Contents

Lesson 3

Method of Moments

A *statistic* is a number that can be calculated from observations. The following table lists examples of statistics and non-statistics.

Example of statistic	Example of non-statistic
First observation	Parameter of underlying distribution
Sample mean	Mean of underlying distribution
Sample variance	Variance of underlying distribution
Sample median	Median of underlying distribution
Percentage of observations less than 100	Probability that an observation is less than 100

The purpose of the field of Statistics is to make inferences regarding the underlying distribution from observed data. Statistics has two branches: estimating parameters and testing hypotheses. This lesson and the next will deal with estimating parameters.

You may have a random variable for which you know or have a strong feeling about the underlying distribution, but for which you need to estimate the parameters of the distribution from observations. Examples are:

- You sell one-year term life insurance to a group of 1000 45-year-old men. The insurance pays a death benefit if the insured dies within one year.

  Let X be the random variable indicating the number of deaths within a year. Then X is a binomial random variable with parameters $m=1000$ and q. You would like to estimate the parameter q.

- You sell group major medical insurance to large groups. The rate you charge per member is based on average aggregate claims per member per year. Since the groups are large, you assume that the Central Limit Theorem provides a good approximation to average aggregate claims per member per year. If X is aggregate claims per member per year, X is then assumed to follow a normal distribution with parameters μ and σ. You would like to estimate μ and σ.

Even if you don't have a good feeling for the underlying distribution, you may wish to try out various distributions and determine which parameters are best. You can then perform tests to determine whether the distribution with its best parameters fits the data well.[1] For example:

- Let X be claim size for an automobile liability coverage. Small claim sizes are more frequent than large claim sizes, but there are a significant number of large claim sizes. In other words, the claim size distribution is heavily skewed to the right; it has a heavy tail. You think that a Pareto or a lognormal distribution may fit the data well, and would like to estimate the parameters of these distributions.

One method for estimating parameters is the *Method of Moments*. In the method of moments, we equate the first n moments of the sample with the first n moments of the fitted distribution, where n is the number of parameters of the fitted distribution. Actually, we can equate *any* n moments, not necessarily the first n. So this method can be used even for distributions for which the first n moments don't exist, like the inverse exponential. However, when not told otherwise, equate the *first* n moments.

[1] Exam 4 deals with fitting parametric distributions to data.

EXAMPLE 3A You sell one-year term life insurance to a group of 1000 45-year old men. Six of them die before their 46th birthdays.

Estimate the probability of death in one year using the method of moments.

ANSWER: You may view the distribution of number of deaths for the group as binomial with $m = 1000$, and you want to estimate the parameter q. Let $\tilde{q}$ be the estimate. The mean of the binomial distribution is $1000q$. The observed value is 6. Equating the two, $1000q = 6$, so $\tilde{q} = \boxed{\mathbf{0.006}}$. □

If you are matching the first two moments, you may instead match the mean and the variance. When matching the variance, use the biased sample variance

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$$

rather than the unbiased sample variance, which would have $n-1$ in the denominator.

EXAMPLE 3B Claim sizes are assumed to follow a gamma distribution.[2] You observe the following claim sizes:

3 5 9 21 32

Determine the estimate of the parameter α using the method of moments..

ANSWER: The gamma distribution has 2 unknown parameters, α and θ. Instead of equating the first 2 moments, we can equivalently equate the mean and variance. When doing so, to make the equating of the variance equivalent to equating the 2nd raw moment, the sample variance is calculated with division by n instead of by $n-1$.

The sample moments are (μ_2' is the sample second raw moment.)

$$\begin{aligned}
\bar{x} &= \frac{3+5+9+21+32}{5} = 14\\
\mu_2' &= \frac{3^2+5^2+9^2+21^2+32^2}{5} = \frac{1580}{5} = 316\\
\hat{\sigma}^2 &= 316 - 14^2 = 120
\end{aligned}$$

For X a gamma random variable, $\mathbf{E}[X] = \alpha\theta$ and $\operatorname{Var}(X) = \alpha\theta^2$. Then

$$\begin{aligned}
\tilde{\alpha}\tilde{\theta} &= 14\\
\tilde{\alpha}\tilde{\theta}^2 &= 120
\end{aligned}$$

and dividing the second into the first squared,

$$\tilde{\alpha} = \frac{196}{120} = \frac{49}{30} = \boxed{\mathbf{1.633333}}$$

It follows that $\tilde{\theta} = 14/(49/30) = 60/7$.

If you are the memorizing type, the general formulas for the fitted parameters of a gamma are:

$$\tilde{\alpha} = \frac{\bar{x}^2}{\hat{\sigma}^2} \qquad \tilde{\theta} = \frac{\hat{\sigma}^2}{\bar{x}}$$ □

Let's try fitting the same claims sizes to different distributions.

EXAMPLE 3C You observe the following claim sizes:

[2] The gamma distribution is described in the distribution tables you get at the exam. Please download them if you haven't yet. See page vii for a link to the tables.

3 5 9 21 32

1. Fit the data to a lognormal distribution using the method of moments.

   Determine the probability that a claim is greater than 20 using the fitted distribution.

2. Fit the data to a two-parameter Pareto[3] distribution using the method of moments.

   Determine the probability that a claim is greater than 20 using the fitted distribution.

