Contents

Lesson 22

Survival Distributions: Fractional Ages

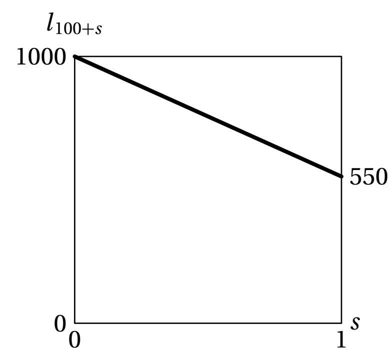
Reading: *Actuarial Mathematics* 3.6 or *Models for Quantifying Risk* (3rd edition) 4.5 or *Models for Quantifying Risk* (4th edition) 6.6.1, 6.6.2

Life tables, such as the Illustrative Life Table, list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

The easiest interpolation method is linear interpolation, or *uniform distribution of deaths* (UDD). This means that the number of lives at age $x+s$, $0 \le s \le 1$, is a weighted average of the number of lives at age x and the number of lives at age $x+1$:

$$l_{x+s} = (1-s)l_x + s\,l_{x+1} = l_x - s\,d_x \tag{22.1}$$

The graph of l_{x+s} is a straight line between $s=0$ and $s=1$ with slope $-d_x$. The graph at the right portrays this for a mortality rate $q_{100}=0.45$ and $l_{100}=1000$.



Contrast UDD with de Moivre. If mortality follows de Moivre's law, l_x as a function of x is a straight line. If UDD is assumed, l_x is a straight line between integral ages, but the slope may vary for different ages. Thus if mortality follows de Moivre's law, UDD holds at all ages, but not conversely.

Using l_{x+s}, we can compute ${}_sq_x$:

$$\begin{aligned} {}_sq_x &= 1 - {}_sp_x \\ &= 1 - \frac{l_{x+s}}{l_x} = 1-(1-sq_x) = sq_x \end{aligned} \tag{22.2}$$

That is one of the most important formulas, so let's state it again:

$$\boxed{{}_sq_x = sq_x} \tag{22.2}$$

More generally, for $0 \le s+t \le 1$,

$$\begin{aligned} {}_sq_{x+t} &= 1 - {}_sp_{x+t} = 1 - \frac{l_{x+s+t}}{l_{x+t}} \\ &= 1 - \frac{l_x-(s+t)d_x}{l_x - t\,d_x} = \frac{s\,d_x}{l_x - t\,d_x} = \frac{sq_x}{1-tq_x} \end{aligned} \tag{22.3}$$

where the last equation was obtained by dividing numerator and denominator by l_x. The important point to pick up is that while ${}_sq_x$ is the proportion of the year s times q_x, the corresponding concept at age $x+t$, ${}_sq_{x+t}$, is *not* sq_x, but is in fact higher than sq_x. The *number* of lives dying in any amount of time is constant, and since there are fewer and fewer lives as the year progresses, the *rate* of death is in fact increasing over the year. The numerator of ${}_sq_{x+t}$ is the proportion of the year being measured s times the death rate, but then this must be divided by 1 minus the proportion of the year that elapsed before the start of measurement.

For most problems involving death probabilities, it will suffice if you remember that l_{x+s} is linearly interpolated. It often helps to create a life table with an arbitrary radix. Try working out the following example before looking at the answer.

EXAMPLE 22A You are given:

(i) $q_x = 0.1$
(ii) Uniform distribution of deaths between integral ages is assumed.

Calculate ${}_{1/2}q_{x+1/4}$.

ANSWER: Let $l_x = 1$. Then $l_{x+1} = l_x(1 - q_x) = 0.9$ and $d_x = 0.1$. Linearly interpolating,

$$l_{x+1/4} = l_x - \tfrac{1}{4}d_x = 1 - \tfrac{1}{4}(0.1) = 0.975$$
$$l_{x+3/4} = l_x - \tfrac{3}{4}d_x = 1 - \tfrac{3}{4}(0.1) = 0.925$$
$${}_{1/2}q_{x+1/4} = \frac{l_{x+1/4} - l_{x+3/4}}{l_{x+1/4}} = \frac{0.975 - 0.925}{0.975} = \boxed{\mathbf{0.051282}}$$

You could also use equation (22.3) to work this example. □

EXAMPLE 22B For two lives age (x) with independent future lifetimes, ${}_{k|}q_x = 0.1(k+1)$ for $k = 0, 1, 2$. Deaths are uniformly distributed between integral ages.

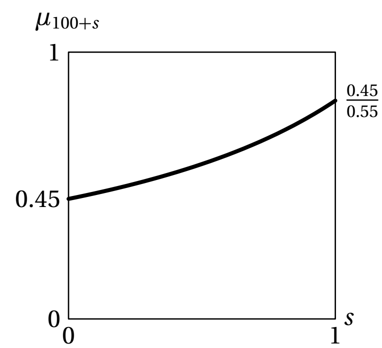
Calculate the probability that both lives will survive 2.25 years.

