surance company, but a pension-plan valuation differs from an insurance-company valuation in many ways—some obvious and others quite subtle—which will reveal themselves as our discussion unfolds. The pension valuation may involve the computation of "liabilities" and the valuation of assets, but its primary purpose is to determine annual cost.

Equation Section (Next)2.2 Unit Credit

Assuming that each employee is entitled to retire at age y with an annual pension (payable monthly) equal to $B(y)$, a properly funded plan should have accumulated for each employee when he reaches age y an amount sufficient to fund his pension, *i.e.*, an amount equal to $B(y)\ddot{a}_y^{(12)}$. This requirement is the first logical premise of the *unit credit* cost method (as well as a number of other methods, as we shall see).

Now the benefit $B(y)$ does not arise suddenly at age y, but is earned or "accrued" in a more or less continuous fashion during the employee's active years of service. Thus, when the employee is hired, say at age w, his accrued benefit $B(w)$ is exactly zero; at retirement age y it is equal to its ultimate value $B(y)$; and at any point in-between, say at age x, it has some intermediate value $B(x)$, which we call his *accrued benefit*.

At any age x, earlier than y, the present value of employee j's accrued benefit is equal to $B^j(x)\,{}_{y-x}p_x v^{y-x}\ddot{a}_y^{(12)}$. Note that the factor ${}_{y-x}p_x$ is computed using a table of q_x's which represents probabilities of termination of employment before age y from all causes—not just from mortality, but also resignation, discharge, disablement, *etc.* This table of q's is called a *service table*—a term parallel to but more general than *mortality table.*

So, if we had assets on hand at all times equal to $\sum_{\mathbf{A_t}} B^j(x)\,{}_{y-x}p_x v^{y-x} \ddot{a}_y^{(12)}$ [1] then no matter what was the distribution of ages among the group $\mathbf{A_t}$ of active employees at time t, we should be assured of having sufficient funds to be able to withdraw $B^j(y)\ddot{a}_y^{(12)}$ as each employee reached age y—even if all employees were the same age and all retired at once. (Of course, we might not actually withdraw money to purchase an annuity, but the philosophy is the same no matter what medium of funding is used. It will make our discussion clearer if we assume for the moment that retirees are removed from both the asset and liability columns of our pension plan. We shall put them back in later on.)

This observation is the source of the second premise of the unit credit cost method, which distinguishes it from all others: The ideal fund balance, or desired amount of assets, on hand at any given time t is equal to $\sum_{\mathbf{A_t}} B^j(x)\frac{D_y}{D_x}\ddot{a}_y^{(12)}$ (remember we are ignoring retirees).[2] This ideal fund balance is called the *accrued liability:*

$$(\text{Accrued liability})_t = AL_t = \sum_{\mathbf{A_t}} B^j(x)\frac{D_y}{D_x}\ddot{a}_y^{(12)} \tag{2.2.1}$$

In other words, under the unit credit cost method, the accrued liability is defined as the *present value of accrued benefits.* This definition distinguishes it from all other cost methods, and carries with it, by implication, a complete definition of the pension cost that should be ascribed to any given year, as we shall now see.

Let us digress for a moment to remark on our peculiar use of the word "liability" to denote a desired level of assets. This oddity, which has caused no end of confusion among accountants,

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[1] The ages x and y should also carry the superscript j, but we omit the superscript to reduce clutter.

[2] The definition of D_x may be found in the Index to Principal Notation.

arises from life-insurance terminology. In ordinary financial accounting, a business records each transaction twice—once on each side of the balance sheet—so its "liabilities" are, roughly speaking, the sum of amounts actually owed to someone else. In life-insurance accounting, by contrast, premiums received are not recorded on both sides of the balance sheet, but only as assets. To a life-insurance company, a "liability" is an actuarially determined amount that has first claim on the invested assets of the company. It is not, strictly speaking, an amount owed to anyone—although it will be if the reserve basis proves true—it is the amount of *assets* to be set aside for whatever the actual claims turn out to be. In the same way, the accrued liability of a pension plan represents a claim on plan assets.

