Solutions Manual for

Mathematics of Investment and Credit

6th Edition

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CHAPTER 1

SECTION 1.1

1.1.1 Balances are $10,000(1.04)=10,400$ after one year,

$10,000(1.04)^2 = 10,816$ after 2 years, and

$10,000(1.04)^3 = 11,248.64$ after 3 years.

Interest amounts are 400 at the end of the 1st year, 416 at the end of the 2nd year, and 432.64 at the end of the 3rd year.

1.1.2 (a) $2500[1+(.04)(10)]=3500$

(b) $2500(1.04)^{10} = 3700.61$

(c) $2500(1.02)^{20} = 3714.87$

(d) $2500(1.01)^{40} = 3722.16$

1.1.3 Balance after 12 months is $10,000(1.01)^3(1.0075)^9 = 11,019.70$. Average monthly interest rate is j, where

$10,000(1+j)^{12} = 11,019.70$.

Solving for j results in .0081244.

1.1.4 There are two (equivalent) ways to approach this problem. We can update the balance in the account at the time of each transaction until we reach the end of 10 years, and set the balance equal to 10,000 to solve for *K*:

Balance at $t = 4$ (after interest and withdrawal) is

$$10{,}000(1.04)^4 - (1.05)K;$$

balance at $t = 5$ is

$$\left[10{,}000(1.04)^4 - (1.05)K\right](1.04) - (1.05)K;$$

balance at $t = 6$ is

$$\left[\left[10{,}000(1.04)^4 - (1.05)K\right](1.04) - (1.05)K\right](1.04) - K;$$

at $t = 7$ is

$$\left[\left[\left[10{,}000(1.04)^4 - (1.05)K\right]\right.\right.$$

$$\left.\left.(1.04) - (1.05)K\right](1.04) - K\right](1.04) - K;$$

at $t = 10$ there is 3 years of compounding from time 7, so that

$$\left[\left[\left[\left[10{,}000(1.04)^4 - (1.05)K\right](1.04) - (1.05)K\right]\right.\right.$$

$$\left.\left.(1.04) - K\right](1.04) - K\right](1.04)^3 = 10{,}000.$$

Solving for *K* from this equation results in $K = 979.93$.

Alternatively, we can accumulate to time 10 the initial deposit and the withdrawals separately. The balance at time 10 is

$$10{,}000(1.04)^{10} - K(1.05)(1.04)^6 - K(1.05)(1.04)^5$$

$$- K(1.04)^4 - K(1.04)^3 = 10{,}000.$$

This is the same equation as in the first approach (and must result in the same value of *K*). In general, when using compound interest for a series of deposits and withdrawals that occur at various points in time, the balance in an account at any given time point is the accumulated values of all deposits minus the accumulated values of all withdrawals to that time point. This is also the idea behind the "dollar-weighted rate of return," which will be discussed later.

1.1.5 (a) Over 5 years the unit value has grown by a factor of $(1.10)(1.16)(1-.07)(1.04)(1.32)=1.629074$. The average annual (compound) growth is $(1.629074)^{1/5}=1.1025$, or average annual growth of 10.25% for 5 years.

(b) Five-year average annual return from January 1, 2006 to December 31, 2015 is j, where $(1+j)^5(1.17)^5=(1.13)^{10}$, so that $j=.0914$. Annual return for 2014 is k, where $(1+k)(1.22)=(1.15)^2$, so that $k=.084$.

(c) Over n years the growth is $(1+i_1)(1+i_2)\cdots(1+i_n)=(1+i)^n$, where the average annual (compound) interest rate is i, so that $i+1$ is the geometric mean of $1+i_1, 1+i_2, \ldots, 1+i_n$. Thus, $1+i \le \frac{(1+i_1)+(1+i_2)+\cdots+(1+i_n)}{n} \rightarrow i \le \frac{i_1+i_2+\cdots+i_n}{n}$.

1.1.6 We equate the accumulated value of Joe's deposits with that of Tina's. Note that it is assumed that for simple interest, each new deposit is considered separately and begins earning simple interest from the time of the new deposit.

$$\begin{aligned} 10\left[1+(.11)\right]+30\left[1+5(.11)\right] &= 67.5 \\ &= 10(1.0915)^{10-n}+30(1.0915)^{10-2n}. \end{aligned}$$

This can be solved by substituting in the possible values of n until the equation is satisfied. Alternatively, the equation can be rewritten as

$$67.5(1.0195)^{2n}-10(1.0915)^{10}(1.0915)^n-30(1.0915)^{10} = 0$$

which is a quadratic equation in $(1.0915)^n$. The solution is

$$(1.0915)^n=\frac{24\pm 141.5}{135}.$$

We ignore the negative root and get $(1.0915)^n=1.226 \rightarrow n=2.3$.

