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CHAPTER 1

1-1 See text page 1.

1-2 See text page 1.

1-3 We are given $R = 10, H = 6,$ and $Z = .25.$ Then by Equation (1.1) we have

$$C = ZR + (1-Z)H = (.25)(10) + (.75)(6) = 7.$$

1-4 We are given $R = 100, H = 200,$ and $Z = .40.$ Then by Equation (1.1) we have

$$C = ZR + (1-Z)H = (.40)(100) + (.60)(200) = 160.$$

1-5 See text page 4.

CHAPTER 2

2-1 The result of Example 2.2 of Section 2.2 is

$$P\left[A_1 \mid U_1\right] = \frac{5}{8}.$$

Since A_1 and A_2 are mutually exclusive and exhaustive events,

$$P[A_1]+P[A_2] = 1,$$

or, equivalently,

$$P[A_2] = 1-P[A_1] = 1-\left(\frac{5}{8}\right) = \frac{3}{8}.$$

2-2 Our goal is to determine $P[A_1 \mid U_1 \text{ and } U_2]$. We are given that

$$P[A_1] = \frac{1}{2} \text{ and } P[A_2] = \frac{1}{2}.$$

Also,

$$P[U_1 \text{ and } U_2 \mid A_1] = P[U_1 \mid A_1]\cdot P[U_2 \mid A_1] = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \frac{25}{36}$$

and

$$P[U_1 \text{ and } U_2 \mid A_2] = P[U_1 \mid A_2]\cdot P[U_2 \mid A_2] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Hence, by the Theorem of Total Probability,

$$\begin{aligned} P[U_1 \text{ and } U_2] &= P[U_1 \text{ and } U_2 \mid A_1]\cdot P[A_1]+P[U_1 \text{ and } U_2 \mid A_2]\cdot P[A_2] \\ &= \left(\frac{25}{36}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{34}{72}. \end{aligned}$$

By Bayes' Theorem,

$$\begin{aligned} P[A_1 \mid U_1 \text{ and } U_2] &= \frac{P[U_1 \text{ and } U_2 \mid A_1]\cdot P[A_1]}{P[U_1 \text{ and } U_2]} = \frac{\left(\frac{25}{36}\right)\left(\frac{1}{2}\right)}{\left(\frac{34}{72}\right)} = \frac{\left(\frac{25}{72}\right)}{\left(\frac{34}{72}\right)} \\ &= \frac{25}{34}. \end{aligned}$$

2-3 Recall from Example 2.2 that for $i = 1, 2\ldots$

$$P[U_i \mid A_1] = \frac{5}{6} \text{ and } P[U_i \mid A_2] = \frac{3}{6}.$$

Given $P[A_1] = \frac{5}{8}$ and $P[A_2] = \frac{3}{8}$, we have, according to the Theorem of Total Probability,

$$\begin{aligned} P[U_2] &= P[A_1 \text{ and } U_2] + P[A_2 \text{ and } U_2] \\ &= P[U_2 \mid A_1] \cdot P[A_1] + P[U_2 \mid A_2] \cdot P[A_2] \\ &= \left(\frac{5}{6}\right)\left(\frac{5}{8}\right) + \left(\frac{3}{6}\right)\left(\frac{3}{8}\right) = \frac{34}{48}. \end{aligned}$$

Finally, by Bayes' Theorem,

$$P[A_1 \mid U_2] = \frac{P[U_2 \mid A_1] \cdot P[A_1]}{P[U_2]} = \frac{\left(\frac{5}{6}\right)\left(\frac{5}{8}\right)}{\left(\frac{34}{48}\right)} = \frac{\left(\frac{25}{48}\right)}{\left(\frac{34}{48}\right)} = \frac{25}{34}.$$

Note that the answers to Exercises 2-2 and 2-3 are identical.