From year to year the accrued liability changes, not only because the ages of the active participants increase, but also because the composition of the active group itself changes. To keep things simple, we shall assume that there are no new entrants into the plan; we shall put them in their own separate pension plan for the moment, and recall them later when we have need of them. Then the active group can never grow, but can only shrink, during the year. Denote by **T** the set of all employees who terminate employment between times t and $t+1$, and by **R** the set of employees who reach retirement age y during the year; then we can write

$$\mathbf{A}_{t+1} = \mathbf{A}_t - \mathbf{T} - \mathbf{R}. \qquad (2.2.2)$$

We now construct the following purely algebraic argument to show the relationship between the accrued liability at time t and the accrued liability at time $t+1$ (using the results of exercise 2.2.1):

$$
\begin{aligned}
(\textbf{Accrued liability})_{\mathbf{t+1}} &\equiv AL_{t+1} \\
&= \sum_{\mathbf{A_{t+1}}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} \\
&= \sum_{\mathbf{A_t}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - \sum_{\mathbf{T+R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} \\
&= \sum_{\mathbf{A_t}} B^j(x+1)\left[\frac{D_y}{D_x}(1+i) + q_x\frac{D_y}{D_{x+1}}\right]\ddot{a}_y^{(12)} \\
&\qquad - \sum_{\mathbf{T+R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} \\
&= \sum_{\mathbf{A_t}}\left[B^j(x) + \Delta B^j\right]\frac{D_y}{D_x}\ddot{a}_y^{(12)}(1+i) \\
&\qquad + \sum_{\mathbf{A_t}} q_x B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} \\
&\qquad - \sum_{\mathbf{T+R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)},
\end{aligned}
$$

where ΔB^j is the increase in j's accrued benefit during the year. This means that

$$
\begin{aligned}
AL_{t+1} &= \left[AL_t + \sum_{\mathbf{A_t}} \Delta B^j \frac{D_y}{D_x}\ddot{a}_y^{(12)}\right](1+i) \\
&\quad - \left[\sum_{\mathbf{T}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - \sum_{\mathbf{A_t}} q_x B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}\right] \qquad (2.2.3) \\
&\quad - \sum_{\mathbf{R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}.
\end{aligned}
$$

Now look at the second bracketed term of (2.2.3). If actual experience during the year were in accord with assumed experience, this term would be zero. That is to say, the expected release of liability on account of termination of employment before age y

from all causes except retirement (the second summation) would exactly offset the actual amount released on account of termination (the first summation). Also, if actual experience is in accord with assumed, the ideal fund balance AL_t will have grown to $AL_t(1+i)$ less $\sum_{\mathbf{R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}$ withdrawn for purchase of annuities. Therefore, if the assumptions work out, an amount equal to

$$NC_t = \sum_{\mathbf{A_t}} \Delta B^j \frac{D_y}{D_x}\ddot{a}_y^{(12)} = \sum_{\mathbf{A_t}} NC_t^j \tag{2.2.4}$$

will have to be added at the beginning of the year in order to raise the desired fund balance to its proper level at time $t+1$. This amount is called the *normal cost* of the plan, because it is the cost of keeping the pension fund at the desired level if the assumptions work out and if fund assets equal the accrued liability – *i.e.*, the cost under "normal" circumstances. This normal cost is the present value of the increase in accrued benefit between time t and time $t+1$, and is a single sum assumed to be paid at time t.

The normal cost is not a complete reflection of the full cost of the plan except in this ideal setting, *i.e.*, except where the fund balance is exactly equal to the accrued liability and where the assumptions are exactly borne out. In real life, (a) actual experience is not exactly in accord with assumptions during a given year, and (b) the fund balance is not equal to the accrued liability—either because when the plan was started past service benefits were granted and the accrued liability started out at some positive value, or because the plan has had good experience, or bad experience, relative to assumptions, over time. Therefore, although the central component of the pension cost is the normal cost, there must be adjustments to the cost to allow for variations from the ideal.

Let us now assume that the fund balance at time t is equal to F_t, abandoning our previous assumption that the fund is equal to AL_t. During the year between times t and $t+1$ the fund will increase by some amount I attributable to investment return and by contributions C, and will be diminished by amounts P withdrawn to "purchase" pensions:

$$F_{t+1} = F_t + I + C - P \tag{2.2.5}$$

The difference $AL_t - F_t$ between the accrued liability and the fund balance is called the *unfunded accrued liability*. A negative unfunded accrued liability is often called a *surplus*, but we shall use the term "unfunded accrued liability" or simply "unfunded" to refer to this quantity whether it is positive or negative.