ANSWER: 1. We already calculated the first two sample moments, 14 and 316, in the previous example. For a lognormal random variable X,

$$\mathbf{E}[X] = e^{\mu+\sigma^2/2} = 14$$
$$\mathbf{E}[X^2] = e^{2\mu+2\sigma^2} = 316$$

Logging the two equations eases backing out the parameters. We will use tilde for estimates.

$$\tilde{\mu} + \frac{\tilde{\sigma}^2}{2} = \ln 14 = 2.639057 \qquad (*)$$
$$2\tilde{\mu} + 2\tilde{\sigma}^2 = \ln 316 = 5.755742 \qquad (**)$$

Multiplying (*) by 4 and subtracting (**),

$$2\tilde{\mu} = 4(2.639057) - 5.755742 = 4.800486$$
$$\tilde{\mu} = \frac{4.800486}{2} = 2.400243$$

Plugging $\tilde{\mu}$ into (*),

$$\frac{\tilde{\sigma}^2}{2} = 2.639057 - 2.400243 = 0.238814$$
$$\tilde{\sigma} = \sqrt{0.477628} = 0.691106$$

The estimated probability that a claim is greater than 20 is[4]

$$\begin{aligned}\widetilde{\Pr}(X > 20) &= 1 - \Phi\left(\frac{\ln 20 - \tilde{\mu}}{\tilde{\sigma}}\right)\\ &= 1 - \Phi\left(\frac{2.995732 - 2.400244}{0.691106}\right)\\ &= 1 - \Phi(0.8616)\\ &= 1 - 0.8056 = \boxed{\mathbf{0.1944}}\end{aligned}$$

If we let m be the sample mean and t the sample second moment, the general formulas are

$$\tilde{\mu} = 2\ln m - \frac{\ln t}{2}$$
$$\tilde{\sigma} = \sqrt{\ln t - 2\ln m}$$

[3]When you are given a Pareto and not told whether it is single- or two- parameter, *always assume it is two parameter*.

[4]As mentioned in the preface, the CAS has not specified rules for evaluating the standard normal distribution. In the Statistics part of the manual, I use interpolated values of $\Phi(z)$. In the rest of the manual, I use SOA rounding rules based on the table they provide.

2. For a Pareto distribution, let's derive the general formulas in terms of the sample mean m and the sample second moment t. Let X be a Pareto random variable.

$$\mathbf{E}[X] = \frac{\theta}{\alpha-1} = m \tag{3.1}$$
$$\mathbf{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = t \tag{3.2}$$

Squaring the first equation and dividing it into the second,

$$\begin{aligned}
\frac{2(\tilde{\alpha}-1)}{\tilde{\alpha}-2} &= 2 + \frac{2}{\tilde{\alpha}-2} = \frac{t}{m^2}\\
\tilde{\alpha}-2 &= \frac{2}{(t/m^2)-2} = \frac{2m^2}{t-2m^2}\\
\tilde{\alpha} &= 2 + \frac{2m^2}{t-2m^2} = \frac{2(t-m^2)}{t-2m^2} \qquad (3.3)\\
\tilde{\alpha}-1 &= 1 + \frac{2m^2}{t-2m^2} = \frac{t}{t-2m^2}
\end{aligned}$$

Substituting into equation (3.1),

$$\tilde{\theta} = m(\tilde{\alpha}-1) = \frac{mt}{t-2m^2}$$

Plugging in $m = 14$ and $t = 316$,

$$\tilde{\alpha} = \frac{2(316-14^2)}{316-2(14^2)} = -3.15789$$

Since α can't be negative, a Pareto distribution cannot be fitted to this data with the method of moments. □

It is possible to use the method of moments to fit some parameters when others are known. For example, you could be asked to fit a two-parameter Pareto with known $\alpha = 3$ using the method of moments, in which case you would equate the means and solve for θ. Another possibility is fitting a single-parameter Pareto. For such a distribution, θ is assumed to be known; attempting to fit it may result in a θ that is higher than a sample value, which is an implausible fit since under the single-parameter Pareto distribution, the probability of a value being less than θ is 0. Let's derive a formula for $\tilde{\alpha}$ in terms of the sample mean m, matching first moments.

$$\begin{aligned}
\frac{\alpha\theta}{\alpha-1} &= m\\
\alpha\theta &= m\alpha - m\\
\tilde{\alpha} &= \frac{m}{m-\theta}
\end{aligned}$$

So far, we have only discussed fitting continuous distributions. Fitting both parameters of a binomial may not work, because the fitted first parameter may not be an integer. For a Poisson, $\hat{\lambda} = \bar{x}$. The next example derives a general formula for a negative binomial.

EXAMPLE 3D The annual number of claims follows a negative binomial distribution. You observe the following claim counts for one year:

Number of Claims	Number of Policyholders
0	80
1	15
2	5
3+	0

Estimate the parameters r and β using the method of moments.

ANSWER: The general formula is derived as follows:

$$\begin{aligned} r\beta &= \bar{x} \\ r\beta(1+\beta) &= \hat{\sigma}^2 \\ 1+\beta &= \frac{\hat{\sigma}^2}{\bar{x}} \\ \tilde{\beta} &= \frac{\hat{\sigma}^2-\bar{x}}{\bar{x}} \\ \tilde{r} &= \frac{\bar{x}^2}{\hat{\sigma}^2-\bar{x}} \end{aligned}$$

In our case, $\bar{x}=0.25$ and $\hat{\sigma}^2=\frac{15(1)+5(4)}{100}-0.25^2=0.2875$. Therefore,

$$\begin{aligned} \tilde{r} &= \frac{0.25^2}{0.0375} = \boxed{\mathbf{1\tfrac{2}{3}}} \\ \tilde{\beta} &= \frac{0.0375}{0.25} = \boxed{\mathbf{0.15}} \end{aligned}$$

□

You may be asked to use the the method of moments to fit a distribution not in the tables. In that case, you may need to integrate to derive the moments.

EXAMPLE 3E For observations $x_1, x_2, \ldots, x_{10}$ of a random variable you are given

- $\sum x_i = 4$
- $\sum x_i^2 = 2$

The random variable's density function is

$$f(x) = \alpha x^{\alpha-1} \qquad 0 \le x \le 1$$

The parameter α is estimated using the method of moments.
Using this estimate, determine the variance of the random variable.

