ACTEX ACADEMIC SERIES

Probability for Risk Management

2nd Edition

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3.5 Bayes' Theorem

3.5.1 Testing a Test: An Example

In Example 2.27, we showed how to list the possible outcomes of a disease test using a tree. In the discussion, we mentioned that disease tests can have their problems. A test can indicate that you have the disease when you don't (a false positive) or indicate that you are free of the disease when you really have it (a false negative). Most of us are subjected to other tests that have similar problems — placement tests, college and graduate school admission tests, and job screening tests are a few examples. Bayes' Theorem and the related probability formulas presented in this section are quite useful in analyzing how well such tests are working, and we will begin discussion of Bayes' Theorem with a continuation of the disease-testing example. (This material has a wide variety of other applications.)

Example 3.23 The outcomes of interest in a disease test, from Example 2.27, are the following:

D: the person tested has the disease

$\sim D$: the person tested does not have the disease

Y: the test is positive

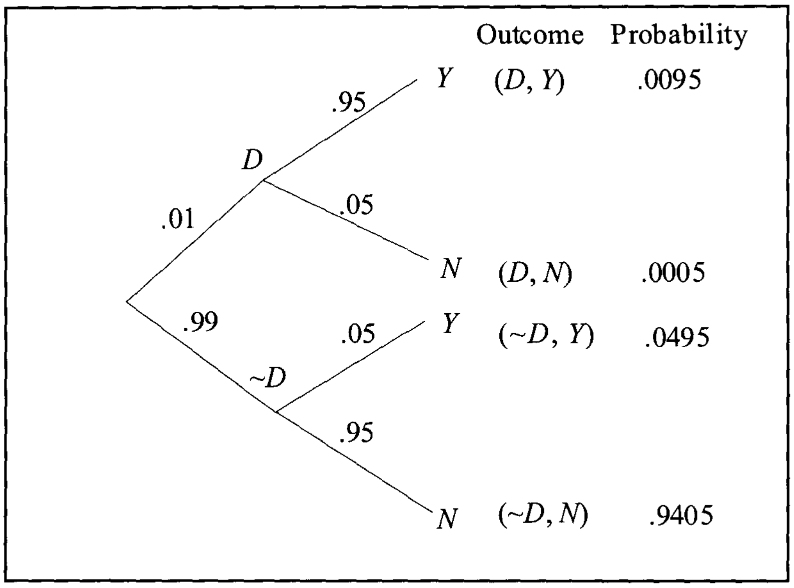
N: the test is negative

In this example, we will consider a hypothetical disease test which most people would think of as "95% accurate", defined as follows:

(a) $P(Y|D) = .95$; in words, if you have the disease there is a .95 probability that the test will be positive.

(b) $P(N|{\sim}D) = .95$; if you don't have the disease the probability is .95 that the test will be negative.

Only 1% of all people actually have the disease, so $P(D) = .01$. The tree for this test (with branch probabilities) is given on the following page.



The tree illustrates that the test is misleading in some cases. 5% of individuals with the disease will test negative, and 5% of the individuals who do not have the disease will test positive. There are two important questions to ask about this test.

(a) What percentage of the population will test positive? This percentage is given by $P(Y)$.

(b) Suppose you know that someone has tested positive for the disease. What is the probability that the person does not actually have the disease? (This probability is $P(\sim D|Y)$.)

Solution

(a) $P(Y)$ is just the sum of the probabilities of all branches ending in Y.

$$P(Y) = P[(D,Y)] + P[(\sim D,Y)] = .0095 + .0495 = .059$$

(b) Note that the event $\sim D \cap Y$ corresponds to the branch $(\sim D, Y)$.

$$P(\sim D|Y) = \frac{P(\sim D \cap Y)}{P(Y)} = \frac{P(\sim D, Y)}{P(Y)} = \frac{.0495}{.0590} \approx .839$$

The practical information here is interesting. The "95% accurate" test will classify 5.9% of the population as positives — a classification which can be alarming and stressful. 83.9% of the individuals who tested positive will not actually have the disease. □

In Example 3.23 we used Bayes' Theorem and the law of total probability without mentioning them by name. In the next section we will state these useful rules.