1.1.7 (a) $1000 = 850\left[1 + i\left(\frac{60}{365}\right)\right] \rightarrow i = 1.0735(107.35\%)$

(b) $1000 = 900\left[1 + i\left(\frac{60}{365}\right)\right] \rightarrow i = .6759(67.59\%)$

(c) $900\left[1 + (.09)\left(\frac{60}{365}\right)\right] = 913.32$

(d) $900\left[1 + (.09)\left(\frac{d}{365}\right)\right] = 1000 \rightarrow d = 451$

1.1.8 It is to Smith's advantage to take the loan of 975 on the 7^{th} day if the amount payable on the 30th day is less than the amount due to the supplier:

$$975\left[1 + i\frac{23}{365}\right] \leq 1000 \rightarrow i \leq .4069.$$

1.1.9 (a) Maturity value of 180-day certificate is

$$100,000\left(1 + .075\left(\frac{180}{365}\right)\right) = 103,698.63.$$

Interim book value after 120 days is

$$100,000\left(1 + .075\left(\frac{120}{365}\right)\right) = 102,465.75.$$

Bank will pay X after 120 days so that

$$X\left(1 + .09\left(\frac{60}{365}\right)\right) = 103,698.63 \rightarrow X = 102,186.82.$$

The penalty charged is $102,465.75 - 102,186.82 = 278.93$.

(b) $1.08 = \left(1 + \frac{.075}{2}\right)\left(1 + \frac{i}{2}\right) \rightarrow i = .0819$

1.1.10 (a) $1000(1.12)^t = 3000 \;\rightarrow\; t = \frac{\ln(3)}{\ln(1.12)} = 9.694$ (9 years and approximately 253 days).

(b) At the end of 9 years the accumulated value is $1000(1.12)^9 = 2773.08$. At time s during the 10^{th} year, the accumulated value based on simple interest within the 10^{th} year is $2773.08(1+.12s)$. Setting this equal to 3000 and solving for s results in $s = \frac{\left(\frac{3000}{2773.08}\right)-1}{.12} = .6819$ years (approximately 249 days) after the end of 9 years.

(c) $1000(1.01)^t = 3000 \rightarrow t = \frac{\ln(3)}{\ln(1.01)} = 110.41$ months (about 9 years and 2 months and 13 days).

(d) $1000(1+i)^{10} = 3000 \;\rightarrow\; i = 3^{1/10} - 1 = .1161$ (11.61% per year).

(e) $1000(1+j)^{120} = 3000 \;\rightarrow\; i = 3^{1/120} - 1 = .009197$ (.9197% per month).

1.1.11 (a) $(1.0075)^{67/17} = 1.0299 < 1.03$

$\left(\text{but } (1.0075)^{68/17} = (1.0075)^4 = 1.0303\right)$

(b) $(1.015)^{67/17} = 1.0604 > 1.06$

1.1.12 (a) Smith buys $\frac{910}{4} = 227.5$ units after the front-end load is paid. Six months later she receives $(227.5)(5)(.985) = 1120.4375$. Smith's 6-month rate of return is 12.04% on her initial 1000.

(b) If unit value had dropped to 3.50, she receives $(227.5)(3.5)(.985) = 784.30625$, which is a 6-month effective rate of -21.57%.

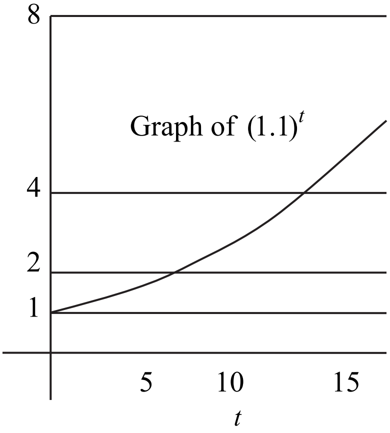
1.1.13 We use the following result from calculus: if f and g are differentiable functions such that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$, then $f(b) < g(b)$.

