zero, p_0^M, is altered by thinning.

Distribution	Result of thinning by a factor of t
Zero-Modified Binomial	$q \to tq$ m remains the same
	$p_0^M \to \dfrac{p_0^M - (1-q)^m + (1-tq)^m - p_0^M (1-tq)^m}{1 - (1-q)^m}$
Zero-Modified Poisson	$\lambda \to t\lambda$
	$p_0^M \to \dfrac{p_0^M - e^{-\lambda} + e^{-t\lambda} - p_0^M e^{-t\lambda}}{1 - e^{-\lambda}}$
Zero-Modified Negative Binomial[107]	$\beta \to t\beta$ r remains the same
	$p_0^M \to \dfrac{p_0^M - (1+\beta)^{-r} + (1+t\beta)^{-r} - p_0^M (1+t\beta)^{-r}}{1 - (1+\beta)^{-r}}$
Zero-Modified Logarithmic	$\beta \to t\beta$
	$p_0^M \to 1 - (1 - p_0^M) \dfrac{\ln[1+t\beta]}{\ln[1+\beta]}$

In each case, the new probability of zero claims is the probability generating function for the original zero-modified distribution at 1 - t, where t is the thinning factor.

For example, for the Zero-Modified Binomial, $P^M(z) = p_0^M + (1 - p_0^M)\, P^T(z) =$

$$p_0^M + (1 - p_0^M) \frac{\{1 + q(z-1)\}^m - (1-q)^m}{1 - (1-q)^m}.$$

$$P(1 - t) = p_0^M + (1 - p_0^M) \frac{\{1 - qt\}^m - (1-q)^m}{1 - (1-q)^m} = \frac{p_0^M - (1-q)^m + (1-tq)^m - p_0^M (1-tq)^m}{1 - (1-q)^m}.$$

[106] See Table 8.3 in Loss Models.
[107] Including the special case the zero-modified geometric.

For example, let us assume we look at only large claims, which are t of all claims.
Then if we have n claims, the probability of zero large claims is: $(1-t)^n$.
Thus the probability of zero large claims is:
Prob[zero claims] $(1-t)^0$ + Prob[1 claim] $(1-t)^1$ + Prob[2 claims] $(1-t)^2$ + Prob[3 claims] $(1-t)^3$ + ...
$E[(1-t)^n] = P(1 - t)$ = p.g.f. for the unthinned distribution at 1 - t.

Exercise: Show that the p.g.f. for the original zero-modified Logarithmic distribution at 1 - t matches the above result for the density at zero for the thinned distribution.
[Solution: For the Zero-Modified Logarithmic, $P(z) = p_0^M + (1 - p_0^M)$ (p.g.f. of Logarithmic) =

$$p_0^M + (1 - p_0^M)\{1 - \frac{\ln[1 - \beta(z-1)]}{\ln[1+\beta]}\}.$$

$$P(1 - t) = p_0^M + (1 - p_0^M)\{1 - \ln[1 - \beta t] / \ln[1+\beta]\} = 1 - (1 - p_0^M)\frac{\ln[1+t\beta]}{\ln[1+\beta]}.$$]

Exercise: The number of losses follows a zero-modified Poisson with $\lambda = 2$ and $p_0^M = 10\%$.
30% of losses are large. What is the distribution of the large losses?
[Solution: Large losses follow a zero-modified Poisson with $\lambda = (30\%)(2) = 0.6$ and

$$p_0^M = \frac{0.1 - e^{-2} + e^{-0.6} - (0.1)e^{-0.6}}{1 - e^{-2}} = 0.5304.$$]

Exercise: The number of members per family follows a zero-truncated Negative Binomial with $r = 0.5$ and $\beta = 4$.
It is assumed that 60% of people have first names that begin with the letters A through M, and that size of family is independent of the letters of the first names of its members.
What is the distribution of the number of family members with first names that begin with the letters A through M?
[Solution: The zero-truncated distribution is mathematically the same as a zero-modified distribution with $p_0^M = 0$.

Thus the thinned distribution is a zero-modified Negative Binomial with $r = 0.5$, $\beta = (60\%)(4) = 2.4$,

and $$p_0^M = \frac{0 - 5^{-0.5} + 3.4^{-0.5} - (0)(3.4^{-0.5})}{1 - 5^{-0.5}} = 0.1721.$$

Comment: While prior to thinning there is no probability of zero members, after thinning there is a probability of zero members with first names that begin with the letters A through M.
Thus the thinned distribution is zero-modified rather than zero-truncated.]

Problems:

Use the following information for the next six questions:
The number of claims per year is given by a Zero-Modified Binomial Distribution with parameters q = 0.3 and m = 5, and with 15% probability of zero claims.

14.1 (1 point) What is the mean number of claims over the coming year?
A. less than 1.4
B. at least 1.4 but less than 1.5
C. at least 1.5 but less than 1.6
D. at least 1.6 but less than 1.7
E. at least 1.7

14.2 (2 points) What is the variance of the number of claims per year?
A. less than 0.98
B. at least 0.98 but less than 1.00
C. at least 1.00 but less than 1.02
D. at least 1.02 but less than 1.04
E. at least 1.04

14.3 (1 point) What is the chance of observing 3 claims over the coming year?
A. less than 13.0%
B. at least 13.0% but less than 13.4%
C. at least 13.4% but less than 13.8%
D. at least 13.8% but less than 14.2%
E. at least 14.2%

14.4 (2 points) What is the 95th percentile of the distribution of the number of claims per year?
A. 1 B. 2 C. 3 D. 4 E. 5

14.5 (2 points) What is the probability generating function at 3?
A. less than 9
B. at least 9 but less than 10
C. at least 10 but less than 11
D. at least 11 but less than 12
E. at least 12

14.6 (2 points) Small claims are 70% of all claims.
What is the chance of observing exactly 2 small claims over the coming year?
A. 20% B. 22% C. 24% D. 26% E. 28%

Use the following information for the next five questions:
The number of claims per year is given by a Zero-Modified Negative Binomial Distribution with parameters r = 4 and β = 0.5, and with 35% chance of zero claims.

14.7 (1 point) What is the mean number of claims over the coming year?
A. less than 1.7
B. at least 1.7 but less than 1.8
C. at least 1.8 but less than 1.9
D. at least 1.9 but less than 2.0
E. at least 2.0

14.8 (2 points) What is the variance of the number of claims year?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

14.9 (1 point) What is the chance of observing 7 claims over the coming year?
A. less than 0.8%
B. at least 0.8% but less than 1.0%
C. at least 1.0% but less than 1.2%
D. at least 1.2% but less than 1.4%
E. at least 1.4%

14.10 (3 points) What is the probability of more than 5 claims in the coming year?
A. less than 1%
B. at least 1%, but less than 3%
C. at least 3%, but less than 5%
D. at least 5%, but less than 7%
E. at least 7%

14.11 (3 points) Large claims are 40% of all claims.
What is the chance of observing more than 1 large claim over the coming year?
A. 10% B. 12% C. 14% D. 16% E. 18%

Use the following information for the next four questions:

The number of claims per year is given by a Zero-Modified Logarithmic Distribution with parameter $\beta = 2$, and a 25% chance of zero claims.

14.12 (1 point) What is the mean number of claims over the coming year?
A. less than 1.0
B. at least 1.0 but less than 1.1
C. at least 1.1 but less than 1.2
D. at least 1.2 but less than 1.3
E. at least 1.3

14.13 (2 points) What is the variance of the number of claims per year?
A. less than 2.0
B. at least 2.0 but less than 2.2
C. at least 2.2 but less than 2.4
D. at least 2.4 but less than 2.6
E. at least 2.6

14.14 (1 point) What is the chance of observing 6 claims over the coming year?
A. less than 1.1%
B. at least 1.1% but less than 1.3%
C. at least 1.3% but less than 1.5%
D. at least 1.5% but less than 1.7%
E. at least 1.7%

14.15 (2 points) Medium sized claims are 60% of all claims.
What is the chance of observing exactly one medium sized claim over the coming year?
A. 31% B. 33% C. 35% D. 37% E. 39%

14.16 (1 point) The number of claims per year is given by a Zero-Modified Negative Binomial Distribution with parameters $r = -0.6$ and $\beta = 3$, and with a 20% chance of zero claims.
What is the chance of observing 5 claims over the coming year?
A. less than 0.8%
B. at least 0.8% but less than 1.0%
C. at least 1.0% but less than 1.2%
D. at least 1.2% but less than 1.4%
E. at least 1.4%

Use the following information for the next seven questions:

The number of claims per year is given by a Zero-Modified Poisson Distribution with parameter $\lambda = 2.5$, and with 30% chance of zero claims.

14.17 (1 point) What is the mean number of claims over the coming year?
A. 1.9 B. 2.0 C. 2.1 D. 2.2 E. 2.3

14.18 (2 points) What is the variance of the number of claims per year?
A. less than 2.7
B. at least 2.7 but less than 2.8
C. at least 2.8 but less than 2.9
D. at least 2.9 but less than 3.0
E. at least 3.0

14.19 (1 point) What is the chance of observing 6 claims over the coming year?
A. less than 2%
B. at least 2% but less than 3%
C. at least 3% but less than 4%
D. at least 4% but less than 5%
E. at least 5%

14.20 (1 point) What is the chance of observing 2 claims over the coming year?
A. 18% B. 20% C. 22% D. 24% E. 26%

14.21 (2 points) What is the chance of observing fewer than 4 claims over the coming year?
A. less than 70%
B. at least 70% but less than 75%
C. at least 75% but less than 80%
D. at least 80% but less than 85%
E. at least 85%

14.22 (2 points) What is the mode of this frequency distribution?
A. 0 B. 1 C. 2 D. 3 E. 4

14.23 (2 points) Large claims are 20% of all claims.
What is the chance of observing exactly one large claim over the coming year?
A. 15% B. 17% C. 19% D. 21% E. 23%

14.24 (3 points) Let p_k denotes the probability that the number of claims equals k for k = 0, 1, ...
If $p_n / p_m = 2.4^{n-m}$ m! / n!, for m ≥ 0, n ≥ 0, then using the corresponding zero-modified claim count distribution with $p_0^M = 0.31$, calculate p_3^M.

(A) 16% (B) 18% (C) 20% (D) 22% (E) 24%

14.25 (3 points) The number of losses follow a zero-modified Poisson Distribution with λ and p_0^M.
Small losses are 70% of all losses.
From first principles determine the probability of zero small losses.

14.26 (3 points) The following data is the number sick days taken at a large company during the previous year.

Number of days:	0	1	2	3	4	5	6	7	8+
Number of employees:	50,122	9190	5509	3258	1944	1160	693	418	621

Is it likely that this data was drawn from a member of the (a, b, 0) class?
Is it likely that this data was drawn from a member of the (a, b, 1) class?

<u>**14.27**</u> (3 points) For a zero-modified Poisson, p_2^M = 27.3%, and p_3^M = 12.7%.

Determine p_0^M.

(A) 11% (B) 12% (C) 13% (D) 14% (E) 15%

14.28 (3 points) X is a discrete random variable with a probability function which is a member of the (a, b, 1) class of distributions.
p_k denotes the probability that X = k.
$p_1 = 0.1637$, $p_2 = 0.1754$, and $p_3 = 0.1503$.
Calculate p_5.

(A) 7.5% (B) 7.7% (C) 7.9% (D) 8.1% (E) 8.3%

14.29 (2 points) The number of claims per year is given by a Zero-Modified Poisson Distribution with λ = 2, and with 25% chance of zero claims.
Where N is the number of claims, determine E[N ∧ 3].
A. 1.50 B. 1.55 C. 1.60 D. 1.65 E. 1.70

14.30 (1 point) What is the probability generating function for a Zero-Modified Poisson Distribution with λ = 0.1 and p_0^M = 60%?

14.31 (7 points) For a zero-truncated Geometric Distribution:

$$P^T(z) = \frac{\{1 - \beta(z-1)\}^{-1} - (1+\beta)^{-1}}{1 - (1+\beta)^{-1}} = \frac{z}{1 + \beta - \beta z}.$$

(a) (1 point) X follows a zero-modified Geometric Distribution with $\beta = 0.25$ and $p_0^M = 40\%$.

Determine the probability generating function for X.

(b) (3 points) Let Y be the sum of two independent, identically distributed such variables X.
With the aid of a computer, using its probability generating function, determine for Y the densities at: 0, 1, 2, 3, 4, 5, and 6.

(c) (3 points) Let Z be the zero-modified Negative Binomial Distribution with r = 2, $\beta = 0.25$, and the same probability of zero claims as Y.
With the aid of a computer, determine for Z the densities at: 0, 1, 2, 3, 4, 5, and 6.

14.32 (3, 5/00, Q.37) (2.5 points) Given:

(i) p_k denotes the probability that the number of claims equals k for k = 0, 1, 2, ...

(ii) $p_n / p_m = m! / n!$, for $m \geq 0$, $n \geq 0$

Using the corresponding zero-modified claim count distribution with $p_0^M = 0.1$, calculate p_1^M.

(A) 0.1 (B) 0.3 (C) 0.5 (D) 0.7 (E) 0.9

Solutions to Problems:

14.1. C. Mean is that of the unmodified Binomial, multiplied by (1 - 0.15) and divided by 1- f(0):
$(0.3)(5)(0.85) / (1 - 0.7^5)$ = **1.533**.

14.2. D. The second moment is that of the unmodified Binomial,
multiplied by (1-.15) and divided by 1 - f(0):
$(1.05 + 1.5^2)(0.85) / (1 - 0.7^5) = 3.372$. Variance = $3.372 - 1.533^2$ = **1.022**.

14.3. C. For an unmodified binomial, $f(3) = (5!/(3!)(2!))\ 0.3^3\ 0.7^2 = 0.1323$.
For the zero-truncated distribution one gets the density by multiplying by (1-.15)
and dividing by 1 - f(0):
$(0.1323)(0.85) / (1 - 0.7^5)$ = **13.5%**.

14.4. C. The 95th percentile is that value corresponding to the distribution function being 95%. For a discrete distribution such as we have here, employ the convention that the 95th percentile is the first value at which the distribution function is greater than or equal to 0.95. F(2) = 0.8334 $<$ 95%, F(3) = 0.9686 $\geq$ 95%, and therefore the 95th percentile is **3**.

Number of Claims	Unmodified Binomial	Zero-Modified Binomial	Cumulative Zero-Modified Binomial
0	16.81%	15.00%	
1	36.02%	36.80%	51.80%
2	30.87%	31.54%	83.34%
3	13.23%	13.52%	96.86%
4	2.83%	2.90%	99.75%
5	0.24%	0.25%	100.00%

14.5. C. As shown in Appendix B of Loss Models, for the zero-truncated Binomial Distribution:

$$P^T(z) = \frac{\{1 + q(z-1)\}^m - (1-q)^m}{1 - (1-q)^m}. \Rightarrow P^T(3) = \frac{\{1 + (0.3)(3-1)\}^5 - (1-0.3)^5}{1 - (1-0.3)^5} = 12.402.$$

The p.g.f. for the zero-modified distribution is:

$P^M(z) = p_0^M + (1 - p_0^M)P^T(z). \Rightarrow P^M(3) = (0.15) + (0.85)(12.402) =$ **10.69**.

Comment: The densities of the zero-modified distribution:

Number of Claims	Unmodified Binomial	Zero-Modified Binomial
0	16.807%	15.000%
1	36.015%	36.797%
2	30.870%	31.541%
3	13.230%	13.517%
4	2.835%	2.897%
5	0.243%	0.248%

$P^T(3)$ is the expected value of 3^n:

$(15\%)(3^0) + (36.797\%)(3^1) + (31.541\%)(3^2) + (13.517\%)(3^3) + (2.897\%)(3^4) + (0.248\%)(3^5) =$ 10.69.

14.6. B. After thinning we get another zero-modified Binomial, with m = 5, but q = (0.7)(0.3) = 0.21, and

$$p_0^M \rightarrow \frac{p_0^M - (1-q)^m + (1-tq)^m - p_0^M (1-tq)^m}{1 - (1-q)^m} = \frac{0.15 - 0.7^5 + 0.79^5 - (0.15)(0.79^5)}{1 - 0.7^5} = 0.2927.$$

The density at two of the new zero-modified Binomial is:

$\frac{1 - 0.2927}{1 - 0.79^5} (10)(0.21^2)(0.79^3) =$ **22.21%**.

Comment: The probability of zero claims for the thinned distribution is the p.g.f. for the original zero-modified distribution at 1 - t, where t is the thinning factor.

14.7. A. Mean is that of the unmodified negative binomial, multiplied by (1-.35) and divided by 1 - f(0): $(4)(0.5)(0.65)/ (1 - 1.5^{-4}) = 2 / 0.8025 =$ **1.62**

14.8. E. The second moment is that of the unmodified negative binomial, multiplied by (1-.35) and divided by 1 - f(0): $(3+2^2)(0.65)/ (1 - 1.5^{-4}) = 5.67$. Variance $= 5.67 - 1.62^2 =$ **3.05**.

14.9. B. For the unmodified negative binomial, $f(7) = (4)(5)(6)(7)(8)(9)(10)\ .5^7 / \{(7!)(1.5)^{11}\} = 1.08\%$. For the zero-truncated distribution one gets the density by multiplying by (1-.35) and dividing by 1 - f(0): $(1.08\%)(0.65) / (1 - 1.5^{-4}) =$ **0.87%.**

14.10. C. The chance of more than 5 claims is: 1 - 0.9656 = **3.44%.**

Number of Claims	Unmodified Neg. Binomial	Zero-Modified Neg. Binomial	Cumulative Zero-Modified Neg. Binomial
0	19.75%	35.00%	
1	26.34%	21.33%	56.33%
2	21.95%	17.78%	74.11%
3	14.63%	11.85%	85.96%
4	8.54%	6.91%	92.88%
5	4.55%	3.69%	96.56%
6	2.28%	1.84%	98.41%
7	1.08%	0.88%	99.29%
8	0.50%	0.40%	99.69%
9	0.22%	0.18%	99.87%

14.11. D. After thinning we get another zero-modified Negative Binomial, with r = 4,
but β = (40%)(0.5) = 0.2, and

$$p_0^M \to \frac{p_0^M - (1+\beta)^{-r} + (1+t\beta)^{-r} - p_0^M (1+t\beta)^{-r}}{1 - (1+\beta)^{-r}} = \frac{0.35 - 1.5^{-4} + 1.2^{-4} - (0.35)(1.2^{-4})}{1 - 1.5^{-4}} =$$

0.5806.
The density at one of the new zero-modified Negative Binomial is:

$$\frac{1 - 0.5806}{1 - 1/1.2^4} \frac{(4)(0.2)}{1.2^5} = 0.2604.$$

Probability of more than one large claim is: 1 - 0.5806 - 0.2604 = **15.90%**.
Comment: The probability of zero claims for the thinned distribution is the p.g.f. for the original zero-modified distribution at 1 - t, where t is the thinning factor.
Similar to Example 8.9 in Loss Models.

14.12. E. Mean of the logarithmic distribution is: $\beta/\ln(1+\beta)$ = 2 / ln(3) = 1.82.
For the zero-modified distribution, the mean is multiplied by 1 - 0.25: (0.75)(1.82) = **1.37**.
Comment: Note the unmodified logarithmic distribution has no chance of zero claims.
Therefore, we need not divide by 1 - f(0) to get to the zero-modified distribution (or alternately we are dividing by 1 - 0 = 1.)

14.13. C. Variance of the unmodified logarithmic distribution is:
$\beta\{1+\beta-\beta/\ln(1+\beta)\}/\ln(1+\beta)$ = 2{3 -1.82}/ ln(3) = 2.15.

Thus the unmodified logarithmic has a second moment of: 2.15 + 1.82^2 = 5.46.

For the zero-modified distribution, the second moment is multiplied by 1 - 0.25:
(0.75)(5.46) = 4.10.

Thus the variance of the zero-modified distribution is: 4.10 - 1.37^2 = **2.22.**

14.14. A. For the unmodified logarithmic distribution, $f(x) = \{\beta/(1+\beta)\}^x / \{x \ln(1+\beta)\}$

$f(6) = (2/3)^6 / \{6\ln(3)\} = 1.33\%$.

For the zero-modified distribution, the density at 6 is multiplied by 1 - 0.25:
(0.75)(1.33%) = **1.00%**.

14.15. D. After thinning we get another zero-modified Logarithmic, with β = (60%)(2) = 1.2, and

$$p_0^M \rightarrow 1 - (1 - p_0^M) \frac{\ln[1+t\beta]}{\ln[1+\beta]} = 1 - (0.75) \frac{\ln[2.2]}{\ln[3]} = 0.4617.$$

The density at one of the new zero-modified Logarithmic is:

$$(1 - 0.4617) \frac{1.2}{2.2 \ln[2.2]} = \mathbf{37.23\%}.$$

Comment: The probability of zero claims for the thinned distribution is the p.g.f. for the original zero-modified distribution at 1 - t, where t is the thinning factor.

14.16. A. For the zero-truncated Negative Binomial Distribution,

$f(5) = r(r+1)(r+2)(r+3)(r+4)\ (\beta/(1+\beta))^x / \{(5!)((1+\beta)^r - 1)\} =$

$(-.6)(0.4)(1.4)(2.4)(3.4)(3/4)^5 / \{(120)(4^{-0.6} - 1) = (-2.742)(.2373) / \{(120)(-0.5647)\} = 0.96\%$.

For the zero-modified distribution, multiply by 1 - 0.2: (0.8)(.96%) = **0.77%**.

Comment: Note this is an extended zero-truncated negative binomial distribution, with $0 > r > -1$. The same formulas apply as when $r > 0$. (As r approaches zero one gets a logarithmic distribution.) For the unmodified negative binomial distribution we must have $r > 0$. So in this case there is no corresponding unmodified distribution.

14.17. A. The mean is that of the non-modified Poisson, multiplied by (1-.3) and divided by 1- f(0): $(2.5)\ (0.7) / (1 - e^{-2.5}) =$ **1.907.**

14.18. E. The second moment is that of the unmodified Poisson, multiplied by (1-.3) and divided by 1-f(0): $(2.5+2.5^2)(0.7) / (1 - e^{-2.5}) = 6.673$. Variance = $6.673 - 1.907^2$ = **3.04**.

14.19. B. For an unmodified Poisson, $f(6) = (2.5^6)e^{-2.5}/6! = 0.0278$.

For the zero-modified distribution one gets the density by multiplying by (1 - 0.3) and dividing by 1 - f(0): $(0.0278)(0.7) / (1 - e^{-2.5})$ = **2.12%.**

14.20. B. For the unmodified Poisson $f(0) = e^{-2.5} = 8.208\%$, and $f(2) = 2.5^2 e^{-2.5}/2 = 25.652\%$.

The zero-modified Poisson has a density at 2 of: (25.652%)(1 - 30%)/(1 - 8.208%) = **19.56%**.

14.21. D. One adds up the chances of 0, 1, 2 and 3 claims, and gets **81.5%.**

Number of Claims	Unmodified Poisson	Zero-Modified Poisson	Cumulative Zero-Modified Poisson
0	8.21%	30.00%	
1	20.52%	15.65%	45.65%
2	25.65%	19.56%	65.21%
3	21.38%	16.30%	**81.51%**
4	13.36%	10.19%	91.70%
5	6.68%	5.09%	96.80%
6	2.78%	2.12%	98.92%
7	0.99%	0.76%	99.68%
8	0.31%	0.24%	99.91%

Comment: We are given a 30% chance of zero claims.
The remaining 70% is spread in proportion to the unmodified Poisson. For example,
(70%)(20.52%)/(1 - 0.0821) = 15.65%, and (70%)(25.65%)/(1 - 0.0821) = 19.56%
Unlike the zero-truncated distribution, the zero-modified distribution has a probability of zero events.

14.22. A. The mode is where the density function is greatest, **0.**

Number of Claims	Unmodified Poisson	Zero-Modified Poisson
0	8.21%	30.00%
1	20.52%	15.65%
2	25.65%	19.56%
3	21.38%	16.30%
4	13.36%	10.19%
5	6.68%	5.09%
6	2.78%	2.12%
7	0.99%	0.76%
8	0.31%	0.24%

Comment: If the mode of the zero-modified and unmodified distribution are ≠ 0, then the zero-modified distribution has the same mode as the unmodified distribution, since all the densities on the positive integers are multiplied by the same factor.

14.23. E. After thinning we get another zero-modified Poisson, with $\lambda = (20\%)(2.5) = 0.5$, and

$$p_0^M \rightarrow \frac{p_0^M - e^{-\lambda} + e^{-t\lambda} - p_0^M e^{-t\lambda}}{1 - e^{-\lambda}} = \frac{0.3 - e^{-2.5} + e^{-0.5} - (0.3)(e^{-0.5})}{1 - e^{-2.5}} = 0.6999.$$

The density at one of the new zero-modified Poisson is:

$$\frac{1 - 0.6999}{1 - e^{-0.5}} (0.5\ e^{-0.5}) = \mathbf{23.13\%}.$$

Comment: The probability of zero claims for the thinned distribution is the p.g.f. for the original zero-modified distribution at 1 - t, where t is the thinning factor.

14.24. A. f(x+1)/ f(x) = 2.4{x!/ (x+1)!} = 2.4/(x+1). Thus this is a member of the (a, b, 0) subclass, f(x+1)/ f(x) = a + b/(x+1), with a = 0 and b = 2.4. This is a Poisson Distribution, with λ = 2.4.

For the unmodified Poisson, the probability of more than zero claims is: $1 - e^{-2.4}$.
After, zero-modification, this probability is: 1- 0.31 = 0.69. Thus the zero-modified distribution is,
$f_M(x) = (0.69/(1 - e^{-2.4}))f(x) = \{0.69/(1 - e^{-2.4})\}\, e^{-2.4}\, 2.4^x /x! = 2.4^x(0.69)/((e^{2.4} - 1)\, x!)$, x≥1.

$f_M(3) = 2.4^3\{0.69/((e^{2.4} - 1)\, 3!\} =$ **0.159**.

# claims	0	1	2	3	4	5	6	7
zero-modified density	0.31	0.1652	0.1983	0.1586	0.0952	0.0457	0.0183	0.0063

Comment: For a Poisson with λ = 2.4, $f(n)/f(m) = (e^{-2.4}\, 2.4^n / n!)/(e^{-2.4}\, 2.4^m / m!) =$
$2.4^{n-m}\, m! / n!$.

14.25. If there are n losses, then the probability that zero of them are small is 0.3^n.
Prob[0 small losses] =
Prob[0 losses] + Prob[1 loss] Prob[loss is big] + Prob[2 losses] Prob[both losses are big] + ... =

$$p_0^M + \left\{\frac{1 - p_0^M}{1 - e^{-\lambda}} \lambda e^{-\lambda}\right\} (0.3) + \left\{\frac{1 - p_0^M}{1 - e^{-\lambda}} \lambda^2 e^{-\lambda} / 2!\right\} (0.3^2) + \left\{\frac{1 - p_0^M}{1 - e^{-\lambda}} \lambda^3 e^{-\lambda} / 3!\right\} (0.3^3) + ... =$$

$$p_0^M + \frac{1 - p_0^M}{1 - e^{-\lambda}} e^{-\lambda} \{0.3\lambda + (0.3\lambda)^2 / 2! + (0.3\lambda)^3 / 3! + ...\} =$$

$$p_0^M + \frac{1 - p_0^M}{1 - e^{-\lambda}} e^{-\lambda} \{e^{0.3\lambda} - 1\} = p_0^M + \frac{1 - p_0^M}{1 - e^{-\lambda}} \{e^{-0.7\lambda} - e^{-\lambda}\} =$$

$$\frac{(1 - e^{-\lambda})\, p_0^M + (1 - p_0^M)(e^{-0.7\lambda} - e^{-\lambda})}{1 - e^{-\lambda}} = \frac{p_0^M - e^{-\lambda} + e^{-0.7\lambda} - p_0^M e^{-0.7\lambda}}{1 - e^{-\lambda}}.$$

Comment: Matches the general formula with t = 0.7: $p_0^M \rightarrow \dfrac{p_0^M - e^{-\lambda} + e^{-t\lambda} - p_0^M e^{-t\lambda}}{1 - e^{-\lambda}}$.

The thinned distribution is also a zero-modified Poisson, with $\lambda^* = 0.7\lambda$.
The probability of zero claims for the thinned distribution is the P.G.F. for the original zero-modified distribution at 1 - t, where t is the thinning factor.

14.26. Calculate (x+1)f(x+1)/f(x) = (x+1) (number with x+1) / (number with x).

Number of Days	Observed	(x+1)f(x+1)/f(x)	Differences
0	50,122	0.183	
1	9,190	1.199	1.016
2	5,509	1.774	0.575
3	3,258	2.387	0.613
4	1,944	2.984	0.597
5	1,160	3.584	0.601
6	693	4.222	0.638
7	418		
8+	621		

The accident profile is not approximately linear starting at zero.
Thus, this is probably not from a member of the (a, b, 0) class.
The accident profile is approximately linear starting at one.
Thus, this is probably from a member of the (a, b, 1) class.
Comment: f(x+1)/f(x) = a + b/(x+1), so (x+1)f(x+1)/f(x) = a(x+1) + b = ax + a + b.
The slope is positive, so a > 0 and we have a Negative Binomial.

The slope, $a \cong 0.6$. The intercept is about 0.6. Thus $a + b \cong 0.6$. Therefore, $b \cong 0$.

For the Negative Binomial $b = (r-1)\beta/(1+\beta)$. Thus b = 0, implies $r \cong 1$.

Thus the data may have been drawn from a Zero-Modified Geometric, with $\beta \cong 0.6$.

14.27. E. $p_2^M = f(2)\ \frac{1 - p_0^M}{1 - f(0)}$. $p_3^M = f(3)\ \frac{1 - p_0^M}{1 - f(0)}$.

Thus $p_3^M / p_2^M = f(3) / f(2) = \frac{\lambda^3 e^{-\lambda} / 6}{\lambda^2 e^{-\lambda} / 2} = \lambda/3$. $\Rightarrow \lambda/3 = 12.7\%/27.3\%$. $\Rightarrow \lambda = 1.396$.

$\Rightarrow 27.3\% = \{(1.396^2)\ e^{-1.396} / 2\}\ \frac{1 - p_0^M}{1 - e^{-1.396}}$. $\Rightarrow p_0^M$ = **14.86%**.

Comment: $p_1^M = 39.11\%$.

14.28. B. Since we have a member of the (a, b, 1) family:

$p_2/p_1 = a + b/2$. ⇒ 2a + b = (2)(0.1754)/0.1637 = 2.1429.

$p_3/p_2 = a + b/3$. ⇒ 3a + b = (3)(0.1503)/0.1754 = 2.5707.

⇒ a = 0.4278. ⇒ b = 1.2873.

$p_4 = (a + b/4)\ p_3$ = (0.4278 + 1.2873/4) (0.1503) = 0.1127.

$p_5 = (a + b/45)\ p_4$ = (0.4278 + 1.2873/5) (0.1127) = **0.0772**.

Comment: Based on a zero-modified Negative Binomial, with r = 4, β = 0.75, and p_0^M = 20%.

14.29. B. E[N ∧ 3] = 0 f(0) + 1 f(1) + 2 f(2) + 3 {1 - f(0) - f(1) - f(2)} =
0.2348 + (2)(0.2348) + (3)(1 - 0.7196) = **1.55**.

Number	Unmodified	Zero-Modified	Cumulative
of Claims	Poisson	Poisson	Zero-Modified
0	13.53%	25.00%	Poisson
1	27.07%	23.48%	48.48%
2	27.07%	23.48%	71.96%
3	18.04%	15.65%	87.61%

14.30. From Appendix B of Loss Models, for the zero-truncated Poisson:

$P^T(z) = (e^{0.1z} - 1) / (e^{0.1} - 1)$.

Therefore, for the zero-modified Poisson:

$P^M(z)$ = $\mathbf{0.6 + 0.4\ (e^{0.1z} - 1) / (e^{0.1} - 1)}$.

14.31. (a) $P^T(z) = \frac{z}{1 + \beta - \beta z} = \frac{z}{1.25 - 0.25z} = \frac{4z}{5 - z}$.

$P^M(z) = 0.4 + 0.6 \frac{4z}{5 - z} = \frac{2 + 2z}{5 - z}$.

(b) The probability generating function of Y is the square of that of the zero-modified Geometric Distribution: $\frac{4 + 8z + 4z^2}{(5 - z)^2}$.

$f(n) = P^n(0) / n!$. Using a computer, the densities at 0 to 6 are:
0.16, 0.384, 0.3072, 0.10752, 0.03072, 0.0079872, 0.00196608.
Note that for Y to be zero, each of the zero-modified Geometric Distribution have to be zero: $0.4^2 = 0.16$.
For Y to be one, one of the zero-modified Geometric Distribution has to be zero, while the other one is 1: $(2)(0.4) \{(0.25/1.25^2)(0.6)/(1 - 1/1.25)\} = (2)(0.4)(0.48) = 0.384$.

(c) The densities from 0 to 6 for a Negative Binomial Distribution with r = 2, $\beta = 0.25$:
0.64, 0.256, 0.0768, 0.02048, 0.00512, 0.0012288, 0.00028672.
To get the densities of the zero-modified distribution for x > 0, we multiply by : 0.84/(1 - 0.64).
Thus the densities from 0 to 6 for the zero-modified Negative Binomial Distribution are:
0.16, 0.597333, 0.1792, 0.0477867, 0.0119467, 0.0028672, 0.000669013.
Comment: The sum of two zero-modified Geometric Distributions is not a zero-modified Negative Binomial Distribution.

14.32. C. $f(x+1)/ f(x) = x!/ (x+1)! = 1/(x+1)$. Thus this is a member of the (a, b, 0) subclass, $f(x+1) / f(x) = a + b/(x+1)$, with a = 0 and b = 1. This is a Poisson Distribution, with $\lambda = 1$.

For the unmodified Poisson, the probability of more than zero claims is: $1 - e^{-1}$.
After, zero-modification, this probability is: 1 - 0.1 = .9. Thus the zero-modified distribution is,
$f_M(x) = \{0.9/(1-e^{-1})\} f(x) = \{0.9/(1-e^{-1})\} e^{-1} 1^x /x! = 0.9/((e - 1) x!)$, $x \geq 1$.

$f_M(1) = 0.9/(e-1) =$ **0.524**.

# claims	0	1	2	3	4	5	6
zero-modified density	0.1	0.5238	0.2619	0.0873	0.0218	0.0044	0.0007

Comment: For a Poisson with $\lambda = 1$, $f(n)/f(m) = (e^{-1} 1^n / n!)/(e^{-1} 1^m / m!) = m! / n!$.

Section 15, Compound Frequency Distributions[108]

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.

For example, assume the number of taxicabs that arrive per minute at the Heartbreak Hotel is Poisson with mean 1.3. In addition, assume that the number of passengers dropped off at the hotel by each taxicab is Binomial with q = 0.4 and m = 5. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab.

Then the aggregate number of passengers dropped off per minute at the Heartbreak Hotel is an example of a compound frequency distribution. It is a compound Poisson-Binomial distribution, with parameters $\lambda = 1.3$, q = 0.4, m = 5.[109]

The distribution function of the primary Poisson is as follows:

1.3 Number of Claims	Probability Density Function	Cumulative Distribution Function
0	27.253%	0.27253
1	35.429%	0.62682
2	23.029%	0.85711
3	9.979%	0.95690
4	3.243%	0.98934
5	0.843%	0.99777
6	0.183%	0.99960

So for example, there is a 3.243% chance that 4 taxicabs arrive; in which case the number passengers dropped off is the sum of 4 independent identically distributed Binomials[110], given by the secondary Binomial Distribution. There is a 27.253% chance there are no taxicabs, a 35.429% chance we take one Binomial, 23.029% chance we sum the result of 2 independent identically distributed Binomials, etc.

[108] See Section 7.1 of Loss Models, not on the syllabus. However, compound distributions are mathematically the same as aggregate distributions. See "Mahler's Guide to Aggregate Distributions." Some of you may better understand the idea of compound distributions by seeing how they are simulated in "Mahler's Guide to Simulation."
[109] In the name of a compound distribution, the primary distribution is listed first and the secondary distribution is listed second.
[110] While we happen to know that the sum of 4 independent Binomials each with q = 0.4, m = 5 is another Binomial with parameters q = 0.4, m = 20, that fact is not essential to the general concept of a compound distribution.

The secondary Binomial Distribution with q = 0.4, m = 5 is as follows:

Number of Claims	Probability Density Function	Cumulative Distribution Function
0	7.776%	0.07776
1	25.920%	0.33696
2	34.560%	0.68256
3	23.040%	0.91296
4	7.680%	0.98976
5	1.024%	1.00000

Thus assuming a taxicab arrives, there is a 34.560% chance that 2 passengers are dropped off.

In this example, the primary distribution determines how many taxicabs arrive, while the secondary distribution determines the number of passengers departing per taxicab. Instead, the primary distribution could be the number of envelopes arriving and the secondary distribution could be the number of claims in each envelope.[111]

Actuaries often use compound distributions when the primary distribution determines how many accidents there are, while for each accident the number of persons injured or number of claimants is determined by the secondary distribution.[112] This particular model, while useful for comprehension, may or may not apply to any particular use of the mathematical concept of compound frequency distributions.

There are number of methods of computing the density of compound distributions, among them the use of convolutions and the use of the Recursive Method (Panjer Algorithm.)[113]

Probability Generating Function of Compound Distributions:

One can get the Probability Generating Function of a compound distribution in terms of those of its primary and secondary distributions:
p.g.f. of compound distribution = p.g.f. of primary distribution[p.g.f. of secondary distribution]

$$\mathbf{P(z) = P_1[P_2(z)]}.$$

[111] See 3, 11/01, Q.30.
[112] See 3, 5/01, Q.36.
[113] Both discussed in “Mahler’s Guide to Aggregate Distributions,” where they are applied to both compound and aggregate distributions.

Exercise: What is the Probability Generating Function of a Compound Geometric-Binomial Distribution, with $\beta = 3$, $q = 0.1$, and $m = 2$.

[Solution: The p.g.f. of the primary Geometric is: $1 / \{1 - 3(z-1)\} = 1 / (4 - 3z)$, $z < 1 + 1/\beta = 4/3$.

The p.g.f. of the secondary Binomial is: $\{1 + (0.1)(z-1)\}^2 = (0.9 + 0.1z)^2 = 0.01z^2 + 0.18z + 0.81$.

$P(z) = P_1[P_2(z)] = 1 / \{4 - 3(0.01z^2 + 0.18z + 0.81)\} = -1/(0.03z^2 + 0.54z - 1.57)$, $z < 4/3$.]

Recall, that for any frequency distribution, $f(0) = P(0)$. Therefore, for a compound distribution, $c(0) = P_c(0) = P_1[P_2(0)] = P_1[s(0)]$.

compound density at 0 = p.g.f. of the primary at density at 0 of the secondary.[114]

For example, in the previous exercise, the density of the compound distribution at zero is its p.g.f. at $z = 0$: $1/1.57 = 0.637$. The density at 0 of the secondary Binomial Distribution is: $0.9^2 = 0.81$. The p.g.f. of the primary distribution at 0.81 is: $1 / \{4 - (3)(0.81)\} = 1/1.57 = 0.637$.

If one takes the p.g.f. of a compound distribution to a power $\rho > 0$, $P(z)^\rho = P_1^\rho [P_2(z)]$.

Thus if the primary distribution is infinitely divisible, i.e., P_1^ρ has the same form as P_1, then P^ρ has the same form as P. If the primary distribution is infinitely divisible, then so is the compound distribution.

Since the Poisson and the Negative Binomial are each infinitely divisible, so are compound distributions with a primary distribution which is either a Poisson or a Negative Binomial (including a Geometric.)

Adding Compound Distributions:

For example, let us assume that taxi cabs arrive at a hotel (primary distribution) and drop people off (secondary distribution.) Assume two independent Compound Poisson Distributions with the same secondary distribution. The first compound distribution represents those cabs whose drivers were born in January through June and has $\lambda = 11$, while the second compound distribution represents those cabs whose drivers were born in July through December and has $\lambda = 9$.

Then the sum of the two distributions represents the passengers from all of the cabs, and is a Compound Poisson Distribution with $\lambda = 11 + 9 = 20$, and the same secondary distribution as each of the individual Compound Distributions.

Note that the parameter of the primary rather than secondary distribution was affected.

[114] This is the first step of the Panjer Algorithm, discussed in “Mahler’s Guide to Aggregate Distributions.”

Exercise: Let X be a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 4.3$, $q = 0.2$, and $m = 5$. Let Y be a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 2.4$, $q = 0.2$, and $m = 5$. What is the distribution of X + Y?

[Solution: A Poisson-Binomial Distribution with $\lambda = 4.3 + 2.4 = 6.7$, $q = 0.2$, and $m = 5$.]

The sum of two independent identically distributed Compound Poisson variables has the same form. The sum of two independent identically distributed Compound Negative Binomial variables has the same form.

Exercise: Let X be a Negative Binomial-Poisson compound frequency distribution with $\beta = 0.7$, $r = 2.5$, and $\lambda = 3$.

What is the distributional form of the sum of two independent random draws from X?

[Solution: A Negative Binomial-Poisson with $\beta = 0.7$, $r = (2)(2.5) = 5$, and $\lambda = 3$.]

Exercise: Let X be a Poisson-Geometric compound frequency distribution with $\lambda = 0.3$ and $\beta = 1.5$.

What is the distributional form of the sum of twenty independent random draws from X?

[Solution: The sum of 20 independent identically distributed variables is of the same form. However, $\lambda = (20)(0.3) = 6$. We get a Poisson-Geometric compound frequency distribution with $\lambda = 6$ and $\beta = 1.5$.]

If one adds independent identically distributed Compound Binomial variables one gets the same form.

Exercise: Let X be a Binomial-Geometric compound frequency distribution with $q = 0.2$, $m = 3$, and $\beta = 1.5$.

What is the distributional form of the sum of twenty independent random draws from X?

[Solution: The sum of 20 independent identically distributed binomial variables is of the same form, with $m = (20)(3) = 60$. We get a Binomial-Geometric compound frequency distribution with $q = 0.2$, $m = 60$, and $\beta = 1.5$.]

Thinning Compound Distributions:

Thinning compound distributions can be done in two different manners, one manner affects the primary distribution, and the other manner affects the secondary distribution.

For example, assume that taxi cabs arrive at a hotel (primary distribution) and drop people off (secondary distribution.) Then we can either select certain types of cabs or certain types of people. Depending on which we select, we affect the primary or secondary distribution.

Assume we select only those cabs that are less than one year old (and assume age of cab is independent of the number of people dropped off and the frequency of arrival of cabs.)
Then this would affect the primary distribution, the number of cabs.

Exercise: Cabs arrive via a Poisson with mean 1.3. The number of people dropped off by each cab is Binomial with $q = 0.2$ and $m = 5$. The number of people dropped off per cab is independent of the number of cabs that arrive. 30% of cabs are less than a year old.
The age of cabs is independent of the number of people dropped off and the frequency of arrival of cabs.
What is the distribution of the number of people dropped off by cabs less than one year old?
[Solution: Cabs less than a year old arrive via a Poisson with $\lambda = (30\%)(1.3) = 0.39$.
There is no effect on the number of people per cab (secondary distribution.)
We get a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 0.39$, $q = 0.2$, and $m = 5$.]

This first manner of thinning affects the primary distribution. For example, it might occur if the primary distribution represents the number of accidents and the secondary distribution represents the number of claims.

For example, assume that the number of accidents is Negative Binomial with $\beta = 2$ and $r = 30$, and the number of claims per accident is Binomial with $q = 0.3$ and $m = 7$. Then the total number of claims is Compound Negative Binomial-Binomial with parameters $\beta = 2$, $r = 30$, $q = 0.3$ and $m = 7$.

Exercise: Accidents are assigned at random to one of four claims adjusters:
Jerry, George, Elaine, or Cosmo.
What is the distribution of the number claims adjusted by George?
[Solution: We are selecting at random 1/4 of the accidents. We are thinning the Negative Binomial Distribution of the number of accidents. Therefore, the number of accidents assigned to George is Negative Binomial with $\beta = 2/4 = 0.5$ and $r = 30$.
The number claims adjusted by George is Compound Negative Binomial-Binomial with parameters $\beta = 0.5$, $r = 30$, $q = 0.3$ and $m = 7$.]

Returning to the cab example, assume we select only female passengers, (and gender of passenger is independent of the number of people dropped off and the frequency of arrival of cabs.). Then this would affect the secondary distribution, the number of passengers.

Exercise: Cabs arrive via a Poisson with mean 1.3. The number of people dropped off by each cab is Binomial with $q = 0.2$ and $m = 5$. The number of people dropped off per cab is independent of the number of cabs that arrive. 40% of the passengers are female.
The gender of passengers is independent of the number of people dropped off and the frequency of arrival of cabs.
What is the distribution of the number of females dropped off by cabs?
[Solution: The distribution of female passengers per cab is Binomial with $q = (0.4)(0.2) = 0.08$ and $m = 5$. There is no effect on the number of cabs (primary distribution.)
We get a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 1.3$, $q = 0.08$, and $m = 5$.]