2-4 By the definition of conditional probability (Definition 2.1),

$$\begin{aligned} &P[1 \text{ red and } 2 \text{ white} \mid \text{at least } 2 \text{ white}] \\ &\qquad = \frac{P[1 \text{ red and } 2 \text{ white}]}{P[\text{at least } 2 \text{ white}]} \\ &\qquad = \frac{\binom{4}{1}\binom{6}{2} \Big/ \binom{10}{3}}{\left[\binom{4}{1}\binom{6}{2} + \binom{4}{0}\binom{6}{3}\right] \Big/ \binom{10}{3}} \\ &\qquad = \frac{(4)(15)}{(4)(15) + (1)(20)} = \frac{60}{80} = .75. \end{aligned}$$

2-5 $P[\text{first item not defective} \mid \geq 99 \text{ are not defective}]$

$$= \frac{P[\text{first item not defective and} \geq 99 \text{ are not defective}]}{P[\geq 99 \text{ are not defective}]}$$

$$= \frac{P[0 \text{ are defective}] + P[\text{one other than first item is defective and first is not defective}]}{P[0 \text{ are defective}] + P[\text{one is defective})]}$$

$$= \frac{(.95)^{100} + (.95)\binom{99}{1}(.05)^1(.95)^{98}}{(.95)^{100} + \binom{100}{1}(.95)^{99}(.05)^1} = \frac{.95 + (99)(.05)}{.95 + (100)(.05)} = \frac{5.90}{5.95}.$$

2-6 By Bayes' Theorem and the Theorem of Total Probability,

$P[\text{Box I} \mid \text{red marble}]$

$$= \frac{P[\text{red marble} \mid \text{Box I}] \cdot P[\text{Box I}]}{P[\text{red marble}]}$$

$$= \frac{P[\text{red marble} \mid \text{Box I}] \cdot P[\text{Box I}]}{P[\text{red marble} \mid \text{Box I}] \cdot P[\text{Box I}] + P[\text{red marble} \mid \text{Box II}] \cdot P[\text{Box II}]}$$

$$= \frac{(.60)\left(\frac{2}{3}\right)}{(.60)\left(\frac{2}{3}\right) + (.30)\left(\frac{1}{3}\right)} = .80$$

2-7 $P[\text{youthful driver}] = \frac{1500}{1500 + 8500} = \frac{1500}{10,000} = .15.$

$P[\text{adult driver}] = 1 - P[\text{youthful driver}] = 1 - .15 = .85.$

By Bayes' Theorem and the Theorem of Total Probability,

$P[\text{youthful driver} \mid 1 \text{ claim}]$

$$= \frac{P[1 \text{ claim} \mid \text{youth}] \cdot P[\text{youth}]}{P[1 \text{ claim}]}$$

$$= \frac{P[1 \text{ claim} \mid \text{youth}] \cdot P[\text{youth}]}{P[1 \text{ claim} \mid \text{youth} \cdot P[y\text{outh}] \cdot P[1 \text{ claim} \mid \text{adult}] \cdot P[\text{adult}]}$$

$$= \frac{(.30)(.15)}{(.30)(.15) + (.15)(.85)} = \frac{.30}{.30 + .85} = .261.$$

2-8 This is an application of Bayes' theorem. For $x = 0,1,\ldots$ and $y = x,$ $x+1,\ldots,$ we have

$$P[Y = y \mid X = x] = \frac{P[X = x \mid Y = y] \cdot P[Y = y]}{P[X = x]}. \tag{S2.1}$$

By our assumptions, the result

$$P[X{=}x \mid Y{=}y] = \binom{y}{x} d^x (1-d)^{y-x}$$

is an application of the binomial distribution. By Theorem 2.2,

$$\begin{aligned} P[X{=}x] &= \sum_{j=x}^{\infty} P[X{=}x \mid Y{=}j] \cdot P[Y{=}j] \\ &= \sum_{j=x}^{\infty} \binom{j}{x} d^x (1-d)^{j-x} \cdot \binom{r+j-1}{j} p^r (1-p)^j . \end{aligned}$$

