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Preface

Starting in May 2001, the Enrolled Actuaries Examinations, administered by the Joint Board for the Enrollment of Actuaries, were offered in a restructured form. The former series of three exams, denoted EA-1A, EA-1B, and EA-2, has now been restructured into a new series of three exams denoted EA-1, EA-2A, and EA-2B. This ACTEX Study Manual has been prepared to assist the student's preparation for the EA-1 exam to be given in May 2006.

The subject matter of the EA-1 exam can be organized into two broad groups, as described in the Joint Board's Examination Program. The first group, denoted "mathematics of compound interest and financial analysis" in the Examination Program, does not require a knowledge of probability or statistics. The second group, denoted "mathematics of life contingencies and demographic analysis" in the Examination Program, does require a familiarity with basic probably and statistics. Accordingly, this study guide is organized first into these two major sections, with each section split into two subsections, as follows:

Section	Topic
I-A	Compound Interest
I-B	Financial Analysis
II-A	Life Contingencies
II-B	Demographic Analysis

Then all of the EA-1 exam topics are organizing into units within each of these four sections, as shown in the Table of Contents. Each section begins with an Introductory Note describing the contents of each unit in that section.

The reader should understand that the anticipated content of the May 2006 EA-1 exam, upon which this ACTEX Study Manual is based, is based on the most recently published Joint Board Examination Program. In addition, in selecting the contents of this manual we have been guided by the nature of the May 2001, May 2002, May 2003, May 2004, and May 2005 EA-1 exams. These five actual exams, with complete solutions, are presented in Section III of this manual. They should be viewed by the candidate as the best information available regarding what might be expected on the upcoming exam.

We hope this ACTEX Study Manual will be of great value to you as you prepare for the EA-1 exam. We would appreciate any feedback that you care to give us concerning the manual, including any errata that you find and any other suggestions for its improvement.

Storrs, Connecticut
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Unit II-A8
Stationary Population Theory

Overview Commentary

Stationary population theory is an interesting application of the life table model. By assuming (1) that the radix of the life table, denoted l_0, represents the *constant* annual number of births into a population, and (2) that the population exactly experiences the mortality rates given in the life table *without change over time*, then it can be shown that the population will reach a *stationary condition* with many interesting properties.

Parmenter's treatment of stationary population theory (see Sections 7.4 and 7.5) is well-written, but somewhat brief. Jordan gives a fuller presentation of the topic, devoting a full chapter (Chapter 8) to it. The best presentation of stationary population is given by Brown in his Chapter 5. Cunningham's presentation in his Section 14.2 essentially duplicates that of Brown. In any case, students have always found stationary population theory to be more conceptually difficult to understand than most other topics in life contingencies. In light of this, we suggest that you might want to study the topic from several texts, for the same reason we suggested this with regard to the life table itself in Unit II-A1. (All four of Brown, Jordan, Parmenter, and Cunningham have useful end-of-chapter exercises on stationary population.)

In addition, the following summary outline of Brown's Chapter 5 may prove to be useful study material.

5.1 INTRODUCTION

A. A Life Table, a Survivorship Group, and a Stationary Population

1. A life table shows the survivorship pattern of a hypothetical birth cohort of size 0; it is an abstract mathematical model
    a. Important features of the life table model
        (1) The cohort of l_0 births is a closed group; no additional persons join the group at later ages
        (2) Persons leave the group only by death; i.e., it is a single-decrement model
    b. The l_0 births did not "occur" in any particular time interval; in fact, they did not actually occur at all, since the model is abstract
2. A survivorship group which begins with l_0 births, and experiences the mortality rates of a particular life table, is an actual group of people whose properties are identical to those of the hypothetical group described by the life table; some properties of the survivorship group are explored in Section 5.2
3. If a population has l_0 births each year, uniformly spread over the year, and experiences the mortality rates of a particular life table without change over time, then that population is said to be stationary; a stationary population has many nice properties, as described in Sections 5.3-5.5

B. Population Analysis

1. Usually assumes no migration in mathematical analysis; migration is handled informally
2. Requires assumed mortality and fertility profiles
3. Examples of population analysis
    a. Survivorship group (Section 5.2)
        (1) Closed group, so births (i.e., fertility) is not a factor
        (2) The analysis depends on mortality only
    b. Stationary population (Sections 5.3-5.5)
        (1) Assumes constant birth rate over time, so no fertility model is needed
        (2) Mortality is constant over time
    c. Stable population (Sections 6.1-6.4)
        (1)Birth rate varies over time
        (2)Mortality is constant over time
    d. Quasi-stable population (Section 6.5)

(1) Birth rate varies over time

(2) Mortality varies over time

5.2 ANALYSIS OF THE SURVIVORSHIP GROUP

A. Preliminary Assumptions

1. Assumed mortality profile (i.e., life table) applies deterministically; the group is assumed to follow the precise survival/mortality pattern given by the table
2. Assume the group has B = l_0 births, where l_0 is precisely the radix of the table; if B $\neq l_0$, then all values for the group are found by proportionalizing the table values by $\frac{B}{l_0}$
3. Assume the group's births occur uniformly over a one-year period, and the same mortality profile applies to all births
4. The pair of assumptions in item 3 implies that the x persons reaching age *x* do so uniformly over a year

B. Aggregate and Expected Future Lifetimes

1. For the *x* persons in the group who survive to age x, the *aggregate* future lifetime (i.e., total number of life-years lived beyond age x) is given by $T_x = \int_0^\infty l_{x+t}dt$
2. Then the average, or expected, future lifetime is $\frac{T_x}{l_x}$, denoted by $\mathring{e}_x$, and called the complete expectation of life at age *x*
3. Then the average, or expected, age at death would be the attained age *x* plus the average future lifetime, or $x + \mathring{e}_x = \frac{x \cdot l_x + T_x}{l_x}$
4. Special cases of the results in items 1-3 for the l_0 newborns
   a. Total (aggregate) future lifetime: T_0
   b. Average (expected) future lifetime: $\mathring{e}_0 = \frac{T_0}{l_0}$
   c. Average (expected) age at death: $0 + \mathring{e}_x = \frac{T_0}{l_0}$
5. Curtate expectation of life (e_x)
   a. Average (expected) number of *whole years* of future lifetime for the l_x group

b. Approximately related to complete expectation by $\overset{\circ}{e}_x = e_x + \frac{1}{2}$ (Example 5.1)

C. Additional Results from Text Examples

1. Average age at death for the l_x group derived by integration (Example 5.2)
2. Average age at death for those dying between ages *x* and *x+n* (Example 5.3)

D. Dual Meanings of L_x (see Figure 5.1)

1. Life-years lived between ages *x* and *x+1*
2. Number of people in the group when they are all between ages *x* and *x+1*

5.3 THE STATIONARY POPULATION

A. Preliminary Assumptions

1. Simple extension of Section 5.2; we now have 0 births every year, uniformly distributed, with the same deterministic mortality profile applying to all births
2. Then the population has identical characteristics over time; we say the population is stationary
   a. Same age composition over time
   b. Same total size of population over time
   c. Same number of total deaths as there are births in any interval of time, however small
      (1) Thus the stationary property holds in a continuous fashion; each death is simultaneously replaced by a birth
      (2) This was immortalized in the ’60’s song by the group Blood, Sweat and Tears: “And when I’m dead, and when I’m gone, there’ll be one child born in the world to carry on.”

B. Properties of a Stationary Population (with l_0 births each year)

1. l_0 deaths each year
2. l_x persons turning age x each year
3. l_x persons dying at ages x and over each year

4. L_x persons alive at age x last birthday at any point in time

5. $T_x = L_x + L_{x+1} + . . .$ persons alive at ages x and over at any point in time

C. Dual Meanings of L_x and T_x

1. L_x

   a. Life-years lived between ages *x* and *x+1* (i.e., life-years lived by the l_x people within the next year)

   b. Total number of people who are age *x l*ast birthday at any point of time

2. T_x

   a. Total (aggregate) future lifetime of the l_x group (i.e., total life-years lived beyond age x)

   b. Total number of people aged x and over in the population at any point of time

3. Note that the meanings of L_x and T_x, given by statements 1.a and 2.a, respectively, hold for a survivorship group as well; the meanings given by statements 1.b and 2.b apply only in a stationary population

4. Thus we see that a stationary population is an extension of the survivorship group; it is the aggregation of many survivorship groups defined by many consecutive annual birth cohorts, each of size l_0

D. The Function of Y_x

1. Y_x is defined as $\int_0^\infty T_{x+t}dt = \int_x^\infty T_y dy$

2. Meanings

   a. Total *future* lifetime of the T_x people in the population aged x and over

   b. Total *past* lifetime since age x of the same T_x people (see Example 5.6)