This second manner of thinning a compound distribution affects the secondary distribution.
It is mathematically the same as what happens when one takes only the large claims in a frequency and severity situation, when the frequency distribution itself is compound.[115]

For example, if frequency is Poisson-Binomial with $\lambda = 1.3$, $q = 0.2$, and $m = 5$, and 40% of the claims are large. The number of large claims would be simulated by first getting a random draw from the Poisson, then simulating the appropriate number of random Binomials, and then for each claim from the Binomial there is a 40% chance of selecting it at random independent of any other claims. This is mathematically the same as thinning the Binomial. Therefore, large claims have a Poisson-Binomial Distribution compound frequency distribution with $\lambda = 1.3$, $q = (0.4)(0.2) = 0.08$ and $m = 5$.

[115] This is what is considered in Section 8.6 of Loss Models.

Exercise: Let frequency be given by a Geometric-Binomial compound frequency distribution with $\beta = 1.5$, $q = 0.2$, and $m = 3$. Severity follows an Exponential Distribution with mean 1000.
Frequency and severity are independent.
What is the frequency distribution of losses of size between 500 and 2000?
[Solution: The fraction of losses that are of size between 500 and 2000 is:
$F(2000) - F(500) = (1-e^{-2000/1000}) - (1-e^{-500/1000}) = e^{-0.5} - e^{-2} = 0.4712$. Thus the losses of size between 500 and 2000 follow a Geometric-Binomial compound frequency distribution with $\beta = 1.5$, $q = (0.4712)(0.2) = 0.0942$, and $m = 3$.]

Proof of Some Thinning Results:[116]

One can use the result for the probability generating function for a compound distribution,
p.g.f. of compound distribution = p.g.f. of primary distribution[p.g.f. of secondary distribution],
in order to determine the results of thinning a Poisson, Binomial, or Negative Binomial Distribution.

Assume one has a Poisson Distribution with mean λ.
Assume one selects at random 30% of the claims.
This is mathematically the same as a compound distribution with a primary distribution that is Poisson with mean λ and a secondary distribution that is Bernoulli with $q = 0.3$.

The p.g.f. of the Poisson is $P(z) = e^{\lambda(z-1)}$.
The p.g.f. of the Bernoulli is $P(z) = 1 + 0.3(z-1)$.
The p.g.f. of the compound distribution is obtained by replacing z in the p.g.f. of the primary Poisson with the p.g.f. of the secondary Bernoulli:
$P(z) = \exp[\lambda\{1 + 0.3(z-1) - 1\}] = \exp[(0.3\lambda)(z - 1)]$.

This is the p.g.f. of a Poisson Distribution with mean 0.3λ.
Thus the thinned distribution is also Poisson, with mean 0.3λ.

In general, when thinning a Poisson by a factor of t, the thinned distribution is also Poisson with mean $t\lambda$.

[116] See Section 8.6 of Loss Models.

Similarly, assume we are thinning a Binomial Distribution with parameters q and m.

The p.g.f. of the Binomial is $P(z) = \{1 + q(z-1)\}^m$.

This is mathematically the same as a compound distribution with secondary distribution a Bernoulli with mean t.

The p.g.f. of this compound distribution is: $\{1 + q(1 + t(z-1) -1)\}^m = \{1 + tq(z-1))\}^m$.

This is the p.g.f. of a Binomial Distribution with parameters tq and m.

In general, when thinning a Binomial by a factor of t, the thinned distribution is also Binomial with parameters tq and m.

Assume we are thinning a Negative Binomial Distribution with parameters β and r.

The p.g.f. of the Negative Binomial is $P(z) = \{1 - \beta(z-1)\}^{-r}$.

This is mathematically the same as a compound distribution with secondary distribution a Bernoulli with mean t.

The p.g.f. of this compound distribution is: $\{1 - \beta(1 + t(z-1) -1)\}^{-r} = \{1 - t\beta((z-1)\}^{-r}$.

This is the p.g.f. of a Negative Binomial Distribution with parameters r and tβ.

In general, when thinning a Negative Binomial by a factor of t, the thinned distribution is also Negative Binomial with parameters tβ and r.[117]

Since thinning is mathematically the same as a compound distribution with secondary distribution a Bernoulli with mean t, and the p.g.f. of the Bernoulli is 1 - t + tz,
the p.g.f. of the thinned distribution is P(1 - t + tz),
where P(z) is the p.g.f. of the original distribution. In general, P(0) = f(0).
Thus the density at zero for the thinned distribution is: P(1 - t + t0) = P(1 - t).
The density of the thinned distribution at zero is the p.g.f. of the original distribution at 1 - t.[118]

Let us assume instead we start with a zero-modified distribution.
Let P(z) be the p.g.f. of the original distribution prior to being zero-modified.

Then $P_{ZM}(z) = p_0^M + (1 - p_0^M)\, P_{ZT}(z) = p_0^M + (1 - p_0^M)\, \dfrac{P(z) - f(0)}{1 - f(0)}$.

Now the density at zero for the thinned version of the original distribution is: P(1 - t).
The density at zero for the thinned version of the original distribution is:

$$p_0^{M*} = P_{ZM}(1 - t) = p_0^M + (1 - p_0^M)\, \frac{P(1-t) - f(0)}{1 - f(0)}. \Rightarrow 1 - p_0^{M*} = (1 - p_0^M)\, \frac{1 - P(1-t)}{1 - f(0)}.$$

[117] Including the special case the Geometric Distribution.
[118] This general result was discussed previously with respect to thinning zero-modified distributions.

The p.g.f. of the thinned zero-modified distribution is:

$$P_{ZM}(1 - t + tz) = p_0^M + (1 - p_0^M)\ \frac{P(1 - t + tz) - f(0)}{1 - f(0)} =$$

$$p_0^M{}^* - (1 - p_0^M)\ \frac{P(1-t) - f(0)}{1 - f(0)} + (1 - p_0^M)\ \frac{P(1 - t + tz) - f(0)}{1 - f(0)} =$$

$$p_0^M{}^* + (1 - p_0^M)\ \frac{P(1 - t + tz) - P(1 - t)}{1 - f(0)} = p_0^M{}^* + (1 - p_0^M{}^*)\ \frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)}.$$

Now, $\frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)} =$

$$\frac{\text{(p.g.f. of thinned non-modified dist.)} - \text{(density at zero of thinned non-modified dist.)}}{1 - \text{(density at zero of thinned non-modified distribution)}}.$$

Therefore, the form of the p.g.f. of the thinned zero-modified distribution:

$$p_0^M{}^* + (1 - p_0^M{}^*)\ \frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)},$$

is the usual form of the p.g.f. of a zero-modified distribution,
with the thinned version of the original distribution taking the place of the original distribution.

Therefore, provided thinning preserves the family of the original distribution, the thinned zero-truncated distribution is of the same family with $p_0^M{}^*$, and with the other parameters as per thinning of the non-modified distribution. Specifically as discussed before:

Distribution	Result of thinning by a factor of t
Zero-Modified Binomial	$q \to tq$ m remains the same $p_0^M \to \frac{p_0^M - (1-q)^m + (1-tq)^m - p_0^M (1-tq)^m}{1 - (1-q)^m}$
Zero-Modified Poisson	$\lambda \to t\lambda$ $p_0^M \to \frac{p_0^M - e^{-\lambda} + e^{-t\lambda} - p_0^M e^{-t\lambda}}{1 - e^{-\lambda}}$
Zero-Modified Negative Binomial	$\beta \to t\beta$ r remains the same $p_0^M \to \frac{p_0^M - (1+\beta)^{-r} + (1+t\beta)^{-r} - p_0^M (1+t\beta)^{-r}}{1 - (1+\beta)^{-r}}$

As discussed previously, things work similarly for a zero-modified Logarithmic.
Let P(z) be the p.g.f. of the original Logarithmic distribution prior to being zero-modified.
Then $P_{ZM}(z) = p_0^M + (1 - p_0^M)\, P(z)$.

Now the density at zero for the thinned version of the original distribution is: P(1 - t).
The density at zero for the thinned version of the zero-modified distribution is:
$p_0^{M*} = P_{ZM}(1 - t) = p_0^M + (1 - p_0^M)\, P(1 - t)$.
$\Rightarrow 1 - p_0^{M*} = (1 - p_0^M)\, \{1 - P(1 - t)\}$.

As before, since the p.g.f. of the secondary Bernoulli is 1 - t + tz,
the p.g.f. of the thinned zero-modified distribution is:
$P_{ZM}(1 - t + tz) = p_0^M + (1 - p_0^M)\, P(1 - t + tz) = p_0^{M*} - (1 - p_0^M)\, P(1 - t) + (1 - p_0^M)\, P(1 - t + tz) =$
$$p_0^{M*} + (1 - p_0^{M*})\, \frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)}.$$

Now, $\frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)} =$
$$\frac{\text{(p.g.f. of thinned non-modified dist.) - (density at zero of thinned non-modified dist.)}}{\text{1 - (density at zero of thinned non-modified distribution)}}.$$

Therefore, the form of the p.g.f. of the thinned zero-modified distribution:
$$p_0^{M*} + (1 - p_0^{M*})\, \frac{P(1 - t + tz) - P(1 - t)}{1 - P(1 - t)},$$
is the usual form of the p.g.f. of a zero-modified distribution,
with the thinned version of the original distribution taking the place of the original distribution.

Therefore, since thinning results in another Logarithmic, the thinned
zero-truncated distribution is of the same family with p_0^{M*}, and the other parameter as per thinning of the non-modified distribution. As discussed before:

Zero-Modified Logarithmic	$\beta \to t\beta$
	$p_0^M \to 1 - (1 - p_0^M)\, \frac{\ln[1+t\beta]}{\ln[1+\beta]}$

Problems:

15.1 (2 points) The number of accidents is Geometric with $\beta = 1.7$.

The number of claims per accident is Poisson with $\lambda = 3.1$.

For the total number of claims, what is the Probability Generating Function, P(z)?

A. $\dfrac{\exp[3.1(z - 1)]}{2.7 - 1.7z}$

B. $\dfrac{1}{2.7 - 1.7\exp[3.1(z - 1)]}$

C. $\exp[3.1(z - 1)] + (2.7 - 1.7z)$

D. $\exp[\dfrac{3.1(z - 1.7)}{2.7 - 1.7z}]$

E. None of the above

15.2 (1 point) Frequency is given by a Poisson-Binomial compound frequency distribution, with $\lambda = 0.18$, q = 0.3, and m = 3.

One third of all losses are greater than $10,000. Frequency and severity are independent.
What is frequency distribution of losses of size greater than $10,000?

A. Compound Poisson-Binomial with $\lambda = 0.18$, q = 0.3, and m = 3.

B. Compound Poisson-Binomial with $\lambda = 0.18$, q = 0.1, and m = 3.

C. Compound Poisson-Binomial with $\lambda = 0.18$, q = 0.3, and m = 1.

D. Compound Poisson-Binomial with $\lambda = 0.06$, q = 0.3, and m = 3.

E. None of the above.

15.3 (1 point) X is given by a Binomial-Geometric compound frequency distribution, with q = 0.15, m = 3, and $\beta = 2.3$. Y is given by a Binomial-Geometric compound frequency distribution, with q = 0.15, m = 5, and $\beta = 2.3$. X and Y are independent.

What is the distributional form of X + Y?

A. Compound Binomial-Geometric with q = 0.15, m = 4, and $\beta = 2.3$

B. Compound Binomial-Geometric with q = 0.15, m = 8, and $\beta = 2.3$

C. Compound Binomial-Geometric with q = 0.15, m = 4, and $\beta = 4.6$

D. Compound Binomial-Geometric with q = 0.15, m = 8, and $\beta = 4.6$

E. None of the above.

15.4 (2 points) A compound claims frequency model has the following properties:
(i) The primary distribution has probability generating function:

$$P(z) = 0.2z + 0.5z^2 + 0.3z^3.$$

(ii) The secondary distribution has probability generating function:

$$P(z) = \exp[0.7(z - 1)].$$

Calculate the probability of no claims from this compound distribution.
(A) 18% (B) 20% (C) 22% (D) 24% (E) 26%

15.5 (1 point) Assume each exposure has a Poisson-Poisson compound frequency distribution, as per Loss Models, with $\lambda_1 = 0.03$ and $\lambda_2 = 0.07$. You insure 20,000 independent exposures. What is the frequency distribution for your portfolio?
A. Compound Poisson-Poisson with $\lambda_1 = 0.03$ and $\lambda_2 = 0.07$
B. Compound Poisson-Poisson with $\lambda_1 = 0.03$ and $\lambda_2 = 1400$
C. Compound Poisson-Poisson with $\lambda_1 = 600$ and $\lambda_2 = 0.07$
D. Compound Poisson-Poisson with $\lambda_1 = 600$ and $\lambda_2 = 1400$
E. None of the above.

15.6 (2 points) Frequency is given by a Poisson-Binomial compound frequency distribution, with parameters $\lambda = 1.2$, $q = 0.1$, and $m = 4$.
What is the Probability Generating Function?
A. $\{1+ 0.1(z - 1)\}^4$
B. $\exp(1.2(z - 1))$
C. $\exp[1.2(\{1 + 0.1(z - 1)\}^4 -1)]$
D. $\{1 + 0.1(\exp[1.2(z - 1)] - 1)\}^4$
E. None of the above

15.7 (1 point) The total number of claims from a book of business with 100 exposures has a Compound Poisson-Geometric Distribution with $\lambda = 4$ and $\beta = 0.8$.
Next year this book of business will have 75 exposures.
Next year, what is the distribution of the total number of claims from this book of business?
A. Compound Poisson-Geometric with $\lambda = 4$ and $\beta = 0.8$.
B. Compound Poisson-Geometric with $\lambda = 3$ and $\beta = 0.8$.
C. Compound Poisson-Geometric with $\lambda = 4$ and $\beta = 0.6$.
D. Compound Poisson-Geometric with $\lambda = 3$ and $\beta = 0.6$.
E. None of the above.

15.8 (2 points) A compound claims frequency model has the following properties:
(i) The primary distribution has probability generating function:

$$P(z) = 1 / (5 - 4z).$$

(ii) The secondary distribution has probability generating function:

$$P(z) = (0.8 + 0.2z)^3.$$

Calculate the probability of no claims from this compound distribution.

(A) 28% (B) 30% (C) 32% (D) 34% (E) 36%

15.9 (1 point) The total number of claims from a group of 50 drivers has a Compound Negative Binomial-Poisson Distribution with $\beta = 0.4$, r = 3, and $\lambda = 0.7$.
What is the distribution of the total number of claims from 500 similar drivers?

A. Compound Negative Binomial-Poisson with $\beta = 0.4$, r = 30, and $\lambda = 0.7$.

B. Compound Negative Binomial-Poisson with $\beta = 4$, r = 3, and $\lambda = 0.7$.

C. Compound Negative Binomial-Poisson with $\beta = 0.4$, r = 3, and $\lambda = 7$.

D. Compound Negative Binomial-Poisson with $\beta = 4$, r = 30, and $\lambda = 7$.

E. None of the above.

15.10 (SOA M, 11/05, Q.27 & 2009 Sample Q.208) (2.5 points)
An actuary has created a compound claims frequency model with the following properties:
(i) The primary distribution is the negative binomial with probability generating function

$$P(z) = [1 - 3(z - 1)]^{-2}.$$

(ii) The secondary distribution is the Poisson with probability generating function

$$P(z) = \exp[\lambda(z - 1)].$$

(iii) The probability of no claims equals 0.067.

Calculate λ.

(A) 0.1 (B) 0.4 (C) 1.6 (D) 2.7 (E) 3.1

Solutions to Problems:

15.1. B. $P(z) = P_1[P_2(z)]$.

The p.g.f of the primary Geometric is: $1/\{1 - \beta(z-1)\} = 1/\{1 - 1.7(z-1)\} = 1/(2.7 - 1.7z)$.

The p.g.f of the secondary Poisson is: $\exp[\lambda(z-1)] = \exp[3.1(z-1)]$.

Thus the p.g.f. of the compound distribution is: $\mathbf{1 / \{2.7 - 1.7\exp[3.1(z-1)]\}}$.

Comment: P(z) only exist for $z < 1 + 1/\beta = 1 + 1/1.7$.

15.2. B. We are taking 1/3 of the claims from the secondary Binomial. Thus the secondary distribution is Binomial with q = 0.3/3 = 0.1 and m = 3. Thus the frequency distribution of losses of size greater than $10,000 is given by a Poisson-Binomial compound frequency distribution, as per Loss Models with $\lambda = 0.18$, q = 0.1, and m = 3.

15.3. B. Provided the secondary distributions are the same, the primary distributions add as they usually would. The sum of two independent Binomials with the same q, is another Binomial with the sum of the m parameters. In this case it is a Binomial with q = 0.15 and

m = 3 + 5 = 8. X + Y is a Binomial-Geometric with q = 0.15, m = 8, and $\beta = 2.3$.

Comment: The secondary distributions determine how many claims there are per accident. The primary distributions determine how many accidents. In this case the Binomial distributions of the number of accidents add.

15.4. E. $P(z) = P_1[P_2(z)]$.

Density at 0 is: $P(0) = P_1[P_2(0)] = P_1[e^{-0.7}] = 0.2e^{-0.7} + 0.5e^{-1.4} + 0.3e^{-2.1} = \mathbf{0.259}$.

Alternately, the primary distribution has 20% probability of 1, 50% probability of 2, and 30% probability of 3, while the secondary distribution is a Poisson with $\lambda = 0.7$.

The density at zero of the secondary distribution is $e^{-.7}$.

Therefore, the probability of zero claims for the compound distribution is:

(0.2)(Prob 0 from secondary) + (0.5)(Prob 0 from secondary)^2 + (0.3)(Prob 0 from secondary)^3

$= 0.2e^{-0.7} + 0.5(e^{-0.7})^2 + 0.3(e^{-0.7})^3 = \mathbf{0.259}$.

15.5. C. One adds up 20,000 independent identically distributed variables. In the case of a Compound Poisson distribution, the primary Poissons add to give another Poisson with $\lambda_1 = (20000)(0.03) = 600$. The secondary distribution stays the same.

The portfolio has a compound Poisson-Poisson with $\lambda_1 = 600$ and $\lambda_2 = 0.07$.

15.6. C. The p.g.f of the primary Poisson is $\exp(\lambda(z-1)) = \exp(1.2(z-1))$.

The p.g.f of the secondary Binomial is $\{1+ q(z-1)\}^m = \{1+ .1(z-1)\}^4$.

Thus the p.g.f. of the compound distribution is $P(z) = P_1[P_2(z)]$ = $\mathbf{exp[1.2(\{1+ .1(z-1)\}^4 -1)]}$.

15.7. B. Poisson-Geometric with $\lambda = (75/100)(4) = 3$ and $\beta = 0.8$.
Comment: One adjusts the primary Poisson distribution in a manner similar to that if one just had a Poisson distribution.

15.8. D. $P(z) = P_1[P_2(z)]$.

Density at 0 is: $P(0) = P_1[P_2(0)] = P_1[.8^3] = 1/\{5 - 4(.8^3)\}$ = **0.339**.

Alternately, the secondary distribution is a Binomial with m = 3 and q = 0.2.
The density at zero of the secondary distribution is $.8^3$.
Therefore, the probability of zero claims for the compound distribution is:
$P_1[.8^3] = 1/\{5 - 4(.8^3)\}$ = **0.339**.

15.9. A. Negative Binomial-Poisson with $\beta = 0.4$, r = (500/50)(3) = 30, and $\lambda = 0.7$.
Comment: One adjusts the primary Negative Binomial distribution in a manner similar to that if one just had a Negative Binomial distribution.

15.10. E. The p.g.f. of the compound distribution is the p.g.f. of the primary distribution at the p.g.f. of the secondary distribution: $P(z) = [1 - 3(\exp[\lambda(z - 1)] - 1)]^{-2}$.

$0.067 = f(0) = P(0) = [1 - 3(\exp[\lambda(0 - 1)] - 1)]^{-2} = [1 - 3(\exp[-\lambda] - 1)]^{-2}$.

$\Rightarrow 1 - 3(\exp[-\lambda] - 1) = 3.8633. \Rightarrow \exp[-\lambda] = .04555. \Rightarrow \lambda =$ **3.089**.

Alternately, the Poisson secondary distribution at zero is $e^{-\lambda}$.

From the first step of the Panjer Algorithm, $c(0) = P_p[s(0)] = [1 - 3(e^{-\lambda} - 1)]^{-2}$. Proceed as before.

Comment: $P(z) = E[z^n] = \Sigma f(n)z^n$. Therefore, letting z approach zero, P(0) = f(0).

The probability generating function of the Negative Binomial only exists for $z < 1 + 1/\beta = 4/3$.

Section 16, Moments of Compound Frequency Distributions[119]

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.

One may find it helpful to think of the secondary distribution as taking the role of a severity distribution in the calculation of aggregate losses.[120] Since the situations are mathematically equivalent, many of the techniques and formulas that apply to aggregate losses apply to compound frequency distributions.

For example, the same formulas for the mean, variance and skewness apply.[121]

Mean of Compound Dist. =
(Mean of Primary Dist.) (Mean of Secondary Dist.)

Variance of Compound Dist. =
(Mean of Primary Dist.) (Variance of Secondary Dist.) +
(Mean of Secondary Dist.)^2 (Variance of Primary Dist.)

Skewness Compound Dist. =
{(Mean of Primary Dist.)(Variance of Second. Dist.)$^{3/2}$(Skewness of Secondary Dist.) +
3(Variance of Primary Dist.)(Mean of Secondary Dist.)(Variance of Second. Dist.) +
(Variance of Primary Dist.)$^{3/2}$(Skewness of Primary Dist.)(Mean of Second. Dist.)^3 } /
(Variance of Compound Dist.)$^{3/2}$

For example, assume the number of taxicabs that arrive per minute at the Heartbreak Hotel is Poisson with mean 1.3. In addition, assume that the number of passengers dropped off at the hotel by each taxicab is Binomial with q = 0.4 and m = 5. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab.

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[119] See Section 7.1 of Loss Models, not on the syllabus. However, since compound distributions are mathematically the same as aggregate distributions, I believe that a majority of the questions in this section would be legitimate questions for your exam. Compound frequency distributions used to be on the syllabus.

[120] In the case of aggregate losses, the frequency distribution determines how many independent identically distributed severity variables we will sum.

[121] The secondary distribution takes the place of the severity, while the primary distribution takes the place of the frequency, in the formulas involving aggregate losses. $\sigma_{agg}^2 = \mu_F \sigma_S^2 + \mu_S^2 \sigma_F^2$.
See “Mahler’s Guide to Aggregate Distributions.”

Then the total number of passengers dropped off in a minute is a compound distribution compound Poisson-Binomial distribution, with parameters λ= 1.3, m = 5, q = 0.4.

Exercise: What are the mean and variance of this compound distribution?
[Solution: The mean and variance of the primary Poisson Distribution are both 1.3.
The mean and variance of the secondary Binomial Distribution are
(0.4)(5) = 2 and (0.4)(0.6)(5) = 1.2.
Thus the mean of the compound distribution is: (1.3)(2) = 2.6.
The variance of the compound distribution is: $(1.3)(1.2) + (2)^2(1.3) = 6.76$.
Comment: Mathematically as if the number of claims is Poisson and the size of each claim is Binomial.]

Thus in the case of the Heartbreak Hotel example, on average 2.6 passengers are dropped off per minute. The variance of the number of passengers dropped off per minute is 6.76.

Exercise: What is the probability of more than 4 passengers being dropped off during the next minute? Use the Normal Approximation with continuity correction.
[Solution: $1 - \Phi[(4.5 - 2.6) / \sqrt{6.76}] = 1 - \Phi[0.73] = 23.27\%$.]

Exercise: Assuming the minutes are independent, what is the probability of more than 40 passengers being dropped off during the next ten minutes?
Use the Normal Approximation with continuity correction.
[Solution: Over ten minutes the mean is (10)(2.6) = 26, and the variance is (10)(6.76) = 67.6.
$1 - \Phi[(40.5 - 26) / \sqrt{67.6}] = 1 - \Phi[1.76] = 3.92\%$.]

Poisson Primary Distribution:

In the case of a Poisson primary distribution with mean λ, the variance of the compound distribution could be rewritten as:

λ(Variance of Secondary Dist.) + (Mean of Secondary Dist.)^2 λ=

λ(Variance of Secondary Dist. + Mean of Secondary Dist.^{2}) =

λ(2nd moment of Secondary Distribution).

It also turns out that the third central moment of a compound Poisson distribution =
λ(3rd moment of Secondary Distribution).

For a Compound Poisson Distribution:

Mean = λ(mean of Secondary Distribution).

Variance = λ(2nd moment of Secondary Distribution).

3rd central moment = λ(3rd moment of Secondary Distribution).

Skewness = $\lambda^{-.5}$(3rd moment of Second. Dist.)/(2nd moment of Second. Dist.)$^{1.5}$.[122]

Exercise: The number of accidents follows a Poisson Distribution with λ = 0.04.
Each accident generates 1, 2 or 3 claimants with probabilities 60%, 30%, and 10%.
Determine the mean, variance, and skewness of the total number of claimants.
[Solution: The secondary distribution has mean 1.5, second moment 2.7, and third moment 5.7.
Thus the mean number of claimants is: (0.04)(1.5) = 0.06.
The variance of the number of claimants is: (0.04)(2.7) = 0.108.
The skewness of the number of claimants is: $(0.04^{-.5})(5.7)/(2.7)^{1.5}$ = 6.42.]

[122] Skewness = (third central moment)/ Variance$^{1.5}$.

Problems:

16.1 (1 point) For a compound distribution:
Mean of primary distribution = 15.
Standard Deviation of primary distribution = 3.
Mean of secondary distribution = 10.
Standard Deviation of secondary distribution = 4.
What is the standard deviation of the compound distribution?
A. 26 B. 28 C. 30 D. 32 E. 34

16.2 (2 points) The number of accidents follows a Poisson distribution with mean 10 per month.
Each accident generates 1, 2, or 3 claimants with probabilities 40%, 40%, 20%, respectively.
Calculate the variance in the total number of claimants in a year.
A. 250 B. 300 C. 350 D. 400 E. 450

Use the following information for the next 3 questions:
The number of customers per minute is Geometric with $\beta = 1.7$.
The number of items sold to each customer is Poisson with $\lambda = 3.1$.
The number of items sold per customer is independent of the number of customers.

16.3 (1 point) What is the mean of the total number of items sold per minute?
A. less than 5.0
B. at least 5.0 but less than 5.5
C. at least 5.5 but less than 6.0
D. at least 6.0 but less than 6.5
E. at least 6.5

16.4 (1 point) What is the variance of the total number of items sold per minute?
A. less than 50
B. at least 50 but less than 51
C. at least 51 but less than 52
D. at least 52 but less than 53
E. at least 53

16.5 (2 points) What is the chance that more than 4 items are sold during the next minute?
Use the Normal Approximation.
A. 46% B. 48% C. 50% D. 52% E. 54%

16.6 (3 points) A dam is proposed for a river which is currently used for salmon breeding.
You have modeled:
(i) For each hour the dam is opened the number of female salmon that will pass through and reach the breeding grounds has a distribution with mean 50 and variance 100.
(ii) The number of eggs released by each female salmon has a distribution with mean of 3000 and variance of 1 million.
(iii) The number of female salmon going through the dam each hour it is open and the numbers of eggs released by the female salmon are independent.
Using the normal approximation for the aggregate number of eggs released, determine the least number of whole hours the dam should be left open so the probability that 2 million eggs will be released is greater than 99.5%.
(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

16.7 (3 points) The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 7$ envelopes per day. For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

Number of Claims	Probability
1	0.60
2	0.30
3	0.10

Using the normal approximation, calculate the 99^{th} percentile of the number of claims received in 5 days.
(A) 73 (B) 75 (C) 77 (D) 79 (E) 81

16.8 (3 points) The number of persons using an ATM per hour has a Negative Binomial Distribution with $\beta = 2$ and $r = 13$. Each hour is independent of the others.
The number of transactions per person has the following distribution:

Number of Transactions	Probability
1	0.30
2	0.40
3	0.20
4	0.10

Using the normal approximation, calculate the 80^{th} percentile of the number of transactions in 5 hours.
A. 300 B. 305 C. 310 D. 315 E. 320

Use the following information for the next 3 questions:

- The number of automobile accidents follows a Negative Binomial distribution with $\beta = 0.6$ and $r = 100$.
- For each automobile accident the number of claimants with bodily injury follows a Binomial Distribution with $q = 0.1$ and $m = 4$.
- The number of claimants with bodily injury is independent between accidents.

16.9 (2 points) Calculate the variance in the total number of claimants.
(A) 33 (B) 34 (C) 35 (D) 36 (E) 37

16.10 (1 point) What is probability that there are 20 or fewer claimants in total?
(A) 22% (B) 24% (C) 26% (D) 28% (E) 30%

16.11 (3 points) The amount of the payment to each claimant follows a Gamma Distribution with $\alpha = 3$ and $\theta = 4000$. The amount of payments to different claimants are independent of each other and are independent of the number of claimants.
What is the probability that the aggregate payment exceeds 300,000?
(A) 44% (B) 46% (C) 48% (D) 50% (E) 52%

16.12 (3 points) The number of batters per half-inning of a baseball game is:
3 + a Negative Binomial Distribution with $\beta = 1$ and $r = 1.4$.
The number of pitches thrown per batter is:
1 + a Negative Binomial Distribution with $\beta = 1.5$ and $r = 1.8$.
What is the probability of more than 30 pitches in a half-inning?
Use the normal approximation with continuity correction.
A. 1/2% B. 1% C. 2% D. 3% E. 4%

16.13 (3 points) The number of taxicabs that arrive per minute at the Gotham City Railroad Station is Poisson with mean 5.6. The number of passengers dropped off at the station by each taxicab is Binomial with $q = 0.3$ and $m = 4$. The number of passengers dropped off by each taxicab is independent of the number of taxicabs that arrive and is independent of the number of passengers dropped off by any other taxicab. Using the normal approximation for the aggregate passengers dropped off, determine the least number of whole minutes one must observe in order that the probability that at least 1000 passengers will be dropped off is greater than 90%.
A. 155 B. 156 C. 157 D. 158 E. 159

16.14 (4 points) At a storefront legal clinic, the number of lawyers who volunteer to provide legal aid to the poor on any day is uniformly distributed on the integers 1 through 4. The number of hours each lawyer volunteers on a given day is Binomial with q = 0.6 and m = 7. The number of clients that can be served by a given lawyer per hour is a Poisson distribution with mean 5.
Determine the probability that 40 or more clients can be served in a day at this storefront law clinic, using the normal approximation.
(A) 69% (B) 71% (C) 73% (D) 75% (E) 77%

Use the following information for the next 3 questions:
The number of persons entering a library per minute is Poisson with λ = 1.2.
The number of books returned per person is Binomial with q = 0.1 and m = 4.
The number of books returned per person is independent of the number of persons.

16.15 (1 point) What is the mean number of books returned per minute?
A. less than 0.5
B. at least 0.6 but less than 0.7
C. at least 0.7 but less than 0.8
D. at least 0.8 but less than 0.9
E. at least 0.9

16.16 (1 point) What is the variance of the number of books returned per minute?
A. less than 0.6
B. at least 0.6 but less than 0.7
C. at least 0.7 but less than 0.8
D. at least 0.8 but less than 0.9
E. at least 0.9

16.17 (1 point) What is the probability of observing more than two books returned in the next minute?
Use the Normal Approximation.
A. less than 0.6%
B. at least 0.6% but less than 0.7%
C. at least 0.7% but less than 0.8%
D. at least 0.8% but less than 0.9%
E. at least 0.9%

16.18 (2 points) Yosemite Sam is panning for gold.
The number of pans with gold nuggets he finds per day is Poisson with mean 3.
The number of nuggets per such pan are: 1, 5, or 25, with probabilities: 90%, 9%, and 1% respectively.
The number of pans and the number of nuggets per pan are independent.
Using the normal approximation with continuity correction, what is the probability that the number of nuggets found by Sam over the next ten day is less than 30?
(A) $\Phi(-1.2)$ (B) $\Phi(-1.1)$ (C) $\Phi(-1.0)$ (D) $\Phi(-0.9)$ (E) $\Phi(-0.8)$

16.19 (3 points) Frequency and seventy are independent.
Frequency and severity are each members of the (a, b, 0) class of distributions.
Mean frequency is 2. Variance of frequency is 4.
Mean aggregate is 3. Variance of aggregate is 13.5.
Determine the probability that the aggregate losses are zero.
A. 34% B. 36% C. 38% D. 40% E. 42%

16.20 (4 points) At a food bank, people volunteer their time on a daily basis.
The number of people who volunteer on any day is a zero-truncated Binomial Distribution with $m = 10$ and $q = 0.3$.
The number of hours that each person helps at the food bank is a zero-truncated Binomial Distribution with $m = 3$ and $q = 0.4$.
The number of volunteers and the number of hours they each help are independent.
Determine the probability that on a day fewer than 4 volunteer hours will be available, using the normal approximation with continuity correction.
A. 24% B. 26% C. 28% D. 30% E. 32%

16.21 (2 points) A compound claims frequency model has the following properties:
(i) The primary distribution has probability generating function:
$P_1(z) = \exp[0.4z - 0.4]$.
(ii) The secondary distribution has probability generating function:
$P_2(z) = \exp[5z - 5]$.
Determine the variance of the compound distribution.
A. 2.8 B. 4 C. 5.4 D. 12 E. 16

16.22 (3 points) Frequency is Poisson with mean 3.
Size of loss is Geometric with mean 10.
There is deductible of 5.
What is the standard deviation of aggregate payments?
A. 17 B. 18 C. 19 D. 20 E. 21

16.23 (3, 11/00, Q.2 & 2009 Sample Q.112) (2.5 points) In a clinic, physicians volunteer their time on a daily basis to provide care to those who are not eligible to obtain care otherwise. The number of physicians who volunteer in any day is uniformly distributed on the integers 1 through 5. The number of patients that can be served by a given physician has a Poisson distribution with mean 30.
Determine the probability that 120 or more patients can be served in a day at the clinic, using the normal approximation with continuity correction.
(A) 1 - $\Phi(0.68)$ (B) 1 - $\Phi(0.72)$ (C) 1 - $\Phi(0.93)$ (D) 1 - $\Phi(3.13)$ (E) 1 - $\Phi(3.16)$

16.24 (3, 5/01, Q.16 & 2009 Sample Q.106) (2.5 points) A dam is proposed for a river which is currently used for salmon breeding. You have modeled:
(i) For each hour the dam is opened the number of salmon that will pass through and reach the breeding grounds has a distribution with mean 100 and variance 900.
(ii) The number of eggs released by each salmon has a distribution with mean of 5 and variance of 5.
(iii) The number of salmon going through the dam each hour it is open and the numbers of eggs released by the salmon are independent.
Using the normal approximation for the aggregate number of eggs released, determine the least number of whole hours the dam should be left open so the probability that 10,000 eggs will be released is greater than 95%.
(A) 20 (B) 23 (C) 26 (D) 29 (E) 32

16.25 (3, 5/01, Q.36 & 2009 Sample Q.111) (2.5 points)
The number of accidents follows a Poisson distribution with mean 12.
Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, 1/6, respectively.
Calculate the variance in the total number of claimants.
(A) 20 (B) 25 (C) 30 (D) 35 (E) 40

16.26 (3, 11/01, Q.30) (2.5 points) The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of λ = 50 envelopes per week.
For any period of time, the number of envelopes and the numbers of claims in the envelopes are independent. The numbers of claims in the envelopes have the following distribution:

Number of Claims	Probability
1	0.20
2	0.25
3	0.40
4	0.15

Using the normal approximation, calculate the 90^{th} percentile of the number of claims received in 13 weeks.
(A) 1690 (B) 1710 (C) 1730 (D) 1750 (E) 1770

16.27 (3, 11/02, Q.27 & 2009 Sample Q.93) (2.5 points) At the beginning of each round of a game of chance the player pays 12.5. The player then rolls one die with outcome N. The player then rolls N dice and wins an amount equal to the total of the numbers showing on the N dice.
All dice have 6 sides and are fair.
Using the normal approximation, calculate the probability that a player starting with 15,000 will have at least 15,000 after 1000 rounds.
(A) 0.01 (B) 0.04 (C) 0.06 (D) 0.09 (E) 0.12

16.28 (CAS3, 5/04, Q.26) (2.5 points) On Time Shuttle Service has one plane that travels from Appleton to Zebrashire and back each day.
Flights are delayed at a Poisson rate of two per month.
Each passenger on a delayed flight is compensated $100.
The numbers of passengers on each flight are independent and distributed with mean 30 and standard deviation 50.
(You may assume that all months are 30 days long and that years are 360 days long.)
Calculate the standard deviation of the annual compensation for delayed flights.
A. Less than $25,000
B. At least $25,000, but less than $50,000
C. At least $50,000, but less than $75,000
D. At least $75,000, but less than $100,000
E. At least $100,000

16.29 (SOA M, 11/05, Q.18 & 2009 Sample Q.205) (2.5 points) In a CCRC, residents start each month in one of the following three states: Independent Living (State #1), Temporarily in a Health Center (State #2) or Permanently in a Health Center (State #3). Transitions between states occur at the end of the month. If a resident receives physical therapy, the number of sessions that the resident receives in a month has a geometric distribution with a mean which depends on the state in which the resident begins the month. The numbers of sessions received are independent. The number in each state at the beginning of a given month, the probability of needing physical therapy in the month, and the mean number of sessions received for residents receiving therapy are displayed in the following table:

State#	Number in state	Probability of needing therapy	Mean number of visits
1	400	0.2	2
2	300	0.5	15
3	200	0.3	9

Using the normal approximation for the aggregate distribution, calculate the probability that more than 3000 physical therapy sessions will be required for the given month.
(A) 0.21 (B) 0.27 (C) 0.34 (D) 0.42 (E) 0.50

16.30 (SOA M, 11/05, Q.39 & 2009 Sample Q.213) (2.5 points) For an insurance portfolio:
(i) The number of claims has the probability distribution

n	p_n
0	0.1
1	0.4
2	0.3
3	0.2

(ii) Each claim amount has a Poisson distribution with mean 3; and
(iii) The number of claims and claim amounts are mutually independent.
Calculate the variance of aggregate claims.
(A) 4.8 (B) 6.4 (C) 8.0 (D) 10.2 (E) 12.4

16.31 (CAS3, 5/06, Q.35) (2.5 points)
The following information is known about a consumer electronics store:

- The number of people who make some type of purchase follows a Poisson distribution with a mean of 100 per day.
- The number of televisions bought by a purchasing customer follows a Negative Binomial distribution with parameters r = 1.1 and β = 1.0.

Using the normal approximation, calculate the minimum number of televisions the store must have in its inventory at the beginning of each day to ensure that the probability of its inventory being depleted during that day is no more than 1.0%.
A. Fewer than 138
B. At least 138, but fewer than 143
C. At least 143, but fewer than 148
D. At least 148, but fewer than 153
E. At least 153

16.32 (SOA M, 11/06, Q.30 & 2009 Sample Q.285) (2.5 points)
You are the producer for the television show Actuarial Idol.
Each year, 1000 actuarial clubs audition for the show.
The probability of a club being accepted is 0.20.
The number of members of an accepted club has a distribution with mean 20 and variance 20.
Club acceptances and the numbers of club members are mutually independent.
Your annual budget for persons appearing on the show equals 10 times the expected number of persons plus 10 times the standard deviation of the number of persons.
Calculate your annual budget for persons appearing on the show.
(A) 42,600 (B) 44,200 (C) 45,800 (D) 47,400 (E) 49,000

Solutions to Problems:

16.1. E. Standard deviation of the compound distribution is:
$\sqrt{(15)(4^2) + (10^2)(3^2)} = \sqrt{1140}$ = **33.8**.

16.2. E. The frequency over a year is Poisson with mean: (12)(10) = 120 accidents.
Second moment of the secondary distribution is: $(40\%)(1^2) + (40\%)(2^2) + (20\%)(3^2) = 3.8$.
Variance of compound distribution is: (120)(3.8) = **456**.
Comment: Similar to 3, 5/01, Q.36**.**

16.3. B. The mean of the primary Geometric Distribution is 1.7. The mean of the secondary Poisson Distribution is 3.1. Thus the mean of the compound distribution is: (1.7)(3.1) = **5.27**.

16.4. A. Geometric acts as frequency. Mean of the Geometric Distribution is 1.7.
Variance of the Geometric is: (1.7) (1 + 1.7) = 4.59.
Poisson acts as severity. Mean of the Poisson Distribution is 3.1.
Variance of the Poisson Distribution is 3.1.
The variance of the compound distribution is: $(1.7)(3.1) + (3.1^2)(4.59)$ = **49.38**.
Comment: The variance of the compound distribution is large compared to its mean. A very large number of items can result if there are a large number of customers from the Geometric combined with some of those customers buying a large numbers of items from the Poisson.
Compound distributions tend to have relatively heavy tails.

16.5. E. From the previous solutions, the mean of the compound distribution is 5.27, and the variance of the compound distribution is 49.38. Thus the standard deviation is 7.03.
$1 - \Phi[(4.5 - 5.27)/7.03)] = 1 - \Phi(-0.11) = \Phi(0.11)$ = **0.5438**.

16.6. C. Over y hours, the number of salmon has mean 50y and variance 100y.
The mean aggregate number of eggs is: (50y)(3000) = 150000y.
The standard deviation of the aggregate number of eggs is:
$\sqrt{(50y)(1000^2) + (3000^2)(100y)} = 30822\sqrt{y}$.
Thus the probability that the aggregate number of eggs is < 2 million is approximately:
$\Phi((1999999.5 - 150000y)/30822\sqrt{y})$.
Since $\Phi(2.576) = .995$, this probability will be 1/2% if:
$(1999999.5 - 150000y)/ 30822\sqrt{y} = - 2.576 \Rightarrow 150000y - 79397\sqrt{y} - 1999999.5 = 0$.
$\sqrt{y} = (79397 \pm \sqrt{79397^2 + (4)(150000)(1999999.5)})/ ((2)(150000)) = 0.2647 \pm 3.6611$.
$\sqrt{y} = 3.926. \Rightarrow y = 15.4$. The smallest whole number of hours is therefore **16**.
Alternately, try the given choices and stop when (Mean - 2 million)/StdDev. > 2.576.

Hours	Mean	Standard Deviation	# of Claims
14	2,100,000	115,325	0.867
15	2,250,000	119,373	2.094
16	2,400,000	123,288	3.244
17	2,550,000	127,082	4.328
18	2,700,000	130,767	5.353

<u>Comment:</u> Similar to 3, 5/01, Q.16.
Note that since the variance over one hour is 100, the variance of the number of salmon over two hours is: (2)(100) = 200.
Number of salmon over two hours = number over the first hour + number over the second hour.
⇒ Var[Number over two hours] = Var[number over first hour] + Var[number over second hour]
= 2 Var[number over an hour]. We are adding independent random variables, rather than multiplying an individual variable by a constant.

16.7. B. The mean frequency over 5 days is: (7)(5) = 35.
Mean number of claims per envelope is: (60%)(1) + (30%)(2) + (10%)(3) = 1.5.
Mean of compound distribution is: (35)(1.5) = 52.5.
Second moment of number of claims per envelope is: $(60\%)(1^2) + (30\%)(2^2) + (10\%)(3^2) = 2.7$.
Variance of compound distribution is: (35)(2.7) = 94.5.
99th percentile ≅ mean + (2.326)(standard deviations) = $52.5 + (2.326)\sqrt{94.5}$ = **75.1**.
<u>Comment:</u> Similar to 3, 11/01, Q.30.

16.8. C. The number of persons has mean: (13)(2) = 26,
and variance: (13)(2)(2 + 1) = 78.
The number of transactions per person has mean:
(30%)(1) + (40%)(2) + (20%)(3) + (10%)(4) = 2.1,
second moment: (30%)(1^2) + (40%)(2^2) + (20%)(3^2) + (10%)(4^2) = 5.3,
and variance: 5.3 - 2.1^2 = 0.89.
The number of transactions in an hour has mean: (26)(2.1) = 54.6,
and variance: (26)(.89) + (2.1^2)(78) = 367.12.
The number of transactions in 5 hours has mean: (5)(54.6) = 273,
and variance: (5)(367.12) = 1835.6.
Φ(0.842) = 80%. 80th percentile $\cong$ 273 + (0.842)$\sqrt{1835.6}$ = **309.1**.

16.9. E. Mean of the Primary Negative Binomial = (100)(0.6) = 60.
Variance of the Primary Negative Binomial = (100)(0.6)(1.6) = 96.
Mean of the Secondary Binomial = (4)(0.1) = 0.4.
Variance of the Secondary Binomial = (4)(0.1)(0.9) = .36.
Variance of the Compound Distribution = (60)(.36) + (0.4^2)(96) = **36.96**.

16.10. D. Mean of the Compound Distribution = (60) (0.4) = 24.
Prob[# claimants ≤ 20] $\cong \Phi$[(20.5 - 24)/$\sqrt{36.96}$] = Φ(-0.58) = 1 - 0.7190 = **28.1%**.

16.11. A. Mean Frequency: 24. Variance of Frequency: 36.96.
Mean Severity: (3)(4000) = 12,000. Variance of Severity: (3)(4000^2) = 48,000,000.
Mean Aggregate Loss = (24)(12000) = 288,000.
Variance of the Aggregate Loss = (24)(48,000,000) + ($12{,}000^2$)(36.96) = 6474 million.
Prob[Aggregate loss > 300000] $\cong$ 1 - Φ((300000 - 288000)/$\sqrt{6474 \text{ million}}$n) =
1 - Φ(0.15) = 1 - 0.5596 = **44%**.

