

TABLE OF CONTENTS

PREFACE iii

**PART ONE:
REVIEW AND BACKGROUND MATERIAL**

CHAPTER ONE: REVIEW OF INTEREST THEORY 3

- 1.1 Interest Measures 3
- 1.2 Level Annuity Functions 5
 - 1.2.1 Immediate Annuity 6
 - 1.2.2 Annuity-due 6
 - 1.2.3 Continuous Annuity 7
- 1.3 Non-Level Annuity Functions 8
 - 1.3.1 Immediate Annuities 8
 - 1.3.2 Annuities-due 10
 - 1.3.3 Continuous Annuities 12
- 1.4 Equation of Value 13

CHAPTER TWO: REVIEW OF PROBABILITY 15

- 2.1 Random Variables and Their Distributions 15
 - 2.1.1 Discrete Random Variables 15
 - 2.1.2 Continuous Random Variables 18
 - 2.1.3 Mixed Random Variables 19
 - 2.1.4 More on Moments of Random Variables 19
- 2.2 Survey of Particular Discrete Distributions 21
 - 2.2.1 The Discrete Uniform Distribution 21
 - 2.2.2 The Binomial Distribution 21
 - 2.2.3 The Negative Binomial Distribution 22
 - 2.2.4 The Geometric Distribution 23
 - 2.2.5 The Poisson Distribution 23
- 2.3 Survey of Particular Continuous Distributions 24
 - 2.3.1 The Continuous Uniform Distribution 24
 - 2.3.2 The Normal Distribution 25
 - 2.3.3 The Exponential Distribution 26
 - 2.3.4 The Gamma Distribution 27
- 2.4 Multivariate Probability 28
 - 2.4.1 The Discrete Case 28
 - 2.4.2 The Continuous Case 30

CHAPTER THREE: REVIEW OF MARKOV CHAINS 33

- 3.1 Discrete-Time Markov Chains 33
 - 3.1.1 Transition Probabilities 34
 - 3.1.2 State Vector 36
 - 3.1.3 Probabilities over Multiple Steps 36
 - 3.1.4 Properties of Homogeneous Discrete-Time Markov Chains 37
 - 3.1.5 The Non-Homogeneous Discrete-Time Model 37
 - 3.1.6 Probability of Remaining in State i 39
- 3.2 Continuous-Time Markov Chains 395
 - 3.2.1 Forces of Transition 40
 - 3.2.2 Formulas for ${}_r p_{ij}^{(t)} = Pr[X(t+r)=j | X(t)=i]$ 41
- 3.3 Payments 43

CHAPTER FOUR: REVIEW OF STOCHASTIC SIMULATION 45

- 4.1 The Simulation Process 45
- 4.2 Multiplicative Congruential Random Number Generators 46
- 4.3 The Inversion Method for Generating Simulated Output Values from Continuous Distributions 47
 - 4.3.1 The Exponential Distribution 48
 - 4.3.2 The Pareto Distribution 49
 - 4.3.3 Additional Continuous Distributions 49
- 4.4 The Table Look-Up Method for Discrete Probability Distributions 50
 - 4.4.1 The Discrete Uniform Distribution 51
 - 4.4.2 The Bernoulli Distribution 51
 - 4.4.3 The Binomial Distribution 51
 - 4.4.4 The Negative Binomial Distribution 52
 - 4.4.5 The Poisson Distribution 53
 - 4.4.6 Applications of the Material in Section 4.4 56
- 4.5 Methods for Generating Simulated Normal Distribution Values 56
 - 4.5.1 The Polar Method 56
 - 4.5.2 The Box-Muller Method 57

PART TWO:
MODELS FOR SURVIVAL-CONTINGENT RISKS

CHAPTER FIVE: SURVIVAL MODELS (CONTINUOUS PARAMETRIC CONTEXT) 61

- 5.1 The Age-at-Failure Random Variable 61
 - 5.1.1 The Cumulative Distribution Function of X 62
 - 5.1.2 The Survival Distribution Function of X 62
 - 5.1.3 The Probability Density Function of X 63
 - 5.1.4 The Hazard Rate Function of X 64
 - 5.1.5 The Moments of the Age-at-Failure Random Variable X 66
 - 5.1.6 Actuarial Survival Models 66
- 5.2 Examples of Parametric Survival Models 67
 - 5.2.1 The Uniform Distribution 67
 - 5.2.2 The Exponential Distribution 68

5.2.3	The Gompertz Distribution	69
5.2.4	The Makeham Distribution	69
5.2.5	Summary of Parametric Survival Models	70
5.3	The Time-to-Failure Random Variable	70
5.3.1	The Survival Distribution Function of T_x	71
5.3.2	The Cumulative Distribution Function of T_x	71
5.3.3	The Probability Density Function of T_x	72
5.3.4	The Hazard Rate Function of T_x	73
5.3.5	Moments of the Future Lifetime Random Variable T_x	73
5.3.6	Discrete Time-to-Failure Random Variables	75
5.4	Select Survival Models	77
5.5	Multi-State Model Interpretation	78
5.6	Exercises	81