3. Then we can evaluate the following concepts for the T_x people

   a. Total lifetime up to age *x*: $x \cdot T_x$

   b. Total lifetime from age x to respective present ages: Y_x

   c. Total past lifetime: $x \cdot T_x + Y_x$

   d. Average past lifetime (i.e., average attained age): $\dfrac{x \cdot T_x + Y_x}{T_x} = x + \dfrac{Y_x}{T_x}$

e. Total future lifetime: Y_x

f. Average future lifetime: $\frac{Y_x}{T_x}$

g. Total lifetime: $x \cdot T_x + 2Y_x$

h. Average total lifetime (i.e., average age at death): $x + \frac{2Y_x}{T_x}$

E. Proportionalizing Aggregate Results

1. Aggregate measurements, such as T_x persons aged *x* and over or Y_x as their total future lifetime, assume that the population is fed by l_x persons turning age *x* each year
2. If the entrant group at age *x* is *E*, rather than l_x, then all aggregate measurements must be proportionalized, by multiplying by $\frac{E}{l_x}$
3. Example: an army with 100,000 annual inductees at age 18, serving to age 24
   a. If there had been l_{18} inductees, then the size of the army would be $T_{18} - T_{24}$
   b. In this case the size is $\frac{100,000}{l_{18}}(T_{18} - T_{24})$
4. Special case for a population fed by *B* births, rather than l_0 births; all aggregate measurements are multiplied by $\frac{B}{l_0}$
5. *Average* measurements, such as average age at death, would have the proportionalizing multiple in both numerator and denominator, and so would disappear
6. The proportionalizing feature is first encountered in Example 5.5, and later in other examples and exercises

F. Stationary Population with Partial Withdrawals

1. A portion of the group reaching a certain age can leave the population, and it will still be stationary
2. For example, if 25% of those reaching age 40 leave the population, then the over-40 population is fed by $.75l_{40}$ each year, instead of l_{40}
3. Then instead of the over-40 population being of size T_{40}, it will be of size $.75T_{40}$
4. All other aggregate measurements of the over-40 population will be multiplied by .75
5. All average measurements, such as average age at death, would be multiplied by .75 in both the numerator and denominator, so it would disappear

6. Populations with partial withdrawal are very common; they are illustrated in Example 5.5 and in many of the exercises

G. Other Results Illustrated in Text Examples

1. Total past lifetime (TPL) of those now between age x and $x+n$ (Example 5.7)
2. TPL since age y for those now between ages x and $x+n$, where $y < x$ (Example 5.8)
3. Average attained age (AAA) of those now between ages x and $x+n$ who will make it to age $x+m$, where $m > n$ (Example 5.9)
4. Average age at death (AAD) of those now between ages x and $x+n$ who will die before age $x+m$, where $m > n$ (Example 5.10)
5. AAD of those now between ages x and $x+n$ who will die between ages $x+r$ and $x+t$, where $t > r > n$ (Example 5.11)
6. Various results for those now between ages x and xn who will die between ages $x+r$ and $x+t$, and within a certain time interval as well (Examples 5.12 and 5.13)

5.4 THE LEXIS DIAGRAM

A. Two-Dimensional Representation of the Population

1. Applied to population problems by Veit
2. Time increases horizontally and age increases vertically; persons flow across the diagram at 45° angle to the southeast (see Figures 5.1– 5.3)
3. Then a vertical slice down the diagram is a point of time, and a horizontal slice across the diagram is a fixed age

B. Solution of Problems

1. Each problem relates to a certain region, defined by age and time parameters
2. Persons flow into or out of the region across vertical or horizontal boundaries
3. If the problem involves a group of deaths, the number can be found as those into the region, minus those out of it
4. Example 5.11R illustrates this very clearly
5. Total lifetime (TL) of a group of persons can easily be found by the Grace-Nesbitt technique
   a. TL of an l_x group is $x \cdot l_x + T_x$

b. TL of a T_x group is $x \cdot T_x + 2Y_x$

c. TL of a group that is a linear combination of l_x and T_x symbols will be the same linear combination of the ($x \cdot l_x + T_x$) and ($x \cdot T_x + 2Y_x$) aggregates (see Examples 5.11R, 5.12R and 5.13R)

6. Total *past* lifetime (total of attained ages) of a described group
   a. The Grace-Nesbitt factors are only for *total lifetime*, not for total past lifetime or total future lifetime separately
   b. These problems can be handled by integrals, or by the reasoning approach illustrated in Example 5.14

5.5 FURTHER APPLICATIONS

A. Financial Applications

1. If a population is stationary (or nearly so), this theory can be used to model various financial schemes
2. These generally do not consider interest

B. Examples

1. Social insurance plans for death benefits or annuity payments (Example 5.15)
2. Salaries, death benefits, or discharge bonuses paid to members of a military group (Example 5.16)
3. Pension plans (Example 5.17)
4. Tax revenue supporting benefit payments in a social insurance program (Example 5.18)
5. Personal allowances to persons in an institution (Example 5.19)

SAMPLE QUESTIONS

294. Actuarial model Stationary population

Actuarial assumptions

New entrants: 500 each year, all age 21

Terminations: 10% of population withdraws at age 26; none at other ages

Mortality table functions

x	l_x	L_x	T_x
20	95,908	95,831	4,819,585
21	95,755	95,674	4,723,754
25	95,106	95,032	4,342,042
26	94,957	94,887	4,247,010
64	67,970	66,902	920,925
65	65,834	64,718	854,023
66	63,603	62,441	789,305

In what range is the number of persons age 65 and over?

(A) Less than 3,865
(B) 3,865 but less than 4,015
(C) 4,015 but less than 4,165
(D) 4,165 but less than 4,315
(E) 4,315 or more

295. Characteristics of a stationary population of 500,000 members of a welfare plan:

Number of members age 65 or older: Twice the number of members under age 25

Number of members reaching age 25 each year: 7000

Complete life expectancy at age 25: 60 years

Annual contributions and benefits per member:

Age	Annual Contribution	Annual Benefit
Less than 25	0	K
25 but less than 65	1000	0
65 or older	0	$2K$

The fund balance is always 0.

In what range is the annual benefit for each member age 65 or older?

A. Less than 600
B. 600 but less than 900
C. 900 but less than 1200
D. 1200 but less than 1500
E. 1500 or more

296. Characteristics of a stationary population of 100 active members:

Entry age into the population: 25
Preretirement terminations other than deaths: None
Retirement age from the population: 65
Number of deaths per year: 2
Average age at death: 55

In what range is the number of new entrants each year?

A. Less than 2.5
B. 2.5 but less than 3.5
C. 3.5 but less than 4.5
D. 4.5 but less than 5.5
E. 5.5 or more

297. Prior to 1/1/70, a stationary population of 600,000 was maintained by 12,500 births each year. 35% of the total population was under age 20.

On 1/1/70, the annual birth rate increased to 13,000 and remained constant at that level thereafter. The mortality rates at all ages were unchanged.

In what range was the total population as of 1/1/90?

(A) Less than 605,000
(B) 605,000 but less than 607,000
(C) 607,000 but less than 609,000
(D) 609,000 but less than 611,000
(E) 611,000 or more

298. *N* is the number of members in a stationary population attaining age 55 or age 56 in a calendar year who will die before attaining age 57.
Selected values:

$l_{55} = 100{,}000$

$e_{55} = 22.245$

$e_{56} = 21.447$

$e_{57} = 20.661$

In what range is *N?*

(A) Less than 1600
(B) 1600 but less than 2200
(C) 2200 but less than 2800
(D) 2800 but less than 3400
(E) 3400 or more

299. Before 1966, a stationary population of 800,000 was maintained by annual births of 15,000. One-half of the total population was under age 30.

After 1965, annual births increased to 16,000. However, mortality rates did not change at any age.

In what range is the total population as of 1/1/96?

(A) Less than 810,000
(B) 810,000 but less than 820,000
(C) 820,000 but less than 830,000
(D) 830,000 but less than 840,000
(E) 840,000 or more

300. A college maintains a stationary population of 15,000 students by annual admissions at ages 18 and 19.
Selected values:

x	l_x	L_x	T_x
18	100,000	97,917	266,668
19	93,750	85,417	168,751
20	75,000	60,417	83,334
21	56,250	22,917	22,917
22	0	0	0

Annual admissions at age 18: 4,000

In what range is the number of annual admissions at age 19?

(A) Less than 2250
(B) 2250 but less than 2500
(C) 2500 but less than 2750
(D) 2750 but less than 3000
(E) 3000 or more

SOLUTIONS TO SAMPLE QUESTIONS

S-294. If l_{21} entered at age 21 then 26 would survive to age 26 and after the withdrawals $.90l_{26}$ would move on so that $.90l_{65}$ would reach age 65 every year. When the population is stationary $.90T_{65}$ is the number of persons age 65 and over. But this follows from l_{21} entrants. Since there are 500 entrants the population over age 65 is a fraction of $.90T_{65}$ in proportion of actual entrants to l_{21}. Thus our answer is

$$\frac{500}{l_{21}}(.90T_{65}) = \frac{(500)(.90)(854,023)}{95,755} = 4013.48, \qquad \text{ANSWER B.}$$

S-295. The fund balance is zero because the contribution equals the benefit. Recall that $T_x - T_y$ represents the number of people between ages *x* and *y*. Then we see that the annual contribution is $1000(T_{25} - T_{65})$, and the annual benefit is $K(T_0 - T_{25}) + 2K(T_{65})..$ Since these are equal we have $K = \frac{1000(T_{25} - T_{65})}{T_0 - T_{25} + 2 \cdot T_{65}}$. The given information tells us the following:

$T_0 = 500,000$

$T_{65} = 2(T_0 - T_{25})$

$\overset{\circ}{e}_{25} = \frac{T_{25}}{l_{25}} = \frac{T_{25}}{7000} = 60$, so $T_{25} = 420,000$.