16.12. E. The number of batters has mean: 3 + (1.4)(1) = 4.4, and variance: (1.4)(1)(1 + 1) = 2.8.
The number of pitches per batter has mean: 1 + (1.8)(1.5) = 3.7,
and variance: (1.8)(1.5)(1 + 1.5) = 6.75.
The number of pitches per half-inning has mean: (4.4)(3.7) = 16.28,
and variance: (4.4)(6.75) + (3.7^2)(2.8) = 68.032.
Prob[# pitches > 30] $\cong$ 1 - Φ[(30.5 - 16.28)/$\sqrt{68.032}$] = 1 - Φ(1.72) = **4.27%**.

16.13. D. Over y minutes, the number of taxicabs has mean 5.6y and variance 5.6y.
The passengers per cab has mean: (0.3)(4) = 1.2, and variance: (0.3)(1 - 0.3)(4) = 0.84.
The mean aggregate number of passengers is: (5.6y)(1.2) = 6.72y.
The standard deviation of the aggregate number of passengers is:
$\sqrt{(5.6y)(0.84) + (1.2^2)(5.6y)} = 3.573\sqrt{y}$.
Thus the probability that the aggregate number of passengers is ≥ 1000 is approximately:
1 - $\Phi[(999.5 - 6.72y)/3.573\sqrt{y}]$. Since $\Phi(1.282) = 0.90$, this probability will be greater than 90% if:
(Mean - 999.5) / StdDev. = $(6.72y - 999.5) / (3.573\sqrt{y}) > 1.282$.
Try the given choices and stop when (Mean - 999.5) / StdDev. > 1.282.

Minutes	Mean	Standard Deviation	# of Claims
155	1,041.6	44.48	0.946
156	1,048.3	44.63	1.094
157	1,055.0	44.77	1.241
158	1,061.8	44.91	1.386
159	1,068.5	45.05	1.531

The smallest whole number of minutes is therefore **158**.

16.14. A. The mean number of lawyers is: 2.5 and the variance is:
$\{(1 - 2.5)^2 + (2 - 2.5)^2 + (3 - 2.5)^2 + (4 - 2.5)^2\}/4 = 1.25$.
The mean number of hours per lawyer is: (7)(.6) = 4.2 and the variance is: (7)(.4)(.6) = 1.68.
Therefore, the total number of hours volunteered per day has mean: (2.5)(4.2) = 10.5 and variance:
$(2.5)(1.68) + (4.2^2)(1.25) = 26.25$.
The number of clients per hour has mean 5 and variance 5.
Therefore, the total number of clients per day has mean: (5)(10.5) = 52.5,
and variance: $(10.5)(5) + (5^2)(26.25) = 708.75$.
Prob[# clients ≥ 40] ≅ 1 - $\Phi[(39.5 - 52.5)/\sqrt{708.75}]$ = 1 - $\Phi(-.49)$ = **68.79%**.
Alternately, the mean number of clients per lawyer is: (4.2)(5) = 21
with variance: $(4.2)(5) + (5^2)(1.68) = 63$.
Therefore, the total number of clients per day has mean: (2.5)(21) = 52.5 and
variance: $(2.5)(63) + (21^2)(1.25) = 708.75$. Proceed as before.
<u>Comment:</u> Similar to 3, 11/00, Q.2.

16.15. A. The mean of the primary Poisson Distribution is 1.2.
The mean of the secondary Binomial Distribution is: (4)(.1) = .4.
Thus the mean of the compound distribution is: (1.2)(.4) = **0.48**.

16.16. B. The mean of the primary Poisson Distribution is 1.2. The mean of the secondary Binomial Distribution is: (4)(0.1) = 0.4. The variance of the primary Poisson Distribution is 1.2. The variance of the secondary Binomial Distribution is: (4)(0.1)(.9) = 0.36.
The variance of the compound distribution is: $(1.2)(0.36) + (0.4)^2(1.2) =$ **0.624**.

16.17. A. The compound distribution has mean of .48 and variance of .624.
Prob[# books > 2] $\cong 1 - \Phi[(2.5 - 0.48)/\sqrt{0.624}] = 1 - \Phi(2.56) = 1 - 0.9948 =$ **0.0052**.

16.18. B. The mean number of nuggets per pan is: (90%)(1) + (9%)(5) + (1%)(25) = 1.6.
2nd moment of the number of nuggets per pan is: $(90\%)(1^2) + (9\%)(5^2) + (1\%)(25^2) = 9.4$.
Mean aggregate over 10 days is: (10)(3)(1.6) = 48.
Variance of aggregate over 10 days is: (10)(3)(9.4) = 282.
Prob[aggregate < 30] $\cong \Phi[(29.5 - 48)/\sqrt{282}] =$ $\mathbf{\Phi(-1.10)}$ = 13.57%.

16.19. A. Variance of frequency is greater than the mean, so of the (a, b, 0) class we must have a Negative Binomial Distribution. $r\beta = 2$, and $r\beta(1+\beta) = 4$. $\Rightarrow$ $r = 2$ and $\beta = 1$.

Let X be severity. Then: 2 E[X] = 3. $\Rightarrow$ E[X] = 1.5.

$13.5 = 2\ \text{Var}[X] + 4\ E[X]^2$. $\Rightarrow$ Var[X] = 2.25.
Variance of severity is greater than the mean, so of the (a, b, 0) class we must have a Negative Binomial Distribution. $r\beta = 1.5$, and $r\beta(1+\beta) = 2.25$. $\Rightarrow$ $r = 3$ and $\beta = 0.5$.
Compound density at zero is the p.g.f. of the primary at density at 0 of the secondary.
In other words, compound density at zero is p.g.f. of the frequency at density at 0 of the discrete severity distribution.
Density at zero of the discrete severity is: $1 / 1.5^3 = 0.2963$.
Probability Generating Function of Frequency is: $\{1 - \beta(z-1)\}^{-r} = \{\{1 - (1)(z-1)\}^{-2} = (2 - z)^{-2}$.
This probability generating function at 0.2963 is: $(2 - 0.2963)^{-2} =$ **34.45%**.

16.20. D. The mean of each zero truncated Binomial is: $\frac{mq}{1 - (1-q)^m}$.

For the number of volunteers: $\frac{(10)(0.3)}{1 - 0.7^{10}} = 3.0872$.

For the hours per volunteer: $\frac{(3)(0.4)}{1 - 0.6^3} = 1.5306$.

Thus the mean number of hours per day is: (3.0872)(1.5306) = 4.7253.

The variance of each zero truncated Binomial is: $\frac{mq \{(1-q) - (1 - q + mq)(1-q)^m\}}{\{1 - (1-q)^m\}^2}$.

Variance for the number of volunteers: $\frac{(10)(0.3)\{(0.7) - (1 - 0.3 + 3)(0.7^{10})\}}{\{1 - 0.7^{10}\}^2} = 1.8918$.

Variance for the hours per volunteer: $\frac{(3)(0.4)\{(0.6) - (1 - 0.4 + 1.2)(0.6^3)\}}{\{1 - 0.6^3\}^2} = 0.4123$.

Thus the variance of the number of hours per day is:

$(3.0872)(0.4123) + (1.5306^2)(1.8918) = 5.7048$.

Prob[fewer than 4 hours] = $\Phi[(3.5 - 4.7253) / \sqrt{5.7048}] = \Phi[-0.51]$ = **30.5%**.

Comment: Similar to 3, 11/00, Q.2 (2009 Sample Q.112).

The number of volunteers acts as frequency, while the number of hours per volunteer acts as severity.

Using a computer and the Panjer algorithm, discussed in "Mahler's Guide to Aggregate Distributions": Prob[1 hour] = 6.86%, Prob[2 hours] = 11.87%, and Prob[3 hours] = 15.34%.

Thus the Prob[fewer than 4 hours] = 6.86% + 11.87% + 15.34% = 34.07%.

16.21. D. The primary distribution is a Poisson with mean 0.4.

The secondary distribution, which acts like severity, is a Poisson with mean 5.

Thus the variance of the compound (or aggregate) distribution is:

$(0.4)(5) + (5^2)(0.4) = $ **12**.

Alternately, the compound distribution has p.g.f.: $P(z) = P_1[P_2(z)] = \exp[0.4\exp(5z - 5) - 0.4]$.

$P'(z) = (0.4)(5)\exp[5z - 5]\exp[0.4\exp(5z - 5) - 0.4] = 2\exp[0.4\exp(5z - 5) + 5z - 5.4]$.

2 = P'(1) = first factorial moment = mean.

$P''(z) = 2\exp[0.4\exp(5z - 5) + 5z - 5.4]\{(0.4)(5)\exp[5z - 5] + 5\}$.

(2)(1)(7) = P''(1) = second factorial moment = E[X(X-1)] = second moment - mean.

Therefore, second moment = 14 + 2 = 16.

Thus the variance of the compound distribution is: $16 - 2^2 = $ **12**.

Comment: The compound distribution is a Neyman Type A, as shown in Appendix B.4 of Loss Models, not on the syllabus.

16.22. D. The probability of a non-zero payment is the survival function of the geometric at 5:
$\{\beta/(1+\beta)\}^6 = (10/11)^6 = 0.5645$.
Thus the number of non-zero payments is Poisson with mean: (0.5645)(3) = 1.6935.
Due to the memoryless property of the Geometric Distribution, the non-zero payments follow a zero-truncated Geometric with $\beta = 10$, with mean 1 + 10 = 11, and variance (10)(11) = 110.
The total payments are the sum of the non-zero payments.
Thus the variance of the total payments is:
(mean freq.) (var. of sev.) + (mean sev.)^2 (var. freq.) =
$(1.6935)(10)(11) + (11^2)(1.6935) = 391.2$.
Standard deviation of aggregate payments is: $\sqrt{391.2}$ = **19.8**.
Alternately, the probability of a non-zero payment is:
$f(6) + f(7) + f(8) + ... = 10^6/11^7 + 10^7/11^8 + 10^8/11^9 + ...$
$= (10^6/11^7) / (1 - 10/11) = (10/11)^6 = 0.5645$.
Given there is a non-zero payment, the probability that it is for example 8 - 5 = 3 is:
$f(8) / (10/11)^6 = (10^8/11^9) / (10/11)^6 = 10^2/11^3$, which is density at 3 of a zero-truncated Geometric Distribution with $\beta = 10$. One can show in a similar manner that the non-zero payments follow a zero-truncated Geometric Distribution with $\beta = 10$. Proceed as before.
Comment: For example, the sizes of loss could be in units of hundreds of dollars.

16.23. A. This is a compound frequency distribution with a primary distribution that is discrete and uniform on 1 through 5 and with secondary distribution which is Poisson with $\lambda = 30$. The primary distribution has mean of 3 and second moment of:
$(1^2 + 2^2 + 3^2 + 4^2 + 5^2)/5 = 11$. Thus the primary distribution has variance: $11 - 3^2 = 2$.
Mean of the Compound Dist. = (Mean of Primary Dist.)(Mean of Secondary Dist.) = (3)(30) = 90.
Variance of the Compound Distribution = (Mean of Primary Dist.)(Variance of Secondary Dist.) + (Mean of Secondary Dist.)^2(Variance of Primary Dist.) = $(3)(30) + (30^2)(2) = 1890$.
Probability of 120 or more patients $\cong 1 - \Phi[(119.5 - 90)/\sqrt{1890}]$ = **1 - Φ(0.68)**.

16.24. B. Over y hours, the number of salmon has mean 100y and variance 900y.
The mean aggregate number of eggs is: (100y)(5) = 500y.
The variance of the aggregate number of eggs is: $(100y)(5) + (5^2)(900y) = 23000y$.
Thus the probability that the aggregate number of eggs is < 10000 is approximately:
$\Phi((9999.5 - 500y)/\sqrt{23000y})$. Since $\Phi(1.645) = 0.95$, this probability will be 5% if:

$(9999.5 - 500y)/\sqrt{23000y} = -1.645 \Rightarrow 500y - 249.98\sqrt{y} - 9999.5 = 0.$

$\sqrt{y} = \{249.48 \pm \sqrt{249.48^2 + (4)(500)(9999.5)}\} / \{(2)(500)\} = 0.24948 \pm 4.479.$

$\sqrt{y} = 4.729. \Rightarrow y = 22.3$. The smallest whole number of hours is therefore **23**.

Alternately, calculate the probability for each of the number of hours in the choices.

Hours	Mean	Variance	Probability of at least 10,000 eggs
20	10,000	460,000	$1 - \Phi((9999.5 - 10000)/\sqrt{460{,}000}) = 1 - \Phi(-0.0007) = 50.0\%$
23	11,500	529,000	$1 - \Phi((9999.5 - 11500)/\sqrt{529{,}000}) = 1 - \Phi(-2.063) = 98.0\%$
26	13,000	598,000	$1 - \Phi((9999.5 - 13000)/\sqrt{598{,}000}) = 1 - \Phi(-3.880) = 99.995\%$

Thus 20 hours is not enough and 23 hours is enough so that the probability is greater than 95%.
Comment: The number of salmon acts as the primary distribution, and the number of eggs per salmon as the secondary distribution. This exam question should have been worded better. They intended to say "so the probability that at least 10,000 eggs will be released is greater than 95%." The probability of exactly 10,000 eggs being released is very small.

16.25. E. The second moment of the number of claimants per accident is:
$(1/2)(1^2) + (1/3(2^2) + (1/6)(3^2) = 3.333$. The variance of a Compound Poisson Distribution is:
λ(2nd moment of the secondary distribution) = (12)(3.333) = **40**.
Alternately, thinning the original Poisson, those accidents with 1, 2, or 3 claimants are independent Poissons. Their means are: (1/2)(12) = 6, (1/3)(12) = 4, and (1/6)(12) = 2.
Number of accidents with 3 claimants is Poisson with mean 2 $\Rightarrow$
The variance of the number of accidents with 3 claimants is 2.
Number of claimants for those accidents with 3 claimants = (3)(# of accidents with 3 claimants) $\Rightarrow$
The variance of the # of claimants for those accidents with 3 claimants is: $(3^2)(2)$.
Due to independence, the variances of the three processes add: $(1^2)(6) + (2^2)(4) + (3^2)(2) =$ **40**.

16.26. B. Mean # claims / envelope = (1)(0.2) + (2)(0.25) + (3)(0.4) + (4)(0.15) = 2.5.
2nd moment # claims / envelope = $(1^2)(0.2) + (2^2)(0.25) + (3^2)(0.4) + (4^2)(0.15) = 7.2$.
Over 13 weeks, the number of envelopes is Poisson with mean: (13)(50) = 650.
Mean of the compound distribution = (650)(2.5) = 1625.
Variance of the aggregate number of claims = Variance of a compound Poisson distribution =
(mean primary Poisson distribution)(2nd moment of the secondary distribution) = (650)(7.2) =
4680. $\Phi(1.282) = 0.90$. Estimated 90th percentile = $1625 + 1.282\sqrt{4680}$ = **1713.**

16.27. E. The amount won per a round of the game is a compound frequency distribution.
Primary distribution (determining how many dice are rolled) is a six-sided die, uniform and <u>discrete</u> on
1 through 6, with mean 3.5, second moment $(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)/6 = 91/6$,
and variance $91/6 - 3.5^2 = 35/12$.
Secondary distribution is also a six-sided die, with mean 3.5 and variance 35/12.
Mean of the compound distribution is: (3.5)(3.5) = 12.25.
Variance of the compound distribution is: $(3.5)(35/12) + (3.5^2)(35/12) = 45.94$.
Therefore, the net result of a round has mean 12.25 - 12.5 = -0.25, and variance 45.94.
1000 rounds have a net result with mean -250 and variance 45,940.
Prob[net result ≥ 0] $\cong 1 - \Phi((-0.5 + 250)/\sqrt{45,940}) = 1 - \Phi(1.16) = 1 - 0.8770 =$ **0.1220**.

16.28. B. The total number of delayed passengers is a compound frequency distribution, with
primary distribution the number of delayed flights, and the secondary distribution the number of
passengers on a flight.
The number of flights delayed per year is Poisson with mean: (2)(12) = 24.
The second moment of the secondary distribution is: $50^2 + 30^2 = 3400$.
The variance of the number of passengers delayed per year is: (24)(3400) = 81,600.
The standard deviation of the number of passengers delayed per year is: $\sqrt{81,600} = 285.66$.
The standard deviation of the annual compensation is: (100)(285.66) = **28,566**.

16.29. D. The mean number of sessions is:
(400)(0.2)(2) + (300)(0.5)(15) + (200)(0.3)(9) = 2950.
For a single resident we have a Bernoulli primary (whether the resident need therapy) and a geometric secondary (how many visits).
This has variance: (mean of primary)(variance of second.) + (mean second.)^2(var. of primary)
$= q\beta(1 + \beta) + \beta^2 q(1 - q)$.
For a resident in state 1, the variance of the number of visits is:
$(0.2)(2)(3) + (3^2)(0.2)(1 - 0.8) = 1.84$.
For state 2, the variance of the number of visits is: $(0.5)(15)(16) + (15^2)(0.5)(1 - 0.5) = 176.25$.
For state 3, the variance of the number of visits is: $(0.3)(9)(10) + (9^2)(0.3)(1 - 0.3) = 44.01$.
The sum of the visits from 400 residents in state 1, 300 in state 2, and 200 in state 3, has variance:
(400)(1.84) + (300)(176.25) + (200)(44.01) = 62,413.
Prob[sessions > 3000] $\cong 1 - \Phi[(3000.5 - 2950)/\sqrt{62413}] = 1 - \Phi[0.20] =$ **0.4207**.

16.30. E. Primary distribution has mean: (0)(0.1) + (1)(0.4) + (2)(0.3) + (3)(0.2) = 1.6,
second moment: $(0^2)(0.1) + (1^2)(0.4) + (2^2)(0.3) + (3^2)(0.2) = 3.4$, and variance: $3.4 - 1.6^2 = 0.84$.
The secondary distribution has mean 3 and variance 3.
The compound distribution has variance: $(1.6)(3) + (3^2)(0.84) =$ **12.36**.

16.31. E. Mean = (mean primary)(mean secondary) = (100)(1.1)(1.0) = 110.
Variance = (mean primary)(variance of secondary) + (mean secondary)^2(variance of primary) =
$(100)(1.1)(1)(1 + 1) + \{(1.1)(1.0)\}^2(100) = 341$. $\Phi(2.326) = 0.99$.
99th percentile: $110 + 2.326\sqrt{341} = 152.95$. Need at least **153** televisions.

16.32. A. The primary distribution is Binomial with m = 1000 and q = .2, with mean 200 and variance 160. The mean of the compound distribution is: (200)(20) = 4000.
The variance of the compound distribution is: $(200)(20) + (20^2)(160) = 68{,}000$.
Annual budget is: $10(4000 + \sqrt{68000}) =$ **42,608**.

Section 17, Mixed Frequency Distributions

One can mix frequency models together by taking a weighted average of different frequency models. This can involve either a discrete mixture of several different frequency distributions or a continuous mixture over a portfolio as a parameter varies.

For example, one could mix together Poisson Distributions with different means.[123]

Discrete Mixtures:

Assume there are four types of risks, each with claim frequency given by a Poisson distribution:

Type	Average Annual Claim Frequency	A Priori Probability
Excellent	1	40%
Good	2	30%
Bad	3	20%
Ugly	4	10%

Recall that for a Poisson Distribution with parameter λ the chance of having n claims is given by:

$f(n) = \lambda^n e^{-\lambda} / n!$.

So for example for an Ugly risk with $\lambda = 4$, the chance of n claims is: $4^n e^{-4} / n!$.

For an Ugly risk the chance of 6 claims is: $4^6 e^{-4} /6! = 10.4\%$.
Similarly the chance of 6 claims for Excellent, Good, or Bad risks are: 0.05%, 1.20%, and 5.04% respectively.

If we have a risk but do not know what type it is, we weight together the 4 different chances of having 6 claims, using the a priori probabilities of each type of risk in order to get the chance of having 6 claims: (0.4)(0.05%) + (0.3)(1.20%) + (0.2)(5.04%) + (0.1)(10.42%) = 2.43%.
The table below displays similar values for other numbers of claims.

The probabilities in the final column represents the assumed distribution of the number of claims for the entire portfolio of risks.[124] This "probability for all risks" is the mixed distribution. While the mixed distribution is easily computed by weighting together the four Poisson distributions, it is not itself a Poisson nor other well known distribution.

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[123] The parameter of a Poisson is its mean. While one can mix together other frequency distributions, for example Binomials or Negative Binomials, you are most likely to be asked about mixing Poissons. (It is unclear what if anything they will ask on this subject.)

[124] Prior to any observations. The effect of observations will be discussed in "Mahler's Guide to Buhlmann Credibility" and "Mahler's Guide to Conjugate Priors."

Number of Claims	Probability for Excellent Risks	Probability for Good Risks	Probability for Bad Risks	Probability for Ugly Risks	Probability for All Risks
0	0.3679	0.1353	0.0498	0.0183	0.1995
1	0.3679	0.2707	0.1494	0.0733	0.2656
2	0.1839	0.2707	0.2240	0.1465	0.2142
3	0.0613	0.1804	0.2240	0.1954	0.1430
4	0.0153	0.0902	0.1680	0.1954	0.0863
5	0.0031	0.0361	0.1008	0.1563	0.0478
6	0.0005	0.0120	0.0504	0.1042	0.0243
7	0.0001	0.0034	0.0216	0.0595	0.0113
8	0.0000	0.0009	0.0081	0.0298	0.0049
9	0.0000	0.0002	0.0027	0.0132	0.0019
10	0.0000	0.0000	0.0008	0.0053	0.0007
11	0.0000	0.0000	0.0002	0.0019	0.0002
12	0.0000	0.0000	0.0001	0.0006	0.0001
13	0.0000	0.0000	0.0000	0.0002	0.0000
14	0.0000	0.0000	0.0000	0.0001	0.0000
SUM	1.0000	1.0000	1.0000	1.0000	1.0000

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

Moments of Mixed Distributions:

The overall (a priori) mean can be computed in either one of two ways.

First one can weight together the means for each type of risks, using their (a priori) probabilities:
(0.4)(1) + (0.3)(2) + (0.2)(3) + (0.1)(4) = 2.

Alternately, one can compute the mean of the mixed distribution:
(0)(0.1995) + (1)(0.2656) + (2)(0.2142) + ... = 2.

In either case, the mean of this mixed distribution is 2.

The mean of a mixed distribution is the mixture of the means for specific values of the parameter λ: $E[X] = E_\lambda[E[X \mid \lambda]]$.

One can calculate the second moment of a mixture in a similar manner.

Exercise: What is the second moment of a Poisson distribution with $\lambda = 3$?
[Solution: Second Moment = Variance + Mean^2 = 3 + 3^2 = 12.]

In general, the second moment of a mixture is the mixture of the second moments.

In the case of this mixture, the second moment is:
(0.4)(2) + (0.3)(6) + (0.2)(12) + (0.1)(20) = 7.

One can verify this second moment, by working directly with the mixed distribution:

Probability for All Risks	Number of Claims	Square of # of Claims
0.1995	0	0
0.2656	1	1
0.2142	2	4
0.1430	3	9
0.0863	4	16
0.0478	5	25
0.0243	6	36
0.0113	7	49
0.0049	8	64
0.0019	9	81
0.0007	10	100
0.0002	11	121
0.0001	12	144
0.0000	13	169
0.0000	14	196
Average	2.000	7.000

Exercise: What is the variance of this mixed distribution?
[Solution: $7 - 2^2 = 3$.]

First one mixes the moments, and then computes the variance of the mixture from its first and second moments.[125]

In general, **the nth moment of a mixed distribution is the mixture of the nth moments** for specific values of the parameter λ: $E[X^n] = E_\lambda[E[X^n \mid \lambda]]$.[126]

There is nothing unique about assuming four types of risks. If one had assumed for example 100 different types of risks, with mean frequencies from 0.1 to 10. There would have been no change in the conceptual complexity of the situation, although the computational complexity would have been increased. This discrete example can be extended to a continuous case.

[125] As discussed in "Mahler's Guide to Buhlmann Credibility," one can split the variance of a mixed distribution into two pieces, the Expected Value of the Process Variance and the Variance of the Hypothetical Means.
[126] Third and higher moments are more likely to be asked about for Loss Distributions.
Mixtures of Loss Distributions are discussed in "Mahler's Guide to Loss Distributions."

Continuous Mixtures:

We have seen how one can mix a discrete number of Poisson Distributions.[127] For a continuous mixture, the mixed distribution is given as the integral of the product of the distribution of the parameter λ times the Poisson density function given λ.[128]

$$g(x) = \int f(x; \lambda)\, u(\lambda)\, d\lambda.$$

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

Exercise: The claim count N for an individual insured has a Poisson distribution with mean λ.

λ is uniformly distributed between 0.3 and 0.8.

Find the probability that a randomly selected insured will have one claim.

[Solution: For the Poisson Distribution, $f(1 \mid \lambda) = \lambda e^{-\lambda}$.

$$(1/0.5) \int_{0.3}^{0.8} \lambda\, e^{-\lambda}\, d\lambda = (2)(-\lambda\, e^{-\lambda} - e^{-\lambda})\Big]_{\lambda = 0.3}^{\lambda = 0.8} = (2)(1.3e^{-0.3} - 1.8\, e^{-0.8}) = 30.85\%.]$$

Continuous mixtures can be performed of either frequency distributions or loss distributions.[129] Such a continuous mixture is called a Mixture Distribution.[130]

Mixture Distribution $\Leftrightarrow$ Continuous Mixture of Models.

Mixture Distributions can be created from other frequency distributions than the Poisson.

For example, if f is a Binomial with fixed m, one could mix on the parameter q:

$$g(x) = \int f(x; q)\, u(q)\, dq.$$

For example, if f is a Negative Binomial with fixed r, one could mix on the parameter β:

$$g(x) = \int f(x; \beta)\, u(\beta)\, d\beta.$$

If f is a Negative Binomial with fixed r, one could instead mix on the parameter $p = 1/(1+\beta)$.

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[127] One can mix other frequency distributions besides the Poisson.

[128] The very important Gamma-Poisson situation is discussed in a subsequent section.

[129] See the section on Continuous Mixtures of Models in "Mahler's Guide to Loss Distributions".

[130] See Section 5.2.4 of Loss Models.

Moments of Continuous Mixtures:

As in the case of discrete mixtures, the nth moment of a continuous mixture is the mixture of the nth moments for specific values of the parameter λ: $E[X^n] = E_\lambda[E[X^n \mid \lambda]]$.

Exercise: What is the mean for a mixture of Poissons?

[Solution: For a given value of lambda, the mean of a Poisson Distribution is λ. We need to weight these first moments together via the density of lambda $u(\lambda)$: $\int \lambda\, u(\lambda)\, d\lambda$ = mean of u.]

If for example, λ were uniformly distributed from 0.1 to 0.5, then the mean of the mixed distribution would be 0.3.

In general, the mean of a mixture of Poissons is the mean of the mixing distribution.[131] For the case of a mixture of Poissons via a Gamma Distribution with parameters α and θ, the mean of the mixed distribution is that of the Gamma, $\alpha\theta$.[132]

Exercise: What is the Second Moment for Poissons mixed via a Gamma Distribution with parameters α and θ?

[Solution: For a given value of lambda, the second moment of a Poisson Distribution is $\lambda + \lambda^2$.

We need to weight these second moments together via the density of lambda: $\lambda^{\alpha-1} e^{-\lambda/\theta} \theta^{-\alpha} / \Gamma(\alpha)$.

$$P(z) = \int_{\lambda=0}^{\infty} \frac{(\lambda + \lambda^2)\, \lambda^{\alpha-1}\, e^{-\lambda/\theta}\, \theta^{-\alpha}}{\Gamma(\alpha)}\, d\lambda = \frac{\theta^{-\alpha}}{\Gamma(\alpha)} \int_{\lambda=0}^{\infty} (\lambda^{\alpha} + \lambda^{\alpha+1})\, e^{-\lambda/\theta}\, d\lambda =$$

$$\frac{\theta^{-\alpha}}{\Gamma(\alpha)} \{\Gamma(\alpha+1)\theta^{\alpha+1} + \Gamma(\alpha+2)\theta^{\alpha+2}\} = \alpha\theta + \alpha(\alpha+1)\theta^2.]$$

Since the mean of the mixed distribution is that of the Gamma, $\alpha\theta$, the variance of the mixed distribution is: $\alpha\theta + \alpha(\alpha+1)\theta^2 - (\alpha\theta)^2 = \alpha\theta + \alpha\theta^2$.

As will be discussed, the mixed distribution is a Negative Binomial Distribution, with $r = \alpha$ and $\beta = \theta$. Thus the variance of the mixed distribution is: $\alpha\theta + \alpha\theta^2 = r\beta + r\beta^2 = r\beta(1+\beta)$, which is in fact that the variance of a Negative Binomial Distribution.

[131] This result will hold whenever the parameter being mixed is the mean, as it was in the case of the Poisson.

[132] The Gamma-Poisson will be discussed in a subsequent section.

Factorial Moments of Mixed Distributions:

The nth factorial moment of a mixed distribution is the mixture of the nth factorial moments for specific values of the parameter ζ:

$E[(X)(X-1) \ldots (X+1-n)] = E_\zeta[E[(X)(X-1) \ldots (X+1-n) \mid \zeta]$.

When we are mixing Poissons, the factorial moments of the mixed distribution have a simple form.

nth factorial moment of mixed Poisson = $E[(X)(X-1) \ldots (X+1-n)] = E_\lambda[E[(X)(X-1) \ldots (X+1-n) \mid \lambda]$ =

E_λ[nth factorial moment of Poisson] = $E_\lambda[\lambda^n]$ = nth moment of the mixing distribution.[133]

Exercise: Given Poissons are mixed via a distribution $u(\theta)$, what are the mean and variance of the mixed distribution?
[Solution: The mean of the mixed distribution = first factorial moment =
mean of the mixing distribution.
The second moment of the mixed distribution =
second factorial moment + first factorial moment =
second moment of the mixing distribution + mean of the mixing distribution.
Variance of the mixed distribution =
second moment of mixed distribution - (mean of mixed distribution)^2 =
second moment of the mixing distribution + mean of the mixing distribution -
(mean of the mixing distribution)^2 =
Variance of the mixing distribution + Mean of the mixing distribution.]

When mixing Poissons, Mean of the Mixed Distribution = Mean of the Mixing Distribution,
and the Variance of the Mixed Distribution =
Variance of the Mixing Distribution + Mean of the Mixing Distribution.

Therefore, **for a mixture of Poissons, the variance of the mixed distribution is always greater than the mean of the mixed distribution**.

For example, for a Gamma mixing distribution, the variance of the mixed Poisson is:
Variance of the Gamma + Mean of the Gamma = $\alpha\theta^2 + \alpha\theta$.

[133] See equation 8.24 in Insurance Risk Models by Panjer & Willmot.

Probability Generating Functions of Mixed Distributions:

The Probability Generating Function of the mixed distribution, is the mixture of the probability generating functions for specific values of the parameter λ:

$$P(z) = \int P(z;\ \lambda)\ u(\lambda)\ d\lambda\ .$$

Exercise: What is the Probability Generating Function for Poissons mixed via a Gamma Distribution with parameters α and θ?

[Solution: For a given value of lambda, the p.g.f. of a Poisson Distribution is $e^{\lambda(z-1)}$.
We need to weight these Probability Generating Functions together via the density of lambda:
$\lambda^{\alpha-1}\, e^{-\lambda/\theta}\theta^{-\alpha} / \Gamma(\alpha)$.

$$P(z) = \int_{\lambda=0}^{\infty} \frac{e^{\lambda(z-1)}\ \lambda^{\alpha-1}\ e^{-\lambda/\theta}\ \theta^{-\alpha}}{\Gamma(\alpha)}\ d\lambda = \frac{\theta^{-\alpha}}{\Gamma(\alpha)} \int_{\lambda=0}^{\infty} \lambda^{\alpha-1}\ e^{-\lambda(1/\theta+1-z)}\ d\lambda$$

$$= \frac{\theta^{-\alpha}}{\Gamma(\alpha)} \{\Gamma(\alpha)\, (1/\theta+1-z)^{-\alpha}\} = \{1+\theta(z-1)\}^{-\alpha}.]$$

This is the p.g.f. of a Negative Binomial Distribution with $r = \alpha$ and $\beta=\theta$. This is one way to establish that when Poissons are mixed via a Gamma Distribution, the mixed distribution is always a Negative Binomial Distribution, with $r = \alpha$ = shape parameter of the Gamma and
$\beta = \theta$ = scale parameter of the Gamma.[134]

[134] The Gamma-Poisson frequency process is the subject of an important subsequent section.

Mixing Poissons:[135]

In the very important case of mixing Poisson frequency distributions, the p.g.f. of the mixed distribution can be put in terms of the Moment Generating Function of the mixing distribution of λ.

The Moment Generating Function of a distribution is defined as: $M_X(t) = E[e^{xt}]$.[136]

For a mixture of Poissons:

$P_{\text{mixed distribution}}(z) = E_{\text{mixing distribution of }\lambda}[P_{\text{Poisson}}(z)] = E_{\text{mixing distribution of }\lambda}[\exp[\lambda(z - 1)]] =$

$M_{\text{mixing distribution of }\lambda}(z - 1)$.

Thus when mixing Poissons, $P_{\text{mixed distribution}}(z) = M_{\text{mixing distribution of }\lambda}(z - 1)$.[137]

Exercise: Apply the above formula for probability generating functions to Poissons mixed via a Gamma Distribution.

[Solution: The m.g.f. of a Gamma Distribution with parameters α and θ is: $(1 - \theta t)^{-\alpha}$.

Therefore, the p.g.f. of the mixed distribution is:

$M_{\text{mixing distribution}}(z - 1) = \{1 - \theta(z - 1)\}^{-\alpha}$.

Comment: This is the p.g.f. of a Negative Binomial Distribution, with $r = \alpha$ and $\beta=\theta$.

Therefore, the mixture of Poissons via a Gamma, with parameters α and θ, is a Negative Binomial Distribution, with $r = \alpha$ and $\beta=\theta$.]

$M_X(t) = E_X[e^{xt}] = E_X[\exp[t]^x] = P_X[e^t]$. Therefore, when mixing Poissons:

$M_{\text{mixed distribution}}(t) = P_{\text{mixed distribution}}(e^t) = M_{\text{mixing distribution of }\lambda}(e^t - 1)$.

Exercise: Apply the above formula for moment generating formulas to Poissons mixed via an Inverse Gaussian Distribution with parameters μ and θ.

[Solution: The m.g.f. of an Inverse Gaussian Distribution with parameters μ and θ is:

$\exp[(\theta / \mu)\ (1 - \sqrt{1 - 2\mu^2 t / \theta})]$.

Therefore, the moment generating function of the mixed distribution is:

$M_{\text{mixing distribution of }\lambda}(e^t - 1) = \exp[(\theta / \mu)\ \{1 - \sqrt{1 - 2\mu^2 (e^t - 1) / \theta}\}]$.]

[135] See Section 7.3.2 of Loss Models, not on the syllabus.

[136] See Definition 3.8 in Loss Models and "Mahler's Guide to Aggregate Distributions."

The moment generating functions of loss distributions are shown in Appendix B, when they exist.

[137] See Equation 7.14 in Loss Models, not on the syllabus.

Exercise: The p.g.f. of the Zero-Truncated Negative Binomial Distribution is:

$$P(z) = \frac{\{1 - \beta(z - 1)\}^{-r} - (1 + \beta)^{-r}}{1 - (1 + \beta)^{-r}} = 1 + \frac{\{1 - \beta(z - 1)\}^{-r} - 1}{1 - (1 + \beta)^{-r}},\ z < 1 + 1/\beta.$$

What is the moment generating function of a compound Poisson-Extended Truncated Negative Binomial Distribution, with parameters $\lambda = (\theta/\mu)\{(1+2\mu^2/\theta)^{.5} - 1\}$, r = -1/2, and $\beta = 2\mu^2/\theta$?

[Solution: The p.g.f. of a Poisson Distribution with parameter λ is: $P(z) = e^{\lambda(z - 1)}$.

For a compound distribution, the m.g.f can be written in terms of the p.g.f. of the primary distribution and m.g.f. of the secondary distribution:

$M_{\text{compound dist.}}(t) = P_{\text{primary}}[M_{\text{secondary}}[t]] = P_{\text{primary}}[P_{\text{secondary}}[e^t]] =$

$$\exp[\lambda\{P_{\text{secondary}}[e^t] - 1\}] = \exp[\lambda \frac{\{1 - \beta(e^t - 1)\}^{-r} - 1}{1 - (1 + \beta)^{-r}}] =$$

$$\exp\left[\frac{(\theta/\mu)\{\sqrt{1 + 2\mu^2/\theta} - 1\}\{\sqrt{1 - 2(\mu^2/\theta)(e^t - 1)} - 1\}}{1 - \sqrt{1 + 2\mu^2/\theta}}\right] =$$

$$\exp[(\theta/\mu)\{1 - \sqrt{1 - 2\mu^2(e^t - 1)/\theta}\}] .$$

Comment: This is the same as the m.g.f. of Poissons mixed via an Inverse Gaussian Distribution with parameters μ and θ.]

Since their moment generating functions are equal, if a Poisson is mixed by an Inverse Gaussian as per Loss Models, with parameters μ and θ, then the mixed distribution is a compound Poisson-Extended Truncated Negative Binomial Distribution as per Loss Models, with parameters: $\lambda = \frac{\theta}{\mu}(\sqrt{1 + 2\mu^2/\theta} - 1)$*, r = -1/2, and* $\beta = 2\mu^2/\theta$.[138]

This is an example of a general result:[139] If one mixes Poissons and the mixing distribution is infinitely divisible,[140] then the resulting mixed distribution can also be written as a compound Poisson distribution, with a unique secondary distribution.

The Inverse Gaussian Mixing Distribution was infinitely divisible and the result of mixing the Poissons was a Compound Poisson Distribution with a particular Extended Truncated Negative Binomial Distribution as a secondary distribution.

[138] See Example 7.17 in Loss Models, not on the syllabus.

[139] See Theorem 7.9 in Loss Models, not on the syllabus.

[140] As discussed previously, if a distribution is infinitely divisible, then if one takes the probability generating function to any positive power, one gets the probability generating function of another member of the same family of distributions. Examples of infinitely divisible distributions include: Poisson, Negative Binomial, Compound Poisson, Compound Negative Binomial, Normal, Gamma, and Inverse Gaussian.

Another example is mixing Poissons via a Gamma. The Gamma is infinitely divisible, and therefore the mixed distribution can be written as a compound distribution. As discussed previously, the mixed distribution is a Negative Binomial. It turns out that the Negative Binomial can also be written as a Compound Poisson with a logarithmic secondary distribution.

Exercise: The logarithmic frequency distribution has:

$$f(x) = \frac{\left(\frac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)}, \quad x = 1, 2, 3,... \qquad P(z) = 1 - \frac{\ln[1 - \beta(z - 1)]}{\ln[1+\beta]}, \quad z < 1 + 1/\beta.$$

Determine the probability generating function of a Compound Poisson with a logarithmic secondary distribution.

[Solution: $P_{\text{compound distribution}}(z) = P_{\text{primary}}[P_{\text{secondary}}[z]] =$

$$\exp[\lambda\{P_{\text{secondary}}[z] - 1\}] = \exp[-\lambda \frac{\ln[1 - \beta(z - 1)]}{\ln[1+\beta]}] = \exp[\frac{-\lambda}{\ln[1+\beta]} \ln[1 - \beta(z - 1)]]$$

$= \{1 - \beta(z - 1)\}^{-\lambda/\ln[1 + \beta]}$.]

The p.g.f. of the Negative Binomial is: $P(z) = \{1 - \beta(z - 1)\}^{-r}$. This is the same form as the probability generating function obtained in the exercise, with $r = \lambda/\ln[1+\beta]$ and $\beta = \beta$.

Therefore, a Compound Poisson with a logarithmic secondary distribution is a Negative Binomial Distribution with parameters $r = \lambda/\ln[1 + \beta]$ and $\beta = \beta$.[141]

When mixing Poisson frequency distributions, the p.g.f. of the mixed distribution can also be put in terms of the p.g.f. of the mixing distribution of λ. For a mixture of Poissons:

$P_{\text{mixed distribution}}(z) = E_{\text{mixing distribution of }\lambda}[P_{\text{Poisson}}(z)] = E_{\text{mixing distribution of }\lambda}[\exp[\lambda(z - 1)]] =$

$E_{\text{mixing distribution of }\lambda}[\exp[z - 1]^{\lambda}] = P_{\text{mixing distribution of }\lambda}(\exp[z - 1])$.

Thus when mixing Poissons, $P_{\text{mixed distribution}}(z) = P_{\text{mixing distribution of }\lambda}(\exp[z - 1])$.

For example, assume that a Poisson is mixed via a Negative Binomial, in other words each insured is Poisson with mean λ, but λ in turn follows a Negative Binomial across a group of insureds.

For the Negative Binomial, $P(z) = \{1 - \beta(z-1)\}^{-r}$.

Thus the mixture has probability generating function: $\{1 - \beta(\exp[z - 1] - 1)\}^{-r}$.

[141] See Example 7.5 in Loss Models, not on the syllabus.

Mixing versus Adding:

The number of accidents Alice has is Poisson with mean 3%.
The number of accidents Bob has is Poisson with mean 5%.
The number of accidents Alice and Bob have are independent.

Exercise: Determine the probability that Alice and Bob have a total of two accidents.
[Solution: Their total number of accidents is Poisson with mean 8%. $0.08^2 e^{-0.08} / 2 = 0.30\%$.
Comment: An example of adding two Poisson variables.]

Exercise: We choose either Alice or Bob at random.
Determine the probability that the chosen person has two accidents.
[Solution: $(50\%)(0.03^2 e^{-0.03} / 2) + (50\%)(0.05^2 e^{-0.05} / 2) = 0.081\%$.
Comment: A 50%-50% mixture of two Poisson Distributions with means 3% and 5%.
Mixing is different than adding.]

Problems:

Use the following information for the next three questions:
Each insured's claim frequency follows a Poisson process.
There are three types of insureds as follows:

Type	A Priori Probability	Mean Annual Claim Frequency (Poisson Parameter)
A	60%	1
B	30%	2
C	10%	3

17.1 (1 point) What is the chance of a single individual having 4 claims in a year?
A. less than 0.03
B. at least 0.03 but less than 0.04
C. at least 0.04 but less than 0.05
D. at least 0.05 but less than 0.06
E. at least 0.06

17.2 (1 point) What is the mean of this mixed distribution?
A. 1.1 B. 1.2 C. 1.3 D. 1.4 E. 1.5

17.3 (2 points) What is the variance of this mixed distribution?
A. less than 2.0
B. at least 2.0 but less than 2.1
C. at least 2.1 but less than 2.2
D. at least 2.2 but less than 2.3
E. at least 2.3

17.4 *(7 points) Each insured has its annual number of claims given by a Geometric Distribution with mean β. Across a portfolio of insureds, β is distributed as follows: $\pi(\beta) = 3/(1+\beta)^4$, $0 < \beta$.*
(a) Determine the algebraic form of the density of this mixed distribution.
(b) List the first several values of this mixed density.
(c) Determine the mean of this mixed distribution.
(d) Determine the variance of this mixed distribution.

17.5 (1 point) Each insured's claim frequency follows a Binomial Distribution, with m = 5.
There are three types of insureds as follows:

Type	A Priori Probability	Binomial Parameter q
A	60%	0.1
B	30%	0.2
C	10%	0.3

What is the chance of a single individual having 3 claims in a year?
A. less than 0.03
B. at least 0.03 but less than 0.04
C. at least 0.04 but less than 0.05
D. at least 0.05 but less than 0.06
E. at least 0.06

Use the following information for the following four questions:
- The claim count N for an individual insured has a Poisson distribution with mean λ.
- λ is uniformly distributed between 0 and 4.

17.6 (2 points) Find the probability that a randomly selected insured will have no claims.
A. Less than 0.22
B. At least 0.22 but less than 0.24
C. At least 0.24 but less than 0.26
D. At least 0.26 but less than 0.28
E. At least 0.28

17.7 (2 points) Find the probability that a randomly selected insured will have one claim.
A. Less than 0.22
B. At least 0.22 but less than 0.24
C. At least 0.24 but less than 0.26
D. At least 0.26 but less than 0.28
E. At least 0.28

17.8 (1 point) What is the mean claim frequency?

17.9 (1 point) What is the variance of the mixed frequency distribution?

17.10 (4 points) For a given value of q, the number of claims is Binomial with parameters m and q.
However, m is distributed via a Negative Binomial with parameters r and β.
What is the mixed distribution of the number of claims?

Use the following information for the next 3 questions:

Assume that given q, the number of claims observed for one risk in m trials is given by a Binomial distribution with mean mq and variance mq(1-q). Also assume that the parameter q varies between 0 and 1 for the different risks, with q following a Beta distribution:

$g(q) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} q^{a-1}(1-q)^{b-1}$, with mean $\frac{a}{a+b}$ and variance $\frac{a\,b}{(a+b)^2\,(a+b+1)}$.

17.11 (2 points) What is the unconditional mean frequency?