ANSWER: You may recognize this random variable as a beta distribution and look it up in the tables. But if you didn't recognize it, you could derive the mean:

$$\mathbf{E}[X] = \int_0^1 \alpha x^\alpha \mathrm{d}x = \left.\frac{\alpha x^{\alpha+1}}{\alpha+1}\right|_0^1 = \frac{\alpha}{\alpha+1}$$

Then

$$\begin{aligned} \frac{\alpha}{\alpha+1} &= \frac{4}{10} = 0.4 \\ \alpha &= \frac{2}{3} \end{aligned}$$

$\sum x_i^2$ is not needed. Now we calculate the variance of the random variable.

$$\begin{aligned} \mathbf{E}[X^2] &= \int_0^1 \alpha x^{\alpha+1} \mathrm{d}x = \left.\frac{\alpha x^{\alpha+2}}{\alpha+2}\right|_0^1 = \frac{\alpha}{\alpha+2} \\ \mathrm{Var}(X) &= \frac{\alpha}{\alpha+2} - \mathbf{E}[X]^2 = \frac{2/3}{8/3} - 0.4^2 = \boxed{\mathbf{0.09}} \end{aligned}$$

□

Table 3.1: Summary of Method of Moments

- To fit a distribution with k parameters using method of moments, match the first k distribution moments to the sample moments.
- The sample variance may be matched to the distribution variance instead of matching raw second moments, but use the biased sample variance, or $\hat{\sigma}^2$.
- It's probably not worthwhile memorizing formulas for method of moments estimators, but formulas we've developed in this lesson are:

Gamma	$\tilde{\alpha} = \bar{x}^2/\hat{\sigma}^2$	$\tilde{\theta} = \hat{\sigma}^2/\bar{x}$
Lognormal	$\tilde{\mu} = 2\ln m - (\ln t)/2$	$\tilde{\sigma} = \sqrt{\ln t - 2\ln m}$
Pareto	$\tilde{\alpha} = \dfrac{2(t-m^2)}{t-2m^2}$	$\tilde{\theta} = \dfrac{mt}{t-2m^2}$
Single-parameter Pareto	$\tilde{\alpha} = m/(m-\theta)$	
Poisson	$\tilde{\lambda} = \bar{x}$	
Negative binomial	$\tilde{r} = (\bar{x}^2)/(\hat{\sigma}^2 - \bar{x})$	$\tilde{\beta} = (\hat{\sigma}^2 - \bar{x})/\bar{x}$
Beta with known b	$a = b\bar{x}/(1-\bar{x})$	
Beta with known a	$b = a(1-\bar{x})/\bar{x}$	

You may want to familiarize yourself with the beta distribution. For a beta with $\theta = 1$ (as defined in the distribution tables), the density function has the form:

$$f(x) = kx^{a-1}(1-x)^{b-1} \quad 0 \le x \le 1$$

where a and b are parameters and k is a constant which makes the density proper, i.e., integrating to 1 over its range. The mean is $a/(a+b)$. A uniform distribution on $[0,1]$ is a special case with $a = b = 1$. On this exam, they'll usually only ask you to fit one parameter, so typically they'll set either $a = 1$ or $b = 1$. The parameters a and b must be positive, but need not be integers. You'll recognize this distribution when the density function has powers of x and $1-x$ and nothing else. Remember to increase the two exponents by 1 to get the parameters a and b.

A general formula for the method of moments estimator of a beta distribution when one of the parameters is known is given in Table 3.1.

Quiz 3-1 Four observations are fitted to a distribution with probability density function:

$$f(x) = (\alpha+1)\alpha x(1-x)^{\alpha-1} \qquad 0 \le x \le 1$$

The observations are: 0.51, 0.32, 0.18, 0.07.

Calculate the method of moments estimate for α.

There is about one method of moments question on each exam. Sometimes the question combines method of moments with maximum likelihood, the topic of the next lesson.

Method of moments appears on Exam 4. Many of the exercises below are taken from that exam, and are harder than what you'll encounter on Exam 3L.

Exercises

3.1. You are given the following observations for a discrete distribution:

$$0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 7$$

The data are fitted to a Poisson distribution using the method of moments.

Determine the estimate of λ.

3.2. You are given the following observations for a discrete distribution:

$$0 \qquad 0 \qquad 0 \qquad 1 \qquad 3$$

The data are fitted to a negative binomial distribution, as parametrized in the distribution tables, using the method of moments.

Determine the estimate of r.

3.3. [**110-W96:26**] Let $X_1,\ldots,X_n$ be a random sample from a discrete distribution with probability function

$$p(x)=\begin{cases}\theta & \text{for } x=1\\ \theta & \text{for } x=2\\ 1-2\theta & \text{for } x=3\end{cases}$$

where $0<\theta<\frac{1}{2}$.

Determine the method of moments estimator of θ.

A. $\dfrac{3-\bar{X}}{3}$ B. $\dfrac{\bar{X}-1}{4}$ C. $\dfrac{2\bar{X}-3}{6}$ D. $\bar{X}$ E. $\dfrac{\bar{X}}{2}$

3.4. Claim sizes are as follows:

$$100 \quad 200 \quad 500 \quad 1000 \quad 1500 \quad 2000 \quad 3700$$

An exponential is fitted to claim sizes using the method of moments.

Let X be the claim size.

Calculate the estimate of $\Pr(X>500)$.

3.5. Claim sizes are as follows:

$$10 \quad 20 \quad 30 \quad 40 \quad 50$$

The parameters α and θ of a single-parameter Pareto distribution is fitted to the claim size using the method of moments.

Determine the estimate of α.

Exercises continue on the next page . . .

3.6. Claim sizes are as follows:

10	20	40	80	100	200	200	500

A gamma distribution is fitted to the claim sizes using the method of moments.

Determine the estimate of θ.

3.7. [**4B-S90:34**] (2 points) The observations 1000, 850, 750, 1100, 1250, and 900 are a random sample taken from a gamma distribution with unknown parameters α and θ. Let $\tilde{\alpha}$ and $\tilde{\theta}$ denote the method of moments estimators of α and θ, respectively.

In what range does $\tilde{\alpha}$ fall?

A. $\tilde{\alpha} < 30$
B. $30 \le \tilde{\alpha} < 40$
C. $40 \le \tilde{\alpha} < 50$
D. $50 \le \tilde{\alpha} < 60$
E. $60 \le \tilde{\alpha}$

3.8. [**4B-S91:46**] (2 points) The following is a sample of 10 claims:

1500	3500	1800	4800	3900
6000	3800	5500	4200	3000

The underlying distribution is assumed to be gamma, with parameters α and θ unknown.

In what range does the method of moments estimator, $\hat{\theta}$, of θ fall?