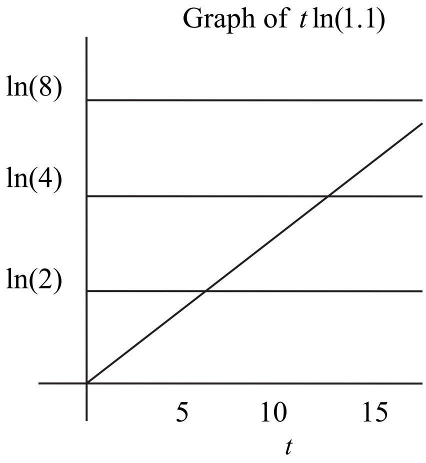
(i) Suppose $0 < t < 1$. Let $f(i) = (1+i)^t$ and $g(i) = 1 + i \cdot t$. Then $f(0) = g(0) = 1$. If we can show that $f'(i) < g'(i)$ for any $i > 0$, then we can use the calculus result above to conclude that $f(i) < g(i)$ for any $i > 0$. First note that $f'(i) = t \cdot (1+i)^{t-1}$ and $g'(i) = t$. Since $i > 0$, it follows that $1 + i > 1$, and since $t < 1$, it follows that $t - 1 < 0$. Then $(1+i)^{t-1} < 1$, so $f'(i) < g'(i)$.

This completes the proof of part (i).

(ii) Suppose that $t > 1$. Let $f(i) = 1 + i \cdot t$ and $g(i) = (1+i)^t$. Again $f(0) = g(0)$. If we can show that $f'(i) < g'(i)$ for any $i > 0$, then we can use the calculus result above to conclude that $f(i) < g(i)$ for any $i > 0$. Since $t > 1$ and $i > 0$ it follows that $t - 1 > 0$ and $1+i > 1$. Thus $(1+i)^{t-1} > 1$, and it follows that $f'(i) = t < t \cdot (1+i)^{t-1} = g'(i)$. This completes the proof of part (ii).

1.1.14 Original graph is $y = (1+i)^t$. New graph is $10^y = (1+i)^t$, or, equivalently, $y = t \cdot \frac{\ln(1+i)}{\ln(10)}$, so that y is now a linear function of t.





SECTIONS 1.2 AND 1.3

1.2.1 Present value is

$$5000\left[\frac{1}{1.06}+\frac{1}{(1.06)^2}+\frac{1}{(1.06)^3}+\frac{1}{(1.06)^4}\right] = 17{,}325.53.$$

1.2.2 Amount now required is

$$25{,}000[v^{17}+v^{15}+v^{12}]+100{,}000[v^{20}+v^{18}+v^{15}] = 75{,}686$$

1.2.3 $28 = 15+16.50v \rightarrow v = .78779 \rightarrow i = .2692$

1.2.4 $1000\cdot v_{.06}^3\cdot v_{.07}^4\cdot v_{.09}^3 = 494.62$

1.2.5 Equation of value on July 1, 2013 is

$$200(1.04)+300v = 100(1.04)^4 + X \rightarrow X = 379.48.$$

1.2.6 $480 = 50+100(v+v^2+v^3+v^4)+Xv^5$, where $v=\frac{1}{1.03}$, so that $X=67.57$. If interest is .01 per month, then $v = \frac{1}{(1.01)^3}$ and $X=67.98$.

1.2.7 $100+200v^n+300v^{2n} = 600v^{10} \rightarrow$
$600v^{10} = 100+200(.75941)+300(.75941)^2$
$\rightarrow v^{10} = .708155 \rightarrow i = (.708155)^{-.1}-1 = .0351.$

1.2.8 (a) $(20)(2000)[v+v^2+v^3+\cdots+v^{48}] = 1{,}607{,}391$ (at .75%)
(b) $1{,}607{,}391+200{,}000v^{48} = 1{,}747{,}114$
(c) $X = 1{,}607{,}391+.15Xv^{48} \rightarrow X = 1{,}795{,}551$

1.2.9 $750 = 367.85\left[1+(1+j)\right] \rightarrow j = .0389$ is the 2-month rate.

1.2.10 With X initially stocked, the number after 4 years is
$X(1.4)^4-5000\left[(1.4)^{1.5}+(1.4)^{.5}\right] = X \rightarrow X = 4997.$

1.2.11 $1000 = \frac{100}{(1+j)^2}+\frac{1000}{(1+j)^3}$, and $1000 = \frac{100}{1+k}+\frac{1000}{(1+k)^3}$. It is not possible that $j=k$, since the two present values could not both be equal to 1000 (unless $j=k=0$, which is not true). If $j>k$, then $(1+j)^2>1+k$ and $(1+j)^3>(1+k)^3$, in which case the first present value would have to be less than the second present value. Since both present values are 1000, it must be the case that $j<k$ ($j=.0333$ and $k=.0345$).