Then we can rewrite the right side of Equation (S2.1) as

$$\begin{aligned} &\frac{\left[\binom{y}{x} d^x (1-d)^{y-x}\right] \cdot \left[\binom{r+y-1}{y} p^r (1-p)^y\right]}{\sum_{j=x}^{\infty} \binom{j}{x} d^x (1-d)^{j-x} \cdot \binom{r+j-1}{j} p^r (1-p)^j} \\ &\qquad = \frac{\left[\binom{y}{x} (1-d)^{y-x}\right] \cdot \left[\binom{r+y-1}{y} (1-p)^y\right]}{\sum_{j=x}^{\infty} \binom{j}{x} (1-d)^{j-x} \cdot \binom{r+j-1}{j} (1-p)^j} \\ &\qquad = \frac{\left[\binom{y}{x} (1-d)^{y-x}\right] \cdot \left[\binom{r+y-1}{y} (1-p)^{y-x}\right]}{\sum_{j=x}^{\infty} \binom{j}{x} (1-d)^{j-x} \cdot \binom{r+j-1}{j} (1-p)^{j-x}} \\ &\qquad = \frac{\binom{y}{x}\binom{r+y-1}{y}\left[(1-d)(1-p)\right]^{y-x}}{\sum_{j=x}^{\infty} \binom{j}{x}\binom{r+j-1}{j}\left[(1-d)(1-p)\right]^{j-x}}. \end{aligned} \tag{S2.2}$$

Using the identities

$$\binom{y}{x}\binom{r+y-1}{y} = \binom{r+x-1}{x}\binom{(r+x)+(y-x)-1}{y-x}$$

and

$$\binom{j}{x}\binom{r+j-1}{j} = \binom{r+x-1}{x}\binom{(r+x)+(j-x)-1}{j-x},$$

Expression (S2.2) can be rewritten as

$$\frac{\binom{r+x-1}{x}\binom{(r+x)+(y-x)-1}{y-x}\left[(1-d)(1-p)\right]^{y-x}}{\sum_{j=x}^{\infty}\binom{r+x-1}{x}\binom{(r+x)+(j-x)-1}{j-x}\left[(1-d)(1-p)\right]^{j-x}}$$

$$= \frac{\binom{(r+x)+(y-x)-1}{y-x}\left[(1-d)(1-p)\right]^{y-x}}{\sum_{j=0}^{\infty}\binom{(r+x)+j-1}{j}\left[(1-d)(1-p)\right]^{j}}. \qquad \text{(S2.3)}$$

If we multiply the numerator and denominator of Expression (S2.3) by $\left[1-(1-d)(1-p)\right]^{r+x}$, the denominator is then equal to 1 since it is the sum of the entire negative binomial distribution with parameters $(r+x)$ and $(1-d)(1-p)$. The result is then just the resulting numerator, which is

$$\binom{(r+x)+(y-x)-1}{y-x}\left[(1-d)(1-p)\right]^{y-x}\cdot\left[1-(1-d)(1-p)\right]^{r+x.}$$

Thus we have shown that for $x = 0,1,\ldots$ and $y = x, x+1,\ldots,$

$$P[Y=y \mid X=x]$$
$$= \binom{(r+x)+(y-x)-1}{y-x}\left[(1-d)(1-p)\right]^{y-x}\cdot\left[1-(1-d)(1-p)\right]^{r+x}$$

which is a negative binomial density function with parameters $(r+x)$ and $\left[1-(1-d)(1-p)\right]$, shifted by x.

2-9 We illustrate the method of solution for $P[Y{=}5\,|\,X{=}3]$. (The solution is similar for the other combinations of x and y.) By Bayes' theorem,

$$P[Y{=}5\,|\,X{=}3] \;=\; \frac{P[X{=}3\,|\,Y{=}5]\cdot P[Y{=}5]}{P[X{=}3]}.$$

By assumption, $P[Y{=}5]=.20$. By the independence assumption,

$$P[X{=}3\,|\,Y{=}5] \;=\; \binom{5}{3}(.50)^3(.50)^2 \;=\; \frac{5}{16}.$$

By the Theorem of Total Probability,

$$\begin{aligned} P[X{=}3] &= \sum_{j=3}^{6} P[X{=}3\,|\,Y{=}j]\cdot P[Y{=}j] \\ &= \binom{3}{3}(.50)^3(.50)^0(.20)+\binom{4}{3}(.50)^3(.50)^1(.20) \\ &\qquad +\binom{5}{3}(.50)^3(.50)^2(.20)+\binom{6}{3}(.50)^3(.50)^3(.20) \\ &= .20. \end{aligned}$$