A. $\frac{m}{a+b}$ B. $m\,\frac{a}{a+b}$ C. $m\,\frac{a\,b}{a+b}$

D. $m\,\frac{a}{(a+b)\,(a+b+1)}$ E. $m\,\frac{a}{(a+b)^2\,(a+b+1)}$

17.12 (4 points) What is the unconditional variance?

A. $m^2\,\frac{a}{a+b}$ B. $m^2\,\frac{a}{(a+b)\,(a+b+1)}$ C. $m^2\,\frac{a\,b}{(a+b)\,(a+b+1)}$

D. $m(m+a+b)\,\frac{a\,b}{(a+b)^2\,(a+b+1)}$ E. $m(m+a+b)\,\frac{a\,b}{(a+b)\,(a+b+1)\,(a+b+2)}$

17.13 (4 points) If a = 2 and b = 4, then what is the probability of observing 5 claims in 7 trials for an individual insured?

A. less than 0.068
B. at least 0.068 but less than 0.070
C. at least 0.070 but less than 0.072
D. at least 0.072 but less than 0.074
E. at least 0.074

17.14 (2 points)
Each insured's claim frequency follows a Negative Binomial Distribution, with r = 0.8.
There are two types of insureds as follows:

Type	A Priori Probability	β
A	70%	0.2
B	30%	0.5

What is the chance of an insured picked at random having 1 claim next year?

A. 13% B. 14% C. 15% D. 16% E. 17%

17.15 (3 points) For a given value of q, the number of claims is Binomial with parameters m and q. However, m is distributed via a Poisson with mean λ.
What is the mixed distribution of the number of claims?

Use the following information for the next two questions:
The number of claims a particular policyholder makes in a year follows a distribution with parameter p: $f(x) = p(1-p)^x$, x = 0, 1, 2, ...
The values of the parameter p for the individual policyholders in a portfolio follow a Beta Distribution, with parameters a = 4, b = 5, and θ = 1: $g(p) = 280\ p^3(1-p)^4, 0 \le p \le 1$.

17.16 (2 points) What is the a priori mean annual claim frequency for the portfolio?
A. less than 1.5
B. at least 1.5 but less than 1.6
C. at least 1.6 but less than 1.7
D. at least 1.7 but less than 1.8
E. at least 1.8

17.17 (3 points) For an insured picked at random from this portfolio, what is the probability of observing 2 claims next year?
A. 9% B. 10% C. 11% D. 12% E. 13%

Use the following information for the next 2 questions:
(i) An individual insured has an annual claim frequency that follow a Poisson distribution with mean λ.
(ii) Across the portfolio of insureds, the parameter λ has probability density function:

$$\Pi(\lambda) = (0.8)(40e^{-40\lambda}) + (0.2)(10e^{-10\lambda}).$$

17.18 (1 point) What is the expected annual frequency?
(A) 3.6% (B) 3.7% (C) 3.8% (D) 3.9% (E) 4.0%

17.19 (2 points) For an insured picked at random, what is the probability that he will have at least one claim in the coming year?
(A) 3.6% (B) 3.7% (C) 3.8% (D) 3.9% (E) 4.0%

17.20 (4 points) For a given value of q, the number of claims is Binomial with parameters m and q. However, m is distributed via a Binomial with parameters 5 and 0.1.
What is the mixed distribution of the number of claims?

Use the following information for the next four questions:

For a given value of q, the number of claims is Binomial distributed with parameters m = 3 and q.

In turn q is distributed uniformly from 0 to 0.4.

17.21 (2 points) What is the chance that zero claims are observed?
A. Less than 0.52
B. At least 0.52 but less than 0.53
C. At least 0.53 but less than 0.54
D. At least 0.54 but less than 0.55
E. At least 0.55

17.22 (2 points) What is the chance that one claim is observed?
A. Less than 0.32
B. At least 0.32 but less than 0.33
C. At least 0.33 but less than 0.34
D. At least 0.34 but less than 0.35
E. At least 0.35

17.23 (2 points) What is the chance that two claims are observed?
A. Less than 0.12
B. At least 0.12 but less than 0.13
C. At least 0.13 but less than 0.14
D. At least 0.14 but less than 0.15
E. At least 0.15

17.24 (2 points) What is the chance that three claims are observed?
A. Less than 0.01
B. At least 0.01 but less than 0.02
C. At least 0.02 but less than 0.03
D. At least 0.03 but less than 0.04
E. At least 0.04

17.25 (2 points) For students at a certain college, 40% do not own cars and do not drive.

For the rest of the students, their accident frequency is Poisson with $\lambda = 0.07$.

Let T = the total number of accidents for a group of 100 students picked at random.
What is the variance of T?

A. 4.0 B. 4.1 C. 4.2 D. 4.3 E. 4.4

Use the following information for the next 7 questions:
On his daily walk, Clumsy Klem loses coins at a Poisson rate.
At random, on half the days, Klem loses coins at a rate of 0.2 per minute.
On the other half of the days, Klem loses coins at a rate of 0.6 per minute.
The rate on any day is independent of the rate on any other day.

17.26 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the sixth minute of today's walk.
(A) 0.21 (B) 0.23 (C) 0.25 (D) 0.27 (E) 0.29

17.27 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first <u>two</u> minutes of today's walk.
A. Less than 32%
B. At least 32%, but less than 34%
C. At least 34%, but less than 36%
D. At least 36%, but less than 38%
E. At least 38%

17.28 (2 points) Let A = the number of coins that Clumsy Klem loses during the first minute of <u>today's</u> walk. Let B = the number of coins that Clumsy Klem loses during the first minute of <u>tomorrow's</u> walk. Calculate Prob[A + B = 1].
(A) 0.30 (B) 0.32 (C) 0.34 (D) 0.36 (E) 0.38

17.29 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of today's walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

17.30 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of <u>tomorrow's</u> walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

17.31 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first four minutes of today's walk and exactly one coin during the first four minutes of <u>tomorrow's</u> walk.
A. Less than 8.5%
B. At least 8.5%, but less than 9.0%
C. At least 9.0%, but less than 9.5%
D. At least 9.5%, but less than 10.0%
E. At least 10.0%

17.32 (3 points) Calculate the probability that Clumsy Klem loses exactly one coin during the first 2 minutes of today's walk, and exactly two coins during the following 3 minutes of today's walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

Use the following information for the next two questions:

Each insured has its accident frequency given by a Poisson Distribution with mean λ.

For a portfolio of insureds, λ is distributed as follows on the interval from a to b:

$$f(\lambda) = \frac{(d+1)\ \lambda^d}{b^{d+1} - a^{d+1}},\ 0 \le a \le \lambda \le b \le \infty.$$

17.33 (2 points) If the parameter d = -1/2, and if a = 0.2 and b = 0.6, what is the mean frequency?
A. less than 0.35
B. at least 0.35 but less than 0.36
C. at least 0.36 but less than 0.37
D. at least 0.37 but less than 0.38
E. at least 0.38

17.34 (2 points) If the parameter d = -1/2, and if a = 0.2 and b = 0.6, what is the variance of the frequency?
A. less than 0.39
B. at least 0.39 but less than 0.40
C. at least 0.40 but less than 0.41
D. at least 0.41 but less than 0.42
E. at least 0.42

17.35 (3 points) Let X be a 50%-50% weighting of two Binomial Distributions.
The first Binomial has parameters m = 6 and q = 0.8.
The second Binomial has parameters m = 6 and q unknown.
For what value of q, does the mean of X equal the variance of X?
A. 0.3 B. 0.4 C. 0.5 D. 0.6 E. 0.7

Use the following information for the next 2 questions:
(i) Claim counts for individual insureds follow a Poisson distribution.
(ii) Half of the insureds have expected annual claim frequency of 4%.
(iii) The other half of the insureds have expected annual claim frequency of 10%.

17.36 (1 point) An insured is picked at random.
What is the probability that this insured has more than 1 claim next year?
(A) 0.21% (B) 0.23% (C) 0.25% (D) 0.27% (E) 0.29%

17.37 (1 point) A large group of such insured is observed for one year.
What is the variance of the distribution of the number of claims observed for individuals?
(A) 0.070 (B) 0.071 (C) 0.072 (D) 0.073 (E) 0.074

Use the following information for the next three questions:

An insurance company sells two types of policies with the following characteristics:

Type of Policy	Proportion of Total Policies	Annual Claim Frequency
I	25%	Poisson with $\lambda = 0.25$
II	75%	Poisson with $\lambda = 0.50$

17.38 (1 point) What is the probability that an insured picked at random will have no claims next year?
A. 50% B. 55% C. 60% D. 65% E. 70%

17.39 (1 point) What is the probability that an insured picked at random will have one claim next year?
A. less than 30%
B. at least 30% but less than 35%
C. at least 35% but less than 40%
D. at least 40% but less than 45%
E. at least 45%

17.40 (1 point) What is the probability that an insured picked at random will have two claims next year?
A. 4% B. 6% C. 8% D. 10% E. 12%

17.41 (3 points) The Spiders sports team will play a best of 3 games playoff series.
They have an 80% chance to win each home game and only a 40% chance to win each road game.
The results of each game are independent of the results of any other game.
It has yet to be determined whether one or two of the three games will be home games for the Spiders, but you assume these two possibilities are equally likely.
What is the chance that the Spiders win their playoff series?
A. 63% B. 64% C. 65% D. 66% E. 67%

17.42 (4 points) The number of claims is modeled as a two point mixture of Poisson Distributions, with weight p to a Poisson with mean λ_1 and weight (1-p) to a Poisson with mean λ_2.

(a) For the mixture, determine the ratio of the variance to the mean as a function of λ_1, λ_2, and p.

(b) With the aid of a computer, for λ_1 = 10% and λ_2 = 20%,

graph this ratio as a function of p for $0 \le p \le 1$.

Use the following information for the next five questions:

For a given value of q, the number of claims is Binomial distributed with parameters m = 4 and q.

In turn q is distributed from 0 to 0.6 via: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

17.43 (3 points) What is the chance that zero claims are observed?

A. 12% B. 14% C. 16% D. 18% E. 20%

17.44 (3 points) What is the chance that one claim is observed?

A. 26% B. 26% C. 28% D. 30% E. 32%

17.45 (3 points) What is the chance that two claims are observed?

A. 26% B. 26% C. 28% D. 30% E. 32%

17.46 (2 points) What is the chance that three claims are observed?

A. 19% B. 21% C. 23% D. 25% E. 27%

17.47 (2 points) What is the chance that four claims are observed?

A. 3% B. 4% C. 5% D. 6% E. 7%

17.48 (3 points) Use the following information:

- There are two types of insurance policies.
- Three quarters are low risk policies, while the remaining one quarter are high risk policies.
- The annual claims from each type of policy are Poisson.
- The mean number of claims from a high risk policy is 0.4.
- The variance of the mixed distribution of the number of claims is 0.2575.

Determine the mean annual claims from a low risk policy.

A. 12% B. 14% C. 16% D. 18% E. 20%

17.49 (8 points) One has a mixture of Poisson Distributions each with mean λ.

The mixing distribution of λ is Poisson with mean μ.

(a) (2 points) Determine the mean and variance of the mixture.

(b) (3 points) Determine the form of Probability Generating Function of the mixture.

(c) (3 points) Use the Probability Generating Function to determine the mean and variance of the mixture, and verify that they match the results in part (a).

17.50 (8.5 points) Use the following information:

- The number of families immigrating to the principality of Genovia per month is a 50%-50% mixture of Poisons Distributions with $\lambda = 4$ and $\lambda = 10$.
- The number of members per family immigrating is a 30%-70% mixture of zero-truncated Binomial Distributions with $m = 2$ and $q = 0.5$, or $m = 8$ and $q = 0.4$.
- The number of families and their sizes are independent.

(a) (2 points) Determine the mean and variance of the distribution of the number of families.
(b) (4 points) Determine the mean and variance of the distribution of the sizes of families.
(c) (1.5 points) Determine the mean and variance of the distribution of the number of people immigrating per month.
(d) (1 point) Using the Normal Approximation with continuity correction, estimate the probability that at least 25 people will immigrate next month.

17.51 (3 points) One has a mixture of Geometric Distributions with mean β.
The mixing distribution of β is Beta with $a = 4$, $b = 3$, and $\theta = 1$.
Determine the mean and variance of the mixture.

Use the following information for the next three questions:

- Frequency is a 50%-50% mixture of Poissons with means 1 and 2.
- Severity is uniform from 0 to 1000.
- Frequency and severity are independent.

17.52 (2 points) What is the probability of exactly 2 claims of size greater than 600?
(There can be any number of small claims.)
A. 10% B. 11% C. 12% D. 13% E. 14%

17.53 (2 points) What is the probability of exactly 2 claims of size less than 600?
(There can be any number of large claims.)
A. 13% B. 14% C. 15% D. 16% E. 17%

17.54 (2 points) What is the probability of exactly 2 claims of size greater than 600 and exactly 2 claims of size less than 600?
A. 1.0% B. 1.2% C. 1.4% D. 1.6% E. 1.8%

17.55 (4, 11/82, Q.48) (3 points)

Let $f(x|\theta)$ = frequency distribution for a particular risk having parameter θ.

$f(x|\theta) = \theta(1-\theta)^x$, where θ is in the interval [p, 1], p is a fixed value such that $0 < p < 1$, and x is a non-negative integer.

$g(\theta)$ = distribution of θ within a given class of risks. $g(\theta) = \frac{-1}{\theta \ln(p)}$, for $p \le \theta \le 1$.

Find the frequency distribution for the class of risks.

A. $\frac{-(x+1)\,(1-p)^x}{p^2 \ln(p)}$ B. $\frac{-p^{x+1}}{(x+1)\ln(p)}$ C. $\frac{-(1-p)^{x+1}}{(x+1)\ln(p)}$ D. $\frac{-(x+1)\,p^x}{\ln(p)}$

E. None A, B, C, or D.

17.56 (2, 5/88, Q.33) (1.5 points) Let X have a binomial distribution with parameters m and q, and let the conditional distribution of Y given X = x be Poisson with mean x.
What is the variance of Y?

A. x B. mq C. mq(1 - q) D. mq^2 E. mq(2 - q)

17.57 (4, 5/88, Q.32) (2 points) Let N be the random variable which represents the number of claims observed in a one year period. N is Poisson distributed with a probability density function with parameter θ: $P[N = n \mid \theta] = e^{-\theta}\,\theta^n/n!$, n = 0, 1, 2, ...

The probability of observing no claims in a year is less than .450.

Which of the following describe possible probability distributions for θ?

1. θ is uniformly distributed on (0, 2).

2. The probability density function of θ is $f(\theta) = e^{-\theta}$ for $\theta > 0$.

3. $P[\theta = 1] = 1$ and $P[\theta \ne 1] = 0$.

A. 1 B. 2 C. 3 D. 1, 2 E. 1, 3

17.58 (3, 11/00, Q.13 & 2009 Sample Q.114) (2.5 points)
A claim count distribution can be expressed as a mixed Poisson distribution.
The mean of the Poisson distribution is uniformly distributed over the interval [0, 5].
Calculate the probability that there are 2 or more claims.
(A) 0.61 (B) 0.66 (C) 0.71 (D) 0.76 (E) 0.81

17.59 (SOA3, 11/04, Q.32 & 2009 Sample Q.130) (2.5 points)

Bob is a carnival operator of a game in which a player receives a prize worth $W = 2^N$ if the player has N successes, N = 0, 1, 2, 3,…

Bob models the probability of success for a player as follows:

(i) N has a Poisson distribution with mean Λ.

(ii) Λ has a uniform distribution on the interval (0, 4).

Calculate E[W].

(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

17.60 (CAS3, 11/06, Q.19) (2.5 points)

In 2006, annual claim frequency follows a negative binomial distribution with parameters β and r.

β follows a uniform distribution on the interval (0, 2) and r = 4.

Calculate the probability that there is at least 1 claim in 2006.

A. Less than 0.85
B. At least 0.85, but less than 0.88
C. At least 0.88, but less than 0.91
D. At least 0.91, but less than 0.94
E. At least 0.94

17.61 (SOA M, 11/06, Q.39 & 2009 Sample Q.288) (2.5 points)

The random variable N has a mixed distribution:

(i) With probability p, N has a binomial distribution with q = 0.5 and m = 2.

(ii) With probability 1 - p, N has a binomial distribution with q = 0.5 and m = 4.

Which of the following is a correct expression for Prob(N = 2)?

(A) $0.125p^2$

(B) $0.375 + 0.125p$

(C) $0.375 + 0.125p^2$

(D) $0.375 - 0.125p^2$

(E) $0.375 - 0.125p$

Solutions to Problems:

17.1. D. Chance of observing 4 accidents is $\theta^4 e^{-\theta} / 24$. Weight the chances of observing 4 accidents by the a priori probability of θ.

	A Priori	Poisson	Chance of
Type	Probability	Parameter	4 Claims
A	0.6	1	0.0153
B	0.3	2	0.0902
C	0.1	3	0.1680
Average			**0.053**

17.2. E. (60%)(1) + (30%)(2) + (10%)(3) = **1.5**.

17.3. A. For a Type A insured, the second moment is: variance + mean^2 = 1 + 1^2 = 2.
For a Type B insured, the second moment is: variance + mean^2 = 2 + 2^2 = 6.
For a Type C insured, the second moment is: variance + mean^2 = 3 + 3^2 = 12.
The second moment of the mixture is: (60%)(2) + (30%)(6) + (10%)(12) = 4.2.
The variance of the mixture is: 4.2 - 1.5^2 = **1.95**.
Alternately, the Expected Value of the Process Variance is:
(60%)(1) + (30%)(2) + (10%)(3) = 1.5.
The Variance of the Hypothetical Means is:
$(60\%)(1 - 1.5)^2 + (30\%)(2 - 1.5)^2 + (10\%)(3 - 1.5)^2 = 0.45$.
Total Variance = EPV + VHM = 1.5 + 0.45 = **1.95**.
Comment: For the mixed distribution, the variance is greater than the mean.

17.4. (a) For the Geometric distribution, $f(x) = \beta^x/(1+\beta)^{x+1}$.

For the mixed distribution, $f(x) = \int_0^\infty f(x;\beta)\,\pi(\beta)\,d\beta = \int_0^\infty \beta^x / (1+\beta)^{x+1}\ 3/(1+\beta)^4\,d\beta = 3\int_0^1 u^3 (1-u)^x\,du$,

where $u = 1/(1+\beta)$, $1 - u = \beta/(1+\beta)$, and $du = -1/(1+\beta)^2$.

This integral is of the Beta variety; its value of $\Gamma(x+1)\ \Gamma(3+1) / \Gamma(x + 1 + 3 + 1)$,

follows from the fact that the density of a Beta Distribution integrates to one over its support.

Therefore, $f(x) = (3)\{\Gamma(x+1)\ \Gamma(3+1) / \Gamma(x + 1 + 3 + 1)\} = (3)(x!)(3!)/(x+4)! =$

$18 / \{(x+1)(x+2)(x+3)(x+4)\}$.

(b) The densities from 0 to 20 are:

3/4, 3/20, 1/20, 3/140, 3/280, 1/168, 1/280, 1/440, 1/660, 3/2860, 3/4004, 1/1820, 3/7280, 3/9520, 1/4080, 1/5168, 1/6460, 1/7980, 3/29260, 3/35420, 1/14168.

(c) The mean of this mixed distribution is: $\int_0^\infty \beta\,\pi(\beta)\,d\beta = \int_0^\infty \beta\ 3/(1+\beta)^4\,d\beta = 3\int_0^1 u(1-u)\,du =$

(3)(1/2 - 1/3) = **1/2**.

(d) The second moment of a Geometric is: variance + mean^2 = $\beta(1+\beta) + \beta^2 = \beta + 2\beta^2$.

$\int_0^\infty \beta^2\,\pi(\beta)\,d\beta = \int_0^\infty \beta^2\ 3/(1+\beta)^4\,d\beta = 3\int_0^1 (1-u)^2\,du = 3/3 = 1$.

Therefore, the second moment of this mixed distribution is: 1/2 + (2)(1) = 2.5.

The variance of this mixed distribution is: $2.5 - 0.5^2$ = **2.25**.

Comment: This is a Yule Distribution as discussed In Example 7.13 of Loss Models, with a = 3.

The mixed density can also be written in terms of a complete Beta function: $3\ \beta[4, x+1]$.

17.5. B. Chance of observing 3 claims is $10q^3 (1-q)^2$. Weight the chances of observing 3 claims by the a priori probability of q.

Type	A Priori Probability	q Parameter	Chance of 3 Claims
A	0.6	0.1	0.0081
B	0.3	0.2	0.0512
C	0.1	0.3	0.1323
Average			**0.033**

17.6. C. The chance of no claims for a Poisson is: $e^{-\lambda}$.

We average over the possible values of λ:

$$(1/4)\int_0^4 e^{-\lambda}\, d\lambda = (1/4)(-e^{-\lambda})\Big]_{\lambda=0}^{\lambda=4} = (1/4)(1 - e^{-4}) = \mathbf{0.245}.$$

17.7. B. The chance of one claim for a Poisson is: $\lambda e^{-\lambda}$.

We average over the possible values of λ:

$$(1/4)\int_0^4 \lambda\, e^{-\lambda}\, d\lambda = (1/4)(-\lambda\, e^{-\lambda} - e^{-\lambda})\Big]_{\lambda=0}^{\lambda=4} = (1/4)(1 - 5e^{-4}) = \mathbf{0.227}.$$

Comment: The densities of this mixed distribution from 0 to 9: 0.245421, 0.227105, 0.190474, 0.141632, 0.0927908, 0.0537174, 0.0276685, 0.0127834, 0.00534086, 0.00203306.

17.8. $E[\lambda] = (0 + 4)/2 = \mathbf{2}$.

17.9. The second moment of a Poisson is: variance + mean^2 = $\lambda + \lambda^2$.

$E[\lambda + \lambda^2] = E[\lambda] + E[\lambda^2]$ = mean of uniform distribution + second moment of uniform distribution

$= 2 + \{2^2 + (4 - 0)^2/12\} = 2 + 4 + 1.333 = 7.333$.

variance = second moment - mean^2 = $7.333 - 2^2 = \mathbf{3.333}$.

17.10. The p.g.f. of each Binomial is: $\{1 + q(z-1)\}^m$.
The p.g.f. of the mixture is the mixture of the p.g.f.s:

$P_{mixture}[z] = \Sigma f(m)\{1 + q(z-1)\}^m$ = p.g.f. of f at: $1 + q(z-1)$.

However, f(m) is Negative Binomial, with p.g.f.: $\{1 - \beta(z-1)\}^{-r}$

Therefore, $P_{mixture}[z] = \{1 - \beta(1 + q(z-1) - 1)\}^{-r} = \{1 - \beta q(z-1)\}^{-r}$.

However, this is the p.g.f. of a Negative Binomial Distribution with parameters r and qβ, which is therefore the mixed distribution.
Alternately, the mixed distribution at k is:

$$\sum_{m=k}^{m=\infty} \text{Prob}[k \mid m]\ \text{Prob}[m] = \sum_{m=k}^{m=\infty} \frac{m!}{(m-k)!\ k!} q^k (1-q)^{m-k} \frac{(r+m-1)!}{(r-1)!\ m!} \frac{\beta^m}{(1+\beta)^{r+m}} =$$

$$\frac{q^k \beta^k}{(1+\beta)^{r+k}} \frac{(r+k-1)!}{(r-1)!\ k!} \sum_{m=k}^{m=\infty} \frac{(r+m-1)!}{(m-k)!\ (r+k-1)!} \left(\frac{(1-q)\ \beta}{1\ +\ \beta}\right)^{m-k} =$$

$$\frac{q^k \beta^k}{(1+\beta)^{r+k}} \frac{(r+k-1)!}{(r-1)!\ k!} \sum_{n=0}^{n=\infty} \frac{(r+k+n-1)!}{n!\ (r+k-1)!} \left(\frac{(1-q)\ \beta}{1\ +\ \beta}\right)^{n} =$$

$$\frac{q^k \beta^k}{(1+\beta)^{r+k}} \frac{(r+k-1)!}{(r-1)!\ k!} \left(\frac{1}{1\ -\ (1-q)\beta / (1+\beta)}\right)^{r+k} =$$

$$\frac{q^k \beta^k}{(1+\beta)^{r+k}} \frac{(r+k-1)!}{(r-1)!\ k!} \frac{(1+\beta)^{r+k}}{(1\ +\ q\beta)^{r+k}} = \frac{(r+k-1)!}{(r-1)!\ k!} \frac{(q\beta)^k}{(1\ +\ q\beta)^{r+k}}.$$

This is a Negative Binomial Distribution with parameters r and qβ.
Comment: The sum was simplified using the fact that the Negative Binomial densities sum to 1:

$$1 = \sum_{i=0}^{i=\infty} \frac{(s+i-1)!}{i!\ (s-1)!} \frac{\alpha^i}{(1+\alpha)^{s+i}}. \Rightarrow \sum_{i=0}^{i=\infty} \frac{(s+i-1)!}{i!\ (s-1)!} \left(\frac{\alpha}{1+\alpha}\right)^i = (1+\alpha)^s.$$

$$\Rightarrow \sum_{i=0}^{i=\infty} \frac{(s+i-1)!}{i!\ (s-1)!} \gamma^i = \frac{1}{(1-\gamma)^s}.$$

Where, $\gamma = \alpha/(1+\alpha)$. $\Rightarrow \alpha = \gamma/(1-\gamma)$. $\Rightarrow 1+ \alpha = 1/(1-\gamma)$.

17.11. B. The conditional mean given q is: mq. The unconditional mean can be obtained by integrating the conditional means versus the distribution of q:

$$E[X] = \int_0^1 E[X \mid q]\, g(q)\, dq = \int_0^1 mq \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} q^{a-1} (1-q)^{b-1}\, dq =$$

$$m \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \int_0^1 q^a (1-q)^{b-1}\, dq = m \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \frac{\Gamma(a+1)\,\Gamma(b)}{\Gamma(a+b+1)} = m \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+1)} = \mathbf{ma / (a+b)}.$$

Alternately,

$$E[X] = \int_0^1 E[X \mid q]\, g(q)\, dq = m \int_0^1 q\, g(q)\, dq = m \text{ (mean of Beta Distribution)} = \mathbf{ma / (a+b)}.$$

Comment: The Beta distribution with $\theta = 1$ has density from 0 to 1 of: $\frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} x^{a-1}(1-x)^{b-1}$.

Therefore, the integral from zero to one of $x^{a-1}(1-x)^{b-1}$ is: $\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$.

17.12. D. The conditional variance given q is: mq(1-q) = mq - mq^2. Thus the conditional second moment given q is: mq - mq^2 + (mq)^2 = mq + (m^2 - m)q^2. The unconditional second moment can be obtained by integrating the conditional second moments versus the distribution of q:

$$E[X^2] = \int_0^1 E[X^2 \mid q]\, g(q)\, dq = \int_0^1 \{mq + (m^2 - m)q^2\} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} q^{a-1}(1-q)^{b-1}\, dq =$$

$$m \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 q^a (1-q)^{b-1}\, dq + (m^2 - m) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 q^{a+1}(1-q)^{b-1}\, dq =$$

$$m \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\,\Gamma(b)}{\Gamma(a+b+1)} + (m^2 - m) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\,\Gamma(b)}{\Gamma(a+b+2)} =$$

$$m \frac{\Gamma(a+1)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+1)} + (m^2 - m) \frac{\Gamma(a+2)}{\Gamma(a)} \frac{\Gamma(a+b)}{\Gamma(a+b+2)} =$$

$m \dfrac{a}{a+b} + (m^2 - m) \dfrac{a\,(a+1)}{(a+b)(a+b+1)}$. Since the mean is $m \dfrac{a}{a+b}$, the variance is:

$$m \frac{a}{a+b} + (m^2 - m) \frac{a\,(a+1)}{(a+b)(a+b+1)} - m^2 \frac{a^2}{(a+b)^2} =$$

$$m \frac{a}{(a+b)^2\,(a+b+1)} \{(a+b+1)(a+b) + (m-1)(a+1)(a+b) - ma(a+b+1)\} =$$

$$m \frac{a}{(a+b)^2\,(a+b+1)} (ab + b^2 + mb) = \mathbf{m \frac{(m+a+b)\, a\, b}{(a+b)^2\,(a+b+1)}}.$$

Alternately, $E[X^2] = \int_0^1 E[X^2 \mid q]\, g(q)\, dq = m \int_0^1 q\, g(q) dq + (m^2 - m) \int_0^1 q^2\, g(q) dq =$

m (mean of Beta Distribution) + (m^2 - m) (second moment of the Beta Distribution) =

$m \dfrac{a}{a+b} + (m^2 - m) \dfrac{a\,(a+1)}{(a+b)(a+b+1)}$. Then proceed as before.

<u>Comment</u>: This is an example of the Beta-Binomial Conjugate Prior Process. See “Mahler’s Guide to Conjugate Priors.” The unconditional distribution is sometimes called a “Beta-Binomial” Distribution. See Example 7.12 in <u>Loss Models</u> or Kendall’s Advanced Theory of Statistics by Stuart and Ord.

17.13. E. The probability density of q is a Beta Distribution with parameters a and b:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} q^{a-1}(1-q)^{b-1}.$$

One can compute the unconditional density via integration:

$$f(n) = \int_0^1 f(n \mid q) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} q^{a-1}(1-q)^{b-1} dq =$$

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{m!}{n!\,(m-n)!} q^n (1-q)^{m-n} q^{a-1}(1-q)^{b-1} dq =$$

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(m+1)}{\Gamma(n+1)\,\Gamma(m+1-n)} \int_0^1 q^{a+n-1} (1-q)^{b+m-n-1} dq =$$

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(m+1)}{\Gamma(n+1)\,\Gamma(m+1-n)} \frac{\Gamma(a+n)\,\Gamma(b+m-n)}{\Gamma(a+b+m)} = \frac{\Gamma(a+b)\,\Gamma(m+1)\,\Gamma(a+n)\,\Gamma(b+m-n)}{\Gamma(a)\,\Gamma(b)\,\Gamma(n+1)\,\Gamma(m+1-n)\,\Gamma(a+b+m)}.$$

For n = 5, a = 2, b = 4, and m = 7.

$$f(5) = \frac{\Gamma(6)\,\Gamma(8)\,\Gamma(7)\,\Gamma(6)}{\Gamma(2)\,\Gamma(4)\,\Gamma(6)\,\Gamma(3)\,\Gamma(13)} = \frac{5!\,7!\,6!\,5!}{1!\,3!\,5!\,2!\,\,12!} = \mathbf{0.07576}.$$

<u>Comment:</u> Beyond what you are likely to be asked on your exam.
The probability of observing other number of claims in 7 trials is as follows:

n	0	1	2	3	4	5	6	7
f(n)	0.15152	0.21212	0.21212	0.17677	0.12626	0.07576	0.03535	0.01010
F(n)	0.15152	0.36364	0.57576	0.75253	0.87879	0.95455	0.98990	1.00000

This is an example of the "Binomial-Beta" distribution with: a = 2, b = 4, and m = 7.

17.14. B. For a Negative Binomial Distribution, $f(1) = \frac{r\beta}{(1 + \beta)^{r+1}}$.

For Type A: $f(1) = (0.8)(0.2) / (1.2^{1.8}) = 11.52\%$.

For Type B: $f(1) = (0.8)(0.5) / (1.5^{1.8}) = 19.28\%$.

(70%) (11.52%) + (30%) (19.28%) = **13.85%**.

17.15. The p.g.f. of each Binomial is: $\{1 + q(z-1)\}^m$.
The p.g.f. of the mixture is the mixture of the p.g.f.s:

$P_{mixture}[z] = \Sigma f(m)\{1 + q(z-1)\}^m$ = p.g.f. of f at: $1 + q(z-1)$.

However, f(m) is Poisson, with p.g.f.: $\exp[\lambda(z-1)]$.

Therefore, $P_{mixture}[z] = \exp[\lambda\{1 + q(z-1) - 1\}] = \exp[\lambda q(z-1)]$.

However, this is the p.g.f. of a Poisson Distribution with mean $q\lambda$, which is therefore the mixed distribution.
Alternately, the mixed distribution at k is:

$$\sum_{m=k}^{\infty} \text{Prob}[k \mid m]\text{Prob}[m] = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!\,k!}\, q^k (1-q)^{m-k} \frac{e^{-\lambda} \lambda^m}{m!}$$

$$= \frac{q^k e^{-\lambda} \lambda^k}{k!} \sum_{m=k}^{\infty} (1-q)^{m-k} \frac{\lambda^{m-k}}{(m-k)!} = \frac{q^k e^{-\lambda} \lambda^k}{k!} \sum_{n=0}^{\infty} (1-q)^n \frac{\lambda^n}{n!} = \frac{q^k e^{-\lambda} \lambda^k}{k!} \exp[(1-q)\lambda]$$

$= (q\lambda)^k e^{-\lambda q} / k!$. This is a Poisson Distribution with mean $q\lambda$.

17.16. C. This is a Geometric Distribution (a Negative Binomial with r =1), parameterized somewhat differently than in Loss Models, with $p = 1/(1 + \beta)$. Therefore for a given value of p the mean is: $\mu(p) = \beta = (1-p)/p$. In order to get the average mean over the whole portfolio we need to take the integral of $\mu(p)\, g(p)\, dp$.

$$\int_0^1 \mu(p)\, g(p)\, dp = \int_0^1 \{(1-p)/p\}\, 280\, p^3 (1-p)^4\, dp = 280 \int_0^1 p^2 (1-p)^5\, dp = 280\, \Gamma(3)\Gamma(6) / \Gamma(3+6)$$

$= 280\, (2!)(5!) / 8! =$ **5/3** .

Comment: Difficult! Special case of mixing a Negative Binomial (for r fixed) via a Beta Distribution. See Example 7.13 in Loss Models, where the mixed distribution is called the Generalized Waring. For the Generalized Waring in general, the a priori mean turns out to be rb/(a-1). For r =1, b = 5 and a = 4, the a priori mean is (1)(5)/3 = 5/3.

17.17. D. The probability density of p is a Beta Distribution with parameters a and b:

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ p^{a-1}(1-p)^{b-1}.$$

One can compute the unconditional density at n via integration:

$$f(n) = \int_0^1 f(n \mid p)\ \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ p^{a-1}(1-p)^{b-1}\ dp = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p(1-p)^x\ p^{a-1}(1-p)^{b-1}\ dp =$$

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^a(1-p)^{b+n-1}\ dp = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ \frac{\Gamma(a+1)\ \Gamma(b+n)}{\Gamma(a+b+n+1)} = a\ \frac{\Gamma(a+b)\ \Gamma(b+n)}{\Gamma(b)\ \Gamma(a+b+n+1)}.$$

For a = 4, b = 5: $f(n) = 4\ \frac{\Gamma(9)\ \Gamma(5+n)}{\Gamma(5)\ \Gamma(10+n)} = 4\ \frac{8!\ (n+4)!}{4!\ (n+9)!} = 6720\ \frac{(n+4)!}{(n+9)!}$.

$f(2) = 6720\ \frac{6!}{11!}$ = **12.1%**.

<u>Comment:</u> The Beta distribution with θ = 1 has density from 0 to 1 of: $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ x^{a-1}(1-x)^{b-1}$.

Therefore, the integral from zero to of $x^{a-1}(1-x)^{b-1}$ is: Γ(a)Γ(b) / Γ(a+b).

This is an example of a Generalized Waring Distribution, with r = 1, a = 4 and b = 5.

See Example 7.13 in <u>Loss Models</u>. The probabilities of observing 0 to 20 claims is as follows:
0.444444, 0.222222, 0.121212, 0.0707071, 0.043512, 0.027972, 0.018648, 0.0128205, 0.00904977, 0.00653595, 0.00481596, 0.00361197, 0.00275198, 0.00212653, 0.00166424, 0.00131752, 0.00105402, 0.000851323, 0.00069367, 0.000569801, 0.000471559.

Since the densities must add to unity:

$$1 = \sum_{n=0}^{\infty} a\ \frac{\Gamma(a+b)\ \Gamma(b+n)}{\Gamma(b)\ \Gamma(a+b+n+1)}. \Rightarrow \sum_{n=0}^{\infty} \frac{\Gamma(b+n)}{\Gamma(a+b+n+1)} = \frac{\Gamma(b)}{a\ \Gamma(a+b)}.$$

17.18. E. E[λ] = the mean of the prior mixed exponential = weighted average of the means of the two exponential distributions = (0.8)(1/40) + (0.2)(1/10) = **4.0%**.

17.19. C. Given λ, $f(0) = e^{-\lambda}$.

$$\int_0^\infty f(0;\lambda)\,\pi(\lambda)\,d\lambda = 32\int_0^\infty e^{-41\lambda}\,d\lambda + 2\int_0^\infty e^{-11\lambda}\,d\lambda = (32/41) + (2/11) = 0.9623.$$

Prob[at least one claim] = 1 - 0.9623 = **3.77%**.

17.20. The p.g.f. of each Binomial is: $\{1 + q(z-1)\}^m$.
The p.g.f. of the mixture is the mixture of the p.g.f.s:

$$P_{mixture}[z] = \sum f(m)\,\{1 + q(z-1)\}^m = \text{p.g.f. of f at: } 1 + q(z-1).$$

However, f(m) is Binomial with parameters 5 and 0.1, with p.g.f.: $\{1 + 0.1(z-1)\}^5$.

Therefore, $P_{mixture}[z] = \{1 + .1(1 + q(z-1) - 1)\}^5 = \{1 + .1q(z-1)\}^5$.

However, this is the p.g.f. of a Binomial Distribution with parameters 5 and 0.1q, which is therefore the mixed distribution.
Alternately, the mixed distribution at $k \le 5$ is:

$$\sum_{m=k}^{\infty} \text{Prob}[k \mid m]\text{Prob}[m] = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!\,k!} q^k (1-q)^{m-k} \frac{5!}{(5-m)!\,m!} 0.1^m\, 0.9^{5-m}$$

$$= q^k\, 0.1^k \frac{5!}{(5-k)!\,k!} \sum_{m=k}^{\infty} \frac{(5-k)!}{(m-k)!\,(5-m)!} (1-q)^{m-k}\, 0.1^{m-k}\, 0.9^{5-m}$$

$$= q^k\, 0.1^k \frac{5!}{(5-k)!\,k!} \sum_{n=0}^{\infty} \frac{(5-k)!}{n!\,(5-k-n)!} (1-q)^n\, 0.1^n\, 0.9^{5-k-n}$$

$$= q^k\, 0.1^k \frac{5!}{(5-k)!\,k!} \{(1-q)(0.1) + 0.9\}^{5-k} = \frac{5!}{(5-k)!\,k!} (0.1q)^k (1 - 0.1q)^{5-k}.$$

This is a Binomial Distribution with parameters 5 and 0.1q.
<u>Comment</u>: The sum was simplified using the Binomial expansion:

$$(x + y)^m = \sum_{i=0}^{m} x^i y^{m-i} \frac{m!}{i!\,(m-i)!}.$$

17.21. D. Given q, we have a Binomial with parameters m = 3 and q. The chance that we observe zero claims is: $(1-q)^3$. The distribution of q is uniform: $\pi(q) = 2.5$ for $0 \le q \le 0.4$.

$$f(0) = \int_0^{0.4} f(0 \mid q)\, \pi(q)\, dq = \int_0^{0.4} (1 - q)^3 (2.5)\, dq = (-2.5/4)(1 - q)^4 \Big]_{q=0}^{q=0.4}$$

$= (-0.625)(0.6^4 - 1^4) = \mathbf{0.544}$.

17.22. B. Given q, we have a Binomial with parameters m = 3 and q.

The chance that we observe one claim is: $3q(1-q)^2 = 3q - 6q^2 + 3q^3$.

$$P(c=1) = \int_0^{0.4} P(c=1 \mid q)\, \pi(q)\, dq = \int_0^{0.4} (3q - 6q^2 + 3q^3)(2.5)\, dq =$$

$$(2.5)(1.5q^2 - 2q^3 + 0.75q^4) \Big]_{q=0}^{q=0.4} = (2.5)(0.24 - 0.128 + 0.0192) = \mathbf{0.328}.$$

17.23. A. Given q, we have a Binomial with parameters m = 3 and q.

The chance that we observe two claims is: $3q^2(1-q) = 3q^2 - 3q^3$.

$$P(c=2) = \int_0^{0.4} P(c=2 \mid q)\, \pi(q)\, dq = \int_0^{0.4} (3q^2 - 3q^3)(2.5)\, dq = (2.5)(q^3 - 0.75q^4) \Big]_{q=0}^{q=0.4}$$

$= (2.5)(0.064 - 0.0192) = \mathbf{0.112}$.

17.24. B. Given q, we have a Binomial with parameters m = 3 and q.

The chance that we observe three claims is: q^3.

$$P(c=3) = \int_0^{0.4} P(c=3 \mid q)\, \pi(q)\, dq = \int_0^{0.4} (q^3)(2.5)\, dq = (2.5)(q^4/4) \Big]_{q=0}^{q=0.4} = (2.5)(0.0064) = \mathbf{0.016}.$$

Comment: Since we have a Binomial with m = 3, the only possibilities are 0, 1, 2 or 3 claims. Therefore, the probabilities for 0, 1, 2 and 3 claims (calculated in this and the prior three questions) add to one: 0.544 + 0.328 + 0.112 + 0.016 = 1.

17.25. D. This is a 40%-60% mixture of zero and a Poisson with λ = .07.

The second moment of the Poisson is: variance + mean^2 = .07 + .07^2 = .0749.

The mean of the mixture is: (40%)(0) + (60%)(.07) = .042.

The second moment of the mixture is: (40%)(0) + (60%)(.0749) = .04494.

The variance of the mixture is: .04494 - .042^2 = .0432, per student.

For a group of 100 students the variance is: (100)(.0432) = **4.32**.

17.26. C. For λ = 0.2, f(1) = 0.2e$^{-0.2}$ = 0.1638. For λ = 0.6, f(1) = .6e$^{-.6}$ = 0.3293.

Prob[1 coin] = (.5)(.1638) + (.5)(.3293) = **24.65%**.

17.27. A. Over two minutes, the mean is either 0.4: f(1) = .4e$^{-.4}$ = 0.2681,

or the mean is 1.2: f(1) = 1.2e$^{-1.2}$ = 0.3614.

Prob[1 coin] = (.5)(.2681) + (.5)(.3614) = **31.48%**.

17.28. C. Prob[0 coins during a minute] = (.5)e$^{-.2}$ + (.5)e$^{-.6}$ = 0.6838.

Prob[1 coin during a minute] = (.5).2e$^{-.2}$ + (.5).6e$^{-.6}$ = 0.2465.

Prob[A + B = 1] = Prob[A= 0]Prob[B] + Prob[A = 1]Prob[B = 0] = (2)(.6838)(.2465) = **33.71%**.

Comment: Since the minutes are on different days, their lambdas are picked independently.

17.29. C.

Prob[1 coin during third minute and 1 coin during fifth minute | λ = 0.2] = (.2e$^{-.2}$)(.2e$^{-.2}$) = 0.0268.

Prob[1 coin during third minute and 1 coin during fifth minute | λ = 0.6] = (.6e$^{-.6}$)(.6e$^{-.6}$) = 0.1084.

(0.5)(0.0268) + (0.5)(0.1084) = **6.76%**.

Comment: Since the minutes are on the same day, they have the same λ, whichever it is.

17.30. B. Prob[1 coin during a minute] = (0.5)0.2e$^{-0.2}$ + (0.5)0.6e$^{-0.6}$ = 0.2465.

Since the minutes are on different days, their lambdas are picked independently.

Prob[1 coin during 1 minute today and 1 coin during 1 minute tomorrow] =

Prob[1 coin during a minute] Prob[1 coin during a minute] = 0.2465^2 = **6.08%**.

17.31. A. Prob[1 coin during 4 minutes] = (0.5)0.8e$^{-0.8}$ + (0.5)2.4e$^{-2.4}$ = 0.2866.

Since the time intervals are on different days, their lambdas are picked independently.

Prob[1 coin during 4 minutes today and 1 coin during 4 minutes tomorrow] =

Prob[1 coin during 4 minutes] Prob[1 coin during 4 minutes] = 0.2866^2 = **8.33%**.

17.32. B. Prob[1 coin during two minute and 2 coins during following 3 minutes | λ = 0.2] =
$(0.4e^{-0.4})\ (0.6^2e^{-0.6}/2) = 0.0265$.
Prob[1 coin during two minute and 2 coins during following 3 minutes | λ = 0.6] =
$(1.2e^{-1.2})\ (1.8^2e^{-1.8}/2) = 0.0968$. $(0.5)(0.0265) + (0.5)(0.0968) =$ **6.17%**.

17.33. E. $$\int_a^b \lambda\, f(\lambda)\, d\lambda = \int_a^b \lambda \frac{(d+\lambda)\,\lambda^d}{b^{d+1} - a^{d+1}}\, d\lambda = \frac{d+1}{b^{d+1} - a^{d+1}} \left.\frac{\lambda^{d+2}}{d+2}\right]_{\lambda=a}^{\lambda=b} =$$

$$\frac{d+1}{d+2}\ \frac{b^{d+2} - a^{d+2}}{b^{d+1} - a^{d+1}} = \frac{0.5}{1.5}\ \frac{0.6^{1.5} - 0.2^{1.5}}{0.6^{0.5} - 0.2^{0.5}} = \mathbf{0.3821}.$$

17.34. B. $$\int_a^b \lambda^2\, f(\lambda)\, d\lambda = \int_a^b \lambda^2 \frac{(d+\lambda)\,\lambda^d}{b^{d+1} - a^{d+1}}\, d\lambda = \frac{d+1}{b^{d+1} - a^{d+1}} \left.\frac{\lambda^{d+3}}{d+3}\right]_{\lambda=a}^{\lambda=b} =$$

$$\frac{d+1}{d+3}\ \frac{b^{d+3} - a^{d+3}}{b^{d+1} - a^{d+1}} = \frac{0.5}{2.5}\ \frac{0.6^{2.5} - 0.2^{2.5}}{0.6^{0.5} - 0.2^{0.5}} = 0.15943.$$

For fixed λ, the second moment of a Poisson is: $\lambda + \lambda^2$.