From these we can find $T_{65} = 2(500,000 - 420,000) = 160,000.$

Finally, $2K = \frac{2000(420,000 - 160,000)}{500,000 - 420,000 + 2(160,000)} = 1300.$ ANSWER D.

S-296. Since all persons die between ages 25 and 65, the average age at death is 25 plus the average number of years lived beyond age 25. The aggregate number of years lived beyond age 25, by those who die before age 65, is given by $T_{25} - T_{65} - 40l_{65}$, and the number of deaths is 2, so the average age at death is

$$25 + \frac{T_{25} - T_{65} - 40l_{65}}{2} = 55,$$

from which we find $T_{25} - T_{65} - 40l_{65} = 60.$

The number of people in the population is given by $T_{25} - T_{65} = 100,$ which shows us that $l_{65} = 1$ is the annual number of retirements.

Finally, since deaths equals entrants minus retirements, and there are 2 deaths, it follows that the number of annual entrants is 3, ANSWER B

S-297. On 1/1/90 the population under age 20 comes from post-1/1/70 births, and the population over age 20 comes from pre-1/1/70 births.

The total population prior to 1/1/70 was $600,000 = \frac{12,500}{l_0} \cdot T_0$, where l_0 and T_0 are taken from the applicable life table. Thus we have $\frac{T_0}{l_0} = 48$. We also know that $T_0 - T_{20} = .35T_0$, so $T_{20} = .65T_0$. The under-20 population on 1/1/90 is $\frac{13,000}{l_0}(T_0 - T_{20})$ and the over-20 population is $\frac{12,500}{l_0} \cdot T_{20}$, for a total of

$$13,000\left(\frac{T_0}{l_0}\right) - 500\left(\frac{T_{20}}{l_0}\right) = 13,000\left(\frac{T_0}{l_0}\right) - 500\left(\frac{.65T_0}{l_0}\right)$$

$$= 12,675\left(\frac{T_0}{l_0}\right)$$

= (12,675)(48) = 608,400, ANSWER C.

S-298. The number attaining age 55 or 56 in a year who will die before age 57 is $N = l_{55} + l_{56} - 2 \cdot l_{57}$. To obtain l_{56} and l_{57} we note that $e_{55} = \frac{1}{l_{55}}(l_{56} + l_{57} + ...)$,
so that $l_{55} \cdot e_{55} = l_{56} + l_{57} + ... = 2,224,500$.
Similarly, $l_{56} \cdot e_{56} = l_{57} + ...$
Subtracting the two equations we find $l_{56} = 2,224,500 - l_{56} \cdot e_{56}$, from which we find $l_{56} = \frac{2,224,500}{1 + e_{56}} = 99,100$. In similar manner we find $l_{57} = \frac{l_{56} \cdot e_{56}}{1 + e_{57}} = 98,121$.
Finally we have $N = 100,000 + 99,100 - 2(98,121) = 2858$, ANSWER D.

S-299. As of 1/1/66, this population had parameters l_0 = 15,000 (annual births), T_0 = 800,000 (total population), and T_{30} = 400,000 (population over age 30). The annual birth rate changed on 1/1/66. Therefore, as of 1/1/96, the under-30 population comes from post-1/1/66 births and the over-30 population comes from pre-1/1/66 births, unaffected by the change in the birth rate. Thus the over-30 population on 1/1/96 continues to be T_{30} = 400,000. The under-30 population would be what it would have been without the birth rate change (i.e., 400,000), proportionalized by the birth rate change. Thus the total 1/1/96 population is

$$400,000+400,000\left(\frac{16,000}{15,000}\right)=826,667, \qquad \text{ANSWER C.}$$

S-300. The population at age 18 last birthday only would be L_{18} if there were l_{18} annual entrants at age 18. Since there were only E_{18} = 4000 instead, then the age 18 last birthday population is $\frac{4000}{l_{18}}\cdot L_{18}=3916.68.$ The population over age 19 would be T_{19} from l_{18} entrants age age 18, plus another T_{19} from l_{19} additional entrants at age 19. Proportionalizing to the actual number of entrants, we find the over-19 population to be $T_{19}\left(\frac{4000}{l_{18}}+\frac{E_{19}}{l_{19}}\right)=6750.04+1.8E_{19}.$ Then the total population is

$$3916.68+6750.04+1.8E_{19}=15,000$$

from which we find $E_{19}=2407.38,$ ANSWER B.

MAY 2002 EA-1 EXAMINATION

1. (3 Points)

Loan repayment period: 5 years
Beginning loan amount: $75,000
Repayment Plan #1: Level annual payments at the beginning of each year.
Repayment Plan #2: Level semi-annual payments at the end of each 6-month period.

A = Annual payment under Repayment Plan #1.
B = Total payments in a year under Repayment Plan #2.
$1000d^{(4)} = 76.225$.

In what range is the absolute value of $(A{-}B)$?

(A) Less than $1,000

(B) $1,000 but less than $1,025

(C) $1,025 but less than $1,050

(D) $1,050 but less than $1,075

(E) $1,075 or more

2. (5 Points)

Annual payments into a fund: $10,000 at the end of year one, increasing by $500 per year in the second through the tenth years. After the tenth year, each payment increases by 3.5% over the prior payment.

Interest Rate: 7% compounded annually

In what range is the accumulated value of the fund at the end of 20 years?

(A) Less than $500,000

(B) $500,000 but less than $550,000

(C) $550,000 but less than $600,000

(D) $600,000 but less than $650,000

(E) $650,000 or more

3. (3 Points)

Given values: $\ddot{s}^{(m)}_{\overline{2n}|} = 180.24943$

$d^{(m)} = 0.08$

In what range is $\ddot{s}^{(m)}_{\overline{4n}|}$?

(A) Less than 2,930

(B) 2,930 but less than 2,970

(C) 2,970 but less than 3,010

(D) 3,010 but less than 3,050

(E) 3,050 or more

4. (3 Points)

A 20-year immediate annuity certain is payable monthly. Immediately after the 43rd payment has been made, the present value of the remaining annuity payments is calculated to be *X*.

N is the number of the payment after which the present value of the remaining annuity payments is less than $\frac{X}{2}$ for the first time.

$d^{(4)} = 0.08.$

What is *N*?

(A) 67

(B) 68

(C) 171

(D) 172

(E) 173

5. (4 Points)

Two $10,000 loans have the following repayment characteristics:

Loan 1: Level quarterly payments at the end of each quarter for five years.

Loan 2: Monthly interest payments on the original loan amount at the end of each month for 48 months plus a balloon repayment of principal at the end of the fourth year. The balloon repayment will be made using the accumulated value of a sinking fund created by level annual deposits at the beginning of each of the four years.

Effective annual interest rate on the loan:	8%
Effective annual interest rate on the sinking fund:	9%

A = Sum of repayments under Loan 1.
B = Sum of interest payments on Loan 2 plus sum of sinking fund payments.

In what range is the absolute value of $(A - B)$?

(A) Less than $875

(B) $875 but less than $950

(C) $950 but less than $1,025

(D) $1,025 but less than $1,100

(E) $1,100 or more

6. (4 Points)

Smith obtains a loan for $10,000 with 40 annual payments at an effective annual interest rate of 7%. The first payment is due one year from now.

A = Sum of *interest* paid in the even-numbered payments.
B = Sum of *principal* paid in the odd-numbered payments.

In what range is $(A + B)$?

(A) Less than $13,800

(B) $13,800 but less than $14,200

(C) $14,200 but less than $14,600

(D) $14,600 but less than $15,000

(E) $15,000 or more

7. (4 Points)

Smith purchases a house for $120,000 and agrees to put 20% down. He takes out a 30-year mortgage, with monthly payments, with the first payment one month after the date of the mortgage. The interest rate is 8% compounded monthly.

Immediately following the 180th payment, Smith refinances the outstanding balance with a new 10-year mortgage, also with monthly payments, with the first payment one month after the date of the new mortgage. The new interest rate is 7.5% compounded monthly.

A = Amount in *interest* paid in the 100th payment of the first mortgage.
B = Amount of *principal* paid in the 100th payment of the refinanced mortgage.

In what range is $(A+B)$?

(A) Less than $1,300

(B) $1,300 but less than $1,325

(C) $1,325 but less than $1,350

(D) $1,350 but less than $1,375

(E) $1,375or more

8. (4 Points)

A serial bond issue bearing 6% annual coupons, payable semiannually, is to be redeemed at par value at annual intervals over a 20-year period. The first redemption will occur at the end of year 10 in the amount of $20,000. Each subsequent annual redemption will be $1,000 less than the preceding one.

In what range is the maximum price an investor would pay for the entire issue to realize an effective annual yield of 7%?

(A) Less than $190,500

(B) $190,500 but less than $191,000

(C) $191,000 but less than $191,500

(D) $191,500 but less than $192,000

(E) $192,000 or more

9. (3 Points)

Face value of a bond:	\$1,000
Redemption value:	\$1,050
Time to maturity:	10 years
Coupon rate:	9.00% per annum, convertible semiannually
Yield rate:	10.25% per annum

The bond is not callable.

In what range is the increase in the book value of the bond during the third year?