A. $\hat{\theta} < 250$
B. $250 \le \hat{\theta} < 300$
C. $300 \le \hat{\theta} < 350$
D. $350 \le \hat{\theta} < 400$
E. $400 \le \hat{\theta}$

3.9. [**Prior exam**] A random sample of death records yields the following exact ages at death: 30, 50, 60, 60, 70, 90. The age at death of the population from which the sample is drawn follows a gamma distribution. The parameters of the gamma distribution are estimated using the method of moments.

Determine the estimate of α.

A. 6.0 B. 7.2 C. 9.0 D. 10.8 E. 12.2

3.10. [**Prior exam**] You are given:

(i) Five lives are observed from time $t = 0$ until death.
(ii) Deaths occur at $t = 3$, 4, 4, 11, and 18.

Assume the lives are subject to the probability function

$$f(t) = \frac{t e^{-t/c}}{c^2}, \qquad t > 0.$$

Determine c by the method of moments.

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2 E. 4

Exercises continue on the next page ...

3.11. [**CAS3-S05:19**] Four losses are observed from a Gamma distribution.

The observed losses are: 200, 300, 350, and 450.

Find the method of moments estimate for α.

A. 0.3 B. 1.2 C. 2.3 D. 6.7 E. 13.0

3.12. Claim sizes are as follows:

100 100 200 200 300 500 1000 2000

A two-parameter Pareto distribution is fitted to the claim sizes using the method of moments. Let X be the claim size.

Determine the estimate of $\Pr(X > 500)$.

3.13. Claim sizes are as follows:

10 20 30 40 50

The parameters α and θ of a single-parameter Pareto distribution is fitted to the claim size using the method of moments.

Determine the estimate of α.

3.14. [**4B-F95:17**] (3 points) You are given the following:

- Losses follow a Pareto distribution with parameters θ (unknown) and $\alpha = 3$.
- 300 losses have been observed.

Determine the variance of $\tilde{\theta}$, the method of moments estimator of θ.

A. $0.0025\theta^2$ B. $0.0033\theta^2$ C. $0.0050\theta^2$ D. $0.0100\theta^2$ E. $0.0133\theta^2$

3.15. [**4B-F97:18**] (2 points) You are given the following:

- The random variable X has the density function

$$f(x) = \alpha x^{-\alpha-1},\ 1 < x < \infty, \alpha > 1.$$

- A random sample is taken of the random variable X.

Determine the limit of $\tilde{\alpha}$, the method of moments estimator of α, as the sample mean goes to infinity.

A. 0 B. 1/2 C. 1 D. 2 E. ∞

3.16. [**4B-S98:5**] (2 points) You are given the following:

- The random variable X has the density function

$$f(x) = \alpha(x+1)^{-\alpha-1},\ 0 < x < \infty,\ \alpha > 0$$

- A random sample of size n is taken of the random variable X.

Assuming $\alpha > 1$, determine $\tilde{\alpha}$, the method of moments estimator of α.

A. $\bar{X}$ B. $\dfrac{\bar{X}}{\bar{X}-1}$ C. $\dfrac{\bar{X}}{\bar{X}+1}$ D. $\dfrac{\bar{X}-1}{\bar{X}}$ E. $\dfrac{\bar{X}+1}{\bar{X}}$

Exercises continue on the next page . . .

3.17. [**4-S00:36**] You are given the following sample of five claims:

4 5 21 99 421

You fit a Pareto distribution using the method of moments.

Determine the 95$^{\text{th}}$ percentile of the fitted distribution.

A. Less than 380
B. At least 380, but less than 395
C. At least 395, but less than 410
D. At least 410, but less than 425
E. At least 425

3.18. The observations 1000, 300, 200, 2500, 9000 are a random sample taken from a lognormal distribution with unknown parameters μ and σ. Let $\tilde{\mu}$ and $\tilde{\sigma}$ be the method of moments estimators for μ and σ respectively.

Determine $\tilde{\mu}$.

3.19. [**110-S90:31**] Let X be a continuous random variable with density function

$$f(x;\theta)=\begin{cases}(\frac{1}{\theta})x^{(1-\theta)/\theta} & \text{for } 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

where $\theta>0$.

Determine the method-of-moments estimator of θ.

A. $\dfrac{1-\bar{X}}{\bar{X}}$ B. $\dfrac{\bar{X}-1}{\bar{X}}$ C. $\dfrac{\bar{X}}{1-\bar{X}}$ D. $\dfrac{\bar{X}}{\bar{X}-1}$ E. $\dfrac{1}{1+\bar{X}}$

3.20. [**4B-S95:5**] (2 points) You are given the following:

- The random variable X has the density function $f(x)=\alpha x^{\alpha-1}$, $0<x<1$, $\alpha>0$.
- A random sample of three observations of X yields the values 0.40, 0.70, 0.90.

Determine the value of $\tilde{\alpha}$, the method of moments estimator of α.

A. Less than 0.5
B. At least 0.5, but less than 1.5
C. At least 1.5, but less than 2.5
D. At least 2.5, but less than 3.5
E. At least 3.5

Exercises continue on the next page . . .

3.21. [**4B-S96:4**] (2 points) You are given the following:

- The random variable X has the density function $f(x)=(2/\theta^2)(\theta-x)$, $0<x<\theta$.
- A random sample of two observations of X yields values 0.50 and 0.90.

Determine $\tilde{\theta}$, the method of moments estimator of θ.

A. Less than 0.45
B. At least 0.45, but less than 0.95
C. At least 0.95, but less than 1.45
D. At least 1.45, but less than 1.95
E. At least 1.95

Use the following information for questions 3.22 and 3.23:

You are given the following:

- The random variable X has the density function

$$f(x)=0.5\left(\frac{1}{\theta_1}e^{-x/\theta_1}\right)+0.5\left(\frac{1}{\theta_2}e^{-x/\theta_2}\right),\ 0<x<\infty,\ 0<\theta_1\le\theta_2.$$

- A random sample taken of the random variable X has mean 1 and variance k.

3.22. [**4B-F98:25**] (3 points) If k is 3/2, determine the method of moments estimate of θ_1.