ANSWER: Since the two lives are independent, the probability of both surviving 2.25 years is the square of ${}_{2.25}p_x$, the probability of one surviving 2.25 years. If we let $l_x = 1$ and use $d_{x+k} = l_x\,{}_{k|}q_x$, we get

$$q_x = 0.1(1) = 0.1 \qquad l_{x+1} = 1 - d_x = 1 - 0.1 = 0.9$$
$${}_{1|}q_x = 0.1(2) = 0.2 \qquad l_{x+2} = 0.9 - d_{x+1} = 0.9 - 0.2 = 0.7$$
$${}_{2|}q_x = 0.1(3) = 0.3 \qquad l_{x+3} = 0.7 - d_{x+2} = 0.7 - 0.3 = 0.4$$

Then linearly interpolating between l_{x+2} and l_{x+3}, we get $l_{x+2.25} = 0.7 - 0.25(0.3) = 0.625$, and ${}_{2.25}p_x = l_{x+2.25}/l_x = 0.625$. Squaring, the answer is $0.625^2 = \boxed{\mathbf{0.390625}}$. □

The probability density function of $T(x)$, ${}_sp_x\,\mu_{x+s}$, is the constant q_x, the derivative of the conditional cumulative distribution function ${}_sq_x = sq_x$ with respect to s. That is another important formula, since the density is needed to compute expected values, so let's repeat it:



$$\boxed{{}_sp_x\,\mu_{x+s} = q_x} \tag{22.4}$$

It follows that the force of mortality is q_x divided by $1 - sq_x$:

$$\mu_{x+s} = \frac{q_x}{{}_sp_x} = \frac{q_x}{1 - sq_x} \tag{22.5}$$

The force of mortality increases over the year, as illustrated in the graph for $q_{100} = 0.45$ to the right.

Complete Expectation of Life Under UDD

If the complete future lifetime random variable T is written as $T = K + S$, where K is the curtate future lifetime and S is the fraction of the last year lived, then K and S are independent. This is usually not true if uniform distribution of deaths is not assumed. Since S is uniform on $[0,1)$, $\mathbf{E}[S] = \frac{1}{2}$ and $\mathrm{Var}(S) = \frac{1}{12}$. It follows from $\mathbf{E}[S] = \frac{1}{2}$ that

$$\mathring{e}_x = e_x + \tfrac{1}{2} \tag{22.6}$$

More common on exams are questions asking you to evaluate the temporary complete expectancy of life under UDD. You can always evaluate the temporary complete expectancy, whether or not UDD is assumed, by integrating ${}_tp_x$, as indicated by formula (20.6) on page 278. For UDD, ${}_tp_x$ is linear between integral ages. Therefore, a rule we learned in Lesson 20 applies for all integral x:

$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x \tag{20.10}$$

This equation will be useful. In addition, the method for generating this equation can be used to work out questions involving temporary complete life expectancies for short periods. The following example illustrates this. This example will be reminiscent of calculating temporary complete life expectancy for de Moivre.

Example 22C You are given

(i) $q_x = 0.1$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x:\overline{0.4}|}$.

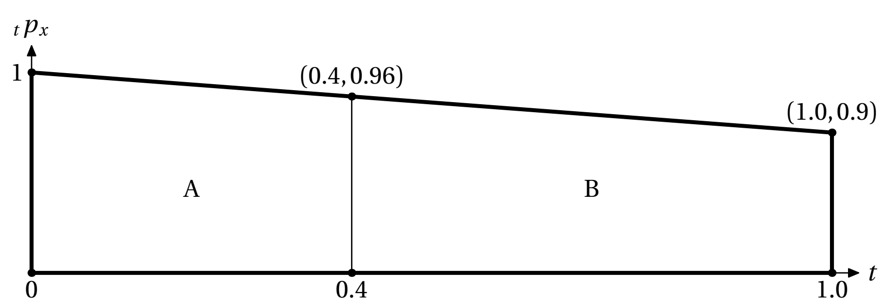
Answer: We will discuss two ways to solve this: an algebraic method and a geometric method.

The algebraic method is based on the double expectation theorem, equation (1.3). It uses the fact that *for a uniform distribution, the mean is the midpoint.* If deaths occur uniformly between integral ages, then those who die within a period contained within a year survive half the period on the average.

In this example, those who die within 0.4 survive an average of 0.2. Those who survive 0.4 survive an average of 0.4 of course. The temporary life expectancy is the weighted average of these two groups, or ${}_{0.4}q_x(0.2) + {}_{0.4}p_x(0.4)$. This is:

$$\begin{aligned} {}_{0.4}q_x &= (0.4)(0.1) = 0.04 \\ {}_{0.4}p_x &= 1 - 0.04 = 0.96 \\ \mathring{e}_{x:\overline{0.4}|} &= 0.04(0.2) + 0.96(0.4) = \boxed{\mathbf{0.392}} \end{aligned}$$

An equivalent geometric method, the trapezoidal rule, is to draw the ${}_tp_x$ function from 0 to 0.4. The integral of ${}_tp_x$ is the area under the line, which is the area of a trapezoid: the average of the heights times the width. The following is the graph (not drawn to scale):



Trapezoid A is the area we are interested in. Its area is $\frac{1}{2}(1 + 0.96)(0.4) = \boxed{\mathbf{0.392}}$. □

Quiz 22-1 As in Example 22C, you are given

(i) $q_x = 0.1$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.4:\overline{0.6}|}$.

Let's now work out an example in which the duration crosses an integral boundary.

EXAMPLE 22D You are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.2$

(iii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

ANSWER: Let's start with the algebraic method. Since the mortality rate changes at $x+1$, we must split the group into those who die before $x+1$, those who die afterwards, and those who survive. Those who die before $x+1$ live 0.25 on the average since the period to $x+1$ is length 0.5. Those who die after $x+1$ live between 0.5 and 1 years; the midpoint of 0.5 and 1 is 0.75, so they live 0.75 years on the average. Those who survive live 1 year.