(A) Less than \$7.00

(B) \$7.00 but less than \$7.50

(C) \$7.50 but less than \$8.00

(D) \$8.00 but less than \$8.50

(E) \$8.50 or more

10. (5 Points)

A bank issues a 20-year loan for \$100,000 on 1/1/2000. Level monthly payments are calculated based on a 7% annual interest rate compounded monthly, with payments due at the end of each month. The borrower can repay the loan in full without penalty on the first day of any year. On 1/1/2002, the bank sells the loan to an investor for \$90,000.

What is the latest full repayment date for which the investor’s yield exceeds 8%, compounded monthly?

(A) 1/1/2009

(B) 1/1/2010

(C) 1/1/2011

(D) 1/1/2012

(E) 1/1/2013

11. (3 Points)

$S1$ = The accumulated value as of 12/31/2002 of $500 invested at the end of each month during 2002 at a nominal interest rate of 8% per year, convertible quarterly.

$A1$ = The present value as of 1/1/2002 of $S1$, at a nominal discount rate of 6% per year, convertible semiannually.

$S2$ = The accumulated value as of 12/31/2002 of $1,500 invested at the end of each quarter during 2002 at a nominal discount rate of 6% per year, convertible monthly.

$A2$ = The present value as of 1/1/2002 of $S2$, at a nominal interest rate of P% per year, convertible once every two years.

In what range is P% such that $A1 = A2$?

(A) Less than 4.60%

(B) 4.60% but less than 4.70%

(C) 4.70% but less than 4.80%

(D) 4.80% but less than 4.90%

(E) 4.90% or more

12. (3 Points)

For a group of lives observed over the age interval $(x, x+1]$, you are given:

(i) 100 lives entered observation at exact age x.
(ii) 40 of these lives are scheduled to leave observation at age $x+.75$.
(iii) 23 deaths were observed.
(iv) No other lives entered or left observation.
(v) The underlying survival distribution is Balducci.

In what range is the moment estimate of q_x?

(A) Less than .249

(B) .249 but less than .254

(C) .254 but less than .259

(D) .259 but less than .264

(E) .264 or more

13. (3 Points)

From Mortality Table A: $\ell_x = 20,000 - 100x - x^2$

Mortality Table B has a constant force of mortality equal to μ_{41} from Mortality Table A. In addition, from Mortality Table B, $\ell_{45} = 100,000$.

In what range is ℓ_{41} from Mortality Table B?

(A) Less than 100,000
(B) 100,000 but less than 105,000
(C) 105,000 but less than 110,000
(D) 110,000 but less than 115,000
(E) 115,000 or more

14. (3 Points)

Assume a uniform distribution of decrement over each interval $[x, x+1]$.

$${}_{.50}q_{40.4} = .025$$

$${}_{.90}p_{41} = .955$$

$$\mu_{42.2} = .05$$

$$\ell_{43} = 100,000$$

In what range is ℓ_{40}?

(A) Less than 116,000
(B) 116,000 but less than 116,500
(C) 116,500 but less than 117,000
(D) 117,000 but less than 117,500
(E) 117,500 or more

15. (5 Points)

Age of retiree on 1/1/2002: 65

Normal form of payment: Single life annuity of $20,000 payable at the beginning of each year.

Optional form of payment: $X payable at the beginning of each year while the retiree is alive

and,

If the retiree dies during 2003, a ten-year decreasing certain annuity starting on January 1, 2004. The initial payment on this date is $X and subsequent annual payments are each 95% of the prior payment.

Selected values:

$i = 7\%$ $\quad p_{65} = .9887$ $\quad p_{66} = .9873$ $\quad \ddot{a}_{65} = 10.3316$ $\quad \ddot{a}_{67} = 9.8614$

The optional form of payment and the single life annuity are actuarially equivalent on 1/1/2002.

In what range is $X?

(A) Less than $18,055

(B) $18,055 but less than $18,655

(C) $18,655 but less than $19,255

(D) $19,255 but less than $19,855

(E) $19,855 or more

16. (3 Points)

For a group of lives, the following is given:

x	ℓ_x	d_x	L_x
35	10,000	300	9,851
36	9,700		9,456
37		600	8,913

${}_2m_{35} = .0404$

There is a constant force of mortality over the interval [36, 37].

In what range is $\ell_{36.5}$?

(A) Less than 9,449

(B) 9,449 but less than 9,452

(C) 9,452 but less than 9,455

(D) 9,455 but less than 9,458

(E) 9,458 or more

17. (4 Points)

$$e_{70:\overline{5}|} = 4.66234$$

$$e_{70:\overline{15}|} = 11.45220$$

$$e_{80} = 8.26871$$

$$e_{75:\overline{10}|} = 7.70883$$

$$e_{75:\overline{5}|} = 4.43230$$

$$e_{80:\overline{5}|} = 4.08531$$

In what range is e_{70} ?

(A) Less than 14.00000

(B) 14.00000 but less than 15.00000

(C) 15.00000 but less than 16.00000

(D) 16.00000 but less than 17.00000

(E) 17.00000 or more

18. (3 Points)

$\mu_x = .10, \quad x > 0.$

N = the average number of years lived between age 60 and age 80 by those who die between age 60 and age 80.

In what range is N?

(A) Less than 7.0

(B) 7.0 but less than 7.7

(C) 7.7 but less than 8.4

(D) 8.4 but less than 9.1

(E) 9.1 or more

19. (5 Points)

An employee age 65 with a spouse age 65 is retiring under one of three actuarially equivalent optional forms of payment, all of which are payable at the beginning of each month.

Option 1: Life annuity of $*X* per month.

Option 2: Life annuity of $*Y* per month with the first 60 months guaranteed.

Option 3: Joint and last survivor annuity with the following monthly payments:

(a) $*Y* during the joint lives of the employee and spouse.

(b) $*X* for the remaining lifetime of the employee if the spouse dies first.

(c) *P*% of $*Y* for the remaining lifetime of the spouse if the employee dies first.

Selected actuarial factors:

$\ddot{a}_{65}^{(12)} = 10.0833$ $\quad {}_{5|}\ddot{a}_{65}^{(12)} = 6.0553$ $\quad \ddot{a}_{65:65}^{(12)} = 9.5833$ $\quad i = 7\%$

In what range is *P*%?

(A) Less than 42.00%

(B) 42.00% but less than 44.00%

(C) 44.00% but less than 46.00%

(D) 46.00% but less than 48.00%

(E) 48.00% or more

20. (3 Points)

Selected values from a double decrement table:

$\ell_{40}^{(T)} = 10{,}000$ $\quad \ell_{42}^{(T)} = 7{,}000$

$q'^{(1)}_{40} = .05$ $\quad q'^{(2)}_{40} = .10$ $\quad q'^{(1)}_{41} = .06$

$q'^{(1)}_{41}$ and $q'^{(2)}_{41}$ are both linear over the interval [41, 42].

In what range is the expected number of decrements due to cause 1 between 41 and 42?

(A) Less than 482

(B) 482 but less than 492

(C) 492 but less than 502

(D) 502 but less than 512

(E) 512 or more

21. (5 Points)

Under a pension plan's actuarial equivalence definition, the interest rate is 7% and $q_x = .04$ for $x \geq 70$. Under the plan, there are two actuarially equivalent forms of payment:

Form 1: 10 years certain and payments for life thereafter.

Form 2:
- Payments of $X while the participant and spouse are both alive.
- Payments of 110% of $X to participant after the death of spouse.
- Payments of 50% of $X to spouse after death of participant.

Payments are made annually at the beginning of each year.

For a participant age 72, with a spouse age 75, the benefit amount under Form 1 is $100.

In what range is $X for this participant?

(A) Less than $86

(B) $86 but less than $92

(C) $92 but less than $98

(D) $98 but less than $104

(E) $104 or more

22. (5 Points)

Age at issue for an insured: 25

Benefit: $200,000 payable at the end of the year of death if death occurs before age 55. $X payable at the end of the year of death if death occurs on or after age 55 but before age 65.

Premiums: A total of 6 premiums consisting of $300 at age 25, doubling each year for the next four premium payments. All premiums are payable at the beginning of the year and one additional premium of $4,000 is payable at age 55.

Selected Values:

$1000A_{25} = 81.6496$

$1000A_{55} = 305.1431$

$1000A_{65} = 439.7965$

$i = 6\%$

${}_{29}p_{26} = .90450$

${}_{40}p_{25} = .78766$

x	p_x
25	.99877
26	.99873
27	.99867
28	.99861
29	.99854

In what range is $X?

(A) Less than $110,000

(B) $110,000 but less than $114,000

(C) $114,000 but less than $118,000

(D) $118,000 but less than $122,000

(E) $122,000 or more

23. (3 Points)

For a study of four automobile engines, you are given:

(i) The engines are subject to a uniform survival distribution over the interval $[0, \omega]$.

(ii) Failures occurred at times 4, 5, and 7; the remaining engine was operational at time *r*.

(iii) The observation period was from time 3 to time *r*.

(iv) The maximum likelihood estimate of ω is 13.67.

In that range is *r*?

(A) Less than 11.3

(B) 11.3 but less than 11.8

(C) 11.8 but less than 12.3

(D) 12.3 but less than 12.8

(E) 12.8 or more

24. A life insurance policy provides for payment of $1,000 at the end of the year of death but provides no payment if death occurs in the first four years.

For a male age 60, the net single premium for this policy is $300.

The interest rate is 7%.