CHAPTER SIX: THE LIFE TABLE (DISCRETE TABULAR CONTEXT) 85

6.1	Definition of the Life Table	85
6.2	The Traditional Form of the Life Table	86
6.3	Other Functions Derived from ℓ_x	88
6.3.1	The Force of Failure	88
6.3.2	The Probability Density Function of X	89
6.3.3	Conditional Probabilities and Densities	91
6.3.4	The Curtate Expectation of Life	93
6.4	Summary of Concepts and Notation	95
6.5	Multi-State Model Interpretation	95
6.6	Methods for Non-Integral Ages	98
6.6.1	Linear Form for ℓ_{x+t}	98
6.6.2	Exponential Form for ℓ_{x+t}	101
6.6.3	Hyperbolic Form for ℓ_{x+t}	102
6.6.4	Summary	104
6.7	Select Life Tables	104
6.8	Life Table Summary	107
6.9	Exercises	108

CHAPTER SEVEN: CONTINGENT PAYMENT MODELS (INSURANCE MODELS) 115

7.1	Discrete Stochastic Models	115
7.1.1	The Discrete Random Variable for Time of Failure	116
7.1.2	The Present Value Random Variable	116
7.1.3	Modifications of the Present Value Random Variable	118
7.1.4	Applications to Life Insurance	123
7.2	Group Deterministic Approach	126
7.3	Continuous Stochastic Models	128
7.3.1	The Continuous Random Variable for Time to Failure	128
7.3.2	The Present Value Random Variable	129
7.3.3	Modifications of the Present Value Random Variable	130
7.3.4	Applications to Life Insurance	130
7.3.5	Continuous Functions Evaluated from Parametric Survival Models	131

- 7.4 Contingent Payment Models with Varying Payments 133
- 7.5 Continuous and m^{thly} Functions Approximated from the Life Table 135
 - 7.5.1 Continuous Contingent Payment Models 135
 - 7.5.2 m^{thly} Contingent Payment Models 138
- 7.6 Multi-State Model Representation 139
 - 7.6.1 Discrete Models 139
 - 7.6.2 Continuous Models 140
 - 7.6.3 Extension to Models with Varying Payments 141
- 7.7 Miscellaneous Examples 141
- 7.8 Exercises 144

CHAPTER EIGHT: CONTINGENT ANNUITY MODELS (LIFE ANNUITIES) 149

- 8.1 Whole Life Annuity Models 150
 - 8.1.1 The Immediate Case 150
 - 8.1.2 The Due Case 154
 - 8.1.3 The Continuous Case 157
- 8.2 Temporary Annuity Models 158
 - 8.2.1 The Immediate Case 158
 - 8.2.2 The Due Case 161
 - 8.2.3 The Continuous Case 164
- 8.3 Deferred Whole Life Annuity Models 166
 - 8.3.1 The Immediate Case 166
 - 8.3.2 The Due Case 168
 - 8.3.3 The Continuous Case 168
- 8.4 Contingent Annuities Payable m^{thly} 170
 - 8.4.1 The Immediate Case 170
 - 8.4.2 The Due Case 171
 - 8.4.3 Random Variable Analysis 172
 - 8.4.4 Numerical Evaluation in the m^{thly} and Continuous Cases 173
- 8.5 Non-Level Payment Annuity Functions 176
- 8.6 Multi-State Model Representation 177
- 8.7 Miscellaneous Examples 179
- 8.8 Exercises 182

**CHAPTER NINE: FUNDING PLANS FOR CONTINGENT CONTRACTS 189
(ANNUAL PREMIUMS)**

- 9.1 Annual Funding Schemes for Contingent Payment Models 190
 - 9.1.1 Discrete Contingent Payment Models 190
 - 9.1.2 Continuous Contingent Payment Models 193
 - 9.1.3 Contingent Annuity Models 194
 - 9.1.4 Non-Level Premium Contracts 196
- 9.2 Random Variable Analysis 196
- 9.3 The Percentile Premium Principle 201
- 9.4 Continuous Payment Funding Schemes 203
 - 9.4.1 Discrete Contingent Payment Models 203
 - 9.4.2 Continuous Contingent Payment Models 204
- 9.5 Funding Schemes with m^{thly} Payments 206
- 9.6 Funding Plans Incorporating Expenses 208