Then we have

$$P[Y{=}5\,|\,X{=}3] \;=\; \frac{P[X{=}3\,|\,Y{=}5]\cdot P[Y{=}5]}{P[X{=}3]} \;=\; \frac{\left(\frac{5}{16}\right)(.20)}{.20} \;=\; \frac{5}{16}.$$

2-10 $$\begin{aligned} E[R_1^2+R_2^2\,|\,R_1{=}r_1] &= E[R_1^2\,|\,R_1{=}r_1]+E[R_2^2\,|\,R_1{=}r_1] \\ &= r_1^2+E[R_2^2] \;=\; r_1^2+2, \end{aligned}$$

since $E[R_2^2]=\int_0^\infty x^2e^{-x}dx=2$, or, more generally, $\int_0^\infty x^ne^{-x}dx=n!$ for n a nonnegative integer. The last integral is, of course, just the Gamma function.

2-11 $E[X^2+Y^3 \mid X=3 \text{ and } Y>0] = E[X^2 \mid X=3] + E[Y^3 \mid Y>0]$

$$= (3)^2 + \int_0^{\infty} y^3 e^{-y}\,dy$$

$$= 9+3! = 9+6 = 15.$$

See the solution to Exercise 2-10 for details on the evaluation of the integral.

2-12 (a) $E_Y[Y \mid X=1] = 1 \cdot P[Y=1 \text{ and } X=1] = \frac{4}{7}$, since

$$P[Y=1 \mid X=1] = \frac{P[Y=1 \mid X=1]}{P[X=1]}$$

$$= \frac{f(1,1)}{f(1,0)+f(1,1)} = \frac{.40}{.30+.40} = \frac{4}{7}.$$

(b) $E_X[X \mid Y=0] = 1 \cdot P[X=1 \mid Y=0]$

$$= \frac{P[X=1, Y=0]}{P[Y=0]}$$

$$= \frac{f(1,0)}{f(0,0)+f(1,0)} = \frac{.30}{.10+.30} = \frac{3}{4}.$$

2-13 (a) For $0<x<y<1$,

$$f(y) = \int_0^y f(x,y)\,dx$$

$$= \int_0^y (6xy+3x^2)\,dx = (3x^2y+x^3)\Big|_0^y = 3y^3+y^3 = 4y^3.$$

(b) For $0<x<y<1$,

$$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{6xy+3x^2}{4y^3} = \left(\frac{3}{2}\right)xy^{-2} + \left(\frac{3}{4}\right)x^2y^{-3}.$$

(c) For $0<y<1$,

$$\begin{aligned} E[X \mid Y=y] &= \int_0^y x \cdot f(x \mid y)\, dx \\ &= \int_0^y \left[\left(\frac{3}{2}\right) x^2 y^{-2} + \left(\frac{3}{4}\right) x^3 y^{-3} \right] dx \\ &= \left(\frac{1}{2}\right) x^3 y^{-2} + \left(\frac{3}{16}\right) x^4 y^{-3} \Big|_0^y \\ &= \left(\frac{1}{2}\right) y + \left(\frac{3}{16}\right) y = \frac{11y}{16}. \end{aligned}$$