The probability of death in each of the first four years is .20% (i.e., q_{60}, q_{61}, q_{62}, and $q_{63} = .002$).

$*P* = the net single premium for a $1 annual life annuity-due with a 3-year certain period for a male age 60.

In what range is $*P*?

(A) Less than $9.50

(B) $9.50 but less than $10.10

(C) $10.10 but less than $10.70

(D) $10.70 but less than $11.30

(E) $11.30 or more

25. (3 Points)

$$s(x) = \frac{(101-x)}{101}, \quad 0 \le x \le 101 \qquad i = 5\%$$

In what range is $\ddot{a}_{80}$?

(A) Less than 8.00

(B) 8.00 but less than 9.00

(C) 9.00 but less than 10.00

(D) 10.00 but less than 11.00

(E) 11.00 or more

26. (4 Points)

Interest Rate: 5% per year, compounded annually.

A term certain and life annuity issued to a person, age 65, provides $500 payable at the end of each month. The load is 8% of the gross premium. Annuity payments are payable at least until a sum equal to the gross single premium has been paid.

Selected Values: $_{13|}a_{65}^{(12)} = 3.31 \qquad _{14|}a_{65}^{(12)} = 2.92 \qquad _{15|}a_{65}^{(12)} = 2.56$

In what range is the gross single premium?

(A) Less than $83,000

(B) $83,000 but less than $85,000

(C) $85,000 but less than $87,000

(D) $87,000 but less than $89,000

(E) $89,000 or more

27. (3 Points)

A group of lives is subject to two decrements over the interval (0,100).

Selected Values: $\mu_x^{(1)} = \mu_x^{(2)} = (100-x)^{-1}, \quad 0 < x < 100$

In what range is $_{10}q_0^{(1)}$?

(A) Less than .093

(B) .093 but less than .096

(C) .096 but less than .099

(D) .099 but less than .102

(E) .102 or more

SOLUTIONS TO THE MAY 2002 EA-1 EXAM

1. First note that the effective quarterly rate of *discount* is $\frac{d^{(4)}}{4} = .019056$, so the effective quarterly rate of *interest* is $\frac{.019056}{1-.019056} = .019426$. Then the corresponding effective annual rate of interest is $(1.019426)^4 - 1 = .08$ and the corresponding effective semiannual rate of interest is $(1.08)^{1/2} - 1 = .03923$.

   Then under Plan 1,

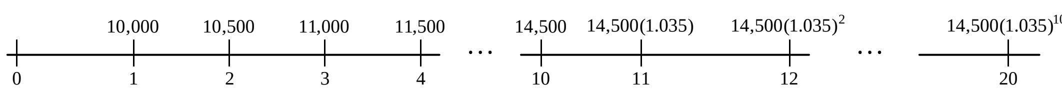
$$A = \frac{75,000}{\ddot{a}_{\overline{5}|.08}} = 17,392.76.$$

   Under Plan 2,

$$\frac{B}{2} = \frac{75,000}{a_{\overline{10}|.03923}} = 9,211.43.$$

   Finally, $|A - B| = |17,392.76 - 2(9,211.43)| = 1030.09$, ANSWER C.

2. The payments are shown on the following diagram:



   The accumulated value of the first ten payments is

$$\begin{aligned}
&\left[9,500s_{\overline{10}|} + 500(Is)_{\overline{10}|}\right](1.07)^{10} \\
&\qquad = \left[(9,500)(13.81643) + (500)(68.33684)\right](1.07)^{10} \\
&\qquad = 325,414.81.
\end{aligned}$$

   The accumulated value of the second ten payments is

$$\begin{aligned}
&14,500\left[(1.035)(1.07)^9 + (1.035)^2(1.07)^8 + \cdots + (1.035)^{10}\right] \\
&\qquad = 14,500(1.035)(1.07)^9\left[1 + \frac{1.035}{1.07} + \cdots + \left(\frac{1.035}{1.07}\right)^9\right] \\
&\qquad = 14,500(1.035)(1.07)^9\left[\frac{1 - \left(\frac{1.035}{1.07}\right)^{10}}{1 - \frac{1.035}{1.07}}\right] = 238,641.81.
\end{aligned}$$

   Then the total accumulated value is

$$325,414.81 + 238,641.81 = 564,056.62, \text{ ANSWER C.}$$

3. Recall that $\ddot{s}^{(m)}_{\overline{2n}|} = \frac{(1+i)^{2n}-1}{d^{(m)}}$. Then we have $\frac{(1+i)^{2n}-1}{.08} = 180.24943,$ so

$$(1+i)^{2n} = (180.24943)(.08)+1 = 15.41995.$$

Then

$$\ddot{s}^{(m)}_{\overline{4n}|} = \frac{(1+i)^{4n}-1}{d^{(m)}} = \frac{(15.41995)^2-1}{.08} = 2959.69, \text{ ANSWER B.}$$

4. The effective quarterly rate of discount is .02, so the effective quarterly rate of interest is $\frac{.02}{1-.02} = .02041$ and the effective monthly rate of interest is $(1.02041)^{1/3}-1 = .00676$. There are 240 monthly payments in the annuity, so there are 197 remaining after 43 have been made. Thus we are given

$$X = P \cdot a_{\overline{197}|.00676}$$

and

$$\frac{X}{2} = P \cdot a_{\overline{240-N}|.00676},$$

which tells us that

$$2a_{\overline{240-N}|} = a_{\overline{197}|} = 108.69699.$$

Then

$$a_{\overline{240-N}|} = \frac{1-v^{240-N}}{.00676} = \frac{108.69699}{2},$$

so

$$v^{240-N} = (1.00676)^{N-240} = .63260.$$

Finally

$$N = \frac{\ln(.63260)}{\ln(1.00676)} + 240 = 172.03,$$

so we need to go to $N = 173$ in order to have $PVFP < \frac{X}{2}$, ANSWER E.

5. Under Loan 1, the effective quarterly rate of interest is $j = (1.08)^{1/4}-1 = .01943,$ so the quarterly payment is

$$P_1 = \frac{10,000}{a_{\overline{20}|j}} = 608.21$$

and the sum of the repayments is $A = 20P_1 = 12,164.13$.

Under Loan 2, the effective monthly rate of interest is $j = (1.08)^{1/12}-1 = .00643$, so the monthly interest payment is $(.00643)(10,000) = 64.30$. The annual sinking fund deposit is

$$D = \frac{10,000}{\ddot{s}_{\overline{4}|.09}} = 2006.13.$$

Then the sum of all payments is

$$B = (48)(64.30)+(4)(2006.13) = 11,110.92,$$

and finally

$$|A-B| = |12,164.13-11,110.92| = 1053.21, \text{ ANSWER D.}$$

6. The annual payment is

$$P = \frac{10{,}000}{a_{\overline{40}|.07}} = 750.09.$$

Recall that the sequence of principal repaid amounts is

$$P \cdot v^{40},\ P \cdot v^{39},\ P \cdot v^{38}, \ldots,\ P \cdot v$$

and the sequence of interest paid amounts is

$$P(1-v^{40}),\ P(1-v^{39}),\ P(1-v^{38}),\ \ldots,\ P(1-v).$$

Then

$$\begin{aligned}
A &= P[(1-v^{39}) + (1-v^{37}) + (1-v^{35}) + \cdots + (1-v)] \\
&= P[20 - (v + v^3 + v^5 + \cdots + v^{39})] \\
&= P\left[20 - v\left(\frac{1-v^{40}}{1-v^2}\right)\right] \\
&= 750.09\left[20 - \frac{1}{1.07}\left(\frac{1-(1.07)^{-40}}{1-(1.07)^{-2}}\right)\right] = 9832.73.
\end{aligned}$$

Similarly

$$\begin{aligned}
B &= P[v^{40} + v^{38} + \cdots + v^2] \\
&= P \cdot v^2(1 + v^2 + \cdots + v^{38}) \\
&= P \cdot v^2\left(\frac{1-v^{40}}{1-v^2}\right) \\
&= (750.09)(1.07)^{-2}\left(\frac{1-(1.07)^{-40}}{1-(1.07)^{-2}}\right) = 4830.91.
\end{aligned}$$

Finally A+B = 9832.73+4830.91 = 14,663.64, ANSWER D.