A. Less than 1/5
B. At least 1/5, but less than 2/5
C. At least 2/5, but less than 3/5
D. At least 3/5, but less than 4/5
E. At least 4/5

3.23. [**4B-F98:26**] (2 points) Determine the values of k for which the method of moments estimates of θ_1 and θ_2 exist.

A. $0<k$
B. $0<k<3$
C. $0<k<2$
D. $1\le k$
E. $1\le k<3$

Exercises continue on the next page . . .

3.24. [**4-F03:24**] You are given:

(i) A sample $x_1, x_2, \ldots, x_{10}$ is drawn from a distribution with probability density function:

$$\frac{1}{2}\left(\frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)+\frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right)\right), \quad 0<x<\infty$$

(ii) $\theta > \sigma$

(iii) $\sum x_i = 150$ and $\sum x_i^2 = 5000$

Estimate θ by matching the first two sample moments to the corresponding population quantities.

A. 9 B. 10 C. 15 D. 20 E. 21

3.25. [**4-S01:39**] You are modeling a claim process as a mixture of two independent distributions A and B.

You are given:

(i) Distribution A is exponential with mean 1.

(ii) Distribution B is exponential with mean 10.

(iii) Positive weight p is assigned to distribution A.

(iv) The standard deviation of the mixture is 2.

Determine p using the method of moments.

A. 0.960 B. 0.968 C. 0.972 D. 0.979 E. 0.983

3.26. [**4B-F99:21**] (2 points) You are given the following:

- The random variable X has the density function

$$f(x) = w f_1(x) + (1-w) f_2(x), \quad 0<x<\infty,\ 0\le w\le 1.$$

- A single observation of the random variable X yields the value 1.
- $\int_0^\infty x f_1(x)\mathrm{d}x = 1$
- $\int_0^\infty x f_2(x)\mathrm{d}x = 2$
- $f_2(1) = 2f_1(1) \neq 0$

Determine the method of moments estimate of w.

A. 0 B. 1/3 C. 1/2 D. 2/3 E. 1

3.27. The number of claims follows a mixture of two Poisson distributions with parameters λ_1 and λ_2, each with weight 1/2, with $0<\lambda_1\le\lambda_2$. Over a five year period, the observed numbers of claims are 2, 2, 2, 7, 2.

Estimate λ_1 using the method of moments.

Exercises continue on the next page . . .

3.28. [**4B-S92:26**] (1 point) The random variable X has the density function with parameter β given by

$$f(x;\beta)=\frac{1}{\beta^2}xe^{-\frac{1}{2}(x/\beta)^2},\quad x>0,\beta>0,$$

where $\mathbf{E}[X]=(\beta/2)\sqrt{2\pi}$ and the variance of X is $2\beta^2-(\pi/2)\beta^2$.

You are given the following observations of X: 4.9, 1.8, 3.4, 6.9, 4.0.

Determine the method of moments estimate of β.

A. Less than 3.00
B. At least 3.00, but less than 3.15
C. At least 3.15, but less than 3.30
D. At least 3.30, but less than 3.45
E. At least 3.45

3.29. [**4-F01:33**] You are given

(i) Claim amounts follow a shifted exponential distribution with probability density function:

$$f(x)=\frac{1}{\theta}e^{-(x-\lambda)/\theta},\quad \lambda<x<\infty$$

(ii) A random sample of claim amounts $X_1,X_2,\ldots,X_{10}$:

5 5 5 6 8 9 11 12 16 23

(iii) $\sum X_i=100$ and $\sum X_i^2=1306$.

Estimate λ using the method of moments.

A. 3.0 B. 3.5 C. 4.0 D. 4.5 E. 5.0

Additional released exam questions: CAS3-S06:1 (negative binomial), CAS3-F06:3 (single-parameter Pareto), CAS3-F07:5 (two-parameter Pareto), CAS3L-S09:17 (lognormal), CAS3L-F09:17 (3-point discrete), CAS3L-F10:20 (gamma), CAS3L-S11:19 (single-parameter Pareto)

Solutions

3.1. The sample mean is $\bigl(3(0)+2(1)+2+3+4+7\bigr)/9=\boxed{\mathbf{2}}$, which is the estimate of λ.

3.2. The sample mean is $(0+0+0+1+3)/5=0.8$. The sample raw second moment is $(1^2+3^2)/5=2$. The biased sample variance is therefore $2-0.8^2=1.36$. Matching moments, we have

$$\begin{aligned}r\beta&=0.8\\ r\beta(1+\beta)&=1.36\\ 1+\beta&=\frac{1.36}{0.8}=1.7\\ \beta&=0.7\\ r&=\frac{0.8}{0.7}=\boxed{\mathbf{1.1429}}\end{aligned}$$

3.3. The expected value of X is $\theta + 2\theta + 3(1-2\theta) = 3 - 3\theta$, so

$$\begin{aligned} 3 - 3\tilde{\theta} &= \bar{X} \\ 3\tilde{\theta} &= 3 - \bar{X} \\ \tilde{\theta} &= \boxed{\dfrac{\mathbf{3 - \bar{X}}}{\mathbf{3}}} \end{aligned} \qquad \textbf{(A)}$$

3.4. For an exponential random variable X, $\mathbf{E}[X] = \theta$.

$$\bar{X} = \frac{9000}{7} = \tilde{\theta}$$

$$\Pr(X > 500) = e^{-(7/9000)(500)} = \boxed{\mathbf{0.6778}}$$

3.5. For a single parameter Pareto, $\mathbf{E}[X] = \alpha\theta/(\alpha - 1)$ and $\mathbf{E}[X^2] = \alpha\theta^2/(\alpha - 2)$.