- 9.7 Miscellaneous Examples 211
- 9.8 Exercises 213

**CHAPTER TEN: CONTINGENT CONTRACT RESERVES
(NET LEVEL PREMIUM BENEFIT RESERVES) 219**

- 10.1 NLP Reserves for Contingent Payment Models with Annual Payment Funding 221
 - 10.1.1 NLP Reserves by the Prospective Method 221
 - 10.1.2 NLP Reserves by the Retrospective Method 223
 - 10.1.3 Additional NLP Terminal Reserve Expressions 225
 - 10.1.4 Random Variable Analysis 227
 - 10.1.5 NLP Reserves for Contingent Contracts
with Immediate Payment of Claims 228
 - 10.1.6 NLP Reserves for Contingent Annuity Models 229
- 10.2 Recursive Relationships for Discrete Models with Annual Premiums 229
- 10.3 NLP Reserves for Contingent Payment Models
with Continuous Payment Funding 233
 - 10.3.1 Discrete Whole Life Contingent Payment Models 233
 - 10.3.2 Continuous Whole Life Contingent Payment Models 233
 - 10.3.3 Random Variable Analysis 236
- 10.4 NLP Reserves for Contingent Payment Models with m^{thly} Payment Funding 237
- 10.5 Multi-State Model Representation 239
- 10.6 Gain and Loss Analysis 239
 - 10.6.1 Contingent Insurance Contracts 239
 - 10.6.2 Contingent Annuity Contracts 241
- 10.7 Miscellaneous Examples 242
- 10.8 Exercises 246

**CHAPTER ELEVEN: CONTINGENT CONTRACT RESERVES
(RESERVES AS FINANCIAL LIABILITIES) 251**

- 11.1 Modified Benefit Reserves 252
 - 11.1.1 Reserve Modification in General 253
 - 11.1.2 Full Preliminary Term Modified Reserves 254
 - 11.1.3 Deficiency Reserves 256
 - 11.1.4 Negative Reserves 256
- 11.2 Benefit Reserves at Fractional Durations 256
- 11.3 Generalization to Non-Level Benefits and Benefit Premiums 258
 - 11.3.1 Discrete Models 258
 - 11.3.2 Continuous Models 260
- 11.4 Incorporation of Expenses 262
- 11.5 Introduction to Universal Life Insurance 264
 - 11.5.1 Universal Life with Variable Failure Benefit 264
 - 11.5.2 Universal Life with Fixed Failure Benefit 266
- 11.6 Introduction to Deferred Variable Annuities 267
- 11.7 Gain and Loss Analysis 268
- 11.8 Exercises 271

**CHAPTER TWELVE: MODELS DEPENDENT ON MULTIPLE SURVIVALS
(MULTI-LIFE MODELS) 275**

- 12.1 The Joint-Life Model 275
 - 12.1.1 The Time-to-Failure Random Variable for a Joint-Life Status 275
 - 12.1.2 The Survival Distribution Function of T_{xy} 276
 - 12.1.3 The Cumulative Distribution Function of T_{xy} 276
 - 12.1.4 The Probability Density Function of T_{xy} 277
 - 12.1.5 The Hazard Rate Function of T_{xy} 278
 - 12.1.6 Conditional Probabilities 278
 - 12.1.7 Moments of T_{xy} 279
- 12.2 The Last-Survivor Model 280
 - 12.2.1 The Time-to-Failure Random Variable for a Last-Survivor Status 280
 - 12.2.2 Functions of the Random Variable $T_{\overline{xy}}$ 281
 - 12.2.3 Relationships Between T_{xy} and $T_{\overline{xy}}$ 283
- 12.3 Contingent Probability Functions 284
- 12.4 Contingent Contracts Involving Multi-Life Statuses 286
 - 12.4.1 Contingent Payment Models 286
 - 12.4.2 Contingent Annuity Models 288
 - 12.4.3 Annual Premiums and Reserves 288
 - 12.4.4 Reversionary Annuities 290
 - 12.4.5 Contingent Insurance Functions 291
- 12.5 Multi-State Model Representation 292
 - 12.5.1 The General Model 292
 - 12.5.2 The Joint-Life Model 293
 - 12.5.3 Reversionary Annuities 294
 - 12.5.4 Contingent Insurance Functions 295
 - 12.5.5 The Last-Survivor Model 295
 - 12.5.6 Solving the Kolmogorov Forward Equation 296
- 12.6 General Random Variable Analysis 297
 - 12.6.1 Marginal Distributions of T_x and T_y 298
 - 12.6.2 The Covariance of T_x and T_y 298
 - 12.6.3 Other Joint Functions of T_x and T_y 300
 - 12.6.4 Joint and Last-Survivor Status Functions 302
- 12.7 Common Shock – A Model for Lifetime Dependency 303
- 12.8 Exercises 307