7. For the first mortgage, the effective monthly rate of interest is $j = .0066\dot{6}$, and the monthly payment is $P_1 = \frac{(.08)(120{,}000)}{a_{\overline{360}|j}} = 704.41$. Then the interest contained in the 100th payment is

$$A = P_1(1 - v^{360-100+1}) = 704.41\left[1 - (1.0066\dot{6})^{-261}\right] = 580.06.$$

The outstanding balance following the 180th payment is

$$OB_{180} = P_1 \cdot a_{\overline{180}|j} = 73{,}710.60,$$

so the monthly payment under the second mortgage is

$$P_2 = \frac{73{,}710.60}{a_{\overline{120}|.075/12}} = 874.96.$$

Then the principal repaid in the 100th payment is

$$B = P_2 \cdot v^{120-100+1} = 874.96(1.00625)^{-21} = 767.65.$$

Finally we have

$$A + B = 580.06 + 767.65 = 1347.71, \quad \text{ANSWER C.}$$

8. The semiannual coupon rate is .03 and the desired effective semiannual yield rate is $i = (1.07)^{1/2} - 1 = .034408$. Serial bonds are best analyzed using Makeham's price formula. Recall that

$$P = \Sigma K + \frac{g}{i}(\Sigma C - \Sigma K).$$

In this case $g = .03$ and $i = .034408$. ΣC denotes the sum of all redemption amounts, so we have

$$\Sigma C = 20{,}000 + 19{,}000 + 18{,}000 + \cdots + 1{,}000 = 210{,}000.$$

ΣK denotes the sum of the present value of redemption amounts. Since the redemptions occur at annual intervals, we go back to the annual yield rate of 7%. Then we have

$$\begin{aligned}
\Sigma K &= 20{,}000v^{10} + 19{,}000v^{11} + 18{,}000v^{12} + \cdots + 1{,}000v^{29} \\
&= 1000v^9 \cdot (Da)_{\overline{20}|} \\
&= 1000(1.07)^{-9}\left(\frac{20 - a_{\overline{20}|.07}}{.07}\right) \\
&= 73{,}089.05.
\end{aligned}$$

Then

$$P = 73{,}089.05 + \frac{.03}{.034408}(210{,}000 - 73{,}089.05) = 192{,}460.37, \quad \text{ANSWER E.}$$

9. The semiannual coupon rate is .045 and the semiannual yield rate is $i = (1.1025)^{1/2} - 1 = .05$. The book value at the end of the second year is

$$BV_4 = 1050v_{.05}^{16} + 45a_{\overline{16}|.05} = 968.72.$$

Similarly the book value at the end of the third year is

$$BV_6 = 1050v_{.05}^{14} + 45a_{\overline{14}|.05} = 975.76.$$

Then the increase in book value during the third year is

$$BV_6 - BV_4 = 7.04, \quad \text{ANSWER B.}$$

10. The level monthly payment under the loan is

$$P = \frac{100,000}{a_{\overline{240}|.07/12}} = 775.30,$$

so the outstanding balance of the loan on 1/1/02 is

$$OB_{24} = 775.30a_{\overline{216}|.07/12} = 95,070.26.$$

The investor pays less than the outstanding balance, so the investor's yield will exceed $i^{(12)} = .07$. If the loan is prepaid early, the investor's yield will exceed $i^{(12)} = .08$. As the prepayment date is delayed, the yield will decrease (although it will always exceed $i^{(12)} = .07$).

For example, if the loan is repaid on1/1/10, the repayment amount will be

$$OB_{120} = 775.30a_{\overline{120}|.07/12} = 66,773.72.$$

For the investor's yield to be $i^{(12)} = .08$, the investor should pay

$$775.30a_{\overline{96}|.08/12} + 66,773.72v_{.08/12}^{96} = 90,127.87.$$

Since the investor pays less than this amount (only 90,000), the yield exceeds $i^{(12)} = .08$. If the loan is repaid on 1/1/11, the repayment amount will be

$$OB_{\overline{132}|} = 775.30a_{\overline{108}|.07/12} = 61,992.99.$$

For the investor's yield to be $i^{(12)} = .08$, the investor should pay

$$775.30a_{\overline{108}|.08/12} + 61,992.99v_{.08/12}^{108} = 89,800.14.$$

Since the investor pays more than this amount, the yield is less than $i^{(12)} = .08$. Therefore the latest repayment date that can realize a yield greater than $i^{(12)} = .08$ is 1/1/10, ANSWER B.

11. For $S1$, the effective quarterly rate is .02, so the effective monthly rate is

$(1.02)^{1/3} - 1 = .00662$. Then $S1 = 500 s_{\overline{12}|.00662} = 6223.57$.

For $A1$, this accumulated value is discounted back to 1/1/02 at $d = .03$, effective semiannually, so $A1 = 6223.57(1-.03)^2 = 5855.76$.

For $S2$, the effective monthly discount rate is .005, so the effective quarterly interest rate is $(1-.005)^{-3} - 1 = .0151511$. Then $S2 = 1500 s_{\overline{4}|.0151511} = 6137.75$.

For $A2$, this accumulated value is discounted back to 1/1/02 at $i^{(1/2)} = p$, which is $2p$ effective per two-year period, so $A2 = 6137.75(1+2p)^{-1/2}$.

Then for $A1$ to equal $A2$, we have $5855.76 = 6137.75(1+2p)^{-1/2}$, which solves for p = .04932, ANSWER E.

12. With 40 of the 100 lives scheduled to leave observation at $x + .75$ and the other 60 scheduled to remain under observation to age $x+1$, then the expected number of deaths is

$$E[D] = 60 \cdot q_x + 40\left[\frac{(.75)q_x}{1-(.25)q_x}\right].$$

The moment estimate results from equating the expected and actual number of deaths, so we have

$$60q + 40\left(\frac{.75q}{1-.25q}\right) = 23,$$

or

$$60q - 15q^2 + 30q = 23 - 5.75q$$

or

$$15q^2 - 95.75q + 23 = 0.$$

Then

$$q = \frac{95.75 - \sqrt{(95.75)^2 - 4(15)(23)}}{30} = .250, \quad \text{ANSWER B.}$$

13. Using Table A, we find

$$\mu_x = \frac{-\frac{d}{dx}\ell_x}{\ell_x} = \frac{100+2x}{20{,}000 - 100x - x^2},$$

so

$$\mu_{41} = \frac{100 + 2(41)}{20{,}000 - 100(41) - (41)^2} = .01280.$$

Under constant force, recall that $p_x = e^{-\mu}$ and ${}_4p_x = e^{-4\mu} = e^{-4(.01280)} = .95008$. Then

$$\ell_{41} = \frac{\ell_{45}}{{}_4p_{41}} = \frac{100{,}000}{.95008} = 105{,}254, \quad \text{ANSWER C.}$$

14. To find ℓ_{40} we need to work backward from $\ell_{43} = 100,000$. Recall that, under UDD,

$$\mu_{x+t} = \frac{q_x}{1-t\cdot q_x} = \frac{d_x}{\ell_x - t\cdot d_x},$$

so

$$\mu_{42.2} = \frac{d_{42}}{\ell_{42}-(.20)d_{42}} = \frac{\ell_{42}-100,000}{\ell_{42}-(.20)(\ell_{42}-100,000)} = .05,$$

which solves for $\ell_{42} = 105,208$.

Next recall that ${}_t p_x = 1 - t\cdot q_x$, so

$${}_{.9}p_{41} = 1-(.9)q_{41} = .955,$$

so $q_{41} = .05$ and $p_{41} = .95$, so $\ell_{41} = \frac{105,208}{.95} = 110,745$.

Next recall that ${}_{.50}q_{40.4} = \frac{(.50)q_{40}}{1-(.40)q_{40}} = .025$, which solves for $q_{40} = .04902$. Then $p_{40} = .95098$ and finally

$$\ell_{40} = \frac{\ell_{41}}{p_{40}} = \frac{110,745}{.95098} = 116,454, \quad \text{ANSWER B.}$$

15. The value of the normal form is $20,000\ddot{a}_{65} = 206,632$. The value of the decreasing annuity certain on 1/1/04 is

$$Y = X\left[1+.95v+(.95)^2v^2+\cdots+(.95)^9v^9\right] = X\left(\frac{1-\left(\frac{.95}{1.07}\right)^{10}}{1-\frac{.95}{1.07}}\right) = 6.20273X,$$

and its value on 1/1/02 is

$$v^2\cdot p_{65}\cdot q_{66}\cdot Y = (1.07)^{-2}(.9887)(1–.9873)(6.20273X) = .068027X.$$

Then the total value of the optional form is

$$10.3316X + .068027X = 10.399627X.$$

This must equal 206,632 so we have $X = 19,869.17$, ANSWER E.

16. Recall that

$$_{2}m_{35} \quad = \quad \frac{d_{35}+d_{36}}{L_{35}+L_{36}} \quad = \quad \frac{300+d_{36}}{19,307} \quad = \quad .0404,$$

so $d_{36}=480.$ Then $\ell_{37}=9,700-480=9,220.$ Under constant force, values of ℓ_{x+t} are found from ℓ_{x} and ℓ_{x+1} by exponential interpolation.

$$\ell_{36.5} \quad = \quad (\ell_{36})^{.50}\cdot(\ell_{37})^{.50} \quad = \quad \left[(9,700)(9,220)\right]^{.50} \quad = \quad 9456.96, \quad \text{ANSWER D.}$$

17. First note that

$$e_{75:\overline{10}|} \quad = \quad \frac{1}{\ell_{75}}(\ell_{76}+\ell_{77}+\cdots+\ell_{85})$$

and

$$e_{75:\overline{5}|} \quad = \quad \frac{1}{\ell_{75}}(\ell_{76}+\cdots+\ell_{80}).$$

Subtracting $e_{75:\overline{10}|}-e_{75:\overline{5}|}$ we find

$$\frac{1}{\ell_{75}}(\ell_{81}+\cdots+\ell_{85}) \quad = \quad 7.70883-4.43230 \quad = \quad 3.27653.$$

Next note that

$$e_{80:\overline{5}|} \quad = \quad \frac{1}{\ell_{80}}(\ell_{81}+\cdots+\ell_{85}) \quad = \quad 4.08531.$$