$$\begin{aligned} \bar{x} = 30 &= \frac{\tilde{\alpha}\tilde{\theta}}{\tilde{\alpha} - 1} \\ \mu_2' = 1100 &= \frac{\tilde{\alpha}\tilde{\theta}^2}{\tilde{\alpha} - 2} \\ \frac{11}{9} &= \frac{(\tilde{\alpha}-1)^2}{\tilde{\alpha}(\tilde{\alpha}-2)} \\ 11\tilde{\alpha}^2 - 22\tilde{\alpha} &= 9\tilde{\alpha}^2 - 18\tilde{\alpha} + 9 \\ 2\tilde{\alpha}^2 - 4\tilde{\alpha} - 9 &= 0 \\ \tilde{\alpha} &= \frac{4 + \sqrt{16 + 72}}{4} = \boxed{\mathbf{3.3452}} \end{aligned}$$

3.6. We equate the first two moments.

$$\begin{aligned} \bar{X} = 143.75 &= \tilde{\alpha}\tilde{\theta} \\ \hat{\sigma}^2 = 22898.4375 &= \tilde{\alpha}\tilde{\theta}^2 \\ \tilde{\theta} &= \frac{22898.4375}{143.75} = \boxed{\mathbf{159.2935}} \end{aligned}$$

3.7. We equate the first two moments.

$$\begin{aligned} \bar{X} &= \frac{5850}{6} = 975 = \tilde{\alpha}\tilde{\theta} \\ \hat{\sigma}^2 &= \tfrac{1}{6}\sum (x_i - \bar{x})^2 = 27291\tfrac{2}{3} = \tilde{\alpha}\tilde{\theta}^2 \\ \tilde{\alpha} &= \frac{975^2}{27291\frac{2}{3}} = \boxed{\mathbf{34.8321}} \end{aligned} \qquad \textbf{(B)}$$

3.8. We equate the first two moments.

$$\begin{aligned} \bar{X} = 3800 &= \hat{\alpha}\hat{\theta} \\ \hat{\sigma}^2 = 1{,}892{,}000 &= \hat{\alpha}\hat{\theta}^2 \\ \hat{\theta} &= \frac{1{,}892{,}000}{3800} = \boxed{\mathbf{497.89}} \end{aligned} \qquad \textbf{(E)}$$

3.9. Equating mean and variance,

$$\begin{aligned}
\alpha\theta &= \frac{1}{6}\sum x_i = 60\\
\alpha\theta^2 &= \frac{1}{6}\sum x_i^2 - 60^2 = 3933\tfrac{1}{3} - 3600 = 333\tfrac{1}{3}\\
\hat{\alpha} &= \frac{60^2}{333\frac{1}{3}} = \boxed{\mathbf{10.8}} \qquad \textbf{(D)}
\end{aligned}$$

3.10. This is a gamma distribution with $\alpha = 2$ and $\theta = c$, so $2c = \bar{x} = 8$, $c = \boxed{\mathbf{4}}$ **(E)**

3.11. The sample mean is $(200+300+350+450)/4 = 325$. The sample second moment is

$$\mu_2' = \frac{200^2+300^2+350^2+450^2}{4} = 113{,}750$$

The sample's variance with division by n is $113{,}750 - 325^2 = 8125$. Therefore

$$\begin{aligned}
\tilde{\alpha}\tilde{\theta} &= 325\\
\tilde{\alpha}\tilde{\theta}^2 &= 8125\\
\tilde{\alpha} &= \frac{325^2}{8125} = \boxed{\mathbf{13}} \qquad \textbf{(E)}
\end{aligned}$$

3.12. Even though we derived the method of moments estimator for a Pareto in the lesson, we'll do this from first principles.

$$\begin{aligned}
\bar{X} = 550 &= \frac{\tilde{\theta}}{\tilde{\alpha}-1}\\
\mu_2' = 680000 &= \frac{2\tilde{\theta}^2}{(\tilde{\alpha}-1)(\tilde{\alpha}-2)}\\
\frac{680000}{550^2} &= \frac{2(\tilde{\alpha}-1)}{\tilde{\alpha}-2}\\
680000\tilde{\alpha} - 2(680000) &= 2(550^2)\tilde{\alpha} - 2(550^2)\\
\tilde{\alpha} &= \frac{755000}{75000} = 10.0667\\
\tilde{\theta} &= 550(\tilde{\alpha}-1) = 4986.67\\
\Pr(X > 500) &= \left(\frac{4986.67}{5486.67}\right)^{10.0667} = \boxed{\mathbf{0.3822}}
\end{aligned}$$

3.13. For a single parameter Pareto, $\mathbf{E}[X] = \alpha\theta/(\alpha-1)$ and $\mathbf{E}[X^2] = \alpha\theta^2/(\alpha-2)$.

$$\begin{aligned}
\bar{x} = 30 &= \frac{\tilde{\alpha}\tilde{\theta}}{\tilde{\alpha}-1}\\
\mu_2' = 1100 &= \frac{\tilde{\alpha}\tilde{\theta}^2}{\tilde{\alpha}-2}\\
\frac{11}{9} &= \frac{(\tilde{\alpha}-1)^2}{\tilde{\alpha}(\tilde{\alpha}-2)}\\
11\tilde{\alpha}^2 - 22\tilde{\alpha} &= 9\tilde{\alpha}^2 - 18\tilde{\alpha} + 9
\end{aligned}$$

$$2\tilde{\alpha}^2 - 4\tilde{\alpha} - 9 = 0$$
$$\tilde{\alpha} = \frac{4+\sqrt{16+72}}{4} = \boxed{\mathbf{3.3452}}$$

3.14. Pareto is always two-parameter unless stated otherwise.
Only one parameter is estimated, so only one moment is matched: the mean.

$$\frac{\tilde{\theta}}{2} = \frac{1}{300}\sum_{i=1}^{300} X_i$$
$$\tilde{\theta} = \frac{1}{150}\sum_{i=1}^{300} X_i$$

The variance of a sum of independent variables is the sum of the variances, and the variance of kY is k^2 times the variance of Y, so

$$\mathrm{Var}(\tilde{\theta}) = \frac{1}{150^2}\sum \mathrm{Var}(X_i) = \frac{300}{150^2}\mathrm{Var}(X_i)$$

The variance of X_i, which follows a two-parameter Pareto, is the second moment minus the first moment squared.