2-14 (a) By the definition of conditional expectation and Bayes' theorem, we obtain

$$\begin{aligned} E[Y \mid X=3] &= \sum_{y=3}^{6} y \cdot P[Y=y \mid X=3] \\ &= \sum_{y=3}^{6} \frac{y \cdot P[X=3 \mid Y=y] \cdot P[Y=y]}{P[X=3]} \\ &= \sum_{y=3}^{6} \frac{y \cdot P[X=3] \mid Y=y] \cdot (.20)}{(.20)} \\ &= \sum_{y=3}^{6} y \cdot P[X=3 \mid Y=y] \\ &= (3)\left(\frac{8}{64}\right) + (4)\left(\frac{16}{64}\right) + (5)\left(\frac{20}{64}\right) + (6)\left(\frac{20}{64}\right) = \frac{77}{16}. \end{aligned}$$

(b) $E[Y \mid X=3] - 3 = \frac{77}{16} - 3 = \frac{29}{16}.$

2-15 Use the result of the second sentence of the hint given in the question, with n replaced by $(r+x)$ and q replaced by $1-(1-d)(1-p)$.

2-16
$$\begin{aligned} E[X] &= E_Y\left[E_X[X \mid Y]\right] \\ &= \sum_{j=0}^{1} 1 \cdot P[X=1 \mid Y=j] \cdot P[Y=j] \\ &= 1 \cdot P[Y=0] \cdot P[X=1 \mid Y=0] + 1 \cdot P[Y=1] \cdot P[X=1 \mid Y=1] \\ &= 1 \cdot P[Y=0, X=1] + 1 \cdot P[Y=1, X=1] \\ &= f(1,0) + f(1,1) \;=\; .30 + .40 \;=\; .70. \end{aligned}$$

2-17 (a) For $0 < x < y < 1$,

$$\begin{aligned} f(x) &= \int_x^1 f(x,y)\,dy \;=\; \int_x^1 (6xy+3x^2)\,dy \\ &= 3xy^2 + 3x^2y\Big|_x^1 \;=\; 3x + 3x^2 - 6x^3. \end{aligned}$$

Elsewhere, $f(x) = 0$.

(b)
$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot f(x)\,dx \;=\; \int_0^1 x(3x+3x^2-6x^3)\,dx \\ &= \int_0^1 (3x^2+3x^3-6x^4)\,dx \\ &= x^3 + \left(\frac{3x^4}{4}\right) - \left(\frac{6x^5}{5}\right)\Bigg|_0^1 \;=\; .55. \end{aligned}$$

2-18
$$\begin{aligned} E_R[R] &= E_\Theta\left[E_R[R \mid \Theta]\right] \\ &= E_\Theta[3\Theta] = \int_0^1 3\theta \cdot g(\theta)\,d\theta \\ &= \int_0^1 3\theta \cdot 6(\theta-\theta^2)\,d\theta \\ &= \int_0^1 18(\theta^2-\theta^3)\,d\theta = 18\left[\left(\frac{\theta^3}{3}\right) - \left(\frac{\theta^4}{4}\right)\right]\Bigg|_0^1 \\ &= 1.50. \end{aligned}$$

CHAPTER 3

3-1 Our goal is to find the value of $\hat{\Theta}$ that minimizes $E_{\Theta}[(\Theta-\hat{\Theta})^2]$. We do this by setting $\frac{d}{d\hat{\Theta}}E_{\Theta}[(\Theta-\hat{\Theta})^2]$ equal to zero and solving for $\hat{\Theta}$. Interchanging the order of integration and differentiation, we obtain

$$\frac{d}{d\hat{\Theta}}E_{\Theta}\left[(\Theta-\hat{\Theta})^2\right] = \frac{d}{d\hat{\Theta}}\int_{-\infty}^{\infty}(\theta-\hat{\Theta})^2\cdot h(\theta)\,d\theta$$

$$= \int_{-\infty}^{\infty}\frac{d}{d\hat{\Theta}}(\theta-\hat{\Theta})^2\cdot h(\theta)\,d\theta$$

$$= \int_{-\infty}^{\infty}-2(\theta-\hat{\Theta})\cdot h(\theta)\,d\theta.$$

Setting the last integral equal to zero, we obtain

$$\int_{-\infty}^{\infty}-2(\theta-\hat{\Theta})\cdot h(\theta)\,d\theta = 0,$$

or

$$\int_{-\infty}^{\infty}\theta\cdot h(\theta)\,d\theta = \int_{-\infty}^{\infty}\hat{\Theta}\cdot h(\theta)\,d\theta,$$

or

$$E_{\Theta}[\Theta] = \hat{\Theta}\int_{-\infty}^{\infty}h(\theta)\,d\theta,$$

or

$$\hat{\Theta} = E_{\Theta}[\Theta],$$

since $\int_{-\infty}^{\infty}h(\theta)\,d\theta = 1$.