Then

$$\frac{e_{75:\overline{10}|}-e_{75:\overline{5}|}}{e_{80:\overline{5}|}} \quad = \quad \frac{3.27653}{4.08531} \quad = \quad .80202 \quad = \quad \frac{\ell_{80}}{\ell_{75}}.$$

Next we note that

$$e_{70:\overline{15}|}-e_{70:\overline{5}|} \quad = \quad \frac{1}{\ell_{70}}(\ell_{76}+\ell_{77}+\cdots+\ell_{85}),$$

so that

$$\frac{e_{70:\overline{15}|}-e_{70:\overline{5}|}}{e_{75:\overline{10}|}} \quad = \quad \frac{11.45220-4.66234}{7.70883} \quad = \quad .88078 \quad = \quad \frac{\ell_{75}}{\ell_{70}}.$$

Then

$$\begin{aligned} e_{70} \quad &= \quad e_{70:\overline{5}|}+\frac{\ell_{75}}{\ell_{70}}\cdot e_{75:\overline{5}|}+\frac{\ell_{75}}{\ell_{70}}\cdot\frac{\ell_{80}}{\ell_{75}}\cdot e_{80} \\ &= \quad 4.66234+(.88078)(4.43230)+(.88078)(.80202)(8.26871) \\ &= \quad 14.40733, \quad \text{ANSWER B.} \end{aligned}$$

18. For ℓ_{60} persons alive at age 60, the future lifetime is T_{60}, of which T_{80} years are lived after age 80, so $T_{60} - T_{80}$ years are lived between ages 60 and 80. Of these, $20 \cdot \ell_{80}$ years are lived by those who survive to age 80, so $T_{60} - T_{80} - 20 \cdot \ell_{80}$ years are lived by those who die between ages 60 and 80. There are $\ell_{60} - \ell_{80}$ such deaths, so

$$N = \frac{T_{60} - T_{80} - 20 \cdot \ell_{80}}{\ell_{60} - \ell_{80}}.$$

We need to evaluate the T_x and ℓ_x functions under the assumption of a constant μ_x at all ages.

Recall that $\ell_x = \ell_0 \cdot e^{-\int_0^x \mu_y \, dy} = \ell_0 \cdot e^{-x \cdot \mu}$ when μ is constant. Thus we have $\ell_{60} = \ell_0 \cdot e^{-(60)(.10)} = \ell_0 \cdot e^{-6}$ and, similarly, $\ell_{80} = \ell_0 \cdot e^{-(80)(.10)} = \ell_0 \cdot e^{-8}$.

Next recall that

$$\begin{aligned} T_{60} - T_{80} &= \int_{60}^{80} \ell_y \, dy \\ &= \int_{60}^{80} \ell_0 \cdot e^{-.10y} \, dy \\ &= \ell_0 \left(-\frac{e^{-.10y}}{.10} \right)\Bigg|_{60}^{80} \\ &= \ell_0 \left(\frac{e^{-6} - e^{-8}}{.10} \right) \\ &= \ell_0 \cdot 10(e^{-6} - e^{-8}). \end{aligned}$$

Then

$$N = \frac{10(e^{-6} - e^{-8}) - 20e^{-8}}{e^{-6} - e^{-8}} = 6.86923, \quad \text{ANSWER A.}$$

19. The equivalent present values are as follows:

Option 1: $PV = 12X \cdot \ddot{a}_{65}^{(12)} = 121X$

Option 2: $PV = Y \cdot \ddot{a}_{\overline{60|}} + 12Y \cdot {}_{5|}\ddot{a}_{65}^{(12)} = Y[51.0541 + 12(6.0553)] = 123.7177Y$

Option 3:
$$\begin{aligned} PV &= 12Y \cdot \ddot{a}_{65:65}^{(12)} + 12X \cdot \ddot{a}_{65|65}^{(12)} + 12p \;\; Y\ddot{a}_{65|65}^{(12)} \\ &= 12\left[Y \cdot \ddot{a}_{65:65}^{(12)} + (X + pY)\,\ddot{a}_{65|65}^{(12)}\right] \\ &= 12\left[Y \cdot \ddot{a}_{65:65}^{(12)} + (X + pY)(\ddot{a}_{65}^{(12)} - \ddot{a}_{65:65}^{(12)})\right] \\ &= 12[9.5833Y + .50(X + pY)] \quad = \quad 115Y + 6(X + pY) \end{aligned}$$

Equating the PV's of Options 1 and 2 gives $X = \frac{123.7177}{121}Y = 1.0225Y$.

Equating the PV's of Options 2 and 3 gives $123.7177Y = 115Y + 6(1.0225Y + pY)$, so

$$123.7177 = 115 + 6(1.0225 + p),$$

which solves for $p = .43045$, ANSWER B.

20. Recall that $p_x^{(T)} = p_x'^{(1)} \cdot p_x'^{(2)}$. Here we have

$$p_{40}^{(T)} = (1-.05)(1-.10) = .855,$$

so

$$\ell_{41}^{(T)} = \ell_{40}^{(T)} \cdot p_{40}^{(T)} = (10{,}000)(.855) = 8{,}550.$$

Next note that $p_{41}^{(T)} = \frac{7{,}000}{8{,}550} = .81871$. But also $p_{41}^{(T)} = p_{41}'^{(1)} \cdot p_{41}'^{(2)} = (1-.06)\,p_{41}'^{(2)} = .81871$, so $p_{41}'^{(2)} = .87096$ and therefore $q_{41}'^{(2)} = .12904$.

Finally, recall that

$$\begin{aligned} q_x^{(1)} &= \int_0^1 {}_t p_x^{(T)} \cdot \mu_{x+t}^{(1)}\, dt = \int_0^1 {}_t p_x'^{(2)} \cdot {}_t p_x'^{(1)} \mu_{x+t}^{(1)}\, dt \\ &= \int_0^1 (1 - t \cdot q_x'^{(2)}) \cdot q_x'^{(1)}\, dt = q_x'^{(1)}\left(1 - \frac{1}{2} \cdot q_x'^{(2)}\right). \end{aligned}$$

Then

$$q_{41}^{(1)} = q_{41}'^{(1)}(1 - \tfrac{1}{2} \cdot q_{41}'^{(2)}) = (.06)\left[1 - \tfrac{1}{2}(.12904)\right] = .05613,$$

so

$$d_{41}^{(1)} = q_{41}^{(1)} \cdot \ell_{41}^{(T)} = (.05613)(8550) = 479.90, \quad \text{ANSWER A.}$$

21. Under Form 1, the present value is

$$PV_1 = 100\left(\ddot{a}_{\overline{10}|.07} + {}_{10|}\ddot{a}_{72}\right).$$

Under Form 2, the equivalent present value is

$$\begin{aligned} PV_2 &= X\left[\ddot{a}_{72:75} + 1.10(\ddot{a}_{72} - \ddot{a}_{72:75}) + .50(\ddot{a}_{75} - \ddot{a}_{72:75})\right] \\ &= X(1.10\ddot{a}_{72} + .50\ddot{a}_{75} - .60\ddot{a}_{72:75}). \end{aligned}$$

Since $q_x = .04$ for all x, then $p_x = .96$ and ${}_t p_x = (.96)^t$ for all x. Thus the needed annuity values are as follows:

$$\ddot{a}_x = \sum_{t=0}^{\infty} v^t \, {}_t p_x = \sum_{t=0}^{\infty} \left(\frac{.96}{1.07}\right)^t = \frac{1}{1 - \frac{.96}{1.07}} = 9.72727$$

$$\ddot{a}_{xy} = \sum_{t=0}^{\infty} v^t ({}_t p_x)({}_t p_y) = \sum_{t=0}^{\infty} \left(\frac{(.96)(.96)}{1.07}\right)^t = \frac{1}{1 - \frac{(.96)^2}{1.07}} = 7.21024$$

$${}_{10|}\ddot{a}_x = \sum_{t=10}^{\infty} v^t \, {}_t p_x = \sum_{t=10}^{\infty} \left(\frac{.96}{1.07}\right)^t = \left(\frac{.96}{1.07}\right)^{10} (9.72727) = 3.28750$$

$$\ddot{a}_{\overline{10}|.07} = 7.51523$$

Then

$$PV_1 = 100(7.51523 + 3.28750) = 1080.27.$$

Since $PV_2 = PV_1$, we finally have

$$X = \frac{1080.27}{1.10(9.72727) + .50(9.72727) - .60(7.21024)} = 96.13, \quad \text{ANSWER C.}$$

22. The present value of the benefit is

$$PVB = 200{,}000 A^1_{25:\overline{30}|} + X \cdot {}_{30}E_{25} \cdot A^1_{55:\overline{10}|}.$$

To evaluate the term insurances, recall that

$$A^1_{x:\overline{n}|} = A_x - {}_n E_x \cdot A_{x+n}.$$

Therefore

$$\begin{aligned} A^1_{25:\overline{30}|} &= A_{25} - {}_{30}E_{25} \cdot A_{55} \\ &= A_{25} - (vp_{25})(v^{29} \, {}_{29}p_{26}) A_{55} \\ &= .0816496 - \left(\frac{.99877}{1.06}\right)\left(\frac{.90450}{(1.06)^{29}}\right)(.3051431) = .033653, \end{aligned}$$