Now let's calculate the probabilities.

$$
\begin{aligned}
{}_{0.5}q_{x+0.5} &= \frac{0.5(0.1)}{1-0.5(0.1)} = \frac{5}{95} \\
{}_{0.5}p_{x+0.5} &= 1 - \frac{5}{95} = \frac{90}{95} \\
{}_{0.5|0.5}q_{x+0.5} &= \left(\frac{90}{95}\right)\bigl(0.5(0.2)\bigr) = \frac{9}{95} \\
{}_{1}p_{x+0.5} &= 1 - \frac{5}{95} - \frac{9}{95} = \frac{81}{95}
\end{aligned}
$$

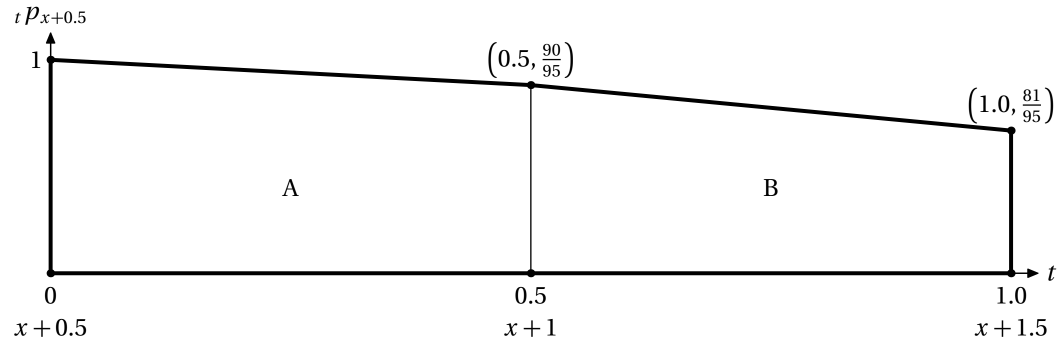
These probabilities could also be calculated by setting up an l_x table with radix 100 at age x and interpolating within it to get $l_{x+0.5}$ and $l_{x+1.5}$. Then

$$
\begin{aligned}
l_{x+1} &= 0.9 l_x = 90 \\
l_{x+2} &= 0.8 l_{x+1} = 72 \\
l_{x+0.5} &= 0.5(90+100) = 95 \\
l_{x+1.5} &= 0.5(72+90) = 81 \\
{}_{0.5}q_{x+0.5} &= 1 - \frac{90}{95} = \frac{5}{95} \\
{}_{0.5|0.5}q_{x+0.5} &= \frac{90-81}{95} = \frac{9}{95} \\
{}_{1}p_{x+0.5} &= \frac{l_{x+1.5}}{l_{x+0.5}} = \frac{81}{95}
\end{aligned}
$$

Either way, we're now ready to calculate $\mathring{e}_{x+0.5:\overline{1}|}$.

$$
\mathring{e}_{x+0.5:\overline{1}|} = \frac{5(0.25)+9(0.75)+81(1)}{95} = \boxed{\mathbf{\frac{89}{95}}}
$$

For the geometric method we draw the following graph:



The heights at $x+1$ and $x+1.5$ are as we computed above. Then we compute each area separately. The area of A is $\frac{1}{2}\left(1+\frac{90}{95}\right)(0.5)=\frac{185}{95(4)}$. The area of B is $\frac{1}{2}\left(\frac{90}{95}+\frac{81}{95}\right)(0.5)=\frac{171}{95(4)}$. Adding them up, we get $\frac{185+171}{95(4)}=\boxed{\frac{89}{95}}$. □

Quiz 22-2 The probability that a battery fails by the end of the kth month is given in the following table:

k	Probability of battery failure by the end of month k
1	0.05
2	0.20
3	0.60

Between integral months, time of failure for the battery is uniformly distributed.
Calculate the expected amount of time the battery survives within 2.25 months.

To calculate $\mathring{e}_{x:\overline{n|}}$ in terms of $e_{x:\overline{n|}}$ when x and n are both integers, note that those who survive n years contribute the same to both. Those who die contribute an average of $\frac{1}{2}$ more to $\mathring{e}_{x:\overline{n|}}$ since they die on the average in the middle of the year. Thus the difference is $\frac{1}{2}\,{}_nq_x$:

$$\mathring{e}_{x:\overline{n|}}=e_{x:\overline{n|}}+0.5\,{}_nq_x \tag{22.7}$$

Example 22E You are given:

(i) $q_x=0.01$ for $x=50,51,\ldots,59$.
(ii) Deaths are uniformly distributed between integral ages.

Calculate $\mathring{e}_{50:\overline{10|}}$.

Answer: As we just said, $\mathring{e}_{50:\overline{10|}}=e_{50:\overline{10|}}+0.5\,{}_{10}q_{50}$. The first summand, $e_{50:\overline{10|}}$, is the sum of ${}_kp_{50}=0.99^k$ for $k=1,\ldots,10$. This sum is a geometric series:

$$e_{50:\overline{10|}}=\sum_{k=1}^{10}0.99^k=\frac{0.99-0.99^{11}}{1-0.99}=9.46617$$

The second summand, the probability of dying within 10 years is ${}_{10}q_{50}=1-0.99^{10}=0.095618$. Therefore

$$\mathring{e}_{50:\overline{10|}}=9.46617+0.5(0.095618)=\boxed{\mathbf{9.51398}}$$ □

The formulas are summarized in Table 22.1.

Table 22.1: Summary of formulas for fractional ages

$$l_{x+s} = l_x - s\,d_x$$
$$_sq_x = sq_x$$
$$_sp_x = 1 - sq_x$$
$$_sq_{x+t} = \frac{sq_x}{1-tq_x}$$
$$\mu_{x+s} = \frac{q_x}{1-sq_x}$$
$$_sp_x\,\mu_{x+s} = q_x$$
$$\mathring{e}_x = e_x + 0.5$$
$$\mathring{e}_{x:\overline{n}|} = e_{x:\overline{n}|} + 0.5\,{}_nq_x$$
$$\mathring{e}_{x:\overline{1}|} = p_x + 0.5q_x$$

Exercises

22.1. [**CAS4-S85:16**] (1 point) Deaths are uniformly distributed between integral ages.

Which of the following represents $_{3/4}p_x + \frac{1}{2}\,{}_{1/2}p_x\,\mu_{x+1/2}$?

