ACTEX ERM Study Manual
Spring 2015 Edition

Please replace pages B-45 through B-54 with the following pages.

Basel Committee – Technical Underpinnings of Aggregation Methods

Reviewer's note: *This resource reviews the most popular aggregation techniques currently in use, VarCovar, distribution-based methods using copulas and scenario based methods. It presumes substantial familiarity with copula mathematics and students are encouraged to refresh their knowledge of the same to get the most out of this paper.*

I. VarCovar Approach

A. Commonly used to combine marginal loss distributions into a single aggregate loss distribution or tail loss estimate. It's main advantages are;

1. Uses a limited number of inputs
2. Can be evaluated formulaically, and
3. Does not require fundamental information about lower-level risks

B. Statistical Foundation of VarCovar

1. The method presumes certain characteristics and relationships among the underlying loss distributions
2. If underlying distributions are completely independent, can simply add the lower-level capital requirements to get aggregate capital requirements
3. If the relationship between the lower level distributions can be represented in a covariance matrix cov(i,j), the aggregate risk can be measured a follows:
4. An expression[50] for aggregate risk under VarCovar is as follows:

$$R = \lambda\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \operatorname{cov}(i,j)} = \sqrt{\sum_{i=1}^{N} w_i^2 r_i^2 + 2\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} w_i w_j r_i r_j corr(i,j)}$$

where R is the aggregate risk or capital requirement, r are the lower-level risks which compose the aggregate risk (evaluated at a fixed confidence level), cov(i,j) is the covariance between variables i and j, corr(i,j) is their correlation, and w(i) are concentration weights for the lower-level risk sources (equal to 1 if lower-level risk is already scaled in the end units).

R can be any tail risk measure consistent with $\boldsymbol{\alpha . f(g,h,..;C) = f(\alpha . g(..), \alpha . h(..),..;C)}$, where f(..) is the aggregate tail risk corresponding to lower-level tail risks g(..), h(..), and so on; and correlation matrix C.

R/λ is the standard deviation of an aggregate loss distribution so long as

(a) each $r(i)/\lambda$ represents the standard deviation of the "i^{th}" lower-level loss distribution,
(b) the correlation matrix contains the true linear correlation coefficients between any two lower-level losses, and
(c) the expected loss in each distribution is assumed to be zero.

λ is the ratio of the tail risk value to the standard deviation; this is specific to the shape of the loss distribution and the choice of risk measure (eg, 99% VaR), but must be jointly applicable to both lower-level and aggregate risks.

C. Perfect linear dependence, independence and diversification
   1. Degree of diversification effect is controlled by the correlation matrix
   2. Assuming a matrix of *1*'s results in summing lower level risks to get aggregate risk (=assumed perfect linear correlation)
   3. Applying the identity matrix (*1*s on the diagonal, *0*s elsewhere) is equivalent to calculating aggregate risk as the square root of the sum of squared lower-level risks (=assumed linear independence)

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E. Correlations within the VarCovar
   1. For elliptical distributions such as normal or Gaussian, the correlation matrix adequately captures the interactions between variables
   2. For other types of distributions, additional information on dependence would be needed
   3. In particular, for capital requirements, the joint behavior in tail losses is more relevant than the correlations over the entire range
   4. Can substitute independently derived tail correlations in the matrix or make other subjective adjustments
   5. Subjective adjustments risk being guided by a desired outcome rather than modeling reality
   6. Alternatively, can use rank correlation measures independently of assumed marginal distributions to reflect greater conservatism in the tail correlation matrix
      a. Rank correlations are used to combine fat-tailed marginal distributions into jointly fat-tailed multivariate distributions that more realistically represent stress outcomes
      b. The calculation of aggregate risk measures using rank information is better suited to simulation via copula functions
   7. Another alternative: Use factor decomposition of lower level risks to determine the correlation
      a. In a pure factor model, the risk factors are orthogonal and the idiosyncratic component is represented by independent Gaussian draws
      b. The interdependence between lower level risks depends on their respective sensitivity to the factors and the variance of the idiosyncratic component

8. Top-down aggregation methods (including VarCovar and copula techniques) can be inadequate where the standalone risks are believed to be integrated (eg. market and credit risk)

F. Conclusions

1. VarCovar imposes a simple dependency structure on what is a more complex web of dependencies
2. Copulas are capable of specifying a full dependence structure with minimal constraints on the underlying distributions
3. The limitations of VarCovar can produce misleading results if the inherent assumptions do not coincide with experience

II. Distribution-based aggregation (copula methods)

A. Use entire loss distributions as inputs and allow direct control over the distributional and dependency assumptions

B. Are analytically complex and do not lend themselves to closed-form solutions

C. Definition

1. A copula is a random vector whose individual components are all random variables uniformly distributed over the interval [0,1]
2. Sklar's Theorem:
   a. Any multivariate distribution is uniquely determined by its marginal distributions and a copula
   b. Any combination of marginal distributions with a copula produces a valid multivariate distribution
3. The theorem can be used to build valid bottom-up multivariate distributions