3.5.2 The Law of Total Probability; Bayes' Theorem

In Example 3.23 we found $P(Y)$ by breaking the event Y into two separate branch outcomes, so

$$Y = \{(D,Y),(\sim D,Y)\},$$

which enabled us to write

$$P(Y) = P[(D,Y)] + P[(\sim D,Y)].$$

Using set notation, we could rewrite the last two identities as

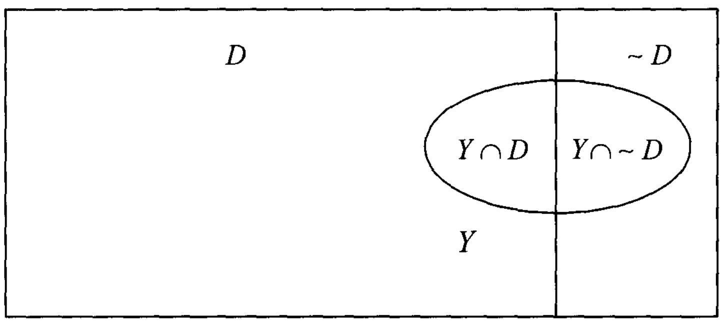
$$Y = (D \cap Y) \cup (\sim D \cap Y)$$

and

$$P(Y) = P(D \cap Y) + P(\sim D \cap Y).$$

Note that $D \cup \sim D = S$. The events D and $\sim D$ partition the sample space into two mutually exclusive pieces. Then the events $(D \cap Y)$ and $(\sim D \cap Y)$ break the event Y into two mutually exclusive pieces. This is illustrated in the following figure.

Sample Space



The events D and $\sim D$ are said to partition the sample space. This is a special case of a more general definition.

Definition 3.5 The events A_1, A_2, $\ldots, A_n$ partition the sample space S if $A_1 \cup A_2 \cup \cdots \cup A_n = S$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

Sample Space

A_1	A_2	$\ldots$	A_n

The **law of total probability** says that a partition of the sample space will lead to a partition of any event E into mutually exclusive pieces.

$$E = (A_1 \cap E) \cup (A_2 \cap E) \cup \cdots \cup (A_n \cap E)$$

Then we can write $P(E)$ as the sum of the probabilities of those pieces.

Law of Total Probability

Let E be an event. If $A_1, A_2, \ldots, A_n$ partition the sample space, then

$$P(E) = P(A_1 \cap E) + P(A_2 \cap E) + \cdots + P(A_n \cap E). \quad (3.7)$$

This is the law we used intuitively when we wrote

$$Y = (D \cap Y) \cup ({\sim}D \cap Y) = \{(D, Y), ({\sim}D, Y)\}$$

and

$$P(Y) = P(D \cap Y) + P({\sim}D \cap Y)$$

In that case $n = 2$, $A_1 = D$, and $A_2 = {\sim}D$.

The law of total probability can be rewritten in a useful way. In the disease testing example, the probabilities $P[(D, Y)]$ and $P[({\sim}D, Y)]$ appeared to be read directly from the tree, but they were actually obtained by multiplying along branches.

$$P(D \cap Y) = P(D) \cdot P(Y|D) \qquad P({\sim}D \cap Y) = P({\sim}D) \cdot P(Y|{\sim}D)$$

Thus when we found $P(Y)$, we were really writing

$$P(Y) = P(D \cap Y) + P({\sim}D \cap Y) = P(D) \cdot P(Y|D) + P({\sim}D) \cdot P(Y|{\sim}D).$$

When we calculated $P(\sim D|Y)$, our reasoning could be summarized as

$$P(\sim D|Y) = \frac{P(\sim D \cap Y)}{P(Y)} = \frac{P(\sim D) \cdot P(Y|\sim D)}{P(D) \cdot P(Y|D) + P(\sim D) \cdot P(Y|\sim D)}.$$

The last expression on the right is referred to as **Bayes' Theorem**. It looks complicated, but can be stated simply in terms of trees.

$$P(\sim D|Y) = \frac{Probability\,for\,(\sim D \cap Y)\,branch}{Sum\,of\,probabilities\,for\,all\,branches\,ending\,in\,Y}$$

The general statement of Bayes' Theorem is simply an extension of the above reasoning for a partition of the sample space into n events.