A. $_{3/4}p_x$ B. $_{3/4}q_x$ C. $_{1/2}p_x$ D. $_{1/2}q_x$ E. $_{1/4}p_x$

22.2. [Based on **150-S88:25**] You are given:

(i) $_{0.25}q_{x+0.75} = 3/31$.
(ii) Mortality is uniformly distributed within age x.

Calculate q_x.

Use the following information for questions 22.3 and 22.4:

You are given:
(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.10$.
(iii) $q_{x+1} = 0.15$.

22.3. Calculate $_{1/2}q_{x+3/4}$.

22.4. Calculate $_{0.3|0.5}q_{x+0.4}$.

22.5. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) Mortality follows the Illustrative Life Table.

Calculate the median future lifetime for (45.5).

Exercises continue on the next page . . .

22.6. You are given:

(i) Deaths are uniformly distributed between integral ages.

(ii)

x	q_x
52	0.10
53	0.11
54	0.13

Calculate ${}_3m_{52}$.

22.7. [**160-F90:5**] You are given:

(i) A survival distribution is defined by

$$l_x = 1000\left(1-\left(\frac{x}{100}\right)^2\right),\ 0 \le x \le 100.$$

(ii) μ_x denotes the actual force of mortality for the survival distribution.

(iii) μ_x^L denotes the approximation of the force of mortality based on the uniform distribution of deaths assumption for l_x, $50 \le x < 51$.

Calculate $\mu_{50.25} - \mu_{50.25}^L$.

A. -0.00016 B. -0.00007 C. 0 D. 0.00007 E. 0.00016

22.8. A survival distribution is defined by

(i) $s(k) = 1/(1+0.01k)^4$ for k a non-negative integer.

(ii) Deaths are uniformly distributed between integral ages.

Calculate ${}_{0.4}q_{20.2}$.

22.9. [**150-82-94:5**] You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) ${}_{0.75}p_x = 0.25$.

Which of the following are true?

I. ${}_{0.25}q_{x+0.5} = 0.5$

II. ${}_{0.5}q_x = 0.5$

III. $\mu_{x+0.5} = 0.5$

A. I and II only B. I and III only C. II and III only D. I, II and III
E. The correct answer is not given by A. , B. , C. , or D. .

Exercises continue on the next page . . .

22.10. [**3-S00:12**] For a certain mortality table, you are given:

(i) $\mu(80.5) = 0.0202$
(ii) $\mu(81.5) = 0.0408$
(iii) $\mu(82.5) = 0.0619$
(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

A. 0.0782 B. 0.0785 C. 0.0790 D. 0.0796 E. 0.0800

22.11. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_x = 0.1$.
(iii) $q_{x+1} = 0.3$.

Calculate $\mathring{e}_{x+0.7:\overline{1}|}$.

22.12. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) $q_{45} = 0.01$.
(iii) $q_{46} = 0.011$.

Calculate $\operatorname{Var}\Big(\min\big(T(45), 2\big)\Big)$.

22.13. You are given:

(i) Deaths are uniformly distributed between integral ages.
(ii) ${}_{10}p_x = 0.2$.

Calculate $\mathring{e}_{x:\overline{10}|} - e_{x:\overline{10}|}$.

22.14. [**4-F86:21**] You are given:

(i) $q_{60} = 0.020$
(ii) $q_{61} = 0.022$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{60:\overline{1.5}|}$.

A. 1.447 B. 1.457 C. 1.467 D. 1.477 E. 1.487

22.15. [**150-F89:21**] You are given:

(i) $q_{70} = 0.040$
(ii) $q_{71} = 0.044$
(iii) Deaths are uniformly distributed over each year of age.

Calculate $\mathring{e}_{70:\overline{1.5}|}$.

A. 1.435 B. 1.445 C. 1.455 D. 1.465 E. 1.475

Exercises continue on the next page . . .

22.16. [**3-S01:33**] For a 4-year college, you are given the following probabilities for dropout from all causes:

$$q_0 = 0.15$$
$$q_1 = 0.10$$
$$q_2 = 0.05$$
$$q_3 = 0.01$$

Dropouts are uniformly distributed over each year.

Compute the temporary 1.5-year complete expected college lifetime of a student entering the second year, $\mathring{e}_{1:\overline{1.5}|}$.

A. 1.25 B. 1.30 C. 1.35 D. 1.40 E. 1.45

22.17. You are given:

(i) Deaths are uniformly distributed between integral ages.

(ii) $\mathring{e}_{x+0.5:\overline{0.5}|} = 5/12$.

Calculate q_x.

22.18. You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) $\mathring{e}_{55.2:\overline{0.4}|} = 0.396$.

Calculate $\mu_{55.2}$.

22.19. [**150-S87:21**] You are given:

(i) $d_x = k$ for $x = 0, 1, 2, \ldots, \omega - 1$

(ii) $\mathring{e}_{20:\overline{20}|} = 18$

(iii) Deaths are uniformly distributed over each year of age.

Calculate ${}_{30|10}q_{30}$.

A. 0.111 B. 0.125 C. 0.143 D. 0.167 E. 0.200

22.20. [**150-S89:24**] You are given:

(i) Deaths are uniformly distributed over each year of age.

(ii) $\mu_{45.5} = 0.5$

Calculate $\mathring{e}_{45:\overline{1}|}$.

A. 0.4 B. 0.5 C. 0.6 D. 0.7 E. 0.8

Exercises continue on the next page ...

22.21. [**CAS3-S04:10**] 4,000 people age (30) each pay an amount, P, into a fund. Immediately after the 1,000$^{\text{th}}$ death, the fund will be dissolved and each of the survivors will be paid \$50,000.

- Mortality follows the Illustrative Life Table, using linear interpolation at fractional ages.
- $i = 12\%$

Calculate P.