Therefore, the second moment of the mixture is: $E[\lambda] + E[\lambda^2] = 0.3821 + 0.15943 = 0.5415$.

Therefore, the variance of the mixture is: $0.5415 - 0.3821^2 =$ **0.3955**.

Alternately, Variance[λ] = Second Moment[λ] - Mean[λ]^2 = $0.15943 - 0.3821^2 = 0.0134$.

The variance of frequency for a mixture of Poissons is: $E[\lambda] + Var[\lambda] = 0.3821 + 0.0134 =$ **0.3955**.

17.35. A. $E[X] = (.5)(6)(.8) + (.5)(6)q = 2.4 + 3q$.
The second moment of a Binomial is: $mq(1 - q) + (mq)^2 = mq - mq^2 + m^2q^2$.
$E[X^2] = (0.5)\ \{(6)(0.8) - (6)(0.8^2) + (6^2)(0.8^2)\} + (0.5)\{6q - 6q^2 + 36q^2\} = 12 + 3q + 15q^2$.
$Var[X] = 12 + 3q + 15q^2 - (2.4 + 3q)^2 = 6.24 - 11.4q + 6q^2$.
$E[X] = Var[X]$. $\Rightarrow 2.4 + 3q = 6.24 - 11.4q + 6q^2$. $\Rightarrow 6q^2 - 14.4q + 3.84 = 0$.
$q = \{14.4 \pm \sqrt{14.4^2 - (4)(6)(3.84)}\} / 12 = \{14.4 \pm 10.7331\} / 12 = 2.094$ or **0.3056**.
Comment: $0 \le q \le 1$. When one mixes distributions, the variance increases. As discussed in "Mahler's Guide to Buhlmann Credibility," $Var[X] = E[Var[X | q]] + Var[E[X | q]] \ge E[Var[X | q]]$.
Since for a Binomial Distribution, the variance is less than the mean, for a mixture of Binomial Distributions, the variance can be either less than, greater than, or equal to the mean.

17.36. D. For $\lambda = 0.04$, Prob[more than 1 claim] = $1 - e^{-0.04} - 0.04\, e^{-0.04} = 0.00077898$.

For $\lambda = 0.10$, Prob[more than 1 claim] = $1 - e^{-0.10} - 0.10\, e^{-0.10} = 0.00467884$.

Prob[more than 1 claim] = (0.5)(0.00077898) + (0.5)(0.00467884) = **0.273%**.

17.37. B. For $\lambda = 0.04$, the mean is 0.04 and the 2nd moment is: $\lambda + \lambda^2 = 0.04 + 0.04^2 = 0.0416$.

For $\lambda = 0.10$, the mean is 0.10 and the second moment is: $\lambda + \lambda^2 = 0.10 + 0.10^2 = 0.11$.

Therefore, the mean of the mixture is: (0.5)(0.04) + (0.5)(0.10) = 0.07, and the second moment of the mixture is: (0.5)(0.0416) + (0.5)(0.11) = 0.0758.

The variance of the mixed distribution is: $0.0758 - 0.07^2 =$ **0.0709**.

Alternately, Variance[λ] = Second Moment[λ] - Mean[λ]^2 =

$(0.5)(0.04^2) + (0.5)(0.1^2) - 0.07^2 = 0.0009$.
The variance of frequency for a mixture of Poissons is:
Expected Value of the Process Variance + Variance of the Hypothetical Means =
E[λ] + Var[λ] = 0.07 + 0.0009 = **0.0709**.

17.38. D. $(25\%)(e^{-0.25}) + (75\%)(e^{-0.5}) =$ **65.0%**.

17.39. A. $(25\%)(0.25\, e^{-0.25}) + (75\%)(0.5\, e^{-0.5}) =$ **27.6%**.

17.40. B. $(25\%)(0.25^2\, e^{-0.25}/2) + (75\%)(0.5^2\, e^{-0.5}/2) =$ **6.3%**.

17.41. D. If there is one home game and two road games, then the distributions of road wins is:
2 @ 16%, 1 @ 48%, 0 @ 36%.
Thus the chance of winning at least 2 games is:
Prob[win 2 road] + Prob[win 1 road] Prob[win one home] = 16% + (48%)(80%) = 0.544.
If instead there is one road game and two home games, then the distributions of home wins is:
2 @ 64%, 1 @ 32%, 0 @ 4%.
Thus the chance of winning at least 2 games is:
Prob[win 2 home] + Prob[win one home] Prob[win 1 road] = 64% + (32%)(40%) = 0.768.
Thus the chance the Spiders win the series is:
(50%)(0.544) + (50%)(0.768) = **65.6%**.
Comment: This is a 50%-50% mixture of two situations.
(Each situation has its own distribution of games won.)
While in professional sports there is a home filed advantage, it is not usually this big.
Note that for m = 3 and q = (0.8 + 0.4)/2 = 0.6, the probability of at least two wins is:
$0.6^3 + (3)(0.6^2)(0.4) = 0.648 \neq 0.656$.

17.42. The mean of the mixture is: $p\lambda_1 + (1 - p)\lambda_2$.

The second moment of a Poisson is: variance + mean^2 = $\lambda + \lambda^2$.

Therefore, the second moment of the mixture is: $p(\lambda_1 + \lambda_1^2) + (1 - p)(\lambda_2 + \lambda_2^2)$.

Variance of the mixture is: $p(\lambda_1 + \lambda_1^2) + (1 - p)(\lambda_2 + \lambda_2^2) - \{p\lambda_1 + (1 - p)\lambda_2\}^2$.

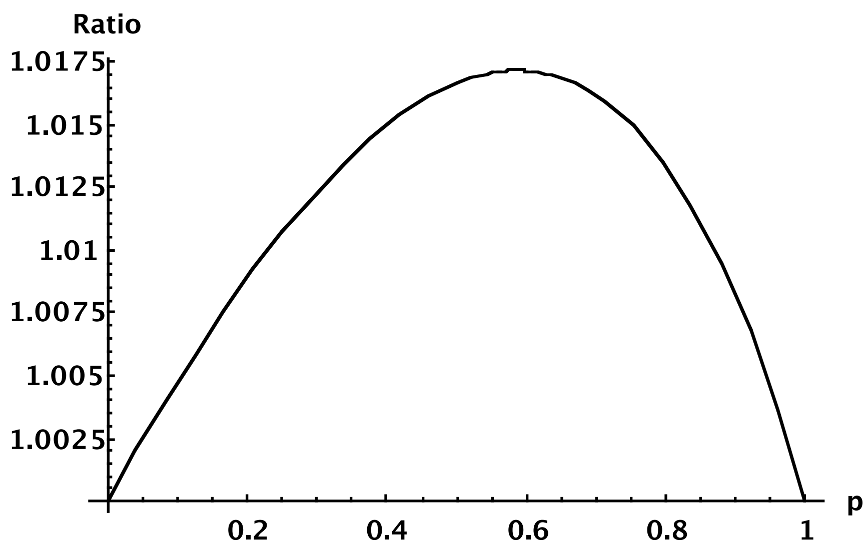
For the mixture, the ratio of the variance to the mean is:

$$\frac{p(\lambda_1 + \lambda_1^2) + (1 - p)(\lambda_2 + \lambda_2^2)}{p\lambda_1 + (1 - p)\lambda_2} - \{p\lambda_1 + (1 - p)\lambda_2\}.$$

For λ_1 = 10% and λ_2 = 20%, the ratio of the variance to the mean is:

$$\frac{p\,0.11 + (1 - p)\,0.24}{p\,0.1 + (1 - p)\,0.2} - \{p\,0.1 + (1 - p)0.2\} = \frac{0.24 - 0.13p}{0.2 - 0.1p} - (0.2 - 0.1p).$$

Here is a graph of the ratio of the variance to the mean as a function of p:



<u>Comment</u>: For either p = 0 or p = 1, this ratio is 1.
For either p = 0 or p = 1, we have a single Poisson and the mean is equal to the variance.
For 0 < p < 1, mixing increases the variance, and the variance of the mixture is greater than its mean.

For example, for p = 80%, $\frac{0.24 - 0.13p}{0.2 - 0.1p} - (0.2 - 0.1p) = 1.013$.

17.43. B. Given q, we have a Binomial with parameters m = 4 and q. The chance that we observe zero claims is: $(1-q)^4$. The distribution of q is: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

$$f(0) = \int_0^{0.6} f(0 \mid q)\, \pi(q)\, dq = \frac{2500}{99} \int_0^{0.6} q^2(1-q)^5\, dq =$$

$$\frac{2500}{99} \int_0^{0.6} q^2 - 5q^3 + 10q^4 - 10q^5 + 5q^6 - q^7\, dq =$$

$$\frac{2500}{99} \{0.6^3/3 - (5)(0.6^4)/4 + (10)(0.6^5)/5 - (10)(0.6^6)/6 + (5)(0.6^7)/7 - 0.6^8/8\} = \mathbf{14.28\%}.$$

17.44. D. Given q, we have a Binomial with parameters m = 4 and q. The chance that we observe one claim is: $4q(1-q)^3$. The distribution of q is: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

$$f(1) = \int_0^{0.6} f(1 \mid q)\, \pi(q)\, dq = \frac{2500}{99} (4) \int_0^{0.6} q^3(1-q)^4\, dq =$$

$$\frac{10{,}000}{99} \int_0^{0.6} q^3 - 4q^4 + 6q^5 - 4q^6 + q^7\, dq =$$

$$\frac{10{,}000}{99} \{0.6^4/4 - (4)(0.6^5)/5 + (6)(0.6^6)/6 - (4)(0.6^7)/7 + 0.6^8/8\} = \mathbf{29.81\%}.$$

17.45. E. Given q, we have a Binomial with parameters m = 4 and q. The chance that we observe two claims is: $6q^2(1-q)^2$. The distribution of q is: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

$$f(2) = \int_0^{0.6} f(2 \mid q)\, \pi(q)\, dq = \frac{2500}{99} (6) \int_0^{0.6} q^4(1-q)^3\, dq = \frac{5000}{33} \int_0^{0.6} q^4 - 3q^5 + 3q^6 - q^7\, dq =$$

$$\frac{5000}{33} \{0.6^5/5 - (3)(0.6^6)/6 + (3)(0.6^7)/7 - 0.6^8/8\} = \mathbf{32.15\%}.$$

17.46. A. Given q, we have a Binomial with parameters m = 4 and q. The chance that we observe three claims is: $4q^3(1-q)$. The distribution of q is: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

$$f(3) = \int_0^{0.6} f(3 \mid q)\, \pi(q)\, dq = \frac{2500}{99} (4) \int_0^{0.6} q^5(1-q)^2\, dq = \frac{10{,}000}{99} \int_0^{0.6} q^5 - 2q^6 + q^7\, dq =$$

$$\frac{10{,}000}{99} \{0.6^6/6 - (2)(0.6^7)/7 + 0.6^8/8\} = \mathbf{18.96\%}.$$

17.47. C. Given q, we have a Binomial with parameters m = 4 and q. The chance that we observe four claims is: q^4. The distribution of q is: $\pi(q) = \frac{2500}{99} q^2(1-q)$.

$$f(4) = \int_0^{0.6} f(4 \mid q)\, \pi(q)\, dq = \frac{2500}{99} \int_0^{0.6} q^6(1-q)\, dq = \frac{2500}{99} \int_0^{0.6} q^6 - q^7\, dq =$$

$$\frac{2500}{99} \{0.6^7/7 - 0.6^8/8\} = \mathbf{4.80\%}.$$

Comment: Since we have a Binomial with m = 4, the only possibilities are 0, 1, 2, 3 or 4 claims.
Therefore, the probabilities for 0, 1, 2, 3, and 4 claims must add to one:
14.28% + 29.81% + 32.15% + 18.96% + 4.80% = 1.

17.48. E. Let x be the mean for the low risk policies.
The mean of the mixture is: $(3/4)x + 0.4/4 = 0.75x + 0.1$
The second of the mixture is the mixture of the second moments:
$(3/4)(x + x^2) + (0.4 + 0.4^2)/4 = 0.75x^2 + 0.75x + 0.14$.
Thus the variance of the mixture is:
$0.75x^2 + 0.75x + 0.14 - (0.75x + 0.1)^2 = 0.1875x^2 + 0.6x + 0.13$.
Thus, $0.2575 = 0.1875x^2 + 0.6x + 0.13$. $\Rightarrow$ $0.1875x^2 + 0.6x - 0.1275 = 0$. $\Rightarrow$

$$x = \frac{-0.6 \pm \sqrt{0.6^2 - (4)(0.1875)(-0.1275)}}{(2)(0.1875)} = \mathbf{0.20}, \text{ taking the positive root.}$$

Comment: You can try the choices and see which one works.

17.49. (a) The mean the mixture is: $E[\lambda] = \mu$.

The second moment of each Poisson is: $\lambda + \lambda^2$.

Thus the second moment of the mixture is: $E[\lambda + \lambda^2] = E[\lambda] + E[\lambda^2] = \mu + \mu + \mu^2 = 2\mu + \mu^2$.

Thus the variance of the mixture is: $2\mu + \mu^2 - \mu^2 = 2\mu$.

Alternately, the process variance given λ is λ.

Thus the Expected Value of the Process Variance is $E[\lambda] = \mu$.

The Variance of the Hypothetical Means is the variance of the distribution of the lambdas, which is μ.

Total variance is: EPV + VHM = $\mu + \mu = 2\mu$.

(b) $E_N[z^n \mid \lambda]$ is the Probability Generating Function of a Poisson with mean λ: $\exp[\lambda(z-1)]$.

$$P[z] = E_N[z^n] = \sum_{\lambda=0}^{\infty} \pi[\lambda]\ E_N[z^n \mid \lambda] = \sum_{\lambda=0}^{\infty} \pi[\lambda] \exp[\lambda(z-1)] = \sum_{\lambda=0}^{\infty} \frac{e^{-\mu}\ \mu^{\lambda}}{\lambda!} \exp[\lambda(z-1)] =$$

$$e^{-\mu} \sum_{\lambda=0}^{\infty} \frac{\{\mu\ e^{z-1}\}^{\lambda}}{\lambda!} = e^{-\mu} \exp[\mu\ e^{z-1}] = \exp[\mu(e^{z-1} - 1)].$$

Alternately, when mixing Poissons, $P_{\text{mixed distribution}}(z) = P_{\text{mixing distribution of } \lambda}(e^{z-1})$.

Thus the mixture has probability generating function: $\exp[\mu(e^{z-1} - 1)]$.

(c) $P'(z) = \exp[\mu(e^{z-1} - 1)]\ \mu\ e^{z-1}$. Mean = $P'(1) = \mu$.

$P''(z) = \exp[\mu(e^{z-1} - 1)]\ \mu\ e^{z-1}\ \mu\ e^{z-1} + \exp[\mu(e^{z-1} - 1)]\ \mu\ e^{z-1}$. $P''(1) = \mu^2 + \mu$.

Second factorial moment = $E[N(N-1)] = P''(1) = \mu^2 + \mu$.

Thus $E[N^2] = \mu^2 + \mu + E[N] = \mu^2 + \mu + \mu = \mu^2 + 2\mu$.

$\text{Var}[N] = E[N^2] - E[N]^2 = \mu^2 + 2\mu - \mu^2 = 2\mu$.

Comment: Parts (b) and (c) are beyond what you should be asked on your exam.
The alternate solution to part (a) is discussed in “Mahler’s Guide to Buhlmann Credibility.”

17.50. (a) mean = (0.5)(4) + (0.5)(10) = 7.

The second moment of each Poisson is its variance of square of its mean.

second moment of the mixture is: $(0.5)(4 + 4^2) + (0.5)(10 + 10^2) = 65$.

Thus the variance of the mixture is: $65 - 7^2 = 16$.

(b) The mean of each zero truncated Binomial is: $\frac{mq}{1 - (1-q)^m}$.

For the first one: $\frac{(2)(0.5)}{1 - 0.5^2} = 1.3333$. For the second one: $\frac{(8)(0.4)}{1 - 0.6^8} = 3.2547$.

mean of the mixture is: (0.3)(1.3333) + (0.7)(3.2547) = 2.6783.

The variance of each zero truncated Binomial is: $\frac{mq \{(1-q) - (1 - q + mq)(1-q)^m\}}{\{1 - (1-q)^m\}^2}$.

Variance for the first one: $\frac{(2)(0.5)\{(0.5) - (1 - 0.5 + 1)(0.5^2)\}}{\{1 - 0.5^2\}^2} = 0.2222$.

Variance for the second one: $\frac{(8)(0.4)\{(0.6) - (1 - 0.4 + 3.2)(0.6^8)\}}{\{1 - 0.6^8\}^2} = 1.7749$.

The second moment of each zero truncated Binomial is its variance of square of its mean.

second moment of the mixture is: $(0.3)(0.2222 + 1.3333^2) + (0.7)(1.7749 + 3.2547^2) = 9.2575$.

Thus the variance of the mixture is: $9.2575 - 2.6783^2 = 2.0842$.

Alternately, the second moment of a Binomial Distribution is: $mq(1-q) + (mq)^2 = mq(1 + mq - q)$.

Thus the second moment of a zero truncated Binomial Distribution is: $\frac{mq(1 + mq - q)}{1 - (1-q)^m}$.

Second moment for the first one: $\frac{(2)(0.5)(1 + 1 - 0.5)}{1 - 0.5^2} = 2$.

Second moment for the second one: $\frac{(8)(0.4)(1 + 3.2 - 0.4)}{1 - 0.6^8} = 12.3678$.

second moment of the mixture is: (0.3)(2) + (0.7)(12.3678) = 9.2575.

Thus the variance of the mixture is: $9.2575 - 2.6783^2 = 2.0842$.

(c) mean of the compound distribution: (7)(2.6783) = 18.748.

variance of the compound distribution is: $(7)(2.0842) + (2.6783^2)(16) = 129.36$.

(Treat size of families as severity in a collective risk model of aggregate losses.)

(d) Prob[at least 25 people] = $1 - \Phi[(24.5 - 18.748) / \sqrt{129.36}] = 1 - \Phi[0.51] = 30.5\%$.

17.51. mean of mixture is: $E[\beta]$ = mean of Beta Distribution = $a/(a+b)$ = **4/7**.

The second moment of the Geometric is: $\beta(1+\beta) + \beta^2 = \beta + 2\beta^2$.

Therefore, the second moment of the mixture is:

$E[\beta + 2\beta^2] = E[\beta] + 2E[\beta^2]$ = mean of Beta Distribution + twice second moment of Beta DIstribution

$= a/(a+b) + 2\ a\ (a+1) / \{(a+b)(a+b+1)\} = 4/7 + (2)(4)(5) / \{(7)(8)\} = 1.2857$.

Therefore, the variance of the mixture is: $1.2857 - (4/7)^2$ = **0.959**.

Alternately, the process variance is $\beta(1+\beta)$.

Thus the expected value of the process variance is: $E[\beta(1+\beta)] = E[\beta] + E[\beta^2] =$

= mean of Beta Distribution + second moment of Beta DIstribution

$= a/(a+b) + 2\ a\ (a+1) / \{(a+b)(a+b+1)\} = 4/7 + (4)(5) / \{(7)(8)\} = 0.92857$.

The variance of the hypothetical means is: $Var[\beta]$ =

second moment of Beta DIstribution - square of mean of Beta Distribution =

$(4)(5) / \{(7)(8)\} - (4/7)^2 = 0.03061$.

Variance of the mixture is: EPV + VHM = 0.92857 + 0.03061 = **0.959**.

Comment: The alternate solution is discussed in "Mahler's Guide to Buhlmann Credibility."

17.52. A. The probability of a claim being large is 40%.

Thus conditional on $\lambda = 1$, the number of large claims is Poisson with $\lambda = 0.4$, while conditional on $\lambda = 2$ the number of large claims is Poisson with $\lambda = 0.8$.

Thus the probability of 2 large claims is:

$(0.5)\ (0.4^2\ e^{-0.4} / 2) + (0.5)\ (0.8^2\ e^{-0.8} / 2)$ = **0.0987**.

17.53. D. The probability of a claim being small is 60%.

Thus conditional on $\lambda = 1$, the number of large claims is Poisson with $\lambda = 0.6$, while conditional on $\lambda = 2$ the number of large claims is Poisson with $\lambda = 1.2$.

Thus the probability of 2 small claims is:

$(0.5)\ (0.6^2\ e^{-0.6} / 2) + (0.5)\ (1.2^2\ e^{-1.2} / 2)$ = **0.1578**.

17.54. E. In order to have 2 large and 2 small claims, one has to have 4 claims in total.
The probability of 4 claims is: $(0.5)(1^4 e^{-1} / 24) + (0.5)(2^4 e^{-2} / 24) = 0.05278$.
Given that one has 4 claims, the probability that 2 are large and 2 are small is:
$(6)(0.4^2)(0.6^2) = 0.3456$.
Thus the probability of 2 large and 2 small claims is: (0.5278)(0.3456) = **0.01824**.
Alternately, Prob[2 small & 2 large | λ = 1] =
(density at 2 for a Poisson with mean 0.6) (density at 2 for a Poisson with mean 0.4) =
$(0.6^2 e^{-0.6} / 2)(0.4^2 e^{-0.4} / 2) = 0.0052975$.
Prob[2 small & 2 large | λ = 2] =
(density at 2 for a Poisson with mean 1.2) (density at 2 for a Poisson with mean 0.8) =
$(1.2^2 e^{-1.2} / 2)(0.8^2 e^{-0.8} / 2) = 0.0311812$.
Thus for the mixture the probability of 2 large and 2 small claims is:
(0.5)(0.0052975) + (0.5)(0.0311812) = **0.01824**.
Comment: While for each Poisson the number of large and small claims are independent, the number of large and small claims are not independent for the mixture.
$(0.0987)(0.1578) = 0.01557 \neq 0.01824$.
Given a lot of large claims, the probability that λ is 2 is bigger and thus the probability of a lot of small claims is also higher.

17.55. C. The frequency distribution for the class =

$$\int_p^1 f(x|\theta)\, g(\theta)\, d\theta = -\int_p^1 (1-\theta)^x / \ln(p)\, d\theta = \left. \frac{(1-\theta)^{x+1}}{(x+1)\ln(p)} \right]_{\theta = p}^{\theta = 1} = \mathbf{\frac{-(1-p)^{x+1}}{(x+1)\ln(p)}}.$$

Comment: 4,11/82, Q.48, rewritten. Note that $f(x|\theta)$ is a geometric distribution.
The mixed frequency distribution for the class is a logarithmic distribution, with
$\beta = 1/p - 1$ and x+1 running from 1 to infinity (so that f(0) is the logarithmic distribution at 1, f(1) is the logarithmic at 2, etc. The support of the logarithmic is 1,2,3,...)

17.56. E. $Var[Y] = E_X[VAR[Y | X]] + VAR_X[E[Y | X]] = E_X[x] + VAR_X[x] = mq + mq(1 - q) =$ $\mathbf{mq(2 - q)}$.
Comment: Total Variance = Expected Value of the Process Variance + Variance of the Hypothetical Means. See “Mahler’s Guide to Buhlmann Credibility.”

17.57. E. $P(Y=0) = \int P(Y = 0 \mid \theta)\, f(\theta)\, d\theta = \int e^{-\theta}\, f(\theta)\, d\theta$.

For the first case, $f(\theta) = 1/2$, for $0 \le \theta \le 2$

$$P(Y=0) = \int_0^2 e^{-\theta} / 2 \; d\theta = (1 - e^{-2})/2 = 0.432.$$

For the second case, $f(\theta) = e^{-\theta}$, for $\theta > 0$ and

$$P(Y=0) = \int_0^\infty e^{-2\theta} \; d\theta = 1/2.$$

For the third case, $P(Y=0) = e^{-1} = 0.368$. In the first and third cases $P(Y=0) < 0.45$.
Comment: Three separate problems in which you need to calculate P(Y=0) given three different distributions of θ.

17.58. A. The chance of zero or one claim for a Poisson distribution is: $e^{-\lambda} + \lambda e^{-\lambda}$.

We average over the possible values of λ:

$$\text{Prob(0 or 1 claim)} = (1/5) \int_0^5 e^{-\lambda} + \lambda e^{-\lambda} \; d\lambda = (1/5)\; (-2e^{-\lambda} - \lambda e^{-\lambda})\Big]_{\lambda=0}^{\lambda=5} = (1/5)\,(2 - 7e^{-5}) = 0.391.$$

Probability that there are 2 or more claims = 1 - Prob(0 or 1 claim) = 1 - 0.391 = **0.609**.

17.59. E. $P(z) \equiv E[z^N]$. The p.g.f. of the Poisson Distribution is: $P(z) = e^{\lambda(z-1)}$.

Therefore, for the Poisson, $E[z^N] = e^{\lambda(z-1)}$. $E[2^N \mid \lambda] = P(2) = e^{\lambda(2-1)} = e^{\lambda}$.

$$E[W] = \int_0^4 E[2^N \mid \lambda]\; (1/4)\; d\lambda = (1/4) \int_0^4 e^{\lambda} \; d\lambda = (1/4)(e^4 - 1) = \mathbf{13.4}.$$

17.60. A. For a Negative Binomial with r = 4, $f(0) = 1/(1+\beta)^4$.

$$\text{Prob[0 claims]} = \int_0^2 1/(1+\beta)^4 (1/2)\, d\beta = -1/\{6(1 + \beta)^3\}\Big]_{\beta=0}^{\beta=2} = (1/6)(1 - 3^3) = .1605.$$

Prob[at least 1 claim] = 1 - 0.1605 = **0.8395**.

17.61. E. For q = 0.5 and m = 2, f(2) = $.5^2$ = .25.

For q = 0.5 and m = 4, f(2) = $\binom{4}{2}$ (0.5^2) (0.5^2) = .375.

Probability that the mixed distribution is 2 is: p(.25) + (1 - p)(.375) = **0.375 - 0.125p**.

<u>Comment</u>: The solution cannot involve p^2, eliminating choices A, C, and D.

Section 18, Gamma Function[142]

The quantity $x^{\alpha-1}e^{-x}$ is finite for $x \geq 0$ and $\alpha \geq 1$.

Since it declines quickly to zero as x approaches infinity, its integral from zero to ∞ exists.
This is the much studied and tabulated (complete) Gamma Function.

$$\boldsymbol{\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt} = \theta^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt \text{ , for } \alpha \geq 0 \text{ , } \theta \geq 0.$$

We prove the equality of these two integrals, by making the change of variables $x = t/\theta$:

$$\int_0^\infty t^{\alpha-1} e^{-t}\, dt = \int_0^\infty (x/\theta)^{\alpha-1} e^{-x/\theta}\, dx/\theta = \theta^{-\alpha} \int_0^\infty x^{\alpha-1} e^{-x/\theta}\, dx.$$

$\boldsymbol{\Gamma(\alpha) = (\alpha-1)\,!}$ $\Gamma(\alpha) = (\alpha\text{-}1)\ \Gamma(\alpha-1)$.

$\Gamma(1) = 1$. $\Gamma(2) = 1$. $\Gamma(3) = 2$. $\Gamma(4) = 6$. $\Gamma(5) = 24$. $\Gamma(6) = 120$. $\Gamma(7) = 720$. $\Gamma(8) = 5040$.

One does <u>not</u> need to know how to compute the complete Gamma Function for noninteger alpha.
Many computer programs will give values of the complete Gamma Function.

$\Gamma(1/2) = \sqrt{\pi}$ $\Gamma(3/2) = 0.5\sqrt{\pi}$ $\Gamma(-1/2) = -2\sqrt{\pi}$ $\Gamma(-3/2) = (4/3)\sqrt{\pi}$.

For $\alpha \geq 10$: $$\ln\Gamma(\alpha) \cong (\alpha - 0.5)\ln\alpha - \alpha + \frac{\ln(2\pi)}{2} + \frac{1}{12\alpha} - \frac{1}{360\alpha^3} + \frac{1}{1260\alpha^5} - \frac{1}{1680\alpha^7} + \frac{1}{1188\alpha^9} - \frac{691}{360{,}360\ \alpha^{11}} + \frac{1}{156\alpha^{13}} - \frac{3617}{122{,}400\alpha^{15}}.$$ [143]

For $\alpha < 10$ use the recursion relationship $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$.
The Gamma function is undefined at the negative integers and zero.

For large α: $\Gamma(\alpha) \cong e^{-\alpha}\alpha^{\alpha-1/2}\sqrt{2\pi}$, which is Sterling's formula.[144]

The ratios of two Gamma functions with arguments that differ by an integer can be computed in terms of a product of factors, just as one would with a ratio of factorials.

[142] See Appendix A of <u>Loss Models.</u> Also see the Handbook of Mathematical Functions, by M. Abramowitz, et. al.
[143] See Appendix A of <u>Loss Models</u>, and the Handbook of Mathematical Functions, by M. Abramowitz, et. al.
[144] See the Handbook of Mathematical Functions, by M. Abramowitz, et. al.

Exercise: What is $\Gamma(7) / \Gamma(4)$?

[Solution: $\Gamma(7) / \Gamma(4) = 6! / 3! = (6)(5)(4) = 120$.]

Exercise: What is $\Gamma(7.2) / \Gamma(4.2)$?

[Solution: $\Gamma(7.2) / \Gamma(4.2) = 6.2! / 3.2! = (6.2)(5.2)(4.2) = 135.4$.]

Note that even when the arguments are not integer, the ratio still involves a product of factors.
The solution of the last exercise depended on the fact that 7.2 - 4.2 = 3 is an integer.

Integrals involving e^{-x} and powers of x can be written in terms of the Gamma function:

$$\int_0^\infty \mathbf{t^{\alpha - 1} e^{-t/\theta} \, dt = \Gamma(\alpha)\theta^\alpha}, \text{ or for integer n: } \int_0^\infty t^n \, e^{-ct} \, dt = n! / c^{n+1}.$$

Exercise: What is the integral from 0 to ∞ of: $t^3 e^{-t/10}$?

[Solution: With $\alpha = 4$ and $\theta = 10$, this integral is: $\Gamma(4)\, 10^4 = (6)(10000) = 60{,}000$.]

This formula for "gamma-type" integrals is very useful for working with anything involving the Gamma distribution, for example the Gamma-Poisson process. It follows from the definition of the Gamma function and a change of variables.

The Gamma density in the Appendix of Loss Models is: $\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$.
Since this probability density function must integrate to unity, the above formula for gamma-type integrals follows. This is a useful way to remember this formula on the exam.

Incomplete Gamma Function:

As shown in Appendix A of Loss Models, the **Incomplete Gamma Function** is defined as:

$$\Gamma(\alpha \,;\, x) = \int_0^x t^{\alpha - 1} e^{-t} \, dt \;/\; \Gamma(\alpha).$$

$\Gamma(\alpha \,;\, 0) = 0$. $\Gamma(\alpha \,;\, \infty) = \Gamma(\alpha)/\Gamma(\alpha) = 1$. As discussed below, the Incomplete Gamma Function with the introduction of a scale parameter θ is the Gamma Distribution.

Computing Incomplete Gamma Functions:

Exercise: Via integration by parts, put $\Gamma(2 ; x)$ in terms of Exponentials and powers of x.

[Solution: $\Gamma(2 ; x) = \int_0^x t e^{-t} dt \,/\, \Gamma(2) = \int_0^x t e^{-t} dt = e^{-t} - t e^{-t}\Big]_{t=0}^{t=x} = 1 - e^{-x} - xe^{-x}$.]

One can prove via integration by parts that $\Gamma(\alpha ; x) = \Gamma(\alpha\text{-}1 ; x) - x^{\alpha-1} e^{-x} / \Gamma(\alpha)$.[145]

This recursion formula for integer alpha is: $\Gamma(n ; x) = \Gamma(n\text{-}1 ; x) - x^{n-1} e^{-x} /(n\text{-}1)!$.

Combined with the fact that $\Gamma(1 ; x) = \int_0^x e^{-t} dt = 1 - e^{-x}$, this leads to the following formula for the

Incomplete Gamma for positive integer alpha:[146]

$$\Gamma(\alpha ; x) = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i e^{-x}}{i!} = \sum_{i=\alpha}^{\infty} \frac{x^i e^{-x}}{i!}.$$

Exercise: Compute $\Gamma[4; 6.5]$.

[Solution: $\Gamma[4; 6.5] = 1 - e^{-6.5} (1 + 6.5 + 6.5^2/2 + 6.5^3/6) = 0.8882$.]

Relationship of the Gamma to Poisson Processes:[147]

In general, assume claims are given by a Poisson Process with claims intensity λ. Then the claims in the time interval from (0, 1) are Poisson Distributed with mean λ. One can calculate the chance that there are least n claims in two different ways.

First, the chance of at least n claims is a sum of Poisson densities:

$$1 - F(n\text{-}1) = 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda} \lambda^i}{i!} = \sum_{i=n}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} .$$

[145] See for example, Formula 6.5.13 in the Handbook of Mathematical Functions, by Abramowitz, et. al.

[146] See Theorem A.1 in Appendix A of Loss Models.

[147] Not on the syllabus. See “Mahler’s Guide to Poisson Processes,” for CAS Exam 3ST.

On the other hand, the times between claims are independent, identically distributed Exponential Distributions, each with mean $\theta = 1/\lambda$.[148]

Thus, the time of the n^{th} claim is a sum of n independent, identically distributed Exponentials with $\theta = 1/\lambda$, and thus a Gamma Distribution with $\alpha = n$ and $\theta = 1/\lambda$.

Thus the n^{th} claim has distribution function at time t of: $\Gamma[\alpha ; t/\theta] = \Gamma[n ; \lambda t]$.

The chance of at least n claims by time one is the probability that the n^{th} claim occurs by time one, which is: $\Gamma[n ; \lambda]$.

Comparing the two results: $\Gamma[n ; \lambda] = 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda} \lambda^i}{i!} = \sum_{i=n}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!}$.

Thus, the Incomplete Gamma Function with positive integer shape parameter α can be written in terms of a sum of Poisson densities:

$$\Gamma[\alpha ; x] = 1 - \sum_{i=0}^{\alpha-1} \frac{x^i e^{-x}}{i!} = \sum_{i=\alpha}^{\infty} \frac{x^i e^{-x}}{i!}.$$

<u>Integrals Involving Exponentials times Powers</u>:

One can use the incomplete Gamma Function to handle integrals involving $te^{-t/\theta}$.

$$\int_0^x t\, e^{-t/\theta}\, dt = \int_0^{x/\theta} \theta s\, e^{-s}\, \theta ds = \theta^2 \int se^{-s}\, ds = \theta^2 \Gamma(2 ; x/\theta)\Gamma(2) = \theta^2\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}.$$

$$\boldsymbol{\int_0^x t\, e^{-t/\theta}\, dt = \theta^2\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}}.$$

Exercise: What is the integral from 0 to 3.4 of: $te^{-t/10}$?

[Solution: $(10^2)\ \{1 - e^{-3.4/10} - (3.4/10)e^{-3.4/10}\} = 4.62$.]

[148] This fact is used to derive the special algorithm to simulate a Poisson Distribution, as discussed in “Mahler’s Guide to Simulation.”

Such integrals can also be done via integration by parts, or as discussed below using the formula for the present value of a continuously increasing annuity, or one can make use of the formula for the Limited Expected Value of an Exponential Distribution:[149]

$$\int_0^x t\, e^{-t/\theta}\, dt = \theta \int_0^x t\, e^{-t/\theta} / \theta\, dt = \theta\{E[X \wedge x] - xS(x)\} =$$

$\theta\{\theta(1 - e^{-x/\theta}) - xe^{-x/\theta}\} = \theta^2\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}$.[150]

When the upper limit is infinity, the integral simplifies:

$$\int_0^\infty t\, e^{-t}\, dt = \theta^2.$$ [151]

In a similar manner, one can use the incomplete Gamma Function to handle integrals involving $t^n e^{-t/\theta}$, for n integer:

$$\int_0^x t^n\, e^{-t/\theta}\, dt = \theta^{n+1} \int_0^{x/\theta} s^n\, e^{-s}\, ds = \theta^{n+1}\Gamma(n+1; x/\theta)\Gamma(n+1) = n!\, \theta^{n+1}\{1 - \sum_{i=0}^{n} \frac{x^i\, e^{-x}}{i!}\}.$$

Exercise: What is the integral from 0 to 3.4 of: $t^3 e^{-t/10}$?

[Solution: $$\int_0^x t^3\, e^{-t/\theta}\, dt = \theta^4 \int_0^{x/\theta} s^3\, e^{-s}\, ds = \theta^4\Gamma(4; x/\theta)\Gamma(4) =$$

$6\theta^4\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta} - (x/\theta)^2 e^{-x/\theta}/2 - (x/\theta)^3 e^{-x/\theta}/6\}$.

For $\theta = 10$ and $x = 3.4$, this is:

$60000\{1 - e^{-0.34} - 0.34e^{-0.34} - 0.34^2 e^{-0.34}/2 - 0.34^3 e^{-0.34}/6\} = 25.49.$*]*

[149] See Appendix A of Loss Models.

[150] $E[X \wedge x]$ is the limited expected value, as discussed in "Mahler's Guide to Loss Distributions."

[151] If one divided by θ, then the integrand would be t times the density of an Exponential Distribution. Therefore, the given integral is θ(mean of an Exponential Distribution) = θ^2.

Continuously Increasing Annuities:

The present value of a continuously increasing annuity of term n, with force of interest δ, is:[152]

$$(\bar{I}\bar{a})_{\overline{n}|} = (\bar{a}_{\overline{n}|} - ne^{-n\delta})/\delta$$

where the present value of a continuous annuity of term n, with force of interest δ, is:

$$\bar{a}_{\overline{n}|} = (1 - e^{-n\delta})/\delta$$

However, the present value of a continuously increasing annuity can also be written as the integral from 0 to n of $te^{-t\delta}$. Therefore,

$$\int_0^n t\, e^{-t\delta}\, dt = \{(1-e^{-n\delta})/\delta - ne^{-n\delta}\}/\delta = (1-e^{-n\delta})/\delta^2 - ne^{-n\delta}/\delta.$$

Those who remember the formula for the present value of an increasing continuous annuity will find writing such integrals involving $te^{-t\delta}$ in terms of increasing annuities to be faster than doing integration by parts.

Exercise: What is the integral from 0 to 3.4 of: $te^{-t/10}$?

[Solution: $\{(1-e^{-3.4/10})/0.1 - (3.4)e^{-3.4/10}\}/0.1 = (2.882 - 2.420)/0.1 = 4.62$.

Comment: Matches the answer gotten above using Incomplete Gamma Functions.

4.62 is the present value of a continuously increasing annuity with term 3.4 years and force of interest 10%.]

[152] See for example, The Theory of Interest by Kellison.

Gamma Distribution:[153]

The **Gamma Distribution** can be defined in terms of the Incomplete Gamma Function,
F(x) = $\Gamma(\alpha; x/\theta)$. Note that $\Gamma(\alpha; \infty) = \Gamma(\alpha)/\Gamma(\alpha) = 1$ and $\Gamma(\alpha; 0) = 0$, so we have as required for a distribution function $F(\infty) = 1$ and $F(0) = 0$.

$$\mathbf{f(x) = \frac{(x/\theta)^{\alpha}\, e^{-x/\theta}}{x\,\Gamma(\alpha)} = \frac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^{\alpha}\,\Gamma(a)}},\ x > \infty.$$

Exercise: What is the mean of a Gamma Distribution?

[Solution: $\int_0^\infty x\, f(x)\, dx = \int_0^\infty x\, \frac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^{\alpha}\,\Gamma(\alpha)}\, dx = \frac{\int_0^\infty x^{\alpha}\, e^{-x/\theta}\, dx}{\theta^{\alpha}\,\Gamma(\alpha)} = \frac{\Gamma(\alpha+1)\,\theta^{\alpha+1}}{\theta^{\alpha}\,\Gamma(\alpha)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\,\theta = \alpha\theta$.]

Exercise: What is the n^{th} moment of a Gamma Distribution?
[Solution:

$$\int_0^\infty x^n\, f(x)\, dx = \int_0^\infty x^n\, \frac{x^{\alpha-1}\, e^{-x/\theta}}{\theta^{\alpha}\,\Gamma(\alpha)}\, dx = \frac{\int_0^\infty x^{n+\alpha-1}\, e^{-x/\theta}\, dx}{\theta^{\alpha}\,\Gamma(\alpha)} = \frac{\Gamma(\alpha+n)\,\theta^{\alpha+n}}{\theta^{\alpha}\,\Gamma(\alpha)} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\theta^n$$

$= (\alpha+n-1)(\alpha+n-2)....(\alpha)\,\theta^n$.
Comment: This is the formula shown in Appendix A of Loss Models.]

Exercise: What is the 3rd moment of a Gamma Distribution with $\alpha = 5$ and $\theta = 2.5$?
[Solution: $(\alpha+n-1)(\alpha+n-2)....(\alpha)\theta^n$ = $(5+3-1)(5+3-2)(5)(2.5^3) = 3281.25$.]

Relation to the Chi-Square Distribution:

A Chi-Square Distribution with ν degrees of freedom is a Gamma Distribution with shape parameter of $\nu/2$ and scale parameter of 2 : $\chi^2{}_\nu(x) = \Gamma(\nu/2\ ;\ x/2)$.

Therefore, one can look up values of the Incomplete Gamma Function (for half integer or integer values of α) by using the cumulative values of the Chi-Square Distribution.

[153] See "Mahler's Guide to Loss Distributions."

For example, $\Gamma(6;10)$ = the Chi-Square Distribution for 2 x 6 = 12 degrees of freedom at a value of 2 x 10 = 20. For the Chi-Square with 12 d.f. there is a 0.067 chance of a value greater than 20.[154]
Thus, the value of the distribution function is: $\chi^2_{12}(20) = 1 - 0.067 = 0.933 = \Gamma(6;10)$.

Relation to the Poisson Distribution:

The distribution function of a Poisson with mean λ is: $F(x) = 1 - \Gamma(x+1 ; \lambda)$.

For example, for $\lambda = 1.745$, $F(3) = 1 - \Gamma(4 ; 1.745)$.
Now a Chi-Square Distribution with 8 degrees of freedom is a Gamma Distribution with $\alpha = 4$ and $\theta = 2$. Thus, $\Gamma(4 ; 1.745) = \chi^2_8(3.490)$. From the Chi-Square Table, $\chi^2_8(3.490) = 0.1$.[155]
Thus for $\lambda = 1.745$, $F(3) = 1 - 0.1 = 0.9$.

One can verify this directly: $F(3) = e^{-1.745} (1 + 1.745 + 1.745^2/2 + 1.745^3/6) = 0.900$.

Inverse Gamma Distribution:[156]

By employing the change of variables y = 1/x, integrals involving $e^{-1/x}$ and powers of 1/x can be written in terms of the Gamma function:

$$\int_0^{\infty} t^{-(\alpha+1)} \, e^{-\theta/t} \, dt = \Gamma(\alpha)\theta^{-\alpha}.$$

The Inverse Gamma Distribution can be defined in terms of the Incomplete Gamma Function,
$F(x) = 1 - \Gamma[\alpha ; (\theta/x)]$.

The density of the Inverse Gamma is: $\dfrac{\theta^{\alpha} \, e^{-\theta/x}}{x^{\alpha+1} \, \Gamma[\alpha]}$, for $0 < x < \infty$.

A good way to remember the result for integrals from zero to infinity of powers of 1/x times Exponentials of 1/x, is that the density of the Inverse Gamma Distribution must integrate to unity.

[154] From the table, one can tell that the survival function at 20 is between 10% and 5%.
[155] I chose the numbers for this example, so that this distribution function value happens to appear in the table.
[156] See "Mahler's Guide to Loss Distributions," and Appendix A of Loss Models.

Problems:

18.1 (1 point) What is the value of the integral from zero to infinity of: $x^5 e^{-8x}$?

A. less than 0.0004
B. at least 0.0004 but less than 0.0005
C. at least 0.0005 but less than 0.0006
D. at least 0.0006 but less than 0.0007
E. at least 0.0007

18.2 (1 point) What is the density at x = 8 of the Gamma distribution with parameters $\alpha = 3$ and $\theta = 10$?

A. less than 0.012
B. at least 0.012 but less than 0.013
C. at least 0.013 but less than 0.014
D. at least 0.014 but less than 0.015
E. at least 0.015

18.3 (1 point) Determine $\int_0^\infty x^{-6} e^{-4/x} dx$.