**CHAPTER THIRTEEN: MULTIPLE-DECREMENT MODELS
(THEORY) 311**

- 13.1 Discrete Multiple-Decrement Models 311
 - 13.1.1 The Multiple-Decrement Table 313
 - 13.1.2 Random Variable Analysis 315
- 13.2 Theory of Competing Risks 316
- 13.3 Continuous Multiple-Decrement Models 317
- 13.4 Uniform Distribution of Decrements 321
 - 13.4.1 Uniform Distribution in the Multiple-Decrement Context 321
 - 13.4.2 Uniform Distribution in the Associated Single-Decrement Tables 323
- 13.5 Miscellaneous Examples 324
- 13.6 Exercises 325

CHAPTER FOURTEEN: MULTIPLE-DECREMENT MODELS (APPLICATIONS) 329

- 14.1 Actuarial Present Value 329
- 14.2 Asset Shares 333
- 14.3 Non-Forfeiture Options 336
 - 14.3.1 Cash Values 336
 - 14.3.2 Reduced Paid-Up Insurance 337
 - 14.3.3 Extended Term Insurance 337
 - 14.3.4 Application to Universal Life Insurance 338
 - 14.3.5 Application to Deferred Variable Annuities 338
- 14.4 Multi-State Model Representation, with Illustrations 338
 - 14.4.1 The General Multiple-Decrement Model 338
 - 14.4.2 The Total and Permanent Disability Model 341
 - 14.4.3 Disability Model Allowing for Recovery 345
 - 14.4.4 Continuing Care Retirement Communities 351
 - 14.4.5 Thiele's Differential Equation in the Multiple-Decrement Case 352
- 14.5 Defined Benefit Pension Plans 357
 - 14.5.1 Normal Retirement Benefits 357
 - 14.5.2 Early Retirement Benefits 360
 - 14.5.3 Withdrawal and Other Benefits 361
 - 14.5.4 Funding and Reserving 362
- 14.6 Gain and Loss Analysis 364
- 14.7 Exercises 365

**PART THREE:
MODELS FOR INTEREST RATE RISKS**

CHAPTER FIFTEEN: MODELS WITH VARIABLE INTEREST RATES 373

- 15.1 Actuarial Present Values Using Variable Interest Rates 373
- 15.2 Deterministic Interest Rate Scenarios 376
- 15.3 Spot Interest Rates and the Term Structure of Interest Rates 378
- 15.4 Forward Interest Rates 381
- 15.5 An Example with Simulated Rates of Return 385
- 15.6 Transferring the Interest Rate Risk 387
- 15.7 Exercises 388

CHAPTER SIXTEEN: UNIVERSAL LIFE INSURANCE 393

- 16.1 Basic Aspects 393
 - 16.1.1 Policyholder Choice 393
 - 16.1.2 Interest Rate Risk 394
 - 16.1.3 Secondary Guarantees 395
- 16.2 Indexed Universal Life Insurance 396
- 16.3 Pricing Considerations 398
 - 16.3.1 Mortality 398
 - 16.3.2 Lapse 399
 - 16.3.3 Expenses 400
 - 16.3.4 Investment Income 401
 - 16.3.5 Pricing for Secondary Guarantees 401

- 16.4 Reserving Considerations 402
 - 16.4.1 Basic Universal Life 402
 - 16.4.2 Variable Universal Life 405
 - 16.4.3 Indexed Universal Life 405
 - 16.4.4 Contracts with Secondary Guarantees 407
- 16.5 Exercises 409

CHAPTER SEVENTEEN: DEFERRED VARIABLE ANNUITIES 411

- 17.1 Background 411
- 17.2 Deferred Annuity Products 412
 - 17.2.1 Fixed Deferred Annuity 412
 - 17.2.2 Variable Deferred Annuity 414
 - 17.2.3 Equity-Indexed Deferred Annuity 417
 - 17.2.4 The Payout Phase of a Deferred Annuity 422
 - 17.2.5 Guaranteed Minimum Death Benefit 423
 - 17.2.6 Guaranteed Minimum Income Benefit 423
 - 17.2.7 Miscellaneous Considerations 424
- 17.3 Immediate Annuity Products 425
 - 17.3.1 Fixed Immediate Annuity 425
 - 17.3.2 Variable Immediate Annuity 425
- 17.4 Pricing Considerations 427
 - 17.4.1 Variable Deferred Annuity 427
 - 17.4.2 Variable Immediate Annuity 427
- 17.5 Reserving Considerations 429
 - 17.5.1 Deferred Annuity Reserve 429
 - 17.5.2 Immediate Annuity Reserve 430
 - 17.5.3 Reserving for a Guaranteed Death Benefit 430
 - 17.5.4 Reserving for a Guaranteed Income Benefit 431
- 17.6 Exercises 431

APPENDIX A USING MICROSOFT EXCEL AND VISUAL BASIC MACROS TO COMPUTE ACTUARIAL FUNCTIONS 435

APPENDIX B SIMULATION ILLUSTRATIONS 449

ANSWERS TO THE EXERCISES 459

BIBLIOGRAPHY 477

INDEX 479