A. Less than 515
B. At least 515, but less than 525
C. At least 525, but less than 535
D. At least 535, but less than 545
E. At least 545

Additional released exam questions: CAS3-S05:31, CAS3-F05:13, CAS3-S06:13, CAS3-F06:13, CAS3-S07:24, CAS3L-S08:16, CAS3L-S09:3, CAS3L-F09:3, CAS3L-S10:4, CAS3L-F10:3, CAS3L-S11:3

Solutions

22.1. In the second summand, ${}_{1/2}p_x\,\mu_{x+1/2}$ is the density function, which is the constant q_x under UDD. The first summand ${}_{3/4}p_x = 1 - \frac{3}{4}q_x$. So the sum is $1 - \frac{1}{4}q_x$, or $\boxed{{}_{1/4}p_x}$. **(E)**

22.2. Using equation (22.3),

$$\begin{aligned}\frac{3}{31} &= {}_{0.25}q_{x+0.75} = \frac{0.25q_x}{1-0.75q_x}\\ \frac{3}{31} - \frac{2.25}{31}q_x &= 0.25q_x\\ \frac{3}{31} &= \frac{10}{31}q_x\\ q_x &= \boxed{0.3}\end{aligned}$$

22.3. We calculate the probability that $(x+\frac{3}{4})$ survives for half a year. Since the duration crosses an integer boundary, we break the period up into two quarters of a year. The probability of $(x+3/4)$ surviving for 0.25 years is, by equation (22.3),

$${}_{1/4}p_{x+3/4} = \frac{1-0.10}{1-0.75(0.10)} = \frac{0.9}{0.925}$$

The probability of $(x+1)$ surviving to $x+1.25$ is

$${}_{1/4}p_{x+1} = 1 - 0.25(0.15) = 0.9625$$

The answer to the question is then the complement of the product of these two numbers:

$${}_{1/2}q_{x+3/4} = 1 - {}_{1/2}p_{x+3/4} = 1 - {}_{1/4}p_{x+3/4}\,{}_{1/4}p_{x+1} = 1 - \left(\frac{0.9}{0.925}\right)(0.9625) = \boxed{0.06351}$$

Alternatively, you could build a life table starting at age x, with $l_x = 1$. Then $l_{x+1} = (1-0.1) = 0.9$ and $l_{x+2} = 0.9(1-0.15) = 0.765$. Under UDD, l_x at fractional ages is obtained by linear interpolation, so

$$l_{x+0.75} = 0.75(0.9) + 0.25(1) = 0.925$$

$$l_{x+1.25} = 0.25(0.765) + 0.75(0.9) = 0.86625$$
$$_{1/2}p_{3/4} = \frac{l_{x+1.25}}{l_{x+0.75}} = \frac{0.86625}{0.925} = 0.93649$$
$$_{1/2}q_{3/4} = 1 - {}_{1/2}p_{3/4} = 1 - 0.93649 = \boxed{\mathbf{0.06351}}$$

22.4. $_{0.3|0.5}q_{x+0.4}$ is $_{0.3}p_{x+0.4} - {}_{0.8}p_{x+0.4}$. The first summand is

$$_{0.3}p_{x+0.4} = \frac{1-0.7q_x}{1-0.4q_x} = \frac{1-0.07}{1-0.04} = \frac{93}{96}$$

The probability that $(x+0.4)$ survives to $x+1$ is, by equation (22.3),

$$_{0.6}p_{x+0.4} = \frac{1-0.10}{1-0.04} = \frac{90}{96}$$

and the probability $(x+1)$ survives to $x+1.2$ is

$$_{0.2}p_{x+1} = 1 - 0.2q_{x+1} = 1 - 0.2(0.15) = 0.97$$

So

$$_{0.3|0.5}q_{x+0.4} = \frac{93}{96} - \left(\frac{90}{96}\right)(0.97) = \boxed{\mathbf{0.059375}}$$

Alternatively, you could use the life table from the solution to the last question, and linearly interpolate:

$$l_{x+0.4} = 0.4(0.9) + 0.6(1) = 0.96$$
$$l_{x+0.7} = 0.7(0.9) + 0.3(1) = 0.93$$
$$l_{x+1.2} = 0.2(0.765) + 0.8(0.9) = 0.873$$
$$_{0.3|0.5}q_{x+0.4} = \frac{0.93 - 0.873}{0.96} = \boxed{\mathbf{0.059375}}$$

22.5. Under uniform distribution of deaths between integral ages, $l_{x+0.5} = \frac{1}{2}(l_x + l_{x+1})$, since the survival function is a straight line between two integral ages. Therefore, $l_{45.5} = \frac{1}{2}(9{,}164{,}051 + 9{,}127{,}426) = 9{,}145{,}738.5$. Median future lifetime occurs when $l_x = \frac{1}{2}(9{,}145{,}738.5) = 4{,}572{,}869$. This happens between ages 77 and 78. We interpolate between the ages to get the exact median:

$$l_{77} - s(l_{77} - l_{78}) = 4{,}572{,}869$$
$$4{,}828{,}182 - s(4{,}828{,}182 - 4{,}530{,}360) = 4{,}572{,}869$$
$$4{,}828{,}182 - 297{,}822s = 4{,}572{,}869$$
$$s = \frac{4{,}828{,}182 - 4{,}572{,}869}{297{,}822} = \frac{255{,}313}{297{,}822} = 0.8573$$

So the median age at death is 77.8573, and median future lifetime is $77.8573 - 45.5 = \boxed{\mathbf{32.3573}}$.