D. Use of copulas for risk aggregation

1. If X is any continuous random variable and F_X is the distribution function of X, then $F_X(X)$ is distributed uniformly on the interval [0,1]
2. if U is a random variable that is uniformly distributed on the interval [0,1], the random variable $F_x^{-1}(U)$ (this is simply the U-th percentile of the random variable X) has the same distribution as X
3. Thus, we can simulate X by drawing random samples from a uniform [0,1] distribution and then evaluating the corresponding percentiles of X, given by the function F_x^{-1}, at the sampled points
4. Steps in using copulas to aggregate distributions
   a. Draw a joint sample of uniform random variables specified by the copula. $(\tilde{u}_1, \ldots, \tilde{u}_n)$ from the distribution
   b. Translate the sample from the copula distribution into a sample from the conjoined loss distribution by calculating the $\tilde{u}$ -th percentile of X_1, the $\tilde{u}$ -th percentile of X_2, etc. (in vector form, this is $\left(F_{x_q}^{-1}(\tilde{u}_1) \ldots F_{x_n}^{-1}(\tilde{u}_n)\right)$

c. Calculate the realized sample for the aggregate loss as the sum of the percentiles drawn from each distribution (ie. $F_{X_q}^{-1}(\tilde{u}_1)+\ldots+F_{X_n}^{-1}(\tilde{u}_n)$)

d. Drawing many samples for the aggregate loss distribution will produce a simulated distribution. Any measure of risk (such as VaR or expected shortfall) can be computed from this simulated distribution

E. Distribution functions of copulas

1. A copula can be described as a function C mapping the Euclidean cube [0,1]n to the interval [0,1]
2. The distribution function of C must satisfy the following conditions:
   a. Non-decreasing in each component
   b. Right continuity
   c. Limits of 0 and 1, and
   d. Rectangle inequality
3. Gaussian copula: Given any multivariate distribution function ***F*** having marginal distribution functions F_1 , $,F_n$, the function:

$$C(u_1,...,u_n)=\mathrm{F}(F_1^{-1}(u_1),....,F_n^{-1}(u_n))$$

4. Archimedean copulas: Defined by $C(u_1,...,u_n)=\varphi^{-1}(\varphi(u_1)+....+\varphi(u_n))$, where $\varphi:[0,1]\rightarrow(0,\infty)$ is a strictly decreasing, surjective, infinitely differentiable convex function
   a. can be simply described in closed form
   b. often require advanced techniques (such as Laplace transforms) to simulate
   c. are highly symmetric (which limits their use to risks that are uniform and homogeneous)

5. Gaussian copulas will not have this symmetry property unless all off-diagonal elements of the correlation matrix are the same

F. Measures of dependence for copulas

1. With copulas, the dependence structure between a set of random variables is encapsulated in the choice of the copula
2. If copula parameters are fit based on the standard correlations observed for a particular set of marginal distributions, the parameters are likely to lead to invalid results
3. To avoid this, we use measures of correlation that depend only on the copula itself, such as Spearman rho and Kendall tau
4. Rank correlation measures from observed data can be used to calibrate a copula directly
5. Large losses, either from different risk types or within the same risk type, tend to strike simultaneously during stress situations. This concept can be formalized through the definition of tail dependence

   Mathematically, this means that if $U1$ and $U2$ are the two uniform copula variables and ν is a value close to zero, the conditional probability:

$$Pr(U_1 \geq v \mid U_2 \geq v)$$

will be higher than v, which is the unconditional probability. Since this conditional probability can be expressed as:

$$\frac{Pr(U_1 \le v \, and U_2 \le v)}{Pr(U_2 \le v)} = \frac{C(v,v)}{v}$$

the coefficient of lower tail dependence for the copula is defined to be:

$$\lim_{v \to 0} \frac{C(v,v)}{v}$$

and the copula is said to exhibit lower tail dependence if this limit is greater than zero

F. Conclusions

1. The characteristics of the various copulas are summarized in the table below

Copula type:	**Gaussian**	**t**	**Archimedean**
Ease of simulation	Easy	Easy	Difficult
Capable of modelling tail dependence?	No	Yes	Yes
Symmetry	Symmetric in 2 dimensions, but generally asymmetric in higher dimensions		Standard construction is symmetric

2. Advantages of the copula approach:

   a) well suited for use in aggregating financial risks because it works directly with the percentile measures of the loss distributions

   b) easy to implement from a computational standpoint

   c) simulated losses can be stored and used for applications beyond aggregate loss modelling

3. Disadvantages

   a) the specification of a copula is very abstract and difficult to interpret

   b) fitting the parameters of a copula is a difficult statistical problem – the estimators used are often complex and not always robust

   c) implementing copulas requires a high level of statistical expertise on the part of the practitioner, and employees who use the output

III. Scenario-based aggregation aggregates risk expressions to common underlying scenarios

A. Determining risk drivers and exposures

1. Developing relevant scenarios requires a thorough knowledge of the business and its exposures
2. Need to identify the risk drivers for these exposures
3. A thorough analysis of the risks is necessary to adequately develop stress scenarios
4. In addition to stress scenarios, some companies use scenario generators to develop a broader spectrum of scenarios

B. Scenario simulation

1. Large series of scenarios are generated by independently drawing large numbers of random variables and processing the random draws through models that describe particular processes or phenomena
2. Three types of models

   a) Models that describe and proxy 'real physical processes, events or natural laws (eg. Pandemics) relying on dynamic modelling over time

   b) Models that describe processes for which there is no physical model (eg. Interest rates, equity prices), relying on a particular theory, calibrated to historical observations

   c) Models that combine the above
3. May use different scenario generators for different portfolios
4. Can combine results of for the different portfolios using VarCovar or copulas
5. In using simulation techniques, must determine how many simulation runs are necessary to adequately measure the risk profile

C. Conclusions

1. Eliminates ad-hoc methods of aggregation by aggregating exposures based on common scenarios
2. Results can be easily and meaningfully be interpreted in a real world context
3. Requires considerable understanding of the risks, extensive risk assessments , identification of risk drivers and exposure to these drivers
4. Strong reliance on expert judgment and qualitative insights
5. Potential for judgmental error or oversight
6. Require considerable computing power and IT resources

Reviewers note: *At the end of Annex G are two exhibits that are worth a careful reading. Box A lays out the properties of and examples of coherent risk measures. Box B compares and contrasts dependence versus correlation. Both exhibits are already in summary form and are not reproduced here.*