We now subtract Equation (2.2.5) from Equation (2.2.3) in order to derive a relationship between the unfunded at time t and its value at time $t+1$:

$$\begin{aligned}
(\text{Unfunded accrued liability})_{t+1} \equiv UAL_{t+1} \\
&= AL_{t+1} - F_{t+1} \\
&= \left(AL_t + NC_t\right)(1+i) \\
&\quad - \left[\sum_{\mathbf{T}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - \sum_{\mathbf{A_t}} q_x B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}\right] \\
&\quad - \sum_{\mathbf{R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - (F_t + I + C - P),
\end{aligned}$$

or,

$$\begin{aligned}
UAL_{t+1} &= UAL_t(1+i) - [I - iF_t] + [NC_t(1+i) - C] \\
&\quad - \left[\sum_{\mathbf{T}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - \sum_{\mathbf{A_t}} q_x B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}\right] \\
&\quad - \left[\sum_{\mathbf{R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - P\right].
\end{aligned} \tag{2.2.6}$$

We should like to be able to say that all bracketed terms but the first in Equation (2.2.6) would be zero if all assumptions worked out and if contributions actually were equal to the normal cost, but a bit of adjustment is needed to maneuver the equation into suitable form. Let I_C represent interest on the actual contributions at assumed rate *i* from the date they were actually made to year-end. For example, if the contribution were made in a single deposit at the beginning of the year, then $I_C = iC$, but if the contribution were made in a single sum half-way through the year, $I_C = [(1+i)^{\frac{1}{2}} - 1]C$, and so forth. Define a corresponding term I_P for pension purchases. Then we can write

$$\begin{aligned} UAL_{t+1} = {} & UAL_t(1+i) - \left[C + I_C - NC_t(1+i)\right] - \left[I - \left(iF_t + I_C - I_P\right)\right] \\ & - \left[\sum_{\mathbf{T}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - \sum_{\mathbf{A}_t} q_x B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)}\right] \\ & - \left[\sum_{\mathbf{R}} B^j(x+1)\frac{D_y}{D_{x+1}}\ddot{a}_y^{(12)} - P - I_P\right]. \end{aligned} \tag{2.2.7}$$

Now look at Equation (2.2.7) and note that if the actual rate of interest earned during the year was *i*, then the third term would equal zero, and that if the actual accrued liability released by those who actually terminated during the year before age *y* worked out exactly as planned, then the fourth term would be exactly zero as well. Likewise, the fifth term would be zero if the amounts withdrawn for retirement were exactly as anticipated.

The unfunded, by definition, measures the deviation of the actual fund balance F_t from its ideal value AL_t, and the sum of the third, fourth, and fifth terms represents the change in unfunded due to deviations of actual from expected *experience* (as distinguished from amount of contributions). We have a name for the sum of these three terms; it is called the *actuarial gain*, and is defined as follows:

$$\text{Gain} = (UAL_t + NC_t)(1+i) - C - I_C - UAL_{t+1}. \qquad (2.2.8)$$

Of course, we could just as well have defined the gain as the sum of the third, fourth, and fifth terms of Equation (2.2.7), but these terms are more difficult to compute. Historically, the gain has always been defined by (2.2.8). "Gain and loss analysis", however, involves the direct computation of the components of the gain using terms similar to the third, fourth, and fifth terms of Equation (2.2.7). A "loss" is just a negative gain.

Finally, looking at the second term of (2.2.7), you can see that the unfunded is not expected to decrease unless the actual contribution to the fund with interest to the end of the year exceeds the normal cost with interest to the end of the year. Any additional contribution, in excess of the normal cost and interest serves to *amortize* the unfunded. Therefore, in order to cause the unfunded to decrease, eventually, to zero, some amortization amount must be added to the normal cost in order to reflect the total cost of the plan. Since the unfunded at any time includes prior gains, if the original amortization period for the initial unfunded is to be achieved, either the amortization amount will have to be adjusted each year to reflect gains, or each year's gain will have to be amortized separately.

Summary

Under the unit credit cost method the pension cost for any year is equal to (1) the normal cost, plus (2) some amortization of the unfunded accrued liability, minus (3) some amortization of the gain (except in the first year of operation), where

- The *unfunded accrued liability* is the present value of accrued benefits, less assets;
- The *normal cost* is the present value of the increase in accrued benefit during the year; and
- The *gain* is equal to last year's unfunded with a year's interest, plus the normal cost with a year's interest, minus contributions with interest, minus the new unfunded.