22.6. We use the fact that for each age, $L_x = \frac{1}{2}(l_x + l_{x+1})$, so $_3L_{52} = \frac{1}{2}(l_{52} + 2l_{53} + 2l_{54} + l_{55})$, and we have

$$_3m_{52} = \frac{_3d_{52}}{\frac{1}{2}(l_{52} + 2l_{53} + 2l_{54} + l_{55})}$$

Dividing numerator and denominator by l_{52},

$$_3m_{52} = \frac{2\,{}_3q_{52}}{1 + 2p_{52} + 2\,{}_2p_{52} + {}_3p_{52}}$$

$$= \frac{2\Big(1-(0.90)(0.89)(0.87)\Big)}{1+2(0.90)+2(0.90)(0.89)+(0.90)(0.89)(0.87)}$$
$$= \frac{0.60626}{5.09887} = \boxed{\mathbf{0.11890}}$$

22.7. ${}_xp_0 = \frac{l_x}{l_0} = 1-\left(\frac{x}{100}\right)^2$. The force of mortality is calculated as the negative derivative of $\ln {}_xp_0$:

$$\mu_x = -\frac{\mathrm{d}\ln {}_xp_0}{\mathrm{d}x} = \frac{2\left(\frac{x}{100}\right)\left(\frac{1}{100}\right)}{1-\left(\frac{x}{100}\right)^2} = \frac{2x}{100^2-x^2}$$
$$\mu_{50.25} = \frac{100.5}{100^2-50.25^2} = 0.0134449$$

For UDD, we need to calculate q_{50}.

$$p_{50} = \frac{l_{51}}{l_{50}} = \frac{1-0.51^2}{1-0.50^2} = 0.986533$$
$$q_{50} = 1-0.986533 = 0.013467$$

so under UDD,

$$\mu^L_{50.25} = \frac{q_{50}}{1-0.25q_{50}} = \frac{0.013467}{1-0.25(0.013467)} = 0.013512.$$

The difference between $\mu_{50.25}$ and $\mu^L_{50.25}$ is $0.013445-0.013512 = \boxed{\mathbf{-0.000067}}$. **(B)**

22.8. $s(20) = 1/1.2^4$ and $s(21) = 1/1.21^4$, so $q_{20} = 1-(1.2/1.21)^4 = 0.03265$. Then

$${}_{0.4}q_{20.2} = \frac{0.4q_{20}}{1-0.2q_{20}} = \frac{0.4(0.03265)}{1-0.2(0.03265)} = \boxed{\mathbf{0.01315}}$$

22.9. First calculate q_x.

$$1-0.75q_x = 0.25$$
$$q_x = 1$$

Then by equation (22.3), ${}_{0.25}q_{x+0.5} = 0.25/(1-0.5) = 0.5$, making I true.
By equation (22.2), ${}_{0.5}q_x = 0.5q_x = 0.5$, making II true.
By equation (22.5), $\mu_{x+0.5} = 1/(1-0.5) = 2$, making III false. **(A)**

22.10. We use equation (22.5) to back out q_x for each age.

$$\mu_{x+0.5} = \frac{q_x}{1-0.5q_x} \Rightarrow q_x = \frac{\mu_{x+0.5}}{1+0.5\mu_{x+0.5}}$$
$$q_{80} = \frac{0.0202}{1.0101} = 0.02$$
$$q_{81} = \frac{0.0408}{1.0204} = 0.04$$
$$q_{82} = \frac{0.0619}{1.03095} = 0.06$$

Then by equation (22.3), ${}_{0.5}p_{80.5} = 0.98/0.99$. $p_{81} = 0.96$, and ${}_{0.5}p_{82} = 1-0.5(0.06) = 0.97$. Therefore

$${}_2q_{80.5} = 1-\left(\frac{0.98}{0.99}\right)(0.96)(0.97) = \boxed{\mathbf{0.0782}} \qquad \textbf{(A)}$$

22.11. To do this algebraically, we split the group into those who die within 0.3 years, those who die between 0.3 and 1 years, and those who survive one year. Under UDD, those who die will die at the midpoint of the interval (assuming the interval doesn't cross an integral age), so we have

Group	Survival time	Probability of group	Average survival time
I	$(0,0.3]$	$1-{}_{0.3}p_{x+0.7}$	0.15
II	$(0.3,1]$	${}_{0.3}p_{x+0.7}-{}_{1}p_{x+0.7}$	0.65
III	$(1,\infty)$	${}_{1}p_{x+0.7}$	1

We calculate the required probabilities.

$$\begin{aligned}
{}_{0.3}p_{x+0.7} &= \frac{0.9}{0.93} = 0.967742 \\
{}_{1}p_{x+0.7} &= \frac{0.9}{0.93}\big(1-0.7(0.3)\big) = 0.764516 \\
1-{}_{0.3}p_{x+0.7} &= 1-0.967742 = 0.032258 \\
{}_{0.3}p_{x+0.7}-{}_{1}p_{x+0.7} &= 0.967742-0.764516 = 0.203226 \\
\mathring{e}_{x+0.7:\overline{1}|} &= 0.032258(0.15)+0.203226(0.65)+0.764516(1) = \boxed{\mathbf{0.901452}}
\end{aligned}$$

Alternatively, we can use trapezoids. We already know from the above solution that the heights of the first trapezoid are 1 and 0.967742, and the heights of the second trapezoid are 0.967742 and 0.764516. So the sum of the area of the two trapezoids is

$$\begin{aligned}
\mathring{e}_{x+0.7:\overline{1}|} &= (0.3)(0.5)(1+0.967742)+(0.7)(0.5)(0.967742+0.764516) \\
&= 0.295161+0.606290 = \boxed{\mathbf{0.901451}}
\end{aligned}$$

22.12. For the expected value, we'll use the recursive formula. (The trapezoidal rule could also be used.)

$$\begin{aligned}
\mathring{e}_{45:\overline{2}|} &= \mathring{e}_{45:\overline{1}|} + p_{45}\,\mathring{e}_{46:\overline{1}|} \\
&= (1-0.005)+0.99(1-0.0055) \\
&= 1.979555
\end{aligned}$$