A. less than 0.02
B. at least 0.02 but less than 0.03
C. at least 0.03 but less than 0.04
D. at least 0.04 but less than 0.05
E. at least 0.05

18.4 (2 points) What is the integral from 6.3 to 8.4 of $x^2 e^{-x} / 2$?
Hint: Use the Chi-Square table.
A. less than 0.01
B. at least 0.01 but less than 0.03
C. at least 0.03 but less than 0.05
D. at least 0.05 but less than 0.07
E. at least 0.07

18.5 (2 points) What is the integral from 4 to 8 of: $xe^{-x/5}$?
A. 7 B. 8 C. 9 D. 10 E. 11

Use the following information for the next 3 questions:

Define the following distribution function in terms of the Incomplete Gamma Function:
$F(x) = \Gamma[\alpha; \ln(x)/\theta]$, $1 < x$.

18.6 (2 points) What is the probability density function corresponding this distribution function?

A. $\dfrac{\theta^\alpha\, x^\alpha\, e^{-\theta/x}}{\Gamma[\alpha]}$

B. $\dfrac{\ln[x]^{\alpha-1}}{\theta^\alpha\, x^{1+1/\theta}\, \Gamma[\alpha]}$

C. $\dfrac{\theta^\alpha\, x^{\alpha+1}\, e^{-\theta/x}}{\Gamma[\alpha]}$

D. $\dfrac{\ln[x]^{\alpha}}{\theta^\alpha\, x^{1+1/\theta}\, \Gamma[\alpha]}$

E. None of the above

18.7 (2 points) What is the mean of this distribution?

A. $\theta/(\alpha-1)$

B. θ/α

C. $\theta(\alpha-1)$

D. $\theta\alpha$

E. None of the above

18.8 (3 points) If $\alpha = 5$ and $\theta = 1/7$, what is the 3rd moment of this distribution?

A. less than 12
B. at least 12 but less than 13
C. at least 13 but less than 14
D. at least 14 but less than 15
E. at least 15

Solutions to Problems:

18.1. B. $\Gamma(5+1) / 8^{5+1} = 5! / 8^6 =$ **0.000458**.

18.2. D. $\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha) = (10^{-3})\ 8^2 e^{-0.8} / \Gamma(3) =$ **0.0144.**

18.3. B. The density of the Inverse Gamma is: $\theta^\alpha e^{-\theta/x} / \{x^{\alpha+1} \Gamma(\alpha)\}$, $0 < x < \infty$.

Since this density integrates to one, $x^{-(\alpha+1)} e^{-\theta/x}$ integrates to $\theta^{-\alpha}\Gamma(\alpha)$.

Thus taking $\alpha = 5$ and $\theta = 4$, $x^{-6} e^{-4/x}$ integrates to: $4^{-5}\ \Gamma(5) = 24 / 4^5 =$ **0.0234**.

Comment: Alternately, one can make the change of variables y = 1/x.

18.4. C. The integrand is that of the Incomplete Gamma Function for $\alpha = 3$:

$x^{\alpha-1} e^{-x} / \Gamma(\alpha) = x^2 e^{-x} / 2$. Thus the integral is: $\Gamma(3; 8.4) - \Gamma(3; 6.3)$.

Since the Chi-Square Distribution with ν degrees of freedom is a Gamma Distribution with shape parameter of $\nu/2$ and scale parameter of 2 : $\chi^2_6(x) = \Gamma(3\ ; x/2)$.

Looking up the Chi-Square Distribution for 6 degrees of freedom, the distribution function is 99% at 16.8 and 95% at 12.6.

99% = $\Gamma(3; 16.8/2) = \Gamma(3; 8.4)$, and 95% = $\Gamma(3; 12.6/2) = \Gamma(3; 6.3)$.

Thus $\Gamma(3; 8.4) - \Gamma(3; 6.3) = 0.99 - 0.95 =$ **0.04**.

Comment: The particular integral can be done via repeated integration by parts.

One gets: $-e^{-x} \{(x^2/2) + x + 1\}$. Evaluating at the limits of 8.4 and 6.3 gives the same result.

18.5. A. $\int_0^x t\, e^{-t/\theta}\, dt = \theta^2\{1 - e^{-x/\theta} - (x/\theta)e^{-x/\theta}\}$. Set $\theta = 5$.

$$\int_{0.4}^{0.8} t\, e^{-t/5}\, dt = \int_0^{0.8} t\, e^{-t/5}\, dt - \int_0^{0.4} t\, e^{-t/5}\, dt = (5^2)\{1 - e^{-x/5} - (x/5)e^{-x/5}\}\Big]_{x=0.4}^{x=0.8} =$$

$(25)\{e^{-0.8} + (0.8)e^{-0.8} - e^{-1.6} - (1.6)e^{-1.6}\} =$ **7.10**.

Comment: Can also be done using integration by parts or the increasing annuity technique.

18.6. B. Let y = ln(x)/θ. If y follows a Gamma Distribution with parameters α and 1, then x follows a LogGamma Distribution with parameters α and θ. If y follows a Gamma Distribution with parameters α and 1, then $f(y) = y^{\alpha-1} e^{-y} / \Gamma(\alpha)$. Then the density of x is given by:

$f(y)(dy/dx) = \{(\ln(x)/\theta)^{\alpha-1} \exp(-\ln(x)/\theta) / \Gamma(\alpha)\} / (x\theta) = \boldsymbol{\theta^{-\alpha}\{\ln(x)\}^{\alpha-1} / \{x^{1+1/\theta}\Gamma(\alpha)\}}$.

Comment: This is called the LogGamma Distribution and bears the same relationship to the Gamma Distribution as the LogNormal bears to the Normal Distribution.

Note that the support for the LogGamma is 1 to ∞, since when y = 0, x = exp(0θ) = 1.

18.7. E. $\int_1^\infty x f(x) dx = \int_1^\infty \frac{\theta^{-\alpha} \{\ln(x)\}^{\alpha-1}}{x^{1/\theta}\Gamma(\alpha)} dx.$

Let y = ln(x)/θ, and thus x = exp(θ y), dx = exp(θy)θ dy, then the integral for the first moment is:

$$\int_0^\infty \frac{\theta^{-\alpha}\{\theta y\}^{\alpha-1}}{\exp(y)\,\Gamma(\alpha)} \exp(\theta y)\theta\, dy = \int_0^\infty \frac{y^{\alpha-1} \exp[-y(1-\theta)]}{\Gamma(\alpha)} dy = \mathbf{(1-\theta)^{-\alpha}}.$$

18.8. E. The formula for the nth moment is derived as follows:

$$\int_1^\infty x^n f(x)\, dx = \int_1^\infty x^n \frac{\theta^{-\alpha}\{\ln(x)\}^{\alpha-1}}{x^{1+1/\theta}\Gamma(\alpha)} dx = \int_1^\infty x^{n-1-\theta} \frac{\theta^{-\alpha}\{\ln(x)\}^{\alpha-1}}{\Gamma(\alpha)} dx.$$

Let y = ln(x)/θ, and thus x = exp(θ y), dx = exp(θy)θ dy, then the integral for the nth moment is:

$$\int_0^\infty \frac{\theta^{-\alpha}\{\theta y\}^{\alpha-1}}{\exp[(n-1-1/\theta)y\theta]\,\Gamma(\alpha)} \exp(\theta y)\theta\, dy = \int_0^\infty \frac{y^{\alpha-1}\exp[-y(1-n\theta)]}{\Gamma(\alpha)} dy = (1-n\theta)^{-\alpha},\ n\theta < 1.$$

Thus the 3rd moment with α = 5 and θ = 1/7 is: $(1 - n\theta)^{-\alpha} = (1 - 3/7)^{-5}$ = **16.41**.

Comment: One could plug in n = 3 and the value of the parameters at any stage in the computation. I have chosen to do so at the very end.

Section 19, Gamma-Poisson Frequency Process[157]

The single most important specific example of mixing frequency distributions, is mixing Poisson Frequency Distributions via a Gamma Distribution. Each insured in a portfolio is assumed to have a Poisson distribution with mean λ. Across the portfolio, λ is assumed to be distributed via a Gamma Distribution. Due to the mathematical properties of the Gamma and Poisson there are some specific relationships. For example, as will be discussed, the mixed distribution is a Negative Binomial Distribution.

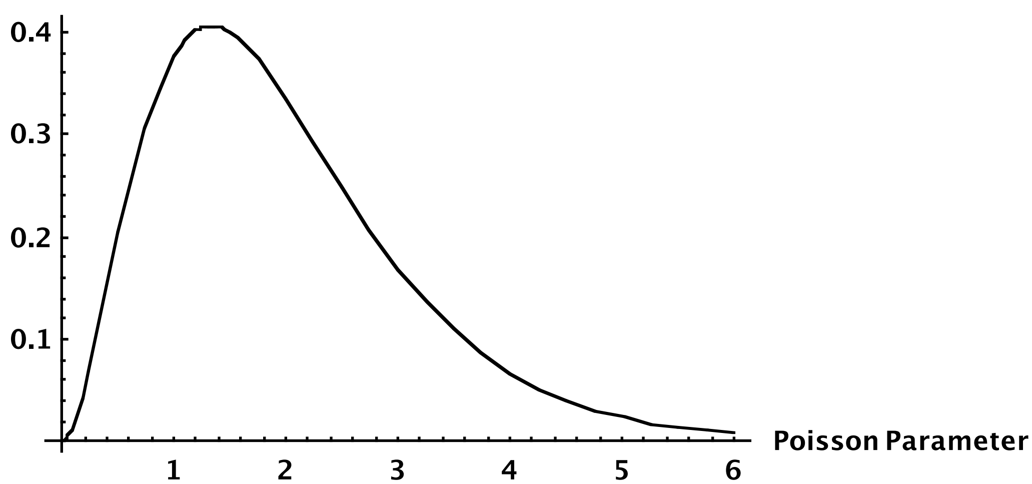
Prior Distribution:

The number of claims a particular policyholder makes in a year is assumed to be Poisson with mean λ. For example, the chance of having 6 claims is given by: $\lambda^6 e^{-\lambda} / 6!$

Assume the λ values of the portfolio of policyholders are Gamma distributed with $\alpha = 3$ and $\theta = 2/3$, and therefore probability density function:[158]

$$f(\lambda) = 1.6875\ \lambda^2 e^{-1.5\lambda} \qquad \lambda \geq 0.$$

This prior Gamma Distribution of Poisson parameters is displayed below:



[157] Section 6.3 of Loss Models.
Additional aspects of the Gamma-Poisson are discussed in "Mahler's Guide to Conjugate Priors."
[158] For the Gamma Distribution, $f(x) = \theta^{-\alpha} x^{\alpha-1} e^{-x/\theta} / \Gamma(\alpha)$.

The Prior Distribution Function is given in terms of the Incomplete Gamma Function:
$F(\lambda) = \Gamma(3; 1.5\lambda)$. So for example, the a priori chance that the μ value lies between 4 and 5 is: $F(5) - F(4) = \Gamma(3; 7.5) - \Gamma(3; 6) = 0.9797 - 0.9380 = 0.0417$.
Graphically, this is the area between 4 and 5 and under the prior Gamma.

Mixed Distribution:

If we have a risk and do not know what type it is, in order to get the chance of having 6 claims, one would weight together the chances of having 6 claims, using the a priori probabilities and integrating from zero to infinity:[159]

$$\int_0^\infty \frac{\lambda^6 e^{-\lambda}}{6!} f(\lambda)\, d\lambda = \int_0^\infty \frac{\lambda^6 e^{-\lambda}}{6!} 1.6875\, \lambda^2 e^{-1.5\lambda}\, d\lambda = 0.00234375 \int_0^\infty \lambda^8 e^{-2.5\lambda}\, d\lambda.$$

This integral can be written in terms of the (complete) Gamma function:

$$\int_0^\infty \lambda^{\alpha-1} e^{-\lambda/\theta}\, d\lambda = \Gamma(\alpha)\theta^\alpha.$$

Thus $\int_0^\infty \lambda^8 e^{-2.5\lambda}\, d\lambda = \Gamma(9)\, 2.5^{-9} = (8!)\,(0.4)^9 \cong 10.57$.

Thus the chance of having 6 claims $\cong$ (0.00234375) (10.57) $\cong$ 2.5%.

More generally, if the distribution of Poisson parameters λ is given by a Gamma distribution $f(\lambda) = \theta^{-\alpha}\lambda^{\alpha-1} e^{-\lambda/\theta} / \Gamma(\alpha)$, and we compute the chance of having n accidents by integrating from zero to infinity:

$$\int_0^\infty \frac{\lambda^n e^{-\lambda}}{n!} f(\lambda)\, d\lambda = \int_0^\infty \frac{\lambda^6 e^{-\lambda}}{6!} \frac{\lambda^{\alpha-1} e^{-\lambda/\theta}}{\theta^\alpha \Gamma(\alpha)}\, d\lambda = \frac{1}{n!\, \theta^\alpha \Gamma(\alpha)} \int_0^\infty \lambda^{n+\alpha-1} e^{-\lambda(1+1/\theta)}\, d\lambda =$$

$$\frac{1}{n!\, \theta^\alpha \Gamma(\alpha)} \frac{\Gamma(n+\alpha)}{(1 + 1/\theta)^{n+\alpha}} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!} \frac{\theta^n}{\theta^{n+\alpha} (1 + 1/\theta)^{n+\alpha}} = \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} \frac{\theta^n}{(1 + \theta)^{n+\alpha}}.$$

The mixed distribution is in the form of the Negative Binomial distribution with parameters $r = \alpha$ and $\beta = \theta$:

Probability of n accidents = $\frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}$.

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[159] Note the way both the Gamma and the Poisson have factors involving powers of λ and $e^{-\lambda}$ and these similar factors combine in the product.

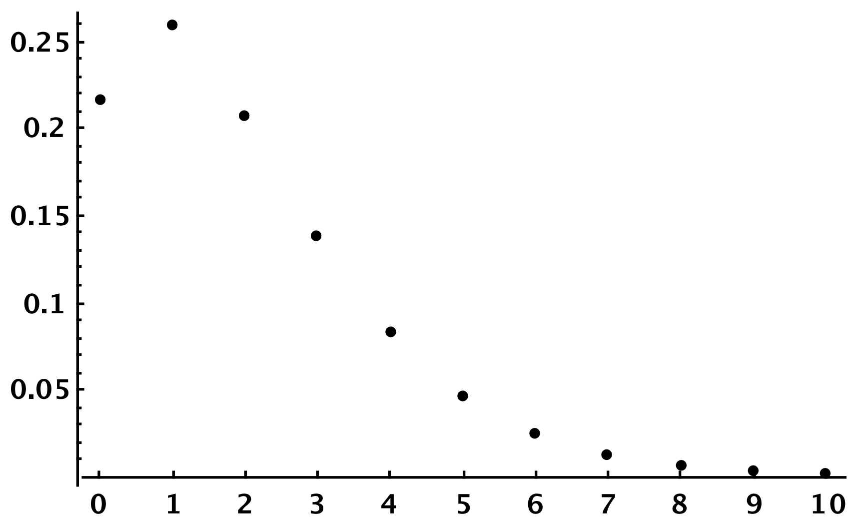
For the specific case dealt with previously: n = 6, α = 3 and θ = 2/3.

Therefore, the mixed Negative Binomial Distribution has parameters r = α = 3 and β = θ = 2/3.

Thus the chance of having 6 claims is: $\frac{(3)(4)(5)(6)(7)(8)}{6!} \frac{(2/3)^6}{(1 + 2/3)^{6+3}}$ = 2.477%.

This is the same result as calculated above.

This mixed Negative Binomial Distribution is displayed below, through 10 claims:



On the exam, one should not go through the calculation above. Rather remember that the mixed distribution is a Negative Binomial.

When Poissons are mixed via a Gamma Distribution,
the mixed distribution is always a Negative Binomial Distribution,
with r = α = shape parameter of the Gamma and
β = θ = scale parameter of the Gamma.

r goes with alpha, beta rhymes with theta.

Note that the overall (a priori) mean can be computed in either one of two ways.
First one can weight together the means for each type of risk, using the a priori probabilities.
This is $E[\lambda]$ = the mean of the prior Gamma = $\alpha\theta = 3(2/3) = 2$.
Alternately, one can compute the mean of the mixed distribution: the mean of a Negative Binomial is $r\beta = 3(2/3) = 2$. Of course the two results match.

Exponential-Poisson:[160]

It is important to note that **the Exponential distribution is a special case of the Gamma distribution, for α = 1**.

For the important special case $\alpha = 1$, we have an Exponential distribution of λ: $f(\lambda) = e^{-\lambda/\theta}/\theta$, $\lambda \geq 0$.
The mixed distribution is a Negative Binomial Distribution with $r = 1$ and $\beta = \theta$.

For the Exponential-Poisson, the mixed distribution is a Geometric Distribution with $\beta = \theta$.

Mixed Distribution for the Gamma-Poisson, When Observing Several Years of Data:

One can observe for a period of time longer than a year. If an insured has a Poisson parameter of λ for each individual year, with λ the same for each year, and the years are independent, then for example one has a Poisson parameter of 7λ for 7 years. The chances of such an insured having a given number of claims over 7 years is given by a Poisson with parameter 7λ. For a portfolio of insureds, each of its Poisson parameters is multiplied by 7. This is mathematically just like inflation.

If before their each being multiplied by 7, the Poisson parameters follow a Gamma distribution with parameter α and θ, then after being multiplied by 7 they follow a Gamma with parameters α and 7θ.[161] Thus the mixed distribution for 7 years of data is given by a Negative Binomial with parameters $r = \alpha$ and $\beta = 7\theta$.

[160] See for example 3/11/01, Q.27.
[161] Under uniform inflation, the scale parameter of the Gamma Distribution is multiplied by the inflation factor. See "Mahler's Guide to Loss Distributions."

In general, **if one observes a Gamma-Poisson situation for Y years, and each insured's Poisson parameter does not change over time, then the distribution of Poisson parameters for Y years is given by a Gamma Distribution with parameters α and $Y\theta$, and the mixed distribution for Y years of data is given by a Negative Binomial Distribution, with parameters $r = \alpha$ and $\beta = Y\theta$.**[162]

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. The distribution of λ over the portfolio of insureds is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
What is the mean annual claim frequency for the portfolio of insureds?
[Solution: The mean annual claims frequency = mean of the (prior) Gamma $= \alpha\theta = (3)(0.01) = 3\%$.]

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. For each insured, λ does not change over time. For each insured, the numbers of claims in one year is independent of the number of claims in another year. The distribution of λ over the portfolio of insureds is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
An insured is picked at random and observed for 9 years.
What is the chance of observing exactly 4 claims from this insured?
[Solution: The mixed distribution for 9 years of data is given by a Negative Binomial Distribution with parameters $r = \alpha = 3$ and $\beta = Y\theta = (9)(0.01) = 0.09$.

$$f(4) = \frac{(4+3-1)!}{4!\ 2!} \frac{0.09^4}{(1 + 0.09)^{3+4}} = 0.054\%.]$$

If Lois has a low expected annual claim frequency, for example 2%, then over 9 years she has a Poisson Distribution with mean 18%. Her chance of having 4 claims during these nine years is:
$0.18^4\ e^{-0.18} / 24 = 0.004\%$.

If Hi has a very high expected annual claim frequency, for example 20%, then over 9 years he has a Poisson Distribution with mean 180%. His chance of having 4 claims during these nine years is:
$1.8^4\ e^{-1.8} / 24 = 7.23\%$.

Drivers such as Lois with a low λ in one year are assumed to have the same low λ every year. Such good drivers have an extremely small chance of having four claims in 9 years.

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[162] "Each insured's Poisson parameter does not change over time." If Alan's lambda is 4% this year, it is 4% next year, and every year. Similarly, if Bonnie's lambda is 3% this year , then it is 3% every year.
Unless stated otherwise, on the exam assume lambda does <u>not</u> vary over time.

Drivers such as Hi with a very high λ in one year are assumed to have the same high λ every year. Such drivers have a significant chance of having four claims in 9 years. It is such very bad drivers which contribute significantly to the 0.054% probability of four claims in 9 years for an insured picked at random.

This situation in which for a given insured λ is the same over time, contrasts with that in which λ changes randomly each year.

Exercise: Assume that the number of claims in a year for each insured has a Poisson Distribution with mean λ. For each insured, λ changes each year at random; the λ in one year is independent of the λ in another year.
The distribution of λ is a Gamma Distribution with parameters $\alpha = 3$ and $\theta = 0.01$.
An insured is picked at random and observed for 9 years.
What is the chance of observing exactly 4 claims from this insured?
[Solution: The mixed distribution for 1 year of data is given by a Negative Binomial Distribution with parameters $r = \alpha = 3$ and $\beta = \theta = 0.01$. Over 9 years, we get a sum of 9 independent Negative Binomials, with $r = (9)(3) = 27$ and $\beta = 0.01$.

$$f(4) = \frac{(4+27-1)!}{4!\ 26!} \frac{0.01^4}{(1\ +\ 0.01)^{27+4}} = 0.00020.]$$

This is different than the Gamma-Poisson process in which we assume that the lambda for an individual insured is the same each year. For the Gamma-Poisson the β parameter is multiplied by Y, while here the r parameter is multiplied by Y. This situation in which instead λ changes each year is mathematically the same as if we assume an insured each year has a Negative Binomial Distribution.

For example, assume an insured has a Negative Binomial with parameters r and β. Assume the numbers of claims in one year is independent of the number of claims in another year. Then over Y years, we add up Y independent identically distributed Negative Binomials; over Y years, the frequency distribution for this insured is Negative Binomial with parameters Yr and β.

Exercise: Assume that the number of claims in a year for an insured has a Negative Binomial Distribution with parameters $r = 3$ and $\beta = 0.01$. What is the mean annual claim frequency?
[Solution: $r\beta = (3)(0.01) = 3\%$.]

Exercise: Assume that the number of claims in a year for an insured has a Negative Binomial Distribution with parameters r = 3 and β = 0.01. The numbers of claims in one year is independent of the number of claims in another year. What is the chance of observing exactly 4 claims over 9 years from this insured?
[Solution: Over 9 years, the frequency distribution for this insured is Negative Binomial with parameters r = (9)(3) = 27 and β = 0.01.

$$f(4) = \frac{(4+27-1)!}{4!\ 26!} \frac{0.01^4}{(1\ +\ 0.01)^{27+4}} = 0.00020.]$$

Even though both situations had a 3% mean annual claim frequency, the probability of observing 4 claims over 9 years was higher in the Gamma-Poisson situation with λ the same each year for a given insured, than when we assumed λ changed each year or equivalently an insured had the same Negative Binomial Distribution each year. In the Gamma-Poisson situation with λ the same each year for a given insured, we were more likely to see extreme results such as 4 claims in 9 years, since there is a small probability of picking at random an insured with a high expected annual claim frequency, such as Hi with λ = 20%.

Thinning a Negative Binomial Distribution:

Since the Gamma-Poisson is one source of the Negative Binomial Distribution, it can be used to aid our understanding of the Negative Binomial Distribution.

For example, assume we have a Negative Binomial Distribution with r = 4 and β = 2.
We can think of that as resulting from a mixture of Poisson Distributions, with λ distributed via a Gamma Distribution with α = 4 and θ = 2.[163]

Assume frequency and severity are independent, and that 30% of losses are “large.”
Then for each insured, his large losses are Poisson with mean 0.3λ. If λ is distributed via a Gamma with α = 4 and θ = 2, then 0.3λ is distributed via a Gamma with α = 4 and θ = (0.3)(2) = 0.6.[164]
The large losses are a Gamma-Poisson Process, and therefore, across the whole portfolio, the distribution of large losses is Negative Binomial, with r = 4 and β = 0.6.

[163] While this may not be real world situation that the Negative Binomial is modeling, since the results are mathematically identical, we can assume it is for the purpose of deriving general mathematical results.
[164] When a variable is Gamma Distributed, then a constant times that variable is also Gamma Distributed, with the same shape parameter, but with the scale parameter multiplied by that constant. See the discussion of uniform inflation in ”Mahler’s Guide to Loss Distributions.”

In this manner one can show, as has been discussed previously, that if losses are Negative Binomial with parameters r and β, then if we take a fraction t of all the losses in a manner independent of frequency, then these selected losses are Negative Binomial with parameters r and $t\beta$.[165]

Returning to the example, the small losses for an individual insured are Poisson with mean 0.7λ. Since λ is Gamma distributed, 0.7λ is distributed via a Gamma with $\alpha = 4$ and $\theta = (0.7)(2) = 1.4$. Therefore, across the whole portfolio, the distribution of small losses is Negative Binomial, with r = 4 and $\beta = 1.4$.

Thus as in the Poisson situation, the overall process has been thinned into two similar processes. However, unlike the Poisson case, these two Negative Binomials are not independent.

If for example, we observe a lot of large losses, such as 5, it is more likely that the observation came from an insured with a large λ. This implies we are more likely to also have observed a higher than average number of small losses. The number of large losses and the number of small losses are positively correlated.[166]

Correlation of Number of Small and Large Losses, Negative Binomial:

Assume the number of losses follow a Negative Binomial Distribution with parameters r and β, and that "large" losses are t of all the losses. As previously, assume each insured is Poisson with mean λ, and λ is distributed via a Gamma with $\alpha = r$ and $\theta = \beta$.

Then the number of large losses is a Gamma-Poisson with $\alpha = r$ and $\theta = t\beta$.
Posterior to observing L large losses, the distribution of the mean frequency for large losses is Gamma with $\alpha = r + L$ and $1/\theta = 1/(t\beta) + 1 \Rightarrow \theta = t\beta/(1+ t\beta)$.[167]
Since the mean frequency of large losses is t times the mean frequency, posterior to observing L large losses, the distribution of the mean frequency is Gamma with $\alpha = r + L$ and $\theta = \beta/(1+ t\beta)$.

Therefore, given we have observed L large losses, the small losses are Gamma-Poisson with $\alpha = r + L$, and $\theta = (1-t)\beta/(1+ t\beta)$.

[165] This can be derived via probability generating functions. See Example 8.8 in Loss Models.
[166] In the case of thinning a Binomial, the number of large and small loses would be negatively correlated.
[167] See "Mahler's Guide to Conjugate Priors."

One computes the correlation between the number of small losses, S, and the number of large losses, L, as follows:

$$E[LS] = E_L[E[LS \mid L]] = E_L[L\, E[S \mid L]] = E_L\left[\frac{L\,(r + L)\,(1-t)\beta}{1 + t\beta}\right] =$$

$$\frac{(1-t)\beta}{1 + t\beta}\{rE_L[L] + E_L[L^2]\} = \frac{(1-t)\beta}{1 + t\beta}\{rrt\beta + rt\beta(1+t\beta)+(rt\beta)^2\} = (1-t)t\beta^2 r(1+r).$$ [168]

$$Cov[L, S] = E[LS] - E[L]E[S] = (1-t)t\beta^2 r(1+r) - rt\beta r(1-t)\beta = \beta^2 rt(1-t).$$

$$Corr[L, S] = \frac{\beta^2\, rt(1-t)}{\sqrt{rt\beta(1+t\beta)r(1-t)\beta\{1+(1-t)\beta\}}} = \frac{1}{\sqrt{\left(1 + \frac{1}{t\beta}\right)\left(1 + \frac{1}{(1-t)\beta}\right)}} > 0.$$

For example, assume we have a Negative Binomial Distribution with $r = 4$ and $\beta = 2$.

Assume frequency and severity are independent, and that 30% of losses are “large.”

Then the number of large losses are Negative Binomial with $r = 4$ and $\beta = 0.6$, and the number of small losses are Negative Binomial with $r = 4$ and $\beta = 1.4$.

The correlation of the number of large and small losses is:

$$\frac{1}{\sqrt{\left(1 + \frac{1}{t\beta}\right)\left(1 + \frac{1}{(1-t)\beta}\right)}} = \frac{1}{\sqrt{(1 + 1/0.6)\,(1 + 1/1.4)}} = 0.468.$$

[168] Large losses are Negative Binomial with parameters r and $t\beta$. Thus, $E_L[L^2] = Var[L] + E[L]^2 = rt\beta(1+t\beta)+(rt\beta)^2$.

Problems:

Use the following information to answer the next 2 questions:

The number of claims a particular insured makes in a year is Poisson with mean λ.

λ for a particular insured remains the same each year.

The values of the Poisson parameter λ (for annual claim frequency) for the insureds in a portfolio follow a Gamma distribution, with parameters $\alpha = 3$ and $\theta = 1/12$.

19.1 (2 points) What is the chance that an insured picked at random from the portfolio will have no claims over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 45% but less than 50%
E. at least 50%

19.2 (2 points) What is the chance that an insured picked at random from the portfolio will have one claim over the next three years?
A. less than 35%
B. at least 35% but less than 40%
C. at least 40% but less than 45%
D. at least 45% but less than 50%
E. at least 50%

19.3 (2 points) The distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.6 and variance 0.9.
The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.
Calculate the variance of this gamma distribution.
(A) 0.20 (B) 0.25 (C) 0.30 (D) 0.35 (E) 0.40

19.4 (2 points) The number of claims a particular policyholder makes in a year has a Poisson distribution with mean μ. The μ-values for policyholders follow a gamma distribution with variance equal to 0.3. The resulting distribution of policyholders by number of claims is a Negative Binomial with parameters r and β such that the variance is equal to 0.7.

What is the value of $r(1+\beta)$?

A. less than 0.90
B. at least 0.90 but less than 0.95
C. at least 0.95 but less than 1.00
D. at least 1.00 but less than 1.05
E. at least 1.05

Use the following information for the next 3 questions:

Assume that the number of claims for an individual insured is given by a Poisson distribution with mean (annual) claim frequency λ and variance λ. Also assume that the parameter λ varies for the different insureds, with λ following a Gamma distribution:

$g(\lambda) = \theta^{-\alpha}\lambda^{\alpha-1} e^{-\lambda/\theta} / \Gamma(\alpha)$, for $0 < \lambda < \infty$, with mean $\alpha\theta$, and variance $\alpha\theta^2$.

19.5 (2 points) An insured is picked at random and observed for one year.
What is the chance of observing 2 claims?

A. $\dfrac{\alpha\theta^2}{(1+\theta)^{\alpha+2}}$ B. $\dfrac{\alpha(\alpha+1)\theta^2}{(1+\theta)^{\alpha+2}}$ C. $\dfrac{\alpha(\alpha+1)\theta^2}{2\,(1+\theta)^{\alpha+2}}$

D. $\dfrac{\alpha^2\,(\alpha+1)\theta^2}{6\,(1+\theta)^{\alpha+2}}$ E. $\dfrac{\alpha^2\,(\alpha+1)\,(\alpha+2)\,\theta^2}{6\,(1+\theta)^{\alpha+2}}$

19.6 (2 points) What is the unconditional mean frequency?

A. $\alpha\theta$ B. $(\alpha-1)\theta$ C. $\alpha(\alpha-1)\theta^2$ D. $\alpha(\alpha-1)\theta^2$ E. $\alpha(\alpha-1)(\alpha+1)\theta^2/2$

19.7 (3 points) What is the unconditional variance?

A. $\alpha\theta^2$ B. $\alpha\theta+\alpha\theta^2$ C. $\alpha\theta+\alpha^2\theta^2$ D. $\alpha^2\theta^2$ E. $\alpha(\alpha+1)\,\theta$

Use the following information for the next 8 questions:
As he walks, Clumsy Klem loses coins at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 0.2 and variance 0.016.
The denominations of coins are randomly distributed: 50% of the coins are worth 5;
30% of the coins are worth 10; and 20% of the coins are worth 25.

19.8 (2 points) Calculate the probability that Clumsy Klem loses exactly one coin during the tenth minute of today's walk.
(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

19.9 (3 points) Calculate the probability that Clumsy Klem loses exactly two coins during the first 10 minutes of today's walk.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

19.10 (4 points) Calculate the probability that the worth of the coins Clumsy Klem loses during his one-hour walk today is greater than 300.
A. 1% B. 3% C. 5% D. 7% E. 9%

19.11 (2 points) Calculate the probability that the sum of the worth of the coins Clumsy Klem loses during his one-hour walks each day for the next 5 days is greater than 900.
A. 1% B. 3% C. 5% D. 7% E. 9%

19.12 (2 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and possibly coins of other denominations?
A. 31% B. 33% C. 35% D. 37% E. 39%

19.13 (3 points) During the first 10 minutes of today's walk, what is the chance that Clumsy Klem loses exactly one coin of worth 5, and no coins of other denominations?
A. 11.6% B. 12.0% C. 12.4% D. 12.8% E. 13.2%

19.14 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. What is the probability that A + B = 3?
A. 0.2% B. 0.4% C. 0.6% D. 0.8% E. 1.0%

19.15 (3 points) Let A be the number of coins Clumsy Klem loses during the first minute of his walk today. Let B be the number of coins Clumsy Klem loses during the first minute of his walk tomorrow. Let C be the number of coins Clumsy Klem loses during the first minute of his walk the day after tomorrow. What is the probability that A + B + C = 2?
A. 8% B. 10% C. 12% D. 14% E. 16%

19.16 (2 points) For an insurance portfolio the distribution of the number of claims a particular policyholder makes in a year is Poisson with mean λ.

The λ-values of the policyholders follow the Gamma distribution, with parameters $\alpha = 4$, and $\theta = 1/9$.

The probability that a policyholder chosen at random will experience x claims is given by which of the following?

A. $\frac{(x+3)!}{x!\ 3!}\ 0.9^4\ 0.1^x$

B. $\frac{(x+3)!}{x!\ 3!}\ 0.1^4\ 0.9^x$

C. $\frac{(x+8)!}{x!\ 8!}\ 0.75^4\ 0.25^x$

D. $\frac{(x+8)!}{x!\ 8!}\ 0.25^4\ 0.75^x$

E. None of A, B, C, or D.

19.17 (2 points) The number of claims a particular policyholder makes in a year has a Poisson distribution with mean λ. The λ-values for policyholders follow a Gamma distribution.

This Gamma Distribution has a variance equal to one quarter that of the resulting Negative Binomial distribution of policyholders by number of claims.

What is the value of the β parameter of this Negative Binomial Distribution?

A. 1/6 B. 1/5 C. 1/4 D. 1/3 E. Can not be determined

19.18 (1 point) Use the following information:

- The random variable representing the number of claims for a single policyholder follows a Poisson distribution.
- For a portfolio of policyholders, the Poisson parameters follow a Gamma distribution representing the heterogeneity of risks within that portfolio.
- The random variable representing the number of claims in a year of a policyholder, chosen at random, follows a Negative Binomial distribution with parameters:

  $r = 4$ and $\beta = 3/17$.

Determine the variance of the Gamma distribution.

(A) 0.110 (B) 0.115 (C) 0.120 (D) 0.125 (E) 0.130

<u>19.19</u> (2 points) Tom will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density $\lambda e^{-\lambda/1.4}/1.96$.

(ii) He then generates x from the Poisson distribution with mean λ.

(iii) He repeats the process 99,999 more times: first generating a value λ, then generating x from the Poisson distribution with mean λ.

Calculate the expected number of Tom's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

<u>19.20</u> (2 points) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

<u>19.21</u> (2 points) Dick will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density $\lambda\ e^{-\lambda/1.4}\ /1.96$.

(ii) He will then generate 100,000 independent values from the Poisson distribution with mean λ.

Calculate the expected number of Dick's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

<u>19.22</u> (2 points) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

19.23 (1 point) Harry will generate via simulation 100,000 values of the random variable X as follows:

(i) He will generate the observed value λ from a distribution with density

$\lambda\ e^{-\lambda/1.4}\ /1.96$.

(ii) He then generates x from the Poisson distribution with mean λ.

(iii) He will then copy 99,999 times this value of x.

Calculate the expected number of Harry's 100,000 simulated values of X that are 6.

(A) 4200 (B) 4400 (C) 4600 (D) 4800 (E) 5000

19.24 (1 point) In the previous question, let V = the variance of a single simulated set of 100,000 values. What is the expected value of V?

A. 0 B. 2.8 C. 3.92 D. 5.6 E. 6.72

Use the following information for the next 7 questions:

- The number of vehicles arriving at an amusement park per day is Poisson with mean λ.
- λ varies from day to day via a Gamma Distribution with $\alpha = 40$ and $\theta = 10$.
- The value of λ on one day is independent of the value of λ on another day.
- The number of people leaving each vehicle is:
  1 + a Negative Binomial Distribution with $r = 1.6$ and $\beta = 6$.
- The amount of money spent at the amusement park by each person is
  LogNormal with $\mu = 5$ and $\sigma = 0.8$.

19.25 (1 point) What is the variance of the number of vehicles that will show up tomorrow at the amusement park?

A. 4,000 B. 4,400 C. 4,800 D. 5,200 E. 5,600

19.26 (1 point) What is the variance of the number of vehicles that will show up over the next 7 days at the amusement park?

A. 25,000 B. 27,000 C. 29,000 D. 31,000 E. 33,000

19.27 (2 points) What is the variance of the number of people that will show up tomorrow at the amusement park?

A. 480,000 B. 490,000 C. 500,000 D. 510,000 E. 520,000

19.28 (1 point) What is the variance of the number of people that will show up over the next 7 days at the amusement park?

A. 2.8 million B. 3.0 million C. 3.2 million D. 3.4 million E. 3.6 million

19.29 (3 points) What is the standard deviation of the money spent tomorrow at the amusement park?

A. 150,000 B. 160,000 C. 170,000 D. 180,000 E. 190,000

19.30 (1 point) What is the standard deviation of the money spent over the next 7 days at the amusement park?

A. 360,000 B. 370,000 C. 380,000 D. 390,000 E. 400,000

19.31 (2 points) You simulate the amount of the money spent over the next 7 days at the amusement park. You run this simulation a total of 1000 times.
How many runs do you expect in which less than 5 million is spent?

A. 1 B. 2 C. 3 D. 4 E. 5

Use the following information for the next 6 questions:

- For each individual driver, the number of accidents in a year follows a Poisson Distribution.
- For each individual driver, the mean of their Poisson Distribution λ is the same each year.
- For each individual driver, the number of accidents each year is independent of other years.
- The number of accidents for different drivers are independent.
- λ varies between drivers via a Gamma Distribution with mean 0.08 and variance 0.0032.
- Moe, Larry, and Curly are each drivers.

19.32 (2 points) What is the probability that Moe has exactly one accident next year?
A. 6.9% B. 7.1% C. 7.3% D. 7.5% E. 7.7%

19.33 (2 points) What is the probability that Larry has exactly 2 accidents over the next 3 years?
A. 2.25% B. 2.50% C. 2.75% D. 3.00% E. 3.25%

19.34 (2 points) What is the probability that Moe, Larry, and Curly have a total of exactly 2 accidents during the next year?
A. 2.25% B. 2.50% C. 2.75% D. 3.00% E. 3.25%

19.35 (2 points) What is the probability that Moe, Larry, and Curly have a total of exactly 3 accidents during the next four years?
A. 5.2% B. 5.4% C. 5.6% D. 5.8% E. 6.0%

19.36 (3 points) What is the probability that Moe has no accidents next year, Larry has exactly one accident over the next two years, and Curly has exactly two accidents over the next three years?
A. 0.3% B. 0.4% C. 0.5% D. 0.6% E. 0.7%

19.37 (9 points) Let M = the number of accidents Moe has next year.
Let L = the number of accidents Larry has over the next two years.
Let C = the number of accidents Curly has over the next three years.
Determine the probability that: M + L + C = 3.
A. 0.9% B. 1.1% C. 1.3% D. 1.5% E. 1.7%

Use the following information to answer the next 3 questions:
The number of claims a particular policyholder makes in a year is Poisson. The values of the Poisson parameter (for annual claim frequency) for the individual policyholders in a portfolio of 10,000 follow a Gamma distribution, with parameters $\alpha = 4$ and $\theta = 0.1$.
You observe this portfolio for one year and divide it into three groups based on how many claims you observe for each policyholder: Group A: Those with no claims.
Group B: Those with one claim. Group C: Those with two or more claims.

19.38 (1 point) What is the expected size of Group A?
(A) 6200 (B) 6400 (C) 6600 (D) 6800 (E) 7000

19.39 (1 point) What is the expected size of Group B?
(A) 2400 (B) 2500 (C) 2600 (D) 2700 (E) 2800

19.40 (1 point) What is the expected size of Group C?
(A) 630 (B) 650 (C) 670 (D) 690 (E) 710

19.41 (3 points) The claims from a particular insured in a time period t are Poisson with mean λt.
The values of λ for the individual insureds in a portfolio follow a Gamma distribution, with parameters $\alpha = 3$ and $\theta = 0.02$.
For an insured picked at random what is the average wait until the first claim?
A. 17 B. 19 C. 21 D. 23 E. 25

19.42 (2 points) Use the following information:

- Frequency for an individual is a 80-20 mixture of two Poissons with means λ and 3λ.
- The distribution of λ is Exponential with a mean of 0.1.

For an insured picked at random, what is the probability of seeing two claims?
A. 1.2% B. 1.3% C. 1.4% D. 1.5% E. 1.6%

19.43 (2 points) Claim frequency follows a Poisson distribution with parameter λ.
λ is distributed according to: $g(\lambda) = 25\, \lambda\, e^{-5\lambda}$.
Determine the probability that there will be at least 2 claims during the next year.
A. 5% B. 7% C. 9% D. 11% E. 13%

Use the following information for the next two questions:

- 60% of claims are small.
- 40% of claims are large.
- The annual number of claims from a particular insured is Poisson with mean λ.
- λ is distributed across a group of insureds via a Gamma with $\alpha = 2$ and $\theta = 0.5$.
- You pick an insured at random and observe for one year.

19.44 (2 points) What is the variance of the number of small claims?
A. 0.78 B. 0.80 C. 0.82 D. 0.84 E. 0.86

19.45 (2 points) What is the variance of the number of large claims?
A. 0.40 B. 0.42 C. 0.44 D. 0.46 E. 0.48

19.46 (CAS9, 11/94, Q.7) (1 point)
For a group of insureds, each insured has a frequency which is Poisson. There are different assumptions for the distribution across this group of the probability of having an accident.
Which of the following statements are true?
1. If the distribution of the probability of having an accident is constant,
then the distribution of the risks by number of accidents is Poisson.
2. If the distribution of the probability of having an accident is Gamma,
then the distribution of risks by number of accidents is Negative Binomial.
3. If the distribution of the probability of having an accident is Poisson,
then the distribution of risks by number of accidents is Negative Binomial.
A. 2 only B. 3 only C. 1 and 2 D. 1 and 3 E. 1, 2, and 3

19.47 (4B, 11/96, Q.15) (2 points) You are given the following:

- The number of claims for a single policyholder follows a Poisson distribution with mean λ.
- λ follows a gamma distribution.
- The number of claims for a policyholder chosen at random follows a distribution with mean 0.10 and variance 0.15.

Determine the variance of the gamma distribution.
A. 0.05 B. 0.10 C. 0.15 D. 0.25 E. 0.30

19.48 (4B, 11/96, Q.26) (2 points) You are given the following:

- The probability that a single insured will produce 0 claims during the next exposure period is $e^{-\lambda}$.
- λ varies by insured and follows a distribution with density function $f(\lambda) = 36\lambda e^{-6\lambda}$, $0 < \lambda < \infty$.