Exercises

2.2.1 Show that

$$\frac{D_y}{D_{x+1}} = \frac{D_y}{D_x}(1+i) + q_x \frac{D_y}{D_{x+1}}. \tag{2.2.9}$$

2.2.2 Look at Equation (2.2.4) and notice that the normal cost is the sum of individual normal costs equal to

$$NC_t^j = \Delta B^j \frac{D_y}{D_x} \ddot{a}_y^{(12)}.$$

(a) Differentiate this equation with respect to x and get

$$\begin{aligned} \frac{1}{NC_t^j} \frac{d}{dx} NC_t^j &= \frac{d}{dx}\left(\log NC_t^j\right) \\ &= \frac{d}{dx}\left(\log \Delta B^j(x)\right) + \mu_x + \delta. \end{aligned} \tag{2.2.10}$$

(b) Suppose that the pension plan is such that a unit of pension is credited each year in the amount of 1% of that year's salary, and the salaries increase at annual rate r. Show that

$$\frac{d}{dx}\left(\log NC_t^j\right) = \mu_x + \delta + \log(1+r). \tag{2.2.11}$$

(c) Now consider the ratio NC_t^j / S_t^j — j's normal cost expressed as a fraction of his salary—and show that

$$\frac{d}{dx}\left[\log\left(\frac{NC_t^j}{S_t^j}\right)\right] = \mu_x + \delta. \qquad (2.2.12)$$

(d) Conclude that if there are no preretirement decrements other than mortality and that if mortality follows Gompertz' Law ($\mu_x = ae^{bx}$) then

$$\frac{d}{dx}\left[\log\left(\frac{NC_t^j}{S_t^j}\right)\right] = ae^{bx} + \delta,$$

which implies that the individual normal cost increases more than exponentially *as a percentage of pay.*

2.2.3 Show that for an individual the accrued liability is equal to the actuarial present value of prior normal costs:

$$AL_t^j = \sum_{k=0}^{t-1} NC_t^j \frac{D_{w+k}}{D_x}. \qquad (2.2.13)$$

2.2.4 Note that in deriving Equation (2.2.3) we required only that the accrued liability equal the present value of accrued benefits at time t and again at time $t+1$, but not necessarily in-between. Suppose we had adopted the more stringent standard that at any age z $(x \le z \le x+1)$ the accrued liability for employee j be defined as

$$\overline{AL}_{t+z-x}^j = B^j(z)\,_{y-z}p_z\, v^{y-z}\ddot{a}_y^{(12)}. \qquad (2.2.14)$$

Then, for any age z and for an arbitrarily small time increment h, we have by definition

$$\overline{AL}^{j}_{t+z-x+h} = B^{j}(z+h)\,{}_{y-z-h}p_z\, v^{y-z-h}\ddot{a}^{(12)}_{y}. \qquad (2.2.15)$$

Show that this line of reasoning leads to the expression

$$\frac{d}{dt}\overline{AL}^{j}_{t} = \overline{NC}^{j}_{t} + (\mu_x + \delta)\overline{AL}^{j}_{t}, \qquad (2.2.16)$$

where

$$NC^{j}_{t} = \frac{d}{dx}\left(B^{j}(x)\,{}_{y-x}p_z\, v^{y-x}\right)\ddot{a}^{(12)}_{y}. \qquad (2.2.17)$$

Give a verbal interpretation of Equations (2.2.16) and (2.2.17).

2.2.5 Show algebraically that, if assumptions work out exactly and if ΔB is constant for each employee over his entire career,

$$AL_{t+1} = \left(AL_t + NC_t\right)(1+i) - P - I_P \qquad (2.2.18)$$

and

$$NC_{t+1} = NC_t(1+i). \qquad (2.2.19)$$

Note that Equation (2.2.18) shows how the accrued liability may be viewed as an ideal fund balance.

2.3 Entry Age Normal

In exercises 2.2.2 and 2.2.5 we saw that under the unit credit method individual normal costs tend to rise more rapidly than salary when the benefit is based on salary. This means that in general the normal cost for the plan as a whole will do likewise, except to the extent that new entrants into the group (being perhaps younger and lower-paid than average) lower the average normal cost. Thus, while it is possible that the normal cost will remain level as a percentage of payroll under the unit credit method, this state is unstable because it depends on the character-