$$\mathrm{Var}(X_i) = \frac{2\theta^2}{2(1)} - \frac{\theta^2}{4} = \frac{3}{4}\theta^2$$
$$\left(\frac{300}{150^2}\right)\left(\frac{3}{4}\right)\theta^2 = \boxed{\mathbf{0.01}\boldsymbol{\theta}^{\mathbf{2}}} \qquad \textbf{(D)}$$

3.15. The distribution is a single parameter Pareto with $\theta = 1$. We set the mean equal to the sample mean:

$$\mathbf{E}[X] = \alpha/(\alpha-1) = \bar{X}$$

As $\bar{X} \to \infty$,

$$\frac{\tilde{\alpha}}{\tilde{\alpha}-1} = 1 + \frac{1}{\tilde{\alpha}-1} \to \infty$$
$$\frac{1}{\tilde{\alpha}-1} \to \infty$$
$$\tilde{\alpha} - 1 \to 0$$
$$\tilde{\alpha} \to \boxed{\mathbf{1}} \qquad \textbf{(C)}$$

3.16. We recognize this as a two-parameter Pareto with $\theta = 1$ and other parameter α. Therefore,

$$\mathbf{E}[X] = \frac{\theta}{\alpha-1} = \frac{1}{\alpha-1} = \bar{X}$$
$$\tilde{\alpha} = 1 + \frac{1}{\bar{X}} = \boxed{\frac{\bar{X}+1}{\bar{X}}} \qquad \textbf{(E)}$$

3.17. We equate the first and second moments.

$$\frac{\theta}{\alpha-1} = \frac{4+5+21+99+421}{5} = 110$$
$$\frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{4^2+5^2+21^2+99^2+421^2}{5} = 37{,}504.8$$

Dividing the second expression by the square of the first

$$\begin{aligned}
\frac{2(\alpha-1)}{\alpha-2} &= 3.09957\\
2+\frac{2}{\alpha-2} &= 3.09957\\
\frac{2}{\alpha-2} &= 1.09957\\
\alpha &= \frac{2}{1.09957}+2 = 3.81889\\
\theta &= 110(\alpha-1) = 310.0782
\end{aligned}$$

To find the 95th percentile, we equate $F(x)=0.95$, or $1-F(x)=0.05$.

$$\begin{aligned}
\left(\frac{\theta}{\theta+x}\right)^\alpha &= 0.05\\
\frac{\theta}{\theta+x} &= \sqrt[\alpha]{0.05}\\
(\theta+x)\sqrt[\alpha]{0.05} &= \theta\\
x &= \frac{\theta(1-\sqrt[\alpha]{0.05})}{\sqrt[\alpha]{0.05}}\\
&= \frac{310.0782(1-0.45637)}{0.45637} = \boxed{\mathbf{369.3659}} \qquad \textbf{(A)}
\end{aligned}$$

3.18. The sample mean is 2600, and the sample raw second moment is $\left(\sum_{i=1}^5 x_i^2\right)\big/5 = 17,676,000$. Matching moments:

$$\begin{aligned}
\mu+0.5\sigma^2 &= \ln 2600 = 7.863\\
2\mu+2\sigma^2 &= \ln 17{,}676{,}000 = 16.688
\end{aligned}$$

Subtract the second equation from 4 times the first.

$$\begin{aligned}
2\mu &= 4(7.863)-16.688 = 14.765\\
\tilde{\mu} &= \boxed{\mathbf{7.383}}
\end{aligned}$$

3.19. If you recognize this as a beta distribution with $a=1/\theta$ and $b=1$, then you know the mean is

$$\frac{a}{a+b} = \frac{1/\theta}{1+1/\theta}$$

Otherwise, you would integrate:

$$\begin{aligned}
\mathbf{E}[X] &= \int_0^1 \frac{1}{\theta} x^{(1-\theta)/\theta+1} \mathrm{d}x \\
&= \int_0^1 \frac{1}{\theta} x^{1/\theta} \mathrm{d}x \\
&= \frac{1}{\theta} \left.\frac{x^{1+1/\theta}}{1+1/\theta}\right|_0^1 \\
&= \frac{1/\theta}{1+1/\theta} = \frac{1}{\theta+1}
\end{aligned}$$

Equating this to the sample mean $\bar{X}$,

$$\begin{aligned}
\bar{X} &= \frac{1}{\tilde{\theta}+1} \\
\tilde{\theta}+1 &= \frac{1}{\bar{X}} \\
\tilde{\theta} &= \frac{1}{\bar{X}} - 1 = \boxed{\frac{\mathbf{1}-\bar{\boldsymbol{X}}}{\bar{\boldsymbol{X}}}} \qquad \textbf{(A)}
\end{aligned}$$

3.20. By now, you know that the mean is $\alpha/(\alpha+1)$. (See Example 3E.)

$$\begin{aligned}
\mathbf{E}[X] &= \frac{\alpha}{\alpha+1} = \frac{2}{3} \\
\tilde{\alpha} &= \boxed{\mathbf{2}} \qquad \textbf{(C)}
\end{aligned}$$

3.21. The distribution is a beta with $a=1$, $b=2$, so the mean is $\theta/3$. If you didn't recognize the distribution, you could calculate the mean by integrating:

$$\begin{aligned}
\mathbf{E}[X] &= \frac{2}{\theta^2} \int_0^\theta x(\theta-x)\mathrm{d}x \\
&= \frac{2\theta}{\theta^2}\frac{\theta^2}{2} - \frac{2}{\theta^2}\frac{\theta^3}{3} = \frac{\theta}{3}
\end{aligned}$$

Setting it equal to 0.7

$$\begin{aligned}
\frac{\theta}{3} &= 0.7 \\
\tilde{\theta} &= 3(0.7) = \boxed{\mathbf{2.1}} \qquad \textbf{(E)}
\end{aligned}$$

3.22. We equate the first two moments.

$$\begin{aligned}
\mathbf{E}(X) &= 0.5\left(\frac{1}{\theta_1}\right)\int_0^\infty x e^{-x/\theta_1}\mathrm{d}x + 0.5\left(\frac{1}{\theta_2}\right)\int_0^\infty x e^{-x/\theta_2}\mathrm{d}x = 0.5(\theta_1+\theta_2) \\
\mathbf{E}(X^2) &= 0.5(2\theta_1^2 + 2\theta_2^2)
\end{aligned}$$