1.2.12 $1000(1{+}i)^2 + 1092 = 2000(1{+}i)$

Solving the quadratic equation for $1+i$ results in no real roots.

1.2.13 (a) $\frac{d}{di}(1{+}i)^n = n(1{+}i)^{n-1}$ (c) $\frac{d}{dn}(1{+}i)^n = (1{+}i)^n \ln(1{+}i)$

(b) $\frac{d}{di}v^n = -nv^{n+1}$ (d) $\frac{d}{dn}v^n = -v^n \ln(1{+}i)$

1.2.14 With an annual yield rate quoted to the nearest .01%, the annual yield i is in the interval $.11065 \le i < .11075$.

Since the quoted annual yield rate is $\frac{365}{182} \cdot \frac{100\text{–Price}}{\text{Price}}$ it follows that $.11065 \le \frac{365}{182} \cdot \frac{100\text{–Price}}{\text{Price}} < .11075$, or, equivalently,

$94.767 \le \text{Price} < 94.771$.

1.2.15 (a) $P = \frac{1000{,}000}{1+(.10)\frac{182}{365}} = 95{,}250.52$

(b) $P = \frac{100{,}000}{1+i\cdot\frac{182}{365}} \rightarrow \frac{dP}{di} = -\frac{100{,}000}{\left(1+i\cdot\frac{182}{365}\right)^2} \cdot \frac{182}{365}$

$= -45{,}239.03$ if $i = .10$

$\frac{dP}{di} \doteq \frac{\Delta P}{\Delta i} \doteq -45{,}239.03 \rightarrow \Delta P \doteq -45.239.03 \cdot \Delta i.$

If $\Delta i = .001$, then $\Delta P \doteq -45.24$.

(c) $P = \frac{100{,}000}{1+i\cdot\frac{91}{365}} \rightarrow \frac{dP}{di} = -\frac{100{,}000}{\left(1+i\cdot\frac{91}{365}\right)^2} \cdot \frac{91}{365} = -23{,}733.34$ if $i{=}.10$.

As the T-bill approaches its due date the $\frac{dP}{di}$ goes to 0.

1.2.16 (a) $B_1 = B_0(1+i) + \sum_{k=1}^{n} a_k\left[1+i(1-t_k)\right]$

$$= B_0 + \sum_{k=1}^{n} a_k + \left[B_0 + \sum_{k=1}^{n} a_k(1-t_k)\right]\cdot i$$

(b) The balance is B_0 for t_1 years, B_0+a_1 for t_2-t_1 years, $B_0+a_1+a_2$ for t_3-t_2 years, $B_0+a_1+a_2+\cdots+a_n$ for $1-t_n$ years. The average balance is

$$\overline{B} = \frac{B_0t_1 + (B_0+a_1)(t_2-t_1) + (B_0+a_1+a_2)(t_3-t_2) + \cdots + (B_0+a_1+a_2+\cdots+a_n)(1-t_n)}{t_1+(t_2-t_1)-(t_3-t_2)+\cdots+(1-t_n)}$$

$$= B_0 + a_1(1-t_1) + a_2(1-t_2) + \cdots + a_n(1-t_n)$$

$$= B_0 + \sum_{k=1}^{n} a_k(1-t_k).$$

(c) Follows directly from (a) and (b).

1.2.17 The difference between the two payment plans is that the first 2 payments are deferred for 2 months, so the saving is

$$30\left[(1+v)-(v^{24}+v^{25})\right] = 12.68.$$

Alternatively, the present value under the current payment plan is

$$30[1+v+v^2+\cdots+v^{23}] = 643.67.$$

The present value under Smith's proposed payment plan is

$$30[v^2+\cdots+v^{25}] = 643.67v^2 = 630.99.$$

Saving is 12.68.

1.2.18 (a) Minimum monthly balance for January 2015 is 2500, for February 2015 is 6000, and for March 2015 is 9500. Interest earned is $(.10)\left(\frac{1}{12}\right)[2500+6000+9500] = 150.$

Balance on March 31 is $2500(4)+1000(3)+150 = 13{,}150.$

(b) Minimum daily balance is 2500 for January 1-15, 3500 for January 16-31, 6000 for February 1-15, 7000 for February 16-28, 9500 for March 1-15, and 10,500 for March 16-31. Interest earned is

$$(.10)\left(\frac{1}{365}\right)\big((2500(15)+3500(16)+6000(15)+7000(13) \\ +9500(15)+10{,}500(16)\big) = 160.27.$$

Balance on March 31 is

$$2500(4)+1000(3)+160.27 = 13{,}160.27.$$

(c) Minimum monthly balance for January 2015 is 2500, so interest on January 31 is $\frac{.10}{12}(2500) = 20.83,$ so balance on January 31 (after deposit and interest) is 6020.83.