We'll use equation (20.7)to calculate the second moment.

$$\begin{aligned}
\mathbf{E}[(T\wedge 2)^2] &= 2\int_0^2 t\,{}_tp_x\,\mathrm{d}t \\
&= 2\left(\int_0^1 t(1-0.01t)\,\mathrm{d}t + \int_1^2 t(0.99)\big(1-0.011(t-1)\big)\,\mathrm{d}t\right) \\
&= 2\left(\frac{1}{2}-0.01\left(\frac{1}{3}\right)+0.99\left(\frac{(1.011)(2^2-1^2)}{2}-0.011\left(\frac{2^3-1^3}{3}\right)\right)\right) \\
&= 2(0.496667+1.475925) = 3.94518
\end{aligned}$$

So the variance is $3.94518-1.979555^2 = \boxed{\mathbf{0.02654}}$.

22.13. As discussed on page 321, by equation (22.7), the difference is

$$\frac{1}{2}\,{}_{10}q_x = \frac{1}{2}(1-0.2) = \boxed{\mathbf{0.4}}$$

22.14. Those who die in the first year survive $\frac{1}{2}$ year on the average and those who die in the first half of the second year survive 1.25 years on the average, so we have

$$p_{60} = 0.98$$
$$_{1.5}p_{60} = 0.98\big(1-0.5(0.022)\big) = 0.96922$$
$$\mathring{e}_{60:\overline{1.5}|} = 0.5(0.02) + 1.25(0.98-0.96922) + 1.5(0.96922) = \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}$$

Alternatively, we use the trapezoidal method. The first trapezoid has heights 1 and $p_{60} = 0.98$ and width 1. The second trapezoid has heights $p_{60} = 0.98$ and $_{1.5}p_{60} = 0.96922$ and width 1/2.

$$\mathring{e}_{60:\overline{1.5}|} = \frac{1}{2}(1+0.98) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(0.98+0.96922)$$
$$= \boxed{\mathbf{1.477305}} \qquad \textbf{(D)}$$

22.15. $p_{70} = 1-0.040 = 0.96$, $_2p_{70} = (0.96)(0.956) = 0.91776$, and by linear interpolation, $_{1.5}p_{70} = 0.5(0.96 + 0.91776) = 0.93888$. Those who die in the first year survive 0.5 years on the average and those who die in the first half of the second year survive 1.25 years on the average. So

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(0.04) + 1.25(0.96-0.93888) + 1.5(0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

Alternatively, we can use the trapezoidal method. The first year's trapezoid has heights 1 and 0.96 and width 1 and the second year's trapezoid has heights 0.96 and 0.93888 and width 1/2, so

$$\mathring{e}_{70:\overline{1.5}|} = 0.5(1+0.96) + 0.5(0.5)(0.96+0.93888) = \boxed{\mathbf{1.45472}} \qquad \textbf{(C)}$$

22.16. First we calculate $_tp_1$ for $t = 1, 2$.

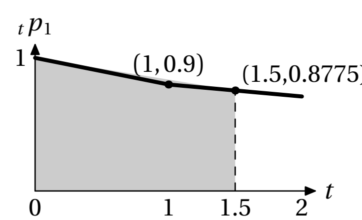
$$p_1 = 1 - q_1 = 0.90$$
$$_2p_1 = (1-q_1)(1-q_2) = (0.90)(0.95) = 0.855$$

By linear interpolation, $_{1.5}p_1 = (0.5)(0.9 + 0.855) = 0.8775$.

The algebraic method splits the students into three groups: first year dropouts, second year (up to time 1.5) dropouts, and survivors. In each dropout group survival on the average is to the midpoint (0.5 years for the first group, 1.25 years for the second group) and survivors survive 1.5 years. Therefore

$$\mathring{e}_{1:\overline{1.5}|} = 0.10(0.5) + (0.90-0.8775)(1.25) + 0.8775(1.5) = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}$$

Alternatively, we could sum the two trapezoids making up the shaded area at the right.

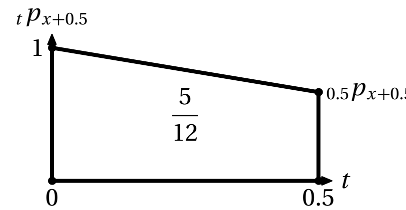


$$\mathring{e}_{1:\overline{1.5}|} = (1)(0.5)(1+0.9) + (0.5)(0.5)(0.90+0.8775)$$
$$= 0.95 + 0.444375 = \boxed{\mathbf{1.394375}} \qquad \textbf{(D)}$$

22.17. Those who die survive 0.25 years on the average and survivors survive 0.5 years, so we have

$$\begin{aligned}
0.25\,{}_{0.5}q_{x+0.5} + 0.5\,{}_{0.5}p_{x+0.5} &= \tfrac{5}{12}\\
0.25\left(\frac{0.5q_x}{1-0.5q_x}\right) + 0.5\left(\frac{1-q_x}{1-0.5q_x}\right) &= \tfrac{5}{12}\\
0.125q_x + 0.5 - 0.5q_x &= \tfrac{5}{12} - \tfrac{5}{24}q_x\\
\frac{1}{2} - \frac{5}{12} &= \left(-\frac{5}{24} + \frac{1}{2} - \frac{1}{8}\right)q_x\\
\frac{1}{12} &= \frac{q_x}{6}\\
q_x &= \boxed{\tfrac{1}{2}}
\end{aligned}$$

Alternatively, complete life expectancy is the area of the trapezoid shown on the right, so



$$\frac{5}{12} = 0.5(0.5)(1 + {}_{0.5}p_{x+0.5})$$

Then ${}_{0.5}p_{x+0.5} = \frac{2}{3}$, from which it follows

$$\begin{aligned}
\frac{2}{3} &= \frac{1-q_x}{1-\frac{1}{2}q_x}\\
q_x &= \boxed{\tfrac{1}{2}}
\end{aligned}$$