Determine the probability that a randomly selected insured will produce 0 claims during the next exposure period.
A. Less than 0.72
B. At least 0.72, but less than 0.77
C. At least 0.77, but less than 0.82
D. At least 0.82, but less than 0.87
E. At least 0.87

<u>19.49</u> (Course 3 Sample Exam, Q.12) The annual number of accidents for an individual driver has a Poisson distribution with mean λ. The Poisson means, λ, of a heterogeneous population of drivers have a gamma distribution with mean 0.1 and variance 0.01.
Calculate the probability that a driver selected at random from the population will have 2 or more accidents in one year.
A. 1/121 B. 1/110 C. 1/100 D. 1/90 E. 1/81

19.50 (3, 5/00, Q.4) (2.5 points) You are given:
(i) The claim count N has a Poisson distribution with mean Λ.
(ii) Λ has a gamma distribution with mean 1 and variance 2.
Calculate the probability that N = 1.
(A) 0.19 (B) 0.24 (C) 0.31 (D) 0.34 (E) 0.37

19.51 (3, 5/01, Q.3 & 2009 Sample Q.104) (2.5 points) Glen is practicing his simulation skills.
He generates 1000 values of the random variable X as follows:
(i) He generates the observed value λ from the gamma distribution with $\alpha = 2$ and $\theta = 1$ (hence with mean 2 and variance 2).
(ii) He then generates x from the Poisson distribution with mean λ.
(iii) He repeats the process 999 more times: first generating a value λ, then generating x from the Poisson distribution with mean λ.
(iv) The repetitions are mutually independent.
Calculate the expected number of times that his simulated value of X is 3.
(A) 75 (B) 100 (C) 125 (D) 150 (E) 175

19.52 (3, 5/01, Q.15 & 2009 Sample Q.105) (2.5 points) An actuary for an automobile insurance company determines that the distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.2 and variance 0.4.
The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.
Calculate the variance of this gamma distribution.
(A) 0.20 (B) 0.25 (C) 0.30 (D) 0.35 (E) 0.40

19.53 (3, 11/01, Q.27) (2.5 points) On his walk to work, Lucky Tom finds coins on the ground at a Poisson rate. The Poisson rate, expressed in coins per minute, is constant during any one day, but varies from day to day according to a gamma distribution with mean 2 and variance 4.
Calculate the probability that Lucky Tom finds exactly one coin during the sixth minute of today's walk.
(A) 0.22 (B) 0.24 (C) 0.26 (D) 0.28 (E) 0.30

19.54 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first <u>two</u> minutes of today's walk.
(A) 0.12 (B) 0.14 (C) 0.16 (D) 0.18 (E) 0.20

19.55 (3 points) In 3, 11/01, Q.27, let A = the number of coins that Lucky Tom finds during the first minute of <u>today's</u> walk. Let B = the number of coins that Lucky Tom finds during the first minute of <u>tomorrow's</u> walk. Calculate Prob[A + B = 1].
(A) 0.09 (B) 0.11 (C) 0.13 (D) 0.15 (E) 0.17

19.56 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the third minute of today's walk and exactly one coin during the fifth minute of today's walk.
A. Less than 4.5%
B. At least 4.5%, but less than 5.0%
C. At least 5.0%, but less than 5.5%
D. At least 5.5%, but less than 6.0%
E. At least 6.0%

19.57 (3 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk, exactly two coins during the second minute of today's walk, and exactly three coins during the third minute of today's walk.
A. Less than 0.2%
B. At least 0.2%, but less than 0.3%
C. At least 0.3%, but less than 0.4%
D. At least 0.4%, but less than 0.5%
E. At least 0.5%

19.58 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first minute of today's walk and exactly one coin during the fifth minute of <u>tomorrow's</u> walk.
(A) 0.05 (B) 0.06 (C) 0.07 (D) 0.08 (E) 0.09

19.59 (2 points) In 3, 11/01, Q.27, calculate the probability that Lucky Tom finds exactly one coin during the first three minutes of today's walk and exactly one coin during the first three minutes of <u>tomorrow's</u> walk.
(A) 0.005 (B) 0.010 (C) 0.015 (D) 0.020 (E) 0.025

19.60 (3, 11/02, Q.5 & 2009 Sample Q.90) (2.5 points) Actuaries have modeled auto windshield claim frequencies. They have concluded that the number of windshield claims filed per year per driver follows the Poisson distribution with parameter λ, where λ follows the gamma distribution with mean 3 and variance 3.
Calculate the probability that a driver selected at random will file no more than 1 windshield claim next year.
(A) 0.15 (B) 0.19 (C) 0.20 (D) 0.24 (E) 0.31

19.61 (CAS3, 11/03, Q.15) (2.5 points)
Two actuaries are simulating the number of automobile claims for a book of business.
For the population that they are studying:
i) The claim frequency for each individual driver has a Poisson distribution.
ii) The means of the Poisson distributions are distributed as a random variable, Λ.
iii) Λ has a gamma distribution.
In the first actuary's simulation, a driver is selected and one year's experience is generated. This process of selecting a driver and simulating one year is repeated N times.
In the second actuary's simulation, a driver is selected and N years of experience are generated for that driver.
Which of the following is/are true?
I. The ratio of the number of claims the first actuary simulates to the number of claims the second actuary simulates should tend towards 1 as N tends to infinity.
II. The ratio of the number of claims the first actuary simulates to the number of claims the second actuary simulates will equal 1, provided that the same uniform random numbers are used.
III. When the variances of the two sequences of claim counts are compared the first actuary's sequence will have a smaller variance because more random numbers are used in computing it.
A. I only B. I and II only C. I and III only D. II and III only E. None of I, II, or III is true

19.62 (CAS3, 5/05, Q.10) (2.5 points) Low Risk Insurance Company provides liability coverage to a population of 1,000 private passenger automobile drivers.
The number of claims during a given year from this population is Poisson distributed.
If a driver is selected at random from this population, his expected number of claims per year is a random variable with a Gamma distribution such that $\alpha = 2$ and $\theta = 1$.
Calculate the probability that a driver selected at random will not have a claim during the year.
A. 11.1% B. 13.5% C. 25.0% D. 33.3% E. 50.0%

19.63 (2 points) In CAS3, 5/05, Q.10, what is the probability that at most 265 of these 1000 drivers will not have a claim during the year?
A. 75% B. 78% C. 81% D. 84% E. 87%

19.64 (2 points) In CAS3, 5/05, Q.10, what is the probability that these 1000 drivers will have a total of more than 2020 claims during the year?
A. 31% B. 33% C. 35% D. 37% E. 39%

19.65 *(4 points) In CAS3, 5/05, Q.10, let A be the number of these 1000 drivers that have one claim during the year and B be the number of these 1000 drivers that have two claims during the year. Determine the correlation of A and B.*
A. -0.32 B. -0.30 C. -0.28 D. -0.26 E. -0.24

Solutions to Problems:

19.1. E. The Poisson parameters over three years are three times those on an annual basis. Therefore they are given by a Gamma distribution with $\alpha = 3$ and $\theta = 3/12 = 1/4$.
(The mean frequency is now 3/4 per three years rather than 3/12 = 1/4 on an annual basis. It might be helpful to recall that θ is the scale parameter for the Gamma Distribution.)
The mixed distribution is a Negative Binomial, with parameters $r = \alpha = 3$ and $\beta = \theta = 1/4$.
$f(0) = 1/(1+\beta)^r = 1/1.25^3 =$ **0.512.**
Comment: Over one year, the mixed distribution is Negative Binomial, with parameters $r = \alpha = 3$ and $\beta = \theta = 1/12$. Thus for a driver picked at random, the probability of no claims next year is: $1/(1 + 1/12)^3 = 0.7865$. Then one might be tempted to think that the probability of no claims over the next three years for a driver picked at random is: $0.7865^3 = 0.4865$.
However, drivers with a low λ in one year are assumed to have the same low λ every year.
Such good drivers have a large chance of having 0 claims in 3 years.
Drivers with a high λ in one year are assumed to have the same high λ every year.
Such drivers have a smaller chance of having 0 claims in 3 years.
As discussed in “Mahler’s Guide to Conjugate Priors,” a driver who has no claims the first year, has a posterior distribution of lambda that is Gamma, but with $\alpha = 3 + 0 = 3$, and $1/\theta = 12 + 1 = 13$.
Therefore for a driver with no claims in year one, the mixed distribution in year two is Negative Binomial with parameters $r = \alpha = 3$ and $\beta=\theta= 1/13$. Thus for a driver with no claims in year one, the probability of no claims in year two is: $1/(1 + 1/13)^3 = 0.8007$.
A driver who has no claims the first two years, has a posterior distribution of lambda that is Gamma, but with $\alpha = 3 + 0 = 3$, and $1/\theta = 12 + 2 = 14$.
Therefore for a driver with no claims in the first two years, the mixed distribution in year two is Negative Binomial with parameters $r = \alpha = 3$ and $\beta=\theta= 1/14$. Thus for a driver with no claims in year one, the probability of no claims in year two is: $1/(1 + 1/14)^3 = 0.8130$.
Prob[0 claims in three years] = (0.7865)(0.8007)(0.8130) = 0.512 ≠ 0.4865.

19.2. A. From the previous solution, $f(1) = r\beta/(1+\beta)^{r+1} = (3)(1/4)/1.25^4 =$ **0.3072.**

19.3. C. The mean of the Negative Binomial is $r\beta = .6$, while the variance is $r\beta(1+\beta) = .9$.

Therefore, $1 + \beta = 0.9/0.6 = 1.5$, and $\beta = 0.5$. Therefore $r = 1.2$.

For a Gamma-Poisson, $\alpha = r = 1.2$ and $\theta = \beta = 0.5$.

Therefore, the variance of the Gamma Distribution is: $\alpha\theta^2 = (1.2)(.5^2) =$ **0.3**.

Comment: Similar to 3, 5/01, Q.15.

19.4. B. For the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to: mean of the Gamma + variance of the Gamma. Thus mean of Gamma + 0.3 = 0.7. Therefore, mean of Gamma = 0.4 = $\alpha\theta$. Variance of Gamma = 0.3 = $\alpha\theta^2$. Therefore, $\theta = 0.3 / 0.4 = 3/4$.

$\alpha = 0.4/\theta = 0.5332$. $r = \alpha = 0.5332$ and $\beta = \theta = 3/4$. $r(1+\beta) = 0.5332\ (7/4) =$ **0.933**.

19.5. C. The conditional chance of 2 claims given λ is $e^{-\lambda}\lambda^2/2$. The unconditional chance can be obtained by integrating the conditional chances versus the distribution of λ:

$$f(2) = \int_0^\infty f(2 \mid \lambda)\, g(\lambda)\, d\lambda = \int_0^\infty \frac{e^{-\lambda}\,\lambda^2}{2}\,\frac{\lambda^{\alpha-1}\,e^{-\lambda/\theta}}{\Gamma(\alpha)\,\theta^\alpha}\,d\lambda = \frac{1}{2\,\Gamma(\alpha)\,\theta^\alpha}\int_0^\infty \lambda^{\alpha+1}\,e^{-(1+1/\theta)/\lambda}\,d\lambda =$$

$$\frac{1}{2\,\Gamma(\alpha)\,\theta^\alpha}\,\frac{\Gamma(\alpha+2)}{(1+1/\theta)^{\alpha+2}} = \mathbf{\frac{\alpha(\alpha+1)\theta^2}{2(1+\theta)^{\alpha+2}}}.$$

Comment: The mixed distribution is a Negative Binomial with $r = \alpha$ and $\beta = \theta$.

$$f(2) = \frac{r(r+1)}{2}\,\frac{\beta^2}{(1+\beta)^{r+2}} = \frac{\alpha(\alpha+1)\theta^2}{2(1+\theta)^{\alpha+2}}.$$

19.6. A. The conditional mean given λ is: λ. The unconditional mean can be obtained by integrating the conditional means versus the distribution of λ:

$$E[X] = \int_0^\infty E[X \mid \lambda]\, g(\lambda)\, d\lambda = \int_0^\infty \lambda\,\frac{\lambda^{\alpha-1}\,e^{-\lambda/\theta}}{\Gamma(\alpha)\,\theta^\alpha}\,d\lambda = \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\int_0^\infty \lambda^\alpha\,e^{-\lambda/\theta}\,d\lambda = \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\,\Gamma(\alpha+1)\,\theta^{\alpha+1}$$

$$= \alpha\theta.$$

Alternately, $E[X] = \int_0^\infty E[X \mid \lambda]\, g(\lambda)\, d\lambda = \int_0^\infty \lambda\, g(\lambda)\, d\lambda$ = Mean of the Gamma Distribution = $\boldsymbol{\alpha\theta}$.

19.7. B. The conditional mean given λ is: λ. The conditional variance given λ is: λ. Thus the conditional second moment given λ is: $\lambda+\lambda^2$. The unconditional second moment can be obtained by integrating the conditional second moments versus the distribution of λ:

$$E[X^2] = \int_0^\infty E[X^2 \mid \lambda]\, g(\lambda)\, d\lambda = \int_0^\infty (\lambda + \lambda^2)\, \frac{\lambda^{\alpha-1}\, e^{-\lambda/\theta}}{\Gamma(\alpha)\, \theta^\alpha}\, d\lambda =$$

$$\frac{1}{\Gamma(\alpha)\, \theta^\alpha} \int_0^\infty \lambda^\alpha\, e^{-\lambda/\theta}\, d\lambda + \frac{1}{\Gamma(\alpha)\, \theta^\alpha} \int_0^\infty \lambda^{\alpha+1}\, e^{-\lambda/\theta}\, d\lambda = \frac{1}{\Gamma(\alpha)\, \theta^\alpha}\, \frac{\Gamma(\alpha+1)}{\theta^{\alpha+1}} + \frac{1}{\Gamma(\alpha)\, \theta^\alpha}\, \frac{\Gamma(\alpha+2)}{\theta^{\alpha+2}} =$$

$= \alpha\theta + \alpha(\alpha+1)\theta^2$. Since the mean is $\alpha\theta$, the variance is: $\alpha\theta + \alpha(\alpha+1)\theta^2 - \alpha^2\theta^2 = \mathbf{\alpha\theta + \alpha\theta^2}$.

Comment: Note that one integrates the conditional second moments in order to obtain the unconditional second moment. If instead one integrated the conditional variance one would obtain the Expected Value of the Process Variance, (in this case $\alpha\theta$), which is only one piece of the total unconditional variance. One would need to also add the Variance of the Hypothetical Means, (which in this case is $\alpha\theta^2$), in order to obtain the total variance of $\alpha\theta+\alpha\theta^2$. The mixed distribution is a Negative Binomial with $r = \alpha$ and $\beta = \theta$. It has variance: $r\beta(1+\beta) = \alpha\theta+\alpha\theta^2$.

19.8. D. For the Gamma, mean = $\alpha\theta$ = 0.2, and variance = $\alpha\theta^2$ = 0.016.

Thus θ = 0.016/0.2 = 0.08 and α = 2.5.

This is a Gamma-Poisson, with mixed distribution a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 0.08$. $f(1) = r\beta/(1+\beta)^{1+r} = (2.5)(0.08) / (1 + 0.08)^{3.5}$ = **0.153**.

Comment: Similar to 3, 11/01, Q.27**.**

19.9. E. Over 10 minutes, the rate of loss is Poisson, with 10 times that for one minute.

λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = 0.08 \Rightarrow$

10λ has a Gamma distribution with $\alpha = 2.5$, and θ = (10)(.08) = 0.8.

The mixed distribution is a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 0.8$.

$f(2) = \{r(r+1)/2\}\, \beta^2/(1+\beta)^{2+r} = \{(2.5)(3.5)/2\}0.8^2/(1.8)^{4.5}$ = **0.199**.

19.10. B. Mean value of a coin is: (50%)(5) + (30%)(10) + (20%)(25) = 10.5.
2nd moment of the value of a coin is: $(50\%)(5^2) + (30\%)(10^2) + (20\%)(25^2) = 167.5$.
Over 60 minutes, the rate of loss is Poisson, with 60 times that for one minute.

λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = .08 \Rightarrow$

60λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = (60)(.08) = 4.8$.

The mixed distribution is a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 4.8$.

Therefore, the mean number of coins: $r\beta = (2.5)(4.8) = 12$,

and the variance of number of coins: $r\beta(1+\beta) = (2.5)(4.8)(5.8) = 69.6$.
The mean worth is: (10.5)(12) = 126.
Variance of worth is:
(mean frequency)(variance of severity) + (mean severity)^2 (variance of frequency) =
$12)(167.5 - 10.5^2) + (10.5^2)(69.6) = 8360.4$.

Prob[worth > 300] $\cong 1 - \Phi[(300.5 - 126)/\sqrt{8360.4}] = 1 - \Phi[1.91] =$ **2.81%**.
Klem loses money in units of 5 cents or more.
Therefore, if he loses more than 300, he loses 305 or more.
Thus it might be better to approximate the probability as:
$1 - \Phi((304.5 - 126)/\sqrt{8360.4}) = 1 - \Phi[1.95] =$ **2.56%**.
Along this same line of thinking, one could instead approximate the probability by taking the probability from 302.5 to infinity: $1 - \Phi[(302.5 - 126)/\sqrt{8360.4}] = 1 - \Phi[1.93] =$ **2.68%**.
Comment: The formula used is for the variance of aggregate losses, which is covered in section 5 of "Mahler's Guide to Aggregate Distributions."

19.11. E. From the previous solution, for a day chosen at random, the worth has mean 126 and variance 8360.4. The worth over five days is the sum of 5 independent variables; the sum of 5 days has mean: (5)(126) = 630 and variance: (5)(8360.4) = 41,802.

Prob[worth > 900] $\cong 1 - \Phi[(900.5 - 630)/\sqrt{41{,}802}] = 1 - \Phi[1.32] =$ **9.34%**.
Klem loses money in units of 5 cents or more.
Therefore, if he loses more than 900, he loses 905 or more.
It might be better to approximate the probability as:
Prob[worth > 900] = Prob[worth ≥ 905] $\cong 1 - \Phi[(904.5 - 630)/\sqrt{41{,}802}] = 1 - \Phi[1.34] =$ **9.01%**.

One might have instead approximated as: $1 - \Phi[(902.5 - 630)/\sqrt{41{,}802}] = 1 - \Phi[1.33] =$ **9.18%**.

19.12. A. 50% of the coins are worth 5, so if the overall process is Poisson with mean λ, then losing coins of worth 5 is Poisson with mean 0.5λ.

Over 10 minutes it is Poisson with mean 5λ.

λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = 0.08 \Rightarrow$

5λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = (5)(.08) = 0.4$.

The mixed distribution is a Negative Binomial, with $r = \alpha = 2.5$ and $\beta = \theta = 0.4$.

$f(1) = r\beta/(1 + \beta)^{r+1} = (2.5)(0.4)/(1.4^{3.5}) =$ **30.8%.**

19.13. D. Losing coins of worth 5, 10, and 25 are three independent Poisson Processes.

Over 10 minutes losing coins of worth 5 is Poisson with mean 5λ.

Over 10 minutes losing coins of worth 10 is Poisson with mean 3λ.

Over 10 minutes losing coins of worth 25 is Poisson with mean 2λ.

Prob[1 coin @ 5]Prob[0 coins @ 10]Prob[0 coins @ 25] = $5\lambda e^{-5\lambda} e^{-3\lambda} e^{-2\lambda} = 5\lambda e^{-10\lambda}$.

λ has a Gamma distribution with $\alpha = 2.5$ and $\theta = 0.08$. $1/\theta = 12.5$.

$\Rightarrow f(\lambda) = 12.5^{2.5} \lambda^{1.5} e^{-12.5\lambda} / \Gamma(2.5)$.

$$\int_0^\infty 5\lambda\, e^{-10\lambda} f(\lambda)\, d\lambda = \int_0^\infty 5\lambda\, e^{-10\lambda}\, 12.5^{2.5} \lambda^{1.5} e^{-12.5\lambda} / \Gamma(2.5)\, d\lambda = \frac{(5)12.5^{2.5}}{\Gamma(2.5)} \int_0^\infty \lambda^{2.5} e^{-22.5\lambda}\, d\lambda =$$

$$\frac{(5)12.5^{2.5}}{\Gamma(2.5)} \frac{\Gamma(3.5)}{22.5^{3.5}} = (5)(2.5)12.5^{2.5} / 22.5^{3.5} = \mathbf{12.8\%}.$$

<u>Comment</u>: While given lambda, each Poisson Process is independent, the mixed Negative Binomials are not independent, since each day we use the same lambda (appropriately thinned) for each denomination of coin.

From the previous solution, the probability of one coin worth 5 is 30.80%.

The distribution of coins worth ten is Negative Binomial with $r = 2.5$ and $\beta = (3)(0.08) = 0.24$.

Therefore, the chance of seeing no coins worth 10 is: $1/1.24^{2.5} = 58.40\%$.

The distribution of coins worth 25 is Negative Binomial with $r = 2.5$ and $\beta = (2)(0.08) = 0.16$.

Therefore, the chance of seeing no coins worth 25 is: $1/1.16^{2.5} = 69.0\%$.

However, (30.80%)(58.40%)(69.00%) = 12.4% ≠ 12.8%, the correct solution.

One can not multiply the three probabilities together, because the three events are not independent. The three probabilities each depend on the same lambda value for the given day.

19.14. E. A is Poisson with mean λ_A, where λ_A is a random draw from a Gamma Distribution with $\alpha = 2.5$ and $\theta = 0.08$. B is Poisson with mean λ_B, where λ_B is a random draw from a Gamma Distribution with $\alpha = 2.5$ and $\theta = 0.08$. Since A and B are from walks on different days, λ_A and λ_B are independent random draws from the same Gamma.

Thus $\lambda_A + \lambda_B$ is from a Gamma Distribution with $\alpha = 2.5 + 2.5 = 5$ and $\theta = 0.08$.

Thus A + B is from a Negative Binomial Distribution with r = 5 and $\beta = .08$.

The density at 3 of this Negative Binomial Distribution is: $\{(5)(6)(7)/3!\}.08^3/1.08^8$ = **0.97%**.

Alternately, A and B are independent Negative Binomials each with r = 2.5 and $\beta = .08$.

Thus A + B is a Negative Binomial Distribution with r = 5 and $\beta = 0.08$. Proceed as before.

Alternately, for A and B the densities for each are:

$f(0) = 1/(1+\beta)^r = 1/1.08^{2.5} = .825$, $f(1) = r\beta/(1+\beta)^{1+r} = (2.5).08/1.08^{3.5} = .153$,

$f(2) = \{r(r+1)/2\}\ \beta^2/(1+\beta)^{2+r} = \{(2.5)(3.5)/2\}.08^2/1.08^{4.5} = .0198$,

$f(3) = \{r(r+1)(r+2)/3!\}\ \beta^3/(1+\beta)^{3+r} = \{(2.5)(3.5)(4.5)/6\}0.08^3/1.08^{5.5} = .00220$.

Prob[A + B = 3] =

Prob[A=0]Prob[B=3] + Prob[A=1]Prob[B=2] + Prob[A=2]Prob[B=1] + Prob[A=3]Prob[B=0] =

(0.825)(0.00220) + (0.153)(0.0198) + (0.0198)(0.153) + (0.00220)(0.825) = **0.97%**.

Comment: For two independent Gamma Distributions with the same θ:

$\text{Gamma}(\alpha_1, \theta) + \text{Gamma}(\alpha_2, \theta) = \text{Gamma}(\alpha_1 + \alpha_2, \theta)$.

19.15. B. $\lambda_A + \lambda_B + \lambda_C$ is from a Gamma Distribution with $\alpha = (3)(2.5) = 7.5$ and $\theta = .08$.

Thus A + B + C is from a Negative Binomial Distribution with r = 7.5 and $\beta = .08$.

The density at 2 of this Negative Binomial Distribution is: $\{(7.5)(8.5)/2!\}.08^2/1.08^{9.5}$ = **9.8%**.

19.16. A. Mixing a Poisson via a Gamma leads to a negative binomial overall frequency distribution. The negative binomial has parameters $r = \alpha = 4$ and $\beta = \theta = 1/9$.

$f(x) = \{r(r+1)...(r+x-1)/x!\}\ \beta^x / (1+\beta)^{x+r} = \{(4)(5)\ ...\ (x + 4)/x!\}\ (1/9)^x / (10/9)^{x+4}$ =

$\mathbf{\{(x+3)! / (x!\ 3!)\}\ 0.9^4\ 0.1^x}$.

19.17. D. For the Gamma-Poisson, $\beta = \theta$ and $r = \alpha$. Therefore, the variance of the Gamma = $\alpha\theta^2$ = $r\beta^2$. Total Variance = Variance of the mixed Negative Binomial = $r\beta(1+\beta)$. Thus for the Gamma-Poisson we have: (Var. of the Gamma)/(Var. of the Negative Binomial) = $\beta/(1+\beta)$ = $1/\{1 + 1/\beta\}$. Thus in this case $1/\{1 + 1/\beta\} = 0.25$. $\Rightarrow \beta =$ **1/3**.

19.18. D. The parameters of the Gamma can be gotten from those of the Negative Binomial, $\alpha = r = 4$, $\theta = \beta = 3/17$. Then the Variance of the Gamma = $\alpha\theta^2$ = **0.125**.
Alternately, the variance of the Gamma is the Variance of the Hypothetical Means =
Total Variance - Expected Value of the Process Variance =
Variance of the Negative Binomial - Mean of the Gamma =
Variance of the Negative Binomial - Mean of Negative Binomial =
$r\beta(1+\beta) - r\beta = r\beta^2 = (4)(3/17)^2 =$ **0.125**.

19.19. D. This is a Gamma-Poisson with $\alpha = 2$ and $\theta = 1.4$. The mixed distribution is Negative Binomial with $r = \alpha = 2$, and $\beta = \theta = 1.4$.
For a Negative Binomial Distribution, $f(6) = \{(r)(r+1)(r+2)(r+3)(r+4)(r+5)/6!\}\beta^6/(1+\beta)^{r+6} =$
$\{(2)(3)(4)(5)(6)(7)/720\}(1.4^6)/(2.4^8) = 0.04788$.
Thus we expect (100,000)(0.04788) = **4788** out of 100,000 simulated values to be 6.
Comment: Similar to 3, 5/01, Q.3. One need know nothing about simulation, in order to answer these questions.

19.20. E. Each year is a random draw from a different Poisson with unknown λ.
The simulated set consists of random draws each from different Poisson Distributions.
Thus each simulated set is a mixed distribution for a Gamma-Poisson, a Negative Binomial Distributions with $r = \alpha = 2$, and $\beta = \theta = 1.4$.
E[V] = variance of this Negative Binomial = (2)(1.4)(1 + 1.4) = **6.72**.
Alternately, Expected Value of the Process Variance is:
$E[P.V. \mid \lambda] = E[\lambda] = \alpha\theta = (2)(1.4) = 2.8$.
Variance of the Hypothetical Means is: $Var[Mean \mid \lambda] = Var[\lambda] = \alpha\theta^2 = (2)(1.4^2) = 3.92$.
Total Variance is: EPV + VHM = 2.8 + 3.92 = **6.72**.
Comment: Difficult! In other words, $Var[X] = E[Var[X \mid Y] + Var[E[X \mid Y]]$.

19.21. D. This is a Gamma-Poisson with $\alpha = 2$ and $\theta = 1.4$. The mixed distribution is Negative Binomial with $r = \alpha = 2$, and $\beta = \theta = 1.4$.
For this Negative Binomial Distribution, 100,000f(6) = **4788**.

19.22. B. Each year is a random draw from the same Poisson with unknown λ.

The simulated set is from this Poisson Distribution with mean λ. $V = \lambda$.

$E[V] = E[\lambda]$ = mean of the Gamma = $\alpha\theta = (2)(1.4) =$ **2.8**.

Comment: Difficult! What Tom did was simulate one year each from 100,000 randomly selected insureds. What Dick did was pick a random insured and simulate 100,000 years for that insured; each year is an independent random draw from the same Poisson distribution with unknown λ. The two situations are different, even though they have the same mean. In Dick's case there is no variance associated with the selection of the parameter lambda; the only variance is associated with the variance of the Poisson Distribution. In Tom's case there is variance associated with the selection of the parameter lambda as well as variance is associated with the variance of the Poisson Distribution.

19.23. D. This is a Gamma-Poisson with $\alpha = 2$ and $\theta = 1.4$.

The mixed distribution is Negative Binomial with $r = \alpha = 2$, and $\beta = \theta = 1.4$.

For this Negative Binomial Distribution, 100000f(6) = **4788**.

19.24. A. Since all 100,000 values in the simulated set are the same, V = **0**. E[V] = 0.

Comment: Contrast Tom, Dick, and Harry's simulations. Even though they all have the same mean, they are simulating somewhat different situations.

19.25. B. The number of vehicles is Negative Binomial with $r = \alpha = 40$ and $\beta = \theta = 10$.

It has variance: $r\beta(1 + \beta) = (40)(10)(11) =$ **4400**.

19.26. D. This is the sum of 7 independent variables, each with variance 4400.
(7)(4400) = **30,800**.

Comment: Although λ is constant on any given day, it varies from day to day. A day picked at random is a Negative Binomial with $r = 40$ and $\beta = 10$. The sum of seven independent Negative Binomials is a Negative Binomial with $r = (7)(40) = 280$ and $\beta = 10$.

This has variance: (280)(10)(11) = 30,800.

If instead λ had been the same for a whole week, the answer would have changed.

In that case, one would get a Negative Binomial with $r = 40$ and $\beta = (7)(10) = 70$, with variance: (40)(70)(71) = 198,800.

19.27. E. The mean number of people per vehicle is: 1 + (1.6)(6) = 10.6.
The variance of the people per vehicle is: (1.6)(6)(1 + 6) = 67.2.

Variance of the number of people is: $(400)(67.2) + (10.6^2)(4400) =$ **521,264**.

19.28. E. This is the sum of 7 independent variables. (7)(521,264) = **3,648,848**.

19.29. A. The number of people has mean: (400)(10.6) = 4240, and variance: 521,264.
The LogNormal has mean: $\exp[5 + 0.8^2/2] = 204.38$, second moment:
$\exp[(2)(5) + (2)(0.8^2)] = 79,221$, and variance: $79221 - 204.38^2 = 37,450$.
Variance of the money spent: $(4240)(37450) + (204.38^2)(521264) = 21,933$ million.
$\sqrt{21,933 \text{ million}}$ = **148,098**.

19.30. D. This is the sum of 7 independent variables, with variance:
(7)(21,933 million) = 153,531 million. $\sqrt{153,531 \text{ million}}$ = **391,830**.

19.31. C. The mean amount spent per day is: (4240)(204.38) = 866,571.
Over 7 days the mean amount spent is: (7)(866,571) = 6,065,997, with variance 153,531 million.
Prob[amount spent < 5 million] $\cong \Phi[(5 \text{ million} - 6.0660 \text{ million})/\sqrt{153,531 \text{ million}}] = \Phi(-2.72) = .33\%$.
So we expect: (1000)(0.33%) = **3** such runs.

19.32. B. For the Gamma, mean = $\alpha\theta = 0.08$, and variance = $\alpha\theta^2 = 0.0032$.
Thus $\theta = 0.04$ and $\alpha = 2$.
This is a Gamma-Poisson, with mixed distribution a Negative Binomial:
with $r = \alpha = 2$ and $\beta = \theta = 0.04$.

$$f(1) = \frac{r\,\beta}{(1 + \beta)^{r+1}} = \frac{(2)(0.04)}{(1 + 0.04)^3} = \mathbf{7.11\%}.$$

Comment: The fact that it is the next year rather than some other year is irrelevant.

19.33. C. For one year, each insureds mean is λ, and is distributed via a Gamma with:
$\theta = 0.04$ and $\alpha = 2$.
Over three years, each insureds mean is 3λ, and is distributed via a Gamma with:
$\theta = (3)(0.04) = 0.12$, and $\alpha = 2$.
This is a Gamma-Poisson, with mixed distribution a Negative Binomial:
with $r = \alpha = 2$ and $\beta = \theta = 0.12$.

$$f(2) = \frac{r\,(r+1)}{2} \frac{\beta^2}{(1 + \beta)^{r+2}} = \frac{(2)\,(3)}{2} \frac{0.12^2}{(1 + 0.12)^4} = \mathbf{2.75\%}.$$

19.34. B. For one year, each insureds mean is λ, and is distributed via a Gamma with:

$\theta = 0.04$ and $\alpha = 2$.

This is a Gamma-Poisson, with mixed distribution a Negative Binomial:

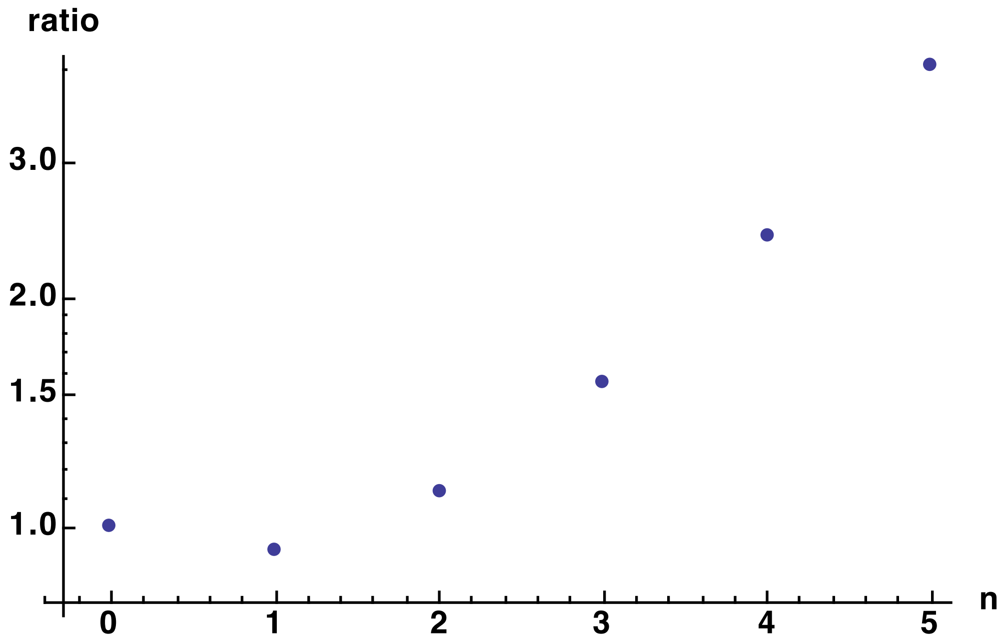
with $r = \alpha = 2$ and $\beta = \theta = 0.04$.

We add up three individual independent drivers and we get a Negative Binomial with:

$r = 2 + 2 + 2 = 6$, and $\beta = 0.04$.

$$f(2) = \frac{r\,(r+1)}{2}\ \frac{\beta^2}{(1 + \beta)^{r+2}} = \frac{(6)\ (7)}{2}\ \frac{0.04^2}{(1 + 0.04)^8} = \textbf{2.46\%}.$$

<u>Comment</u>: The Negative Binomial Distributions here and in the previous solution have the same mean, however the densities are not the same. Here is a graph of the ratios of the densities of the Negative Binomial in the previous solution and those of the Negative Binomial here:



19.35. E. For one year, each insureds mean is λ, and is distributed via a Gamma with:

$\theta = 0.04$ and $\alpha = 2$.

Over four years, each insureds mean is 4λ, and is distributed via a Gamma with:

$\theta = (4)(0.04) = 0.16$, and $\alpha = 2$.

This is a Gamma-Poisson, with mixed distribution a Negative Binomial:

with $r = \alpha = 2$ and $\beta = \theta = 0.16$.

We add up three individual independent drivers and we get a Negative Binomial with:

$r = 2 + 2 + 2 = 6$, and $\beta = 0.16$.

$$f(3) = \frac{r\,(r+1)\,(r+2)}{6}\,\frac{\beta^3}{(1 + \beta)^{r+3}} = \frac{(6)\,(7)\,(8)}{6}\,\frac{0.16^3}{(1 + 0.16)^9} = \mathbf{6.03\%}.$$

19.36. A. The number of accidents Moe has over one year is Negative Binomial:

with $r = \alpha = 2$ and $\beta = \theta = 0.04$.

$$f(0) = \frac{1}{(1 + \beta)^r} = \frac{1}{(1 + 0.04)^2} = 0.9246.$$

The number of accidents Larry has over two years is Negative Binomial:

with $r = \alpha = 2$ and $\beta = 2\theta = 0.08$.

$$f(1) = \frac{r\,\beta}{(1 + \beta)^{r+1}} = \frac{(2)(0.08)}{(1 + 0.08)^3} = 0.1270.$$

The number of accidents Curly has over three years is Negative Binomial:

with $r = \alpha = 2$ and $\beta = 3\theta = 0.12$.

$$f(2) = \frac{r\,(r+1)}{2}\,\frac{\beta^2}{(1 + \beta)^{r+2}} = \frac{(2)\,(3)}{2}\,\frac{0.12^2}{(1 + 0.12)^4} = 0.0275.$$

Prob[Moe = 0, Larry = 1, and Curly = 2)] = (0.9246)(0.1270)(0.0275) = **0.32%**.

19.37. D. The number of accidents Moe has over one year is Negative Binomial:
with $r = \alpha = 2$ and $\beta = \theta = 0.04$.
f(0) = 0.9246. f(1) = 0.0711. f(2) = 0.0041. f(3) = 0.0002.
The number of accidents Larry has over two years is Negative Binomial:
with $r = \alpha = 2$ and $\beta = 2\theta = 0.08$.
f(0) = 0.8573. f(1) = 0.1270. f(2) = 0.0141. f(3) = 0.0014.
The number of accidents Curly has over three years is Negative Binomial:
with $r = \alpha = 2$ and $\beta = 3\theta = 0.12$.
f(0) = 0.7972. f(1) = 0.1708. f(2) = 0.0275. f(3) = 0.0039.
We need to list all of the possibilities:
Prob[M = 0, L = 0, C = 3] + Prob[M = 0, L = 1, C = 2] + Prob[M = 0, L = 2, C = 1] +
Prob[M = 0, L = 3, C = 0] + Prob[M = 1, L = 0, C = 2] + Prob[M = 1, L = 1, C = 1] +
Prob[M = 1, L = 2, C = 0] + Prob[M = 2, L = 0, C = 1] + Prob[M = 2, L = 1, C = 0] +
Prob[M = 3, L = 0, C = 0] =
(0.9246) {(0.8573)(0.0039) + (0.1270)(0.0275) + (0.0141)(0.1708) + (0.0014)(0.7972)} +
(0.0711) {(0.8573)(0.0275) + (0.1270)(0.1708) + (0.0141)(0.7972)} +
(0.0041) {(0.8573)(0.1708) + (0.1270)(0.7972)} + (0.0002)(0.8573)(0.7972) = **1.475%**.
Comment: Adding up the three independent drivers, M + L + C does not follow a Negative Binomial, since the betas are not the same.
Note that the solution to the previous question is one of the possibilities here.

19.38. D, 19.39. B, & 19.40. D.
The mixed distribution is a Negative Binomial with $r = \alpha = 4$ and $\beta = \theta = 0.1$.
$f(0) = (1+\beta)^{-r} = 1.1^{-4} = 0.6830$. Expected size of group A: **6830**.
$f(1) = r\beta(1+\beta)^{-(r+1)} = (4)(.1)1.1^{-5} = 0.2484$. Expected size of group B: **2484**.
Expected size of group C: 10000 - (6830 + 2484) = **686**.

19.41. E. For an individual insured, the probability of no claims by time t is the density at zero of a Poisson Distribution with mean λt: $\exp[-\lambda t]$.

In other words, the probability the first claim occurs by time t is: $1 - \exp[-\lambda t]$.

This an Exponential Distribution with mean $1/\lambda$.

Thus, for an individual the average wait until the first claim is $1/\lambda$.
(This is a general result for Poisson Processes.)

For a Gamma Distribution, $E[X^{-1}] = \theta^{-1}\ \Gamma(\alpha + k) / \Gamma(\alpha) = 1/ \{\theta(\alpha-1)\}$, $\alpha > 1$.

Lambda is Gamma Distributed, thus $E[1/\lambda] = 1/ \{\theta(\alpha-1)\} = 1 / \{(0.02)(3 - 1)\} =$ **25**.

19.42. C. There is an 80% chance we get a random draw from the Poisson with mean λ.

In which case, we have a Gamma-Poisson with $\alpha = 1$ and $\theta = 0.1$.

The mixed distribution is Negative Binomial with r = 1 and $\beta = 0.1$. $f(2) = 0.1^2 / 1.1^3 = 0.751\%$.

There is a 20% chance we get a random draw from the Poisson with mean 3λ.

3λ follows an Exponential with mean 0.3.

We have a Gamma-Poisson with $\alpha = 1$ and $\theta = 0.3$.

The mixed distribution is Negative Binomial with r = 1 and $\beta = 0.3$. $f(2) = 0.3^2 / 1.3^3 = 4.096\%$.

Thus the overall probability of two claims is: (0.8)(0.751%) + (0.2)(4.096%) = **1.420%**.

19.43. B. This is a Gamma-Poisson with $\alpha = 2$ and $\theta = 1/5$.

Thus the mixed distribution is Negative Binomial with r = 2 and $\beta = 1/5$.

For this Negative Binomial: $f(0) = 1/(1 + 1/5)^2 = 25/36$. $f(1) = (2)(1/5)/(1 + 1/5)^3 = 25/108$.

Probability of at least 2 claims is: 1 - 25/36 - 25/108 = 8/108 = 2/27 = **7.41%**.

19.44. A. The mixed distribution is Negative Binomial with r = 2 and $\beta = 0.5$.

Thinning, small claims are Negative Binomial with r = 2 and $\beta = (60\%)(0.5) = 0.3$.

Variance of the number of small claims is: (2)(0.3)(1.3) = **0.78**.

Alternately, for each insured, the number of small claims is Poisson with mean: 0.6λ.

0.6λ follows a Gamma Distribution with $\alpha = 2$ and $\theta = (0.6)(0.5) = 0.3$.

Thus the mixed distribution for small claims is Negative Binomial with r = 2 and $\beta = 0.3$.

Variance of the number of small claims is: (2)(0.3)(1.3) = **0.78**.

19.45. E. The mixed distribution is Negative Binomial with r = 2 and $\beta = 0.5$.

Thinning, large claims are Negative Binomial with r = 2 and $\beta = (40\%)(0.5) = 0.2$.

Variance of the number of large claims is: (2)(0.2)(1.2) = **0.48**.

Alternately, for each insured, the number of large claims is Poisson with mean: 0.2λ.

0.2λ follows a Gamma Distribution with $\alpha = 2$ and $\theta = (0.4)(0.5) = 0.2$.

Thus the mixed distribution for large claims is Negative Binomial with r = 2 and $\beta = 0.2$.

Variance of the number of large claims is: (2)(0.2)(1.2) = **0.48**.

Comment: The number of small and large claims is positively correlated.

The distribution of claims of all sizes is Negative Binomial with r = 2 and $\beta = 0.5$;

it has a variance of: (2)(0.5)!9.5) = 1.5 > 1.26 = 0.78 + 0.48.

19.46. C. Statements 1 and 2 are true, while #3 is false.
The mixture of a Poisson by a Poisson is not a Negative Binomial Distribution, but a much more complicated distribution.

19.47. A. For the Gamma-Poisson, the mixed distribution is a Negative Binomial with mean $r\beta$ and variance = $r\beta(1+\beta)$. Thus we have $r\beta = 0.1$ and $0.15/0.1 = 1+\beta$. Thus $\beta = 0.5$, and $r = 0.1/0.5 = 0.2$.
The parameters of the Gamma follow from those of the Negative Binomial: $\alpha = r = 0.2$ and $\theta = \beta = 0.5$. The variance of the Gamma is $\alpha\theta^2$ = **0.05**.
Alternately, the total variance is 0.15.
For the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to: mean of the Gamma + variance of the Gamma.
Therefore, the variance of the Gamma = 0.15 - 0.10 = 0.05.

19.48. B. $\int_0^\infty e^{-\lambda} f(\lambda)\, d\lambda = \int_0^\infty e^{-\lambda}\, 36\lambda\, e^{-6\lambda}\, d\lambda = 36 \int_0^\infty \lambda\, e^{-7\lambda}\, d\lambda = (36)\{\Gamma(2)/7^2\} = (36)(1/49) =$

0.735.
Alternately, assume that the frequency for a single insured is given by a Poisson with a mean of λ. (This is consistent with the given information that the chance of 0 claims is $e^{-\lambda}$.) In that case one would have a Gamma-Poisson process and the mixed distribution is a Negative Binomial. The given Gamma distribution of θ has $\alpha = 2$ and $\theta = 1/6$. The mixed Negative Binomial has $r = \alpha = 2$ and $\beta = \theta = 1/6$, and $f(0) = (1+\beta)^{-r} = (1 + 1/6)^{-2} = 36/49$.
Comment: Note that while the situation described is consistent with a Gamma-Poisson, it need not be a Gamma-Poisson.

19.49. A. One can solve for the parameters of the Gamma, $\alpha\theta = 0.1$, and $\alpha\theta^2 = 0.01$, therefore $\theta = 0.1$ and $\alpha = 1$.
The mixed distribution is a Negative Binomial with parameters $r = \alpha = 1$ and $\beta = \theta = 0.1$, a Geometric Distribution. $f(0) = 1/(1+\beta) = 1/1.1 = 10/11$. $f(1) = \beta/(1+\beta)^2 = 0.1/1.1^2 = 10/121$.
The chance of 2 or more accidents is: 1 - f(0) - f(1) = 1 - 10/11 - 10/121 = **1/121**.

19.50. A. mean of Gamma = $\alpha\theta$ = 1 and variance of Gamma = $\alpha\theta^2$ = 2.

Therefore, θ = 2 and α = 1/2.

The mixed distribution is a Negative Binomial with r = α = 1/2 and $\beta = \theta$ = 2.

f(1) = $r\beta/(1+\beta)^{1+r}$ = $(1/2)(2)/(3^{3/2})$ = **0.192**.

Alternately, f(1) =

$$\int_0^\infty f(1 \mid \lambda)\, g(\lambda)\, d\lambda = \int_0^\infty \lambda\, e^{-\lambda}\, \frac{(\lambda/2)^{1/2}\, e^{-\lambda/2}}{\lambda\, \Gamma(1/2)}\, d\lambda = \frac{1}{\Gamma(1/2)\,\sqrt{2}} \int_0^\infty \lambda^{1/2}\, e^{-3\lambda/2}\, d\lambda =$$

$$\frac{1}{\Gamma(1/2)\,\sqrt{2}}\, \Gamma(3/2)\, (2/3)^{3/2} = 2\, \frac{\Gamma(3/2)}{\Gamma(1/2)}\, 3^{-3/2} = 2(1/2)\, 3^{-3/2} = \mathbf{0.192}.$$

19.51. C. This is a Gamma-Poisson with α = 2 and θ = 1.

The mixed distribution is Negative Binomial with r = α = 2 , and $\beta = \theta$ = 1.

For a Negative Binomial Distribution,

f(3) = $\{(r)(r+1)(r+2)/3!\}\beta^3/(1+\beta)^{r+3}$ = $\{(2)(3)(4)/6\}(1^3)/(2^5)$ = 1/8.

Thus we expect (1000)(1/8) = **125** out of 1000 simulated values to be 3.

19.52. A. The mean of the Negative Binomial is $r\beta$ = 0.2, while the variance is: $r\beta(1+\beta)$ = 0.4.

Therefore, $1 + \beta = 2 \Rightarrow \beta = 1$ and r = 0.2. For a Gamma-Poisson, α = r = 0.2 and $\theta = \beta$ = 1.

Therefore, the variance of the Gamma Distribution is: $\alpha\theta^2 = (0.2)(1^2)$ = **0.2**.