3-2 Under the squared error loss function, according to Corollary 3.1, the (Bayesian) point estimator $\hat{\Theta}$ which minimizes the expected value of the loss function is the mean of the posterior distribution of the parameter. Since we are told to assume that the mean of the posterior distribution is equal to its sample value, the desired point estimate is the sample mean:

$$\frac{1+(-3)+4+0+(-1)}{5} = .20.$$

3-3 $$E\left[L(\Theta,\hat{\Theta})\right] = \int_{-\infty}^{\infty} (\theta-\hat{\Theta})^2 \cdot f(\theta)\, d\theta$$

$$= \int_{-1}^{8} (\theta-.20)^2 \cdot \frac{1}{9}\, d\theta = \left.\frac{(\theta-.20)^3}{27}\right|_{-1}^{8} = 17.64.$$

3-4 The arithmetic mean of $\{.30, 2.10, 6.40, -.70, 6.90\}$ is

$$\frac{.30+2.10+6.40+(-.70)+6.90)}{5} = 3.$$

Hence,

$$E\left[L(\Theta,\hat{\Theta})\right] = \int_{-\infty}^{\infty} L(\theta,\hat{\Theta}) \cdot f(\theta)\, d\theta$$

$$= \int_{-1}^{8} \frac{(\theta-3)^2}{9}\, d\theta$$

$$= \left.\frac{(\theta-3)^3}{27}\right|_{-1}^{8} = \frac{125-(-64)}{27} = \left(\frac{189}{27}\right) = 7.$$

3-5 Under the absolute error loss function, according to Corollary 3.2, the (Bayesian) point estimator which minimizes the expected value of the loss function is the median of the posterior distribution of the parameter. Since we are told to assume that the median of the posterior distribution is equal to its sample value, the desired point estimate is the median of the observation set $\{1, -3, 4, 0, -1\}$, which is zero.

3-6 From Example 3.3, the probability density function is

$$f(\theta) = \begin{cases} \frac{1}{9} & -1 \le \theta \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

From Exercise 3-5, we obtain $\hat{\Theta} = 0$. Then

$$\begin{aligned} E\left[L(\Theta,\hat{\Theta})\right] &= \int_{-\infty}^{\infty} |\theta - \hat{\Theta}| \cdot f(\theta)\, d\theta \\ &= \int_{-1}^{8} |\theta| \cdot \frac{1}{9}\, d\theta \\ &= \int_{-1}^{0} -\theta \cdot \frac{1}{9}\, d\theta + \int_{0}^{8} \theta \cdot \frac{1}{9}\, d\theta = \left.\frac{-\theta^2}{18}\right|_{-1}^{0} + \left.\frac{\theta^2}{18}\right|_{0}^{8} \\ &= \frac{65}{18}. \end{aligned}$$

3-7 From Equation (3.8) with $c = 2$, we obtain the loss function

$$L(\Theta,\hat{\Theta}) = \begin{cases} 2 & \hat{\Theta} \ne \Theta \\ 0 & \hat{\Theta} = \Theta \end{cases}.$$