22.18. Survivors live 0.4 years and those who die live 0.2 years on the average, so

$$0.396 = 0.4_{0.4}p_{55.2} + 0.2_{0.4}q_{55.2}$$

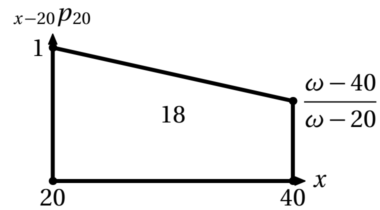
Using the formula ${}_{0.4}q_{55.2} = 0.4q_{55}/(1-0.2q_{55})$ (equation (22.3)), we have

$$\begin{aligned}
0.4\left(\frac{1-0.6q_{55}}{1-0.2q_{55}}\right) + 0.2\left(\frac{0.4q_{55}}{1-0.2q_{55}}\right) &= 0.396\\
0.4 - 0.24q_{55} + 0.08q_{55} &= 0.396 - 0.0792q_{55}\\
0.0808q_{55} &= 0.004\\
q_{55} &= \frac{0.004}{0.0808} = 0.0495\\
\mu_{55.2} &= \frac{q_{55}}{1-0.2q_{55}} = \frac{0.0495}{1-0.2(0.0495)} = \boxed{\mathbf{0.05}}
\end{aligned}$$

22.19. Since d_x is constant for all x and deaths are uniformly distributed within each year of age, mortality follows de Moivre's law. We back out ω using equation (20.9), $\mathring{e}_{x:\overline{n}|} = {}_np_x(n) + {}_nq_x(n/2)$:

$$\begin{aligned}
10\,{}_{20}q_{20} + 20\,{}_{20}p_{20} &= 18\\
10\left(\frac{20}{\omega-20}\right) + 20\left(\frac{\omega-40}{\omega-20}\right) &= 18\\
200 + 20\omega - 800 &= 18\omega - 360\\
2\omega &= 240\\
\omega &= 120
\end{aligned}$$

Alternatively, we can back out ω using the trapezoidal rule. Complete life expectancy is the area of the trapezoid shown to the right.



$$\begin{aligned}\mathring{e}_{20:\overline{20}|} &= 18 = (20)(0.5)\left(1+\frac{\omega-40}{\omega-20}\right)\\ 0.8 &= \frac{\omega-40}{\omega-20}\\ 0.8\omega-16 &= \omega-40\\ 0.2\omega &= 24\\ \omega &= 120\end{aligned}$$

Once we have ω, we compute

$$_{30|10}q_{30} = \frac{10}{\omega-30} = \frac{10}{90} = \boxed{\mathbf{0.1111}} \qquad \textbf{(A)}$$

22.20. We use equation (22.5) to obtain

$$\begin{aligned}0.5 &= \frac{q_x}{1-0.5q_x}\\ q_x &= 0.4\end{aligned}$$

Then $\mathring{e}_{45:\overline{1}|} = 0.5\big(1+(1-0.4)\big) = \boxed{\mathbf{0.8}}$. **(E)**

22.21. According to the Illustrative Life Table, $l_{30} = 9{,}501{,}381$, so we are looking for the age x such that $l_x = 0.75(9{,}501{,}381) = 7{,}126{,}036$. This is between 67 and 68. Using linear interpolation, since $l_{67} = 7{,}201{,}635$ and $l_{68} = 7{,}018{,}432$, we have

$$x = 67 + \frac{7{,}201{,}635-7{,}126{,}036}{7{,}201{,}635-7{,}018{,}432} = 67.4127$$

This is 37.4127 years into the future. $\frac{3}{4}$ of the people collect 50,000. We need $50{,}000\left(\frac{3}{4}\right)\left(\frac{1}{1.12^{37.4127}}\right) = \boxed{\mathbf{540.32}}$ per person. **(D)**

Quiz Solutions

22-1. The algebraic method goes: those who die will survive 0.3 on the average, and those who survive will survive 0.6.

$$\begin{aligned}{}_{0.6}q_{x+0.4} &= \frac{0.6(0.1)}{1-0.4(0.1)} = \frac{6}{96}\\ {}_{0.6}p_{x+0.4} &= 1-\frac{6}{96} = \frac{90}{96}\\ \mathring{e}_{x+0.4:\overline{0.6}|} &= \frac{6}{96}(0.3)+\frac{90}{96}(0.6) = \frac{55.8}{96} = \boxed{\mathbf{0.58125}}\end{aligned}$$

The geometric method goes: we need the area of a trapezoid having height 1 at $x+0.4$ and height 90/96 at $x+1$, where 90/96 is $_{0.6}p_{x+0.4}$, as calculated above. The width of the trapezoid is 0.6. The answer is therefore $0.5(1+90/96)(0.6) = \boxed{\mathbf{0.58125}}$.

22-2. Batteries failing in month 1 survive an average of 0.5 month, those failing in month 2 survive an average of 1.5 months, and those failing in month 3 survive an average of 2.125 months (the average of 2 and 2.25). By linear interpolation, ${}_{2.25}q_0 = 0.25(0.6) + 0.75(0.2) = 0.3$. So we have

$$\begin{aligned}\mathring{e}_{0:\overline{2.25}|} &= q_0(0.5) + {}_{1|}q_0(1.5) + {}_{2|0.25}q_0(2.125) + {}_{2.25}p_0(2.25)\\ &= (0.05)(0.5) + (0.20 - 0.05)(1.5) + (0.3 - 0.2)(2.125) + 0.70(2.25) = \boxed{\mathbf{2.0375}}\end{aligned}$$