Alternately, for the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to:
mean of the Gamma + variance of the Gamma. Variance of the Gamma =
Variance of the Negative Binomial - Mean of the Gamma =
Variance of the Negative Binomial - Overall Mean =
Variance of the Negative Binomial - Mean of the Negative Binomial = 0.4 - 0.2 = **0.2**.

19.53. A. For the Gamma, mean = $\alpha\theta$ = 2, and variance = $\alpha\theta^2$ = 4. Thus θ = 2 and α = 1.

This is a Gamma-Poisson, with mixed distribution a Negative Binomial, with r = α = 1 and $\beta = \theta$ = 2.

This is a Geometric with $f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9$ = **0.222**.

Alternately, λ is distributed via an Exponential with mean 2, $f(\lambda) = e^{-\lambda/2}/2$.

$$\text{Prob[1 claim]} = \int \text{Prob[1 claim} \mid \lambda] \, f(\lambda) \, d\lambda = \int \lambda \, e^{-\lambda} \, e^{-\lambda/2} / 2 \, d\lambda = (1/2) \int_0^\infty \lambda \, e^{-3\lambda/2} \, d\lambda$$

$= (1/2)\,(2/3)^2\,\Gamma(2) = (1/2)(4/9)(1!) = 2/9$ = **0.222**.

Alternately, for the Gamma-Poisson, the variance of the mixed Negative Binomial = total variance = $E[Var[N \mid \lambda]] + Var[E[N \mid \lambda]] = E[\lambda] + Var[\lambda]$ = mean of the Gamma + variance of the Gamma = 2 + 4 = 6. The mean of the mixed Negative Binomial = overall mean = $E[\lambda]$ = mean of the Gamma = 2.

Therefore, $r\beta = 2$ and $r\beta(1+\beta) = 6$. $\Rightarrow$ r =1 and β= 2.

$f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9$ = **0.222**.

Comment: The fact that it is the sixth rather than some other minute is irrelevant.

19.54. C. Over two minutes (on the same day) we have a Poisson with mean 2λ.

$\lambda \sim \text{Gamma}(\alpha, \theta)$ = Gamma (1, 2).

$2\lambda \sim \text{Gamma}(\alpha, 2\theta)$ = Gamma (1, 4), as per inflation.

Mixed Distribution is Negative Binomial, with r = α = 1 and $\beta = \theta$ = 4.

$f(1) = \beta/(1 + \beta)^2 = 4/(1 + 4)^2$ = **16%**.

Comment: If one multiplies a Gamma variable by a constant, one gets another Gamma with the same alpha and with the new theta equal to that constant times the original theta.

19.55. D. A ~ Negative Binomial with r = 1 and β = 2.

B ~ Negative Binomial with r = 1 and β = 2.

A + B ~ Negative Binomial with r = 2 and β = 2.

$f(1) = r\,\beta / (1 + \beta)^{1+r} = (2)(2) / (1 + 2)^3 =$ **14.8%**.

Alternately, the number of coins found in the minutes are independent Poissons with means λ_1 and λ_2. Total number found is Poisson with mean $\lambda_1 + \lambda_2$.

$\lambda_1 + \lambda_2 \sim \text{Gamma}(2\alpha, \theta) = \text{Gamma}(2, 2)$.

Mixed Negative Binomial has r = 2 and β = 2. Proceed as before.

Alternately, P[A + B = 1] = P[A = 1]P[B = 0] + P[A = 0]P[B = 1] = (2/9)(1/3) + (1/3)(2/9) = **14.8%**.

<u>Comment</u>: The sum of two independent Gamma variables with the same theta, is another Gamma with the same theta and with the new alpha equal to the sum of the alphas.

19.56. E. Prob[1 coin during minute 3 | λ] = $\lambda e^{-\lambda}$. Prob[1 coin during minute 5 | λ] = $\lambda e^{-\lambda}$.

The Gamma has θ = 2 and α = 1, an Exponential. $\pi(\lambda) = e^{-\lambda/2}/2$.

Prob[1 coin during minute 3 and 1 coin during minute 5] =

$$\int \text{Prob[1 coin during minute 3 | } \lambda] \text{ Prob[1 coin during minute 5 | } \lambda]\ \pi(\lambda)\ d\lambda =$$

$$\int_0^\infty (\lambda e^{-\lambda})\ (\lambda e^{-\lambda})\ (e^{-\lambda/2}/ 2)\ d\lambda = \int_0^\infty \lambda^2 e^{-2.5\lambda} / 2\ d\lambda = \Gamma(3)\ (1/2.5)^3 / 2 = (1/2)(2/2.5^3) = \mathbf{6.4\%}.$$

<u>Comment</u>: It is true that Prob[1 coin during minute 3] = Prob[1 coin during minute 5] = 2/9. (2/9)(2/9) = 4.94%. However, since the two probabilities both depend on the same lambda, they are <u>not</u> independent.

19.57. D. Prob[1 coin during minute 1 | λ] = $\lambda e^{-\lambda}$. Prob[2 coins during minute 2 | λ] = $\lambda^2 e^{-\lambda}/2$.

Prob[3 coins during minute 3 | λ] = $\lambda^3 e^{-\lambda}/6$.

The Gamma has θ = 2 and α = 1, an Exponential. $\pi(\lambda) = e^{-\lambda/2}/2$.

Prob[1 coin during minute 1, 2 coins during minute 2, and 3 coins during minute 3] =

$$\int \text{Prob[1 coin minute 1 } | \ \lambda] \ \text{Prob[2 coins minute 2 } | \ \lambda] \ \text{Prob[3 coins minute 3 } | \ \lambda] \ \pi(\lambda) \ d\lambda =$$

$$\int_0^\infty (\lambda e^{-\lambda}) \ (\lambda^2 e^{-\lambda}/2) \ (\lambda^3 e^{-\lambda}/6) \ (e^{-\lambda/2}/2) \ d\lambda = \int_0^\infty \lambda^6 e^{-3.5\lambda} / 24 \ d\lambda = \Gamma(7) \ (1/3.5)^7 / 24 =$$

$(720/24)/3.5^7$ = **0.466%**.

Comment: Prob[1 coin during minute 1] = 2/9. Prob[2 coins during minute 2] = 4/27.
Prob[3 coins during minute 3] = 8/81. (2/9)(4/27)(8/81) = 0.325%. However, since the three probabilities depend on the same lambda, they are not independent.

19.58. A. From a previous solution, for one minute, the mixed distribution is Geometric with β = 2.

$f(1) = \beta/(1+\beta)^2 = 2/(1+2)^2 = 2/9 = 0.2222$.

Since the minutes are on different days, their lambdas are picked independently.
Prob[1 coin during 1 minute today and 1 coin during 1 minute tomorrow] =
Prob[1 coin during a minute] Prob[1 coin during a minute] = 0.2222^2 = **4.94%**.

19.59. C. Over three minutes (on the same day) we have a Poisson with mean 3λ.

λ ~ Gamma(α, θ) = Gamma (1, 2).

3λ ~ Gamma(α, 3θ) = Gamma (1, 6).

Mixed Distribution is Negative Binomial, with r = α = 1 and β = θ = 6.

$f(1) = \beta/(1 + \beta)^2 = 6/(1 + 6)^2 = 0.1224$.

Since the time intervals are on different days, their lambdas are picked independently.
Prob[1 coin during 3 minutes today and 1 coin during 3 minutes tomorrow] =
Prob[1 coin during 3 minutes] Prob[1 coin during 3 minutes] = 0.1224^2 = **1.50%**.

19.60. E. Gamma has mean = αθ = 3 and variance = $\alpha\theta^2 = 3 \Rightarrow \theta = 1$ and α = 3.

The Negative Binomial mixed distribution has r = α = 3 and β = θ = 1.

$f(0) = 1/(1+\beta)^3 = 1/8$. $f(1) = r\beta/(1+\beta)^4 = 3/16$. F(1) = 1/8 + 3/16 = 5/16 = **0.3125**.

19.61. E. Assume the prior Gamma, used by both actuaries, has parameters α and θ.
The first actuary is simulating N drivers from a Gamma-Poisson frequency process.
The number of claims from a random driver is Negative Binomial with $r = \alpha$ and $\beta = \theta$.
The total number of claim is a sum of N independent, identically distributed Negative Binomials, which is Negative Binomial with parameters $r = N\alpha$ and $\beta = \theta$.
The second actuary is simulating N years for a single driver.
An individual who is Poisson with mean λ, over N years is Poisson with mean $N\lambda$.

I. The Negative Binomial Distribution simulated by the first actuary has mean $N\alpha\theta$.

The Poisson simulated by the second actuary has mean $N\lambda$, where λ depends on which driver the second actuary has picked at random. There is no reason why the mean number of claims simulated by the two actuaries should be the same. Thus statement I is not true.
II. The number of claims simulated will usually be different, since they are from two different distributions. Thus statement II is not true.
III. The first actuary's Negative Binomial has variance $\alpha\theta(1 + \theta)$. The second actuary's simulated sequence has an expected variance of λ, where λ depends on which driver the second actuary has picked at random. The expected variance for the second actuary's simulated sequence could be higher or lower than the first actuary's, depending on which driver he has picked. Thus statement III is not true.

19.62. C. Gamma-Poisson. The mixed distribution is Negative Binomial with $r = \alpha = 2$ and $\beta = \theta = 1$. $f(0) = 1/(1 + \beta)^r = 1/(1 + 1)^2 =$ **1/4**.

19.63. E. From the previous solution, the probability that each driver does not have a claim is 1/4.
Thus for 1000 independent drivers, the number of drivers with no claims is Binomial with m = 1000 and q = 1/4. This Binomial has mean mq = 250, and variance
mq(1 - q) = 187.5. Using the Normal Approximation with continuity correction,
Prob[At most 265 claim-free drivers] $\cong \Phi[(265.5 - 250)/\sqrt{187.5}] = \Phi[1.13] =$ **87.08%**.

19.64. D. The distribution of number of claims from a single driver is Negative Binomial with $r = 2$ and $\beta = 1$. The distribution of the sum of 1000 independent drivers is Negative Binomial with $r = (1000)(2) = 2000$ and $\beta = 1$. This Negative Binomial has mean $r\beta = 2000$, and variance $r\beta(1 + \beta) = 4000$. Using the Normal Approximation with continuity correction,

Prob[more than 2020 claims] $\cong 1 - \Phi[(2020.5 - 2000)/\sqrt{4000}] = 1 - \Phi[0.32] =$ **37.45%**.

Alternately, the mean of the sum of 1000 independent drivers is 1000 times the mean of single driver: (1000) (2) = 2000.
The variance of the sum of 1000 independent drivers is 1000 times the variance of single driver: (1000) (2) (1) (1+1) = 4000. Proceed as before.

19.65. C. The distribution of number of claims from a single driver is Negative Binomial with $r = 2$ and $\beta = 1$. $f(0) = 1/4$. $f(1) = r\beta/(1 + \beta)^{r+1} = (2)(1)/(1 + 1)^3 = 1/4$.

$f(2) = \{r(r + 1)/2\}\beta^2/(1 + \beta)^{r+2} = \{(2)(3)/2\}(1^2)/(1 + 1)^4 = 3/16$.

The number of drivers with given numbers of claims is a multinomial distribution, with parameters 1000, f(0), f(1), f(2), ... = 1000, 1/4, 1/4, 3/16, ...
The covariance of the number of drivers with 1 claim and the number with 2 claims is: -(1000)(1/4)(3/16) = -46.875.
The variance of the number of drivers with 1 claim is: (1000)(1/4)(1 - 1/4) = 187.5.
The variance of the number of drivers with 2 claims is: (1000)(3/16)(1 - 3/16) = 152.34.
The correlation of the number of drivers with 1 claim and the number with 2 claims is:

$-46.875/\sqrt{(187.5)(152.34)} =$ **-0.277**.

<u>Comment</u>: Well beyond what you are likely to be asked on your exam!
The multinomial distribution is discussed in <u>A First Course in Probability</u> by Ross.

The correlation is: $-\sqrt{\frac{f(1)\, f(2)}{\{1 - f(1)\}\{1 - f(2)\}}} = -1/\sqrt{13} = -0.277$.

Section 20, Tails of Frequency Distributions

Actuaries are sometimes interested in the behavior of a frequency distribution as the number of claims gets very large.[169] The question of interest is how quickly the density and survival function go to zero as x approaches infinity. If the density and survival function go to zero more slowly, one describes that as a "heavier-tailed distribution."

Those frequency distributions which are heavier-tailed than the Geometric distribution are often considered to have heavy tails, while those lighter-tailed than Geometric are consider to have light tails.[170] There are number of general methods by which one can distinguish which distribution or empirical data set has the heavier tail. Lighter-tailed distributions have more moments that exist. For the frequency distributions on the exam all of the moments exist.

Nevertheless, the three common frequency distributions differ in their tail behavior. Since the Binomial has finite support, $f(x) = 0$ for $x > n$, it is very light-tailed. The Negative Binomial has its variance greater than its mean, so that the Negative Binomial is heavier-tailed than the Poisson which has its variance equal to its mean.

From lightest to heaviest tailed, the frequency distribution in the (a,b,0) class are:
Binomial, Poisson, Negative Binomial $r > 1$, Geometric, Negative Binomial $r < 1$.

Skewness:

The larger the skewness, the heavier-tailed the distribution. The Binomial distribution for $q > 0.5$ is skewed to the left (has negative skewness.) The Binomial distribution for $q < 0.5$, the Poisson distribution, and the Negative Binomial distribution are skewed to the right (have positive skewness); they have a few very large values and many smaller values. A symmetric distribution has zero skewness. Therefore, the Binomial Distribution for $q = 0.5$ has zero skewness.

Mean Residual Lives/ Mean Excess Loss:

As with loss distributions one can define the concept of the mean residual life.
The Mean Residual Life, e(x) is defined as:
e(x) = (average number of claims for those insureds with more than x claims) - x.
Thus we only count those insureds with more than x and only that part of each number of claims greater than x.[171] Heavier-tailed distributions have their mean residual life increase to infinity, while lighter-tailed distributions have their mean residual life approach a constant or decline to zero.

[169] Actuaries are more commonly concerned with the tail behavior of loss distributions, as discussed in "Mahler's Guide to Loss Distributions."
[170] See Section 6.3 of Loss Models.
[171] Thus the Mean Residual Life is the mean of the frequency distribution truncated and shifted at x.

One complication is that for discrete distributions this definition is discontinuous at the integers. For example, assume we are interested in the mean residual life at 3. As we take the limit from below we include those insureds with 3 claims in our average; as we approach 3 from above, we don't include insureds with 3 claims in our average.

Define $e(3^-)$ as the limit as x approaches 3 from below of e(x). Similarly, one can define $e(3^+)$ as the limit as x approaches 3 from above of e(x). Then it turns out that $e(0^-)$ = mean, in analogy to the situation for continuous loss distributions. For purposes of comparing tail behavior of frequency distributions, one can use either $e(x^-)$ or $e(x^+)$. I will use the former, since the results using $e(x^-)$ are directly comparable to those for the continuous size of loss distributions. At integer values of x:

$$e(x^-) = \frac{\sum_{i=x}^{\infty} (i-x)\, f(i)}{\sum_{i=x}^{\infty} f(i)} = \frac{\sum_{i=x}^{\infty} (i-x)\, f(i)}{S(x)}.$$

One can compute the mean residual life for the Geometric Distribution, letting $q = \beta/(1+\beta)$ and thus $1 - q = 1/(1+\beta)$:

$$e(x^-)\, S(x-1) = \sum_{i=x+1}^{\infty} (i-x)\, f(i) = \sum_{i=x+1}^{\infty} (i-x)\, \beta^i / (1+\beta)^{i+1} = \frac{1}{1+\beta} \sum_{i=x+1}^{\infty} (i-x)\, q^i =$$

$$\frac{1}{1+\beta}\left\{\sum_{i=x+1}^{\infty} q^i + \sum_{i=x+2}^{\infty} q^i + \sum_{i=x+3}^{\infty} q^i + \ldots\right\} = (1-q)\left\{\frac{q^{x+1}}{1-q} + \frac{q^{x+2}}{1-q} + \frac{q^{x+3}}{1-q} + \ldots\right\}$$

$$= q^{x+1} + q^{x+2} + q^{x+3} + \ldots = q^{x+1}/(1-q) = \{\beta/(1+\beta)\}^{x+1}(1+\beta) = \beta^{x+1}/(1+\beta)^x.$$

In a previous section, the survival function for the geometric distribution was computed as: $S(x) = \{\beta/(1+\beta)\}^{x+1}$. Therefore, $S(x-1) = \{\beta/(1+\beta)\}^x$.

Thus $e(x^-) = \dfrac{\beta^{x+1} / (1+\beta)^x}{\{\beta / (1+\beta)\}^x} = \beta.$

The mean residual life for the Geometric distribution is constant.[172] As discussed previously, the Geometric distribution is the discrete analog of the Exponential distribution which also has a constant mean residual life.[173]

[172] $e(x^-) = \beta = E[X]$.

[173] The Exponential and Geometric distributions have constant mean residual lives due to their memoryless property as discussed in Section 6.3 of Loss Models.

As discussed previously, the Negative Binomial is the discrete analog of the Gamma Distribution. The tail behavior of the Negative Binomial is analogous to that of the Gamma.[174] The mean residual life for a Negative Binomial goes to a constant. For r < 1, $e(x^-)$ increases to β, the mean of the corresponding Geometric, while for r>1, $e(x^-)$ decreases to β as x approaches infinity. For r = 1, one has the Geometric Distribution with $e(x^-)$ constant.

Using the relation between the Poisson Distribution and the Incomplete Gamma Function, it turns out that for the Poisson $e(x^-) = (\lambda - x) + \dfrac{\lambda^x e^{-\lambda}}{\Gamma(x)\,\Gamma(x;\ \lambda)}$.[175] The mean residual life $e(x^-)$ for the Poisson Distribution declines to zero as x approaches infinity.[176] [177] This is another way of seeing that the Poisson has a lighter tail than the Negative Binomial Distribution.

Summary:

Here are the common frequency distributions, arranged from lightest to heaviest righthand tail:

Frequency Distribution	Skewness	Righthand Tail Behavior	Tail Similar to
Binomial, q > 0.5	negative	Finite Support	
Binomial, q = 0.5	zero	Finite Support	
Binomial, q < 0.5	positive	Finite Support	
Poisson	positive	$e(x^-) \to 0$, approximately as 1/x	Normal Distribution
Negative Binomial, r >1	positive	$e(x^-)$ decreases to β	Gamma, $\alpha > 1$
Geometric	positive	$e(x^-)$ constant = β	Exponential
(Negative Binomial, r =1)			(Gamma, $\alpha = 1$)
Negative Binomial, r < 1	positive	$e(x^-)$ increases to β	Gamma, $\alpha < 1$

[174] See "Mahler's Guide to Loss Distributions", for a discussion of the mean residual life for the Gamma and other size of loss distributions. For a Gamma Distribution with α>1, e(x) decreases towards a horizontal asymptote θ.
For a Gamma Distribution with α<1, e(x) increases towards a horizontal asymptote θ.

[175] For the Poisson $F(x) = 1 - \Gamma(x+1; \lambda)$.

[176] It turns out that $e(x^-) \cong \lambda / x$ for very large x. This is similar to the tail behavior for the Normal Distribution.
While $e(x^-)$ declines to zero, $e(x^+)$ for the Poisson Distribution declines to one as x approaches infinity.

[177] This follows from the fact that the Poisson is a limit of Negative Binomial Distributions. For a sequence of Negative Binomial distributions with $r\beta = \lambda$ as $r \to \infty$ (and $\beta \to 0$), in the limit one approaches a Poisson Distribution with the mean λ. The tails of each Negative Binomial have $e(x^-)$ decreasing to β as x approaches infinity.
As $\beta \to 0$, the limits of $e(x^-) \to 0$.

Skewness and Kurtosis of the Poisson versus the Negative Binomial:[178]

The Poisson has skewness: $\frac{1}{\sqrt{\lambda}}$.

The Negative Binomial has skewness: $\frac{1 + 2\beta}{\sqrt{r\beta(1 + \beta)}}$.

Therefore, if a Poisson and Negative Binomial have the same mean, $\lambda = r\beta$, then the ratio of the skewness of the Negative Binomial to that of the Poisson is: $\frac{1 + 2\beta}{\sqrt{1 + \beta}} > 1$.

The Poisson has kurtosis: $3 + 1/\lambda$.

The Negative Binomial has kurtosis: $3 + \frac{6\beta^2 + 6\beta + 1}{r\beta(1+\beta)}$.

Therefore, if a Poisson and Negative Binomial have the same mean, $\lambda = r\beta$, then the ratio of the kurtosis minus 3 of the Negative Binomial to that of the Poisson is:[179]

$$\frac{6\beta^2 + 6\beta + 1}{1 + \beta} > 1.$$

Tails of Compound Distributions:

Compound frequency distributions can have longer tails than either their primary or secondary distribution. If the primary distribution is the number of accidents, and the secondary distribution is the number of claims, then one can have a large number of claims either due to a large number of accidents, or an accident with a large number of claims, or a combination of the two. Thus there is more chance for an unusually large number of claims.

Generally the longer-tailed the primary distribution and the longer-tailed the secondary distribution, the longer-tailed the compound distribution. The skewness of a compound distribution can be rather large.

[178] *See "The Negative Binomial and Poisson Distributions Compared," by Leroy J. Simon, PCAS 1960.*
[179] *The kurtosis minus 3 is sometimes called the excess.*

Tails of Mixed Distributions:

Mixed distributions can also have long tails. For example, the Gamma Mixture of Poissons is a Negative Binomial, with a longer tail than the Poisson. As with compound distributions, with mixed distributions there is more chance for an unusually large number of claims. One can either have a unusually large number of claims for a typical value of the parameter, have an unusual value of the parameter which corresponds to a large expected claim frequency, or a combination of the two. Generally the longer tailed the distribution type being mixed and the longer tailed the mixing distribution, the longer tailed the mixed distribution.

Tails of Aggregate Loss Distributions:

Actuaries commonly look at the combination of frequency and severity. This is termed the aggregate loss distribution. The tail behavior of this aggregate distribution is determined by the behavior of the heavier-tailed of the frequency and severity distributions.[180]

Since the common frequency distributions have tails that are similar to the Gamma Distribution or lighter and the common severity distributions for Casualty Insurance have tails at least as heavy as the Gamma, actuaries working on liability or workers compensation insurance are usually most concerned with the heaviness of the tail of the severity distribution. It is the rare extremely large claims that then are of concern.

However, natural catastrophes such as hurricanes or earthquakes can be examples where a large number of claims can be the concern.[181] (Tens of thousands of homeowners claims, even limited to for example 1/4 million dollars each, can add up to a lot of money!) In that case the tail of the frequency distribution could be heavier than a Negative Binomial.

[180] See for example Panjer & Willmot, Insurance Risk Models.

[181] Natural catastrophes are now commonly modeled using simulation models that incorporate the science of the particular physical phenomenon and the particular distribution of insured exposures.

Problems:

20.1 (1 point) Which of the following frequency distributions have positive skewness?
1. Negative Binomial Distribution with r = 3, β = 0.4.
2. Poisson Distribution with λ = 0.7.
3. Binomial Distribution with m = 3, q = 0.7.

A. 1, 2 only
B. 1, 3 only
C. 2, 3 only
D. 1, 2, and 3
E. The correct answer is not given by A, B, C, or D.

Use the following information for the next five questions:

Five friends: Oleg Puller, Minnie Van, Bob Alou, Louis Liu, and Shelly Fish, are discussing studying for their next actuarial exam. They've counted 10,000 pages worth of readings and agree that on average they expect to find about 2000 "important ideas". However, they are debating how many of these pages there are expected to be with 3 or more important ideas.

20.2 (2 points) Oleg assumes the important ideas are distributed as a Binomial with q = 0.04 and m = 5.
How many pages should Oleg expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.3 (2 points) Minnie assumes the important ideas are distributed as a Poisson with λ= 0.20.
How many pages should Minnie expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.4 (2 points) Bob assumes the important ideas are distributed as a Negative Binomial with $\beta = 0.1$ and $r = 2$. How many pages should Bob expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.5 (3 points) Louis assumes the important ideas are distributed as a compound Poisson-Poisson distribution, with $\lambda_1 = 1$ and $\lambda_2 = 0.2$.

How many pages should Louis expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.6 (3 points) Shelly assumes the important ideas are distributed as a compound Poisson-Poisson distribution, with $\lambda_1 = 0.2$ and $\lambda_2 = 1$.

How many pages should Shelly expect to find with 3 or more important ideas?
A. Less than 10
B. At least 10 but less than 20
C. At least 20 but less than 40
D. At least 40 but less than 80
E. At least 80

20.7 (3 points) Define Riemann's zeta function as: $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, $s > 1$.

Let the zeta distribution be: $f(x) = \frac{1}{x^{\rho+1}\,\zeta(\rho+1)}$, $x = 1, 2, 3, \ldots$, $\rho > 0$.

Determine the moments of the zeta distribution.

20.8 (4B, 5/99, Q.29) (2 points) A Bernoulli distribution, a Poisson distribution, and a uniform distribution each has mean 0.8. Rank their skewness from smallest to largest.
A. Bernoulli, uniform, Poisson
B. Poisson, Bernoulli, uniform
C. Poisson, uniform, Bernoulli
D. uniform, Bernoulli, Poisson
E. uniform, Poisson, Bernoulli

Solutions to Problems:

20.1. A. 1. True. The skewness of any Negative Binomial Distribution is positive.
2. True. The skewness of any Poisson Distribution is positive. 3. False. The skewness of the Binomial Distribution depends on the value of q. For q > .5, the skewness is negative.

20.2. A. $f(x) = \frac{5!}{x!\,(5-x)!}\,0.04^x\,0.96^{5-x}$.

One needs to sum the chances of having x = 0, 1, and 2 :

n	0	1	2
f(n)	0.81537	0.16987	0.01416
F(n)	0.81537	0.98524	0.99940

Thus the chance of 3 or more important ideas is: 1 - 0.99940 = 0.00060.
Thus we expect: (10000)(0.00060) = **6.0** such pages.

20.3. B. $f(x) = e^{-0.2}\,0.2^x/x!$.

One needs to sum the chances of having x = 0, 1, and 2 :

n	0	1	2
f(n)	0.81873	0.16375	0.01637
F(n)	0.81873	0.98248	0.99885

Thus the chance of 3 or more important ideas is: 1 - 0.99885 = 0.00115.
Thus we expect: (10000)(0.00115) = **11.5** such pages.

20.4. C. $f(x) = \frac{(x+2-1)!}{x!\,(2-1)!}\,(0.1)^x/(1.1)^{x+2} = (x+1)(10/11)^2\,(1/11)^x$.

One needs to sum the chances of having x = 0, 1, and 2:

n	0	1	2
f(n)	0.82645	0.15026	0.02049
F(n)	0.82645	0.97671	0.99720

Thus the chance of 3 or more important ideas is: 1 - 0.99720 = 0.00280.
Thus we expect: (10000)(0.00280) = **28.0** such pages.
Comment: Note that the distributions of important ideas in these three questions all have a mean of .2. Since the Negative Binomial has the longest tail, it has the largest expected number of pages with lots of important ideas. Since the Binomial has the shortest tail, it has the smallest expected number of pages with lots of important ideas.

20.5. D. For the Primary Poisson a = 0 and b = λ_1 = 1. The secondary Poisson has density at zero of $e^{-0.2}$ = 0.8187. The p.g.f of the Primary Poisson is P(z) = $e^{(z-1)}$. The density of the compound distribution at zero is the p.g.f. of the primary distribution at 0.8187: $e^{(0.8187-1)}$ = 0.83421.

The densities of the secondary Poisson Distribution with λ = 0.2 are:

n	s(n)
0	0.818731
1	0.163746
2	0.016375
3	0.001092
4	0.000055
5	0.000002

Use the Panjer Algorithm, $c(x) = \frac{1}{1 - a\, s(0)} \sum_{j=1}^{x} (a + jb/x)\, s(j)\, c(x-j) = (1/x) \sum_{j=1}^{x} j\, s(j)\, c(x-j)$.

c(1) = (1/1) (1) s(1) c(0) = (.16375)(.83421) = .13660.
c(2) = (1/2) {(1)s(1) c(1) + (2)s(2)c(0)} = (1/2){(0.16375)(**0**.13660) + (2)(0.01638)(0.83421)} = 0.02485. Thus c(0) + c(1) + c(2) = 0.83421 + 0.13660 + 0.02485 = 0.99566.
Thus the chance of 3 or more important ideas is: 1 - 0.99566 = 0.00434.
Thus we expect: (10000)(0.00434) = **43.4** such pages.
<u>Comment</u>: The Panjer Algorithm (recursive method) is discussed in "Mahler's Guide to Aggregate Distributions."

20.6. E. For the Primary Poisson a = 0 and b = λ_1 = 0.2.

The secondary Poisson has density at zero of e^{-1} = 0.3679.

The p.g.f of the Primary Poisson is P(z) = $e^{0.2(z-1)}$.
The density of the compound distribution at zero is the p.g.f. of the primary distribution at 0.3679 :
$e^{0.2(0.3679-1)}$ = 0.88124.

The densities of the secondary Poisson Distribution with λ = 1 are:

n	s(n)
0	0.367879
1	0.367879
2	0.183940
3	0.061313
4	0.015328
5	0.003066

Use the Panjer Algorithm, $c(x) = \frac{1}{1 - a\,s(0)} \sum_{j=1}^{x} (a + jb/x)\, s(j)\, c(x-j) = (0.2/x) \sum_{j=1}^{x} j\, s(j)\, c(x-j)$.

c(1) = (0.2/1) (1) s(1) c(0) = (0.2)(0.3679)(0.88124) = 0.06484.
c(2) = (0.2/2) {(1)s(1) c(1) + (2)s(2)c(0)} = (.1){(0.3679)(0.06484)+(2)(0.1839)(0.88124)} = 0.03480. Thus c(0) + c(1) + c(2) = 0.88124 + 0.06484 + 0.03480 = 0.98088.
Thus the chance of 3 or more important ideas is: 1 - 0.98088 = 0.01912.
Thus we expect: (10000)(0.01912) = **191.2** such pages.
Comment: This Poisson-Poisson has a mean of .2, but an even longer tail than the previous Poisson-Poisson which has the same mean. Note that it has a variance of $(0.2)(1) + (1)^2(0.2)$ = 0.40, while the previous Poisson-Poisson has a variance of $(1)(0.2) + (0.2)^2(1)$ = 0.24.
The Negative Binomial has a variance of (2)(0.1)(1.1) = 0.22.
The variance of the Poisson is 0.20. The variance of the Binomial is (5)(0.04)(0.96) = 0.192.

20.7.

$$E[X^n] = \sum_{x=1}^{\infty} x^n (1/x^{\rho+1}) / \zeta(\rho + 1) = \sum_{x=1}^{\infty} (1/x^{\rho+1-n}) / \zeta(\rho + 1) = \zeta(\rho + 1 - n) / \zeta(\rho + 1),\ n < \rho.$$

Comment: You are extremely unlikely to be asked about the zeta distribution!
The zeta distribution is discrete and has a heavy righthand tail similar to a Pareto Distribution or a Single Parameter Pareto Distribution, with only some of its moments existing.
The zeta distribution is mentioned in Exercise A.12 at the end of Section 7 in Loss Models.

$\zeta(2) = \pi^2/6$. $\zeta(4) = \pi^4/90$. See the Handbook of Mathematical Functions.

20.8. A. The uniform distribution is symmetric, so it has a skewness of zero.
The Poisson has a positive skewness.
The Bernoulli has a negative skewness for $q = 0.8 > 0.5$.
Comment: For the Poisson with mean μ, the skewness is $1/\sqrt{\mu}$.

For the Bernoulli, the skewness is: $\frac{1 - 2q}{\sqrt{q(1 - q)}} = \frac{1 - (2)(0.8)}{\sqrt{(0.8)(1 - 0.8)}} = -1.5.$

If one computes for this Bernoulli, the third central moment $E[(X-0.8)^3] =$

$0.2(0 - 0.8)^3 + 0.8(1 - 0.8)^3 = -0.096$. Thus the skewness is: $\frac{-0.096}{\sqrt{(0.8)(1 - 0.8)}} = -1.5.$

Section 21, Important Formulas and Ideas

Here are what I believe are the most important formulas and ideas from this study guide to know for the exam.

Basic Concepts (Section 2)

The mean is the average or expected value of the random variable.
The mode is the point at which the density function reaches its maximum.
The median, the 50th percentile, is the first value at which the distribution function is ≥ 0.5.
The 100pth percentile as the first value at which the distribution function ≥ p.
Variance = second central moment = $E[(X - E[X])^2] = E[X^2] - E[X]^2$.
Standard Deviation = Square Root of Variance.

Binomial Distribution (Section 3)

$$f(x) = f(x) = \binom{m}{x} q^x (1-q)^{m-x} = \frac{m!}{x!\,(m-x)!} q^x (1-q)^{m-x},\ 0 \le x \le m.$$

Mean = mq **Variance = mq(1-q)**

Probability Generating Function: $P(z) = \{1 + q(z-1)\}^m$
The Binomial Distribution for m =1 is a Bernoulli Distribution.
X is Binomial with parameters q and m_1, and Y is Binomial with parameters q and m_2,
X and Y independent, then X + Y is Binomial with parameters q and $m_1 + m_2$.

Poisson Distribution (Section 4)

$$f(x) = \lambda^x e^{-\lambda} / x!,\ x \ge 0$$

Mean = λ **Variance = λ**

Probability Generating Function: $P(z) = e^{\lambda(z-1)}$, $\lambda > 0$.

A Poisson is characterized by a constant independent claim intensity and vice versa.
The sum of two independent variables each of which is Poisson with parameters λ_1 and λ_2 is also Poisson, with parameter $\lambda_1 + \lambda_2$.

If frequency is given by a Poisson and severity is independent of frequency, then the number of claims above a certain amount (in constant dollars) is also a Poisson.

Geometric Distribution (Section 5)

f(x) $= \frac{\beta^x}{(1+\beta)^{x+1}}$.

Mean = β **Variance = β(1+β)**

Probability Generating Function: $P(z) = \frac{1}{1 - \beta(z-1)}$.

For a Geometric Distribution, for n > 0, the chance of at least n claims is: $\left(\frac{\beta}{1+\beta}\right)^n$.

For a series of independent identical Bernoulli trials, the chance of the first success following x failures is given by a Geometric Distribution with mean
β = (chance of a failure) / (chance of a success).

The Geometric shares with the Exponential distribution, the "memoryless property." If one were to truncate and shift a Geometric Distribution, then one obtains the same Geometric Distribution.

Negative Binomial Distribution (Section 6)

f(x) $= \frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}$. **Mean = rβ** **Variance = rβ(1+β)**

Negative Binomial for r = 1 is a Geometric Distribution.

For the Negative Binomial Distribution with parameters β and r, with r integer, can be thought of as the sum of r independent Geometric distributions with parameter β.

If X is Negative Binomial with parameters β and r_1, and Y is Negative Binomial with parameters β and r_2, X and Y independent, then X + Y is Negative Binomial with parameters β and $r_1 + r_2$.

For a series of independent identical Bernoulli trials, the chance of success number r following x failures is given by a Negative Binomial Distribution with parameters r and
β = (chance of a failure) / (chance of a success).

Normal Approximation (Section 7)

In general, let μ be the mean of the frequency distribution, while σ is the standard deviation of the frequency distribution, then the chance of observing at least i claims and not more than j claims is approximately:

$$\Phi[\frac{(j + 0.5) - \mu}{\sigma}] - \Phi[\frac{(i - 0.5) - \mu}{\sigma}].$$

Normal Distribution

$F(x) = \Phi[(x-\mu)/\sigma]$

$$f(x) = \phi[(x-\mu)/\sigma] / \sigma = \frac{\exp[-\frac{(x-\mu)^2}{2\sigma^2}]}{\sigma\sqrt{2\pi}}, -\infty < x < \infty. \qquad \phi(x) = \frac{\exp[-x^2/2]}{\sqrt{2\pi}}, -\infty < x < \infty.$$

Mean = μ **Variance = σ^2**

Skewness = 0 (distribution is symmetric) Kurtosis = 3

Skewness (Section 8)

Skewness = third central moment /STDDEV^3 = $E[(X - E[X])^3]$/STDDEV^3

$$= \{E[X^3] - 3\bar{X}E[X^2] + 2\bar{X}^3\} / \text{Variance}^{3/2}.$$

A symmetric distribution has zero skewness.

Binomial Distribution with $q < 1/2 \Leftrightarrow$ positive skewness $\Leftrightarrow$ skewed to the right.

Binomial Distribution $q = 1/2 \Leftrightarrow$ symmetric $\Rightarrow$ zero skewness.

Binomial Distribution $q > 1/2 \Leftrightarrow$ negative skewness $\Leftrightarrow$ skewed to the left.

Poisson and Negative Binomial have positive skewness.

Probability Generating Function (Section 9)

Probability Generating Function, p.g.f.:

$P(z)$ = Expected Value of $z^n = E[z^n] = \sum_{n=0}^{\infty} f(n)\, z^n$.

The Probability Generating Function of the sum of independent frequencies is the product of the individual Probability Generating Functions.
The distribution determines the probability generating function and vice versa.

$f(n) = \left(\frac{d^n P(z)}{dz^n}\right)_{z=0} / n!.$ $f(0) = P(0).$ $P'(1) = \text{Mean}.$

If a distribution is infinitely divisible, then if one takes the probability generating function to any positive power, one gets the probability generating function of another member of the same family of distributions. Examples of infinitely divisible distributions include: Poisson, Negative Binomial, Compound Poisson, Compound Negative Binomial, Normal, Gamma.

Factorial Moments (Section 10)

nth factorial moment $= \mu_{(n)} = E[X(X-1) \ldots (X+1-n)]$.

$\mu_{(n)} = \left(\frac{d^n P(z)}{dz^n}\right)_{z=1}.$ $P'(1) = E[X].$ $P''(1) = E[X(X-1)].$

(a, b, 0) Class of Distributions (Section 11)

For each of these three frequency distributions: $\mathbf{f(x+1) / f(x) = a + \{b / (x+1)\}}$, x = 0, 1, ...
where a and b depend on the parameters of the distribution:

Distribution	a	b	f(0)
Binomial	$-q/(1-q)$	$(m+1)q/(1-q)$	$(1-q)^m$
Poisson	0	λ	$e^{-\lambda}$
Negative Binomial	$\beta/(1+\beta)$	$(r-1)\beta/(1+\beta)$	$1/(1+\beta)^r$

Distribution	Mean	Variance	Variance Over Mean	
Binomial	mq	mq(1-q)	$1-q < 1$	**Variance < Mean**
Poisson	λ	λ	1	**Variance = Mean**
Negative Binomial	$r\beta$	$r\beta(1+\beta)$	$1+\beta > 1$	**Variance > Mean**

Distribution	Thinning by factor of t	Adding n independent, identical copies
Binomial	$q \to tq$	$m \to nm$
Poisson	$\lambda \to t\lambda$	$\lambda \to n\lambda$
Negative Binomial	$\beta \to t\beta$	$r \to nr$

For X and Y independent:

X	Y	X+Y
Binomial(q, m_1)	**Binomial(q, m_2)**	**Binomial(q, $m_1 + m_2$)**
Poisson(λ_1)	**Poisson(λ_2)**	**Poisson($\lambda_1 + \lambda_2$)**
Negative Binomial(β, r_1)	**Negative Bin.(β, r_2)**	**Negative Bin.(β, $r_1 + r_2$)**

Accident Profiles (Section 12)

For the Binomial, Poisson and Negative Binomial Distributions:
(x+1) f(x+1) / f(x) = a(x + 1) + b, where a and b depend on the parameters of the distribution.
a < 0 for the Binomial, a = 0 for the Poisson, and a > 0 for the Negative Binomial Distribution.

Thus if data is drawn from one of these three distributions, then we expect (x+1) f(x+1) / f(x) for this data to be approximately linear with slope a; the sign of the slope, and thus the sign of a, distinguishes between these three distributions of the (a, b, 0) class.

Zero-Truncated Distributions (Section 13)

In general if f is a distribution on 0, 1, 2, 3,..., then $p_k^T = \frac{f(k)}{1 - f(0)}$ is a distribution on 1, 2, 3, ...

Distribution	Density of the Zero-Truncated Distribution	
Binomial	$\dfrac{\frac{m!\, q^x (1-q)^{m-x}}{x!\,(m-x)!}}{1 - (1-q)^m}$	x = 1, 2, 3,..., m
Poisson	$\dfrac{e^{-\lambda} \lambda^x / x!}{1 - e^{-\lambda}}$	x = 1, 2, 3,...
Negative Binomial	$\dfrac{\frac{r(r+1)...(r+x-1)}{x!} \frac{\beta^x}{(1+\beta)^{x+r}}}{1 - 1/(1+\beta)^r}$	x = 1, 2, 3,...

The moments of a zero-truncated distribution are given in terms of those of the corresponding untruncated distribution, f, by: $E^{Truncated}[X^n] = \frac{E_f[X^n]}{1 - f(0)}$.

The Logarithmic Distribution has support on the positive integers: $f(x) = \frac{\left(\frac{\beta}{1+\beta}\right)^x}{x \ln(1+\beta)}$.

The **(a,b,1) class of frequency distributions** is a generalization of the (a,b,0) class.

As with the (a,b,0) class, the recursion formula applies: $\frac{\text{density at } x+1}{\text{density at } x} = a + \frac{b}{x+1}$.

However, it need only apply now for $x \geq 1$, rather than $x \geq 0$.

Members of the (a,b,1) family include: all the members of the (a,b,0) family, the zero-truncated versions of those distributions: Zero-Truncated Binomial, Zero-Truncated Poisson, Extended Truncated Negative Binomial, and the Logarithmic Distribution.
In addition the (a,b,1) class includes the zero-modified distributions corresponding to these.

<u>Zero-Modified Distributions (Section 14)</u>

If f is a distribution on 0, 1, 2, 3,..., and $0 < p_0^M < 1$,

then probability at zero is p_0^M, $p_k^M = f(k) \frac{1 - p_0^M}{1 - f(0)}$, k = 1, 2, 3,... is a distribution on 0, 1, 2, 3, ...

The moments of a zero-modified distribution are given in terms of those of f by:

$E^{\text{Modified}}[X^n] = (1 - p_0^M) \frac{E_f[X^n]}{1 - f(0)} = (1 - p_0^M)\, E^{\text{Truncated}}[X^n]$.

<u>Compound Frequency Distributions (Section 15)</u>

A compound frequency distribution has a primary and secondary distribution, each of which is a frequency distribution. The primary distribution determines how many independent random draws from the secondary distribution we sum.

p.g.f. of compound distribution = p.g.f. of primary dist.[p.g.f. of secondary dist.]

$P(z) = P_1[P_2(z)]$.

compound density at 0 = p.g.f. of the primary at the density at 0 of the secondary.

<u>Moments of Compound Distributions (Section 16)</u>

Mean of Compound Dist. = (Mean of Primary Dist.)(Mean of Sec. Dist.)
Variance of Compound Dist. = (Mean of Primary Dist.)(Var. of Sec. Dist.)
+ (Mean of Secondary Dist.)^2(Variance of Primary Dist.

In the case of a Poisson primary distribution with mean λ, the variance of the compound distribution could be rewritten as: λ(2nd moment of Second. Dist.).

The third central moment of a compound Poisson distribution = λ(3rd moment of Sec. Dist.).

Mixed Frequency Distributions (Section 17)

The density function of the mixed distribution, is the mixture of the density function for specific values of the parameter that is mixed.

The nth moment of a mixed distribution is the mixture of the nth moments.
First one mixes the moments, and then computes the variance of the mixture from its first and second moments.

The Probability Generating Function of the mixed distribution, is the mixture of the probability generating functions for specific values of the parameter.
For a mixture of Poissons, the variance is always greater than the mean.

Gamma Function (Section 18)

The (complete) **Gamma Function** is defined as:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt = \theta^{-\alpha} \int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt, \text{ for } \alpha \ge 0, \theta \ge 0.$$

$\Gamma(\alpha) = (\alpha-1)!$ $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

$$\int_0^\infty t^{\alpha-1} e^{-t/\theta}\, dt = \Gamma(\alpha)\theta^{\alpha}.$$

The **Incomplete Gamma Function** is defined as:

$$\Gamma(\alpha; x) = \int_0^x t^{\alpha-1} e^{-t}\, dt \ / \ \Gamma(\alpha).$$

<u>Gamma-Poisson Frequency Process (Section 19)</u>

If one mixes Poissons via a Gamma, then the mixed distribution is in the form of the Negative Binomial distribution with $r = \alpha$ and $\beta = \theta$.

If one mixes Poissons via a Gamma Distribution with parameters α and θ, then over a period of length Y, the mixed distribution is Negative Binomial with $r = \alpha$ and $\beta = Y\theta$.

For the Gamma-Poisson, the variance of the mixed Negative Binomial is equal to:
mean of the Gamma + variance of the Gamma.

$\text{Var}[X] = E[\text{Var}[X \mid \lambda]] + \text{Var}[E[X \mid \lambda]]$. **Mixing increases the variance**.

<u>Tails of Frequency Distributions (Section 20)</u>

From lightest to heaviest tailed, the frequency distribution in the (a,b,0) class are:
Binomial, Poisson, Negative Binomial $r > 1$, Geometric, Negative Binomial $r < 1$.