Since $\hat{\Theta} = 4$, the loss function becomes

$$L(\Theta,4) = \begin{cases} 2 & \Theta \ne 4 \\ 0 & \Theta = 4 \end{cases}.$$

The expected loss is

$$\begin{aligned} E_{\Theta}\left[L(\Theta,4)\right] &= \sum_{k=0}^{\infty} L(\Theta,4) \cdot P[\Theta = k] \\ &= \sum_{\substack{k=0 \\ k\ne 4}}^{\infty} c \cdot P[\theta = k] \\ &= c\left[1 - P[\Theta = 4]\right] = 2\left[1 - \binom{8}{4} p^4 (1-p)^4\right] \\ &= 2\left[1 - 70 p^4 (1-p)^4\right]. \end{aligned}$$

3-8 From Example 3.3 we have

$$f(\theta) = \begin{cases} \frac{1}{9} & -1 \le \theta \le 8 \\ 0 & \text{elsewhere} \end{cases}.$$

From Example 3.5 with $a = 1$ and $b = 5$, we have

$$L(\Theta, \hat{\Theta}) = \begin{cases} \Theta - \hat{\Theta} & \Theta > \hat{\Theta} \\ 5(\hat{\Theta} - \Theta) & \Theta < \hat{\Theta} \end{cases}.$$

Since we are given that $\hat{\Theta} = 0$, we can rewrite $L(\Theta, \hat{\Theta})$ as

$$L(\Theta, \hat{\Theta}) = \begin{cases} \Theta & \Theta > 0 \\ -5\Theta & \Theta < 0 \end{cases}.$$

So,

$$E\left[L(\Theta, \hat{\Theta})\right] = \int_{-1}^{0} \left(\frac{-5\theta}{9}\right) d\theta + \int_{0}^{8} \left(\frac{\theta}{9}\right) d\theta$$

$$= \left(\frac{-5}{9}\right)\left(\frac{\theta^2}{2}\right)\Bigg|_{-1}^{0} + \frac{\theta^2}{18}\Bigg|_{0}^{8} = \frac{23}{6}.$$

3-9 From Example 3.7 we obtain

$$E_{\Theta}\left[L\left(\Theta, \delta(\Theta)\right)\right] = 1 - P[a < \Theta < b],$$

where Θ has a standard normal distribution. We use tabulated results for the standard normal distribution to obtain

$$P[a < \Theta < b] = P[-1.96 < \Theta < 2.33] = .965.$$

Then the expected loss is

$$1 - P[-1.96 < \Theta < 2.33] = 1 - .965 = .035.$$

3-10 From Example 3.5 the loss function is

$$L(\Theta,\hat{\Theta}) = \begin{cases} a(\Theta-\hat{\Theta}) & \Theta > \hat{\Theta} \\ b(\hat{\Theta}-\Theta) & \Theta < \hat{\Theta} \end{cases}.$$

Integrating by parts (or applying Theorem 3.1 of Bowers, et al. [1986] as in the proof of Theorem 3.2 in the text), it follows that

$$E_\theta\left[L(\Theta,\hat{\Theta})\right] = \int_{\hat{\Theta}}^{\infty} a(\theta-\hat{\Theta})\, dF(\theta) + \int_{-\infty}^{\hat{\Theta}} b(\hat{\Theta}-\theta)\, dF(\theta),$$

where $F(\theta)$ denotes the posterior distribution of Θ. In order to determine the value of $\hat{\Theta}$ which minimizes the last expression, we first differentiate the right side of the last equation with respect to $\hat{\Theta}$ and set the result to zero, obtaining

$$-a\left[1-F(\hat{\Theta})\right]+b\cdot F(\hat{\Theta}) = 0.$$

or

$$(a+b)\cdot F(\hat{\Theta}) - a = 0,$$

or

$$F(\hat{\Theta}) = \frac{a}{a+b}.$$

Thus we have shown that the Bayesian point estimator which minimizes the expected value of the loss function is the $\frac{100a}{a+b}$ percentile of the (posterior) distribution of Θ.

3-11 Apply the result of Exercise 3-10 with $a=1$ and $b=2$, and note that

$$\frac{a}{a+b} = \frac{1}{1+2} = \frac{1}{3}.$$

3-12 I. True.

II. False. Differences that are not statistically significant may often be important practically.

III. False. It is preferable if the selected model does best under several different loss functions.