Equating the means,

$$\theta_1 + \theta_2 = 2 \Rightarrow \theta_2 = 2 - \theta_1$$

Equating the second moments,

$$0.5(2\theta_1^2+2\theta_2^2)=\frac{3}{2}+1^2=\frac{5}{2}$$
$$\theta_1^2+\theta_2^2=2.5$$

Substituting $\theta_2=2-\theta_1$,

$$\theta_1^2+4-4\theta_1+\theta_1^2=2.5$$
$$2\theta_1^2-4\theta_1+1.5=0$$
$$\theta_1=\frac{4-\sqrt{16-12}}{4}=\frac{2}{4}=\boxed{\frac{1}{2}}\qquad \textbf{(C)}$$

3.23. Generalizing the quadratic of the previous question:

$$2\theta_1^2-4\theta_1+(3-k)=0$$

The discriminant must be at least 0 to obtain real solutions, but less than 16, since both solutions should be positive.

$$\begin{aligned}16-4(2)(3-k)&\ge 0 \qquad & 16-4(2)(3-k)&<16\\ 16-24+8k&\ge 0 & 16-24+8k&<16\\ k&\ge 1 & k&<3\end{aligned}$$
$$\boxed{1\le k<3}\qquad \textbf{(E)}$$

3.24. This distribution is a mixture of two exponentials with means θ and σ. We equate the first two moments.

$$\frac{1}{2}(\theta+\sigma)=\frac{150}{10}=15$$
$$\frac{1}{2}(2\theta^2+2\sigma^2)=\frac{5000}{10}=500$$

From the first equation, we substitute $\sigma=30-\theta$.

$$\theta^2+(30-\theta)^2=500$$
$$\theta^2-30\theta+200=0$$
$$(\theta-10)(\theta-20)=0$$
$$\theta=10,20$$

But since $\theta>\sigma$, θ must be $\boxed{20}$. **(D)**

3.25. You are only given the standard deviation, so you have to match that to the fitted standard deviation, or equivalently, you have to match the variances, rather than the first moments. To calculate the variance of the mixture, you calculate the first two moments, and then use $\text{Var}(X)=\mathbf{E}[X^2]-\mathbf{E}[X]^2$. The first moment is

$$\mathbf{E}(X)=p(1)+(1-p)(10)=10-9p$$

The second moment is the weighted average of the second moments of the exponentials. For an exponential, the second moment is $2\theta^2$, where θ is the mean.

$$\mathbf{E}(X^2)=p(2)+(1-p)(200)=200-198p$$
$$\text{Var}(X)=(200-198p)-(10-9p)^2=-81p^2-18p+100$$

We set this equal to the square of the standard deviation, 4.

$$-81p^2 - 18p + 100 = 4$$
$$-81p^2 - 18p + 96 = 0$$
$$p = \frac{18 - \sqrt{18^2 + 4(96)(81)}}{2(-81)} = \boxed{\mathbf{0.9832}} \qquad \textbf{(E)}$$

3.26. The mean of a mixture is the weighted average of the means, which are 1 and 2, so

$$\mathbf{E}[X] = w + 2(1-w) = 2 - w = 1$$
$$w = \boxed{\mathbf{1}} \qquad \textbf{(E)}$$

3.27. Let X be the random variable. The mean and variance of a Poisson with parameter λ are λ, so the second moment is $\sigma^2 + \mu^2 = \lambda + \lambda^2$. The moments of a mixture are the weighted averages of the moments of the mixed distributions, so

$$\mathbf{E}[X] = 0.5(\lambda_1 + \lambda_2) = \bar{x} = 3$$
$$\mathbf{E}[X^2] = 0.5\left(\lambda_1 + \lambda_1^2 + \lambda_2 + \lambda_2^2\right) = \frac{\sum x_i^2}{5} = 13$$

From the first equation, $\lambda_2 = 6 - \lambda_1$. Plugging this into the second equation,

$$\lambda_1 + \lambda_1^2 + (6-\lambda_1) + (6-\lambda_1)^2 = 26$$
$$\lambda_1 + \lambda_1^2 + 6 - \lambda_1 + 36 - 12\lambda_1 + \lambda_1^2 = 26$$
$$2\lambda_1^2 - 12\lambda_1 + 16 = 0$$
$$\lambda_1^2 - 6\lambda_1 + 8 = 0$$
$$\lambda_1 = \frac{6 \pm \sqrt{36-32}}{2} = 2, 4$$

Since λ_1 is the lower value, $\lambda_1 = \boxed{\mathbf{2}}$ and $\lambda_2 = 4$.

3.28. Since $\bar{X} = (4.9 + 1.8 + 3.4 + 6.9 + 4.0)/5 = 4.2$, it follows that

$$\hat{\beta} = \frac{2\bar{X}}{\sqrt{2\pi}} = \frac{2(4.2)}{\sqrt{2\pi}} = \boxed{\mathbf{3.3511}} \qquad \textbf{(D)}$$

Note that the sample variance is not needed since only one parameter is being estimated.

3.29. The distribution is an exponential with mean θ that is shifted to the right by λ. Shifting a distribution by λ adds λ to the mean, but has no effect on the variance. We will equate the mean and the variance (rather than the second moment) so that we can take advantage of this.

$$\mathbf{E}[X] = \theta + \lambda = \frac{100}{10} = 10$$
$$\mathrm{Var}(X) = \theta^2 = \frac{1306}{10} - 10^2 = 30.6$$
$$\theta = \sqrt{30.6} = 5.5317$$
$$\lambda = 10 - 5.5317 = \boxed{\mathbf{4.4683}} \qquad \textbf{(D)}$$

Quiz Solutions

3-1. The sample mean is 0.27. The distribution is beta with $a=2$, $b=\alpha$, so its mean is $2/(2+\alpha)$. Therefore

$$\frac{2}{2+\alpha}=0.27$$
$$\alpha=\frac{2-0.54}{0.27}=\boxed{\mathbf{5.4074}}$$