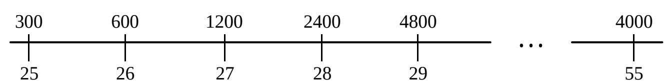
and

$$\begin{aligned} {}_{30}E_{25} \cdot A^{1}_{55:\overline{10|}} &= {}_{30}E_{25}(A_{55} - {}_{10}E_{55} \cdot A_{65}) \\ &= {}_{30}E_{25} \cdot A_{55} - {}_{40}E_{25} \cdot A_{65} \\ &= \left(\frac{.99877}{1.06}\right)\left(\frac{.90450}{(1.06)^{29}}\right)(.3051431) - \left(\frac{.78766}{(1.06)^{40}}\right)(.4397965) \\ &= .047995 - .033678 \quad = \quad .014316. \end{aligned}$$

Then the present value of the benefit is

$$PVB \quad = \quad 200,000(.033653) + X(.014316).$$

The stream of premium payments is as follows:



The present value of the premiums is

$$\begin{aligned} PVP &= 300 + 600vp_{25} + 1200v^{2}\,{}_{2}p_{25} + 2400v^{3}\,{}_{3}p_{25} + 4800v^{4}\,{}_{4}p_{25} + 4000v^{3}\,{}_{30}p_{25} \\ &= 300 + 600\left(\frac{.99877}{1.06}\right) + 1200\left(\frac{(.99877)(.99873)}{(1.06)^{2}}\right) \\ &\qquad + 2400\left(\frac{(.99877)(.99873)(.99867)}{(1.06)^{3}}\right) \\ &\qquad + 4800\left(\frac{(.99877)(.99873)(.99867)(.99861)}{(1.06)^{4}}\right) \\ &\qquad + 4000\left(\frac{.99877}{1.06}\right)\left(\frac{.90450}{(1.06)^{29}}\right) \\ &= 300 + 600(.94223) + 1200(.88777) \\ &\qquad + 2400(.83640) + 4800(.78796) + 4000(.15728) \quad = \quad 8349.39 \end{aligned}$$

Finally,

$$X \quad = \quad \frac{8349.39 - 200,000(.033653)}{.014316} \quad = \quad 113,075.26, \quad \text{ANSWER B.}$$

23. Since observation begins at $t=3$, then all functions are conditional on $T>3$. The value of $Pr(T>3)$ is $S(3)=\frac{\omega-3}{\omega}$. The value of the conditional density for failure at time t is

$$f(t\,|\,T>3) \quad = \quad \frac{f(t)}{S(3)} \quad = \quad \frac{1/\omega}{(\omega-3)/\omega} \quad = \quad \frac{1}{\omega-3},$$

and the conditional probability for survival to time $t=r$ is

$$S(r\,|\,T>3) \quad = \quad \frac{S(r)}{S(3)} \quad = \quad \frac{\omega-r}{\omega-3}.$$

Then the likelihood is

$$L \quad = \quad \left(\frac{1}{\omega-3}\right)^3\left(\frac{\omega-r}{\omega-3}\right) \quad = \quad \frac{\omega-r}{(\omega-3)^4}.$$

The log-likelihood is

$$\ln L \quad = \quad \ln(\omega-r)-4\cdot\ln(\omega-3)$$

and the likelihood equation is

$$\frac{d}{d\omega}\ln L \quad = \quad \frac{1}{\omega-r}-\frac{4}{\omega-3} \quad = \quad 0.$$

But we know that $\omega=13.67$ satisfies this equation, so we have

$$\frac{1}{13.67-r} \quad = \quad \frac{4}{13.67-3},$$

which solves for $r=11.0025$, ANSWER A.

24. With no death benefit in the first four years, we have a four-year deferred insurance. Then

$$NSP \quad = \quad 300 \quad = \quad 1000({}_4E_{60}\cdot A_{64}),$$

where

$${}_4E_{60} \quad = \quad v^4\,{}_4p_{60} \quad = \quad (1.07)^{-4}(1-.002)^4 \quad = \quad .75681.$$

Then $A_{64}=\frac{.30}{.75681}=.39640$. Recall the identity $A_x=1-d\cdot\ddot{a}_x$, so

$$\ddot{a}_{64} \quad = \quad \frac{1-A_{64}}{d} \quad = \quad \frac{1-.39640}{.07/1.07} \quad = \quad 9.22646.$$

Next we are given that

$$P \quad = \quad \ddot{a}_{\overline{3}|}+v^3\,{}_3p_{60}\cdot\ddot{a}_{63} \quad = \quad 2.80799+\left(\frac{.998}{1.07}\right)^3\cdot\ddot{a}_{63},$$

where $\ddot{a}_{63} \; = \; 1+vp_{63}\cdot\ddot{a}_{64} \; = \; 1+\left(\frac{.998}{1.07}\right)(9.22646) \; = \; 9.60561.$

Finally,

$$P \quad = \quad 2.80799+\left(\frac{.998}{1.07}\right)^3(9.60561) \quad = \quad 10.60208, \quad \text{ANSWER C.}$$

25. Recall that ${}_tp_{80} = \frac{S(80+t)}{S(80)} = \frac{21-t}{21} = 1-\frac{t}{21}$. Then

$$\begin{aligned}\ddot{a}_{80} &= \sum_{t=0}^{20} v^t \cdot {}_tp_{80} \\ &= \sum_{t=0}^{20} v^t\left(1-\frac{t}{21}\right) = \ddot{a}_{\overline{21}|} - \frac{1}{21}(Ia)_{\overline{20}|} \\ &= \ddot{a}_{\overline{21}|.05} - \frac{1}{21}\left(\frac{\ddot{a}_{\overline{20}|.05} - 20v_{.05}^{20}}{.05}\right) \\ &= 13.46220 - \frac{1}{21}\left(\frac{13.08533-7.53778}{.05}\right) = 8.17882,\end{aligned}$$

ANSWER B.

26. If the gross premium is G, then the relationship between the gross and net single premiums is $G = NSP + .08G$, or $NSP = .92G$. The certain period extends until the gross premium has been returned in payments.

If the certain period is 13 years, or 156 months, then

$$(13)(12)(500) = 78,000$$

is the amount of the guaranteed return. Since this should equal the gross premium, then the net premium would be $(.92)(78,000) = 71,760$.

But we know that

$$NSP = 500 \cdot a_{\overline{156}|j} + (12)(500) \cdot {}_{13|}a_{65}^{(12)} = (500)(115.32) + (12)(500)(3.31) = 77,518.28,$$

where $j = (1.05)^{1/12} - 1$. Since the calculated NSP exceeds the assumed NSP based on the assumed certain period of 13 years, then the certain period exceeds 13.

If the certain period is 14 years, or 168 months, then

$$(14)(12)(500) = 84,000$$

is the amount of the guaranteed return. Since this should equal the gross premium, then the net premium would be $(.92)(84,000) = 77,280$.

But we know that

$$NSP = 500 \cdot a_{\overline{168}|j} + (12)(500) \cdot {}_{14|}a_{65}^{(12)} = (500)(121.52) + (12)(500)(2.92) = 78,280.$$

Again the calculated NSP exceeds the assumed NSP based on the assumed certain period, so the certain period exceeds 14. If the certain period is 15 years, or 180 months, then

$$(15)(12)(500) = 90,000$$

is the amount of the guaranteed return. Since this should equal the gross premium, then the net premium would be $(.92)(90,000) = 82,800$.

But we know that

$$NSP \;=\; 500 \cdot a_{\overline{180}|j} + (12)(500) \cdot {}_{15|}a_{65}^{(12)} \;=\; (500)(127.43) + (12)(500)(2.56) \;=\; 79{,}073.36.$$

Now the calculated NSP is less than the assumed NSP based on the assumed certain period, so the certain period is less than 15.

Then we conclude that the certain period is approximately 14.25 years, so the gross premium is approximately 85,500.00, ANSWER C.

27. Recall that

$$_{10}q_0^{(1)} \quad = \quad \int_0^{10} {}_t p_0^{(T)} \cdot \mu_{0+t}^{(1)}\, dt.$$

First we find

$$\begin{aligned}
{}_t p_0'^{(1)} \quad &= \quad e^{-\int_0^t \mu_{0+r}^{(1)}\, dr} \\
&= \quad e^{-\int_0^t (100-r)^{-1}\, dr} \\
&= \quad e^{\ln(100-r)|_0^t} \\
&= \quad e^{\ln(100-t)\, -\, \ln 100} \\
&= \quad \frac{100-t}{100}.
\end{aligned}$$

Since $\mu_x^{(1)} = \mu_x^{(2)}$, then ${}_t p_0'^{(2)} \quad = \quad \dfrac{100-t}{100}$ as well.

Then

$${}_t p_0^{(T)} \quad = \quad {}_t p_0'^{(1)} \cdot {}_t p_0'^{(2)} \quad = \quad \left(\frac{100-t}{100}\right)^2.$$

Then

$$\begin{aligned}
{}_{10}q_0^{(1)} \quad &= \quad \int_0^{10} \left(\frac{100-t}{100}\right)^2 \cdot \frac{1}{100-t}\, dt \\
&= \quad \frac{1}{(100)^2} \int_0^{10} (100-t)\, dt \\
&= \quad \frac{100t - \frac{1}{2}t^2}{(100)^2}\Bigg|_0^{10} \\
&= \quad .095, \quad \text{ANSWER B.}
\end{aligned}$$