Bayes' Theorem

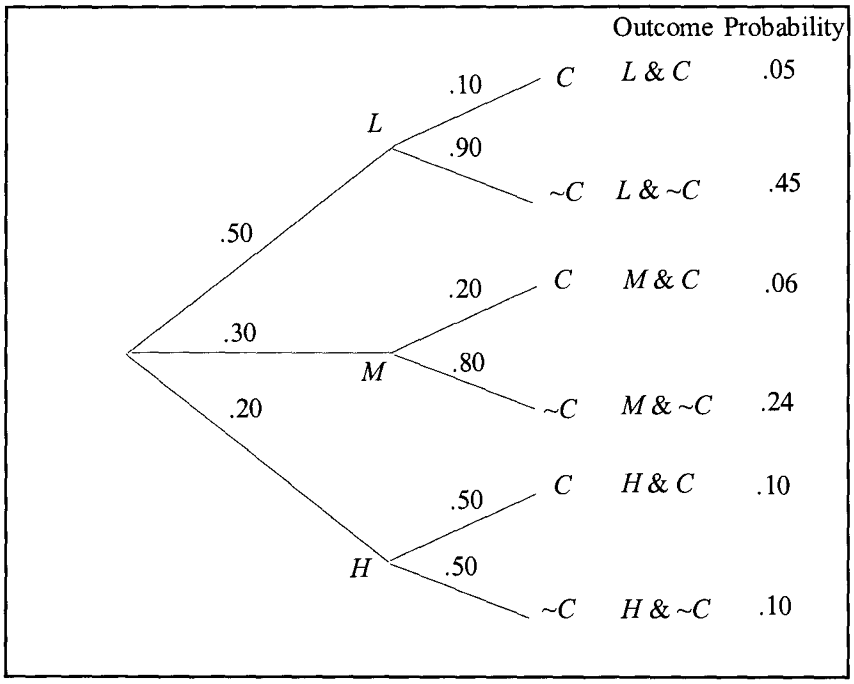
Let E be an event. If $A_1, A_2, \ldots, A_n$ partition the sample space, then

$$\begin{aligned} P(A_i|E) &= \frac{P(E \cap A_i)}{P(E)} \\ &= \frac{P(A_i) \cdot P(E|A_i)}{P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n)}. \end{aligned} \tag{3.8}$$

We illustrate the use of Bayes' Theorem for a partition of the sample space into 3 events in the next example.

Example 3.24 An insurer has three types of auto insurance policyholders. 50% of the policyholders are low risk (L). The probability that a low-risk policyholder will file a claim in a given year is .10. Another 30% of the policyholders are moderate risk (M). The probability that a moderate-risk policyholder will file a claim in a given year is .20. Finally, 20% of the policyholders are high risk (H). The probability that a high-risk policyholder will file a claim in a given year is .50. A policyholder files a claim this year. Find the probability that he is a high-risk policyholder.

Solution The given probabilities lead to the following tree.



$$P(H|C) = \frac{P(H \cap C)}{P(C)} = \frac{.10}{.05 + .06 + .10} \approx .476$$

This shows that approximately 47.6% of the claims are filed by high-risk drivers. □

Note that in a typical problem it is simpler to draw the tree and use branch probabilities than it is to memorize the formula and try to substitute numbers into it. For many people the tree provides the intuition to understand and memorize the formula.

About the Text

This textbook differs from most intermediate probabilitytexts in that it focuses the theory directly on applications in the general field of financial risk management, including insurance, economics, and finance. It will be appropriate for a first course in probability.

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