Minimum monthly balance for February 2015 is 6020.83, so interest on February 28 is $\frac{.10}{12}(6020.83) = 50.17,$ so balance on February 28 is 9571.00.

Minimum monthly balance for March 2015 is 9571, so interest on March 31 is $\frac{.10}{12}(9571) = 79.76,$ so balance on March 31 is 13,150.76.

(d) Minimum daily balance is 2500 for January 1-15 and 3500 for January 16-31, so interest on January 31 is 25.62.

Minimum daily balance is 6025.62 for February 1-15 and 7025.62 for February 16-28, so interest on February 28 is 49.79.

Minimum daily balance is 9575.41 for March 1-15 and 10,575.41 for March 16-31, so interest on March 31 is 85.71.

Balance on March 31 is 13,161.12.

SECTION 1.4

1.4.1 $m=1$ implies interest convertible annually (m=1 time per year), which implies the effective annual interest rate $i^{(1)}=i=.12$. We use Equation (1.5) to solve for i for the other values of m, as shown below.

m (Effective Period)	$\frac{1}{m}$-year effective interest rate $\frac{i^{(m)}}{m}$	$i=\left[1+\frac{i^{(m)}}{m}\right]^m-1$
1 (1 year)	$\frac{i^{(1)}}{1}=\frac{.12}{1}=.12$	$(1.12)^1-1=.12$
2 (6 months)	$\frac{i^{(2)}}{2}=\frac{.12}{2}=.06$	$(1.06)^2-1=.1236$
3 (4 months)	$\frac{i^{(3)}}{3}=\frac{.12}{3}=.04$	$(1.04)^3-1=.124864$
4 (3 months)	$\frac{i^{(4)}}{4}=\frac{.12}{4}=.03$	$(1.03)^4-1=.125509$
6 (2 months)	$\frac{i^{(6)}}{6}=\frac{.12}{6}=.02$	$(1.02)^6-1=.126162$
8 (1.5 months)	$\frac{i^{(8)}}{8}=\frac{.12}{8}=.015$	$(1.015)^8-1=.126593$
12 (1 months)	$\frac{i^{(12)}}{12}=\frac{.12}{12}=.01$	$(1.01)^{12}-1=.126825$
52 (1 week)	$\frac{i^{(52)}}{52}=\frac{.12}{52}=.0023$	$\left(1+\frac{.12}{52}\right)^{52}-1=.12689$
365 (1 day)	$\frac{i^{(365)}}{365}=\frac{.12}{365}=.00033$	$\left(1+\frac{.12}{365}\right)^{365}-1=.127475$
∞	$\lim_{y\to\infty}\left(1+\frac{.12}{y}\right)^y-1=e^{.12}-1=.127497$	

1.4.2 (a) $1000v_{.045}^{20}=414.64$

(b) $1000v_{.015}^{60}=409.30$

(c) $1000v_{.0075}^{120}=407.94$

1.4.3 Equivalent effective annual rates are

Mountain Bank: $(1.075)^2 - 1 = .155625$

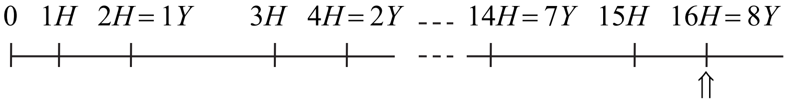
River Bank:

$$\left(1+\frac{i^{(365)}}{365}\right)^{365} - 1 \ge .155625$$

$$\to \left(1+\frac{i^{(365)}}{365}\right) \ge (1.155625)^{1/365} = 1.000396356$$

$$\to i^{(365)} \ge .144670$$

1.4.4 The last 6 months of the 8^{th} year is the time from the end of the 15^{th} to the end of the 16^{th} half-year.



The amount of interest earned in Eric's account in the 16^{th} half-year is the change in balance from time $15H$ to time $16H$.

The balances at those points are $X\left(1+\frac{i}{2}\right)^{15}$ and $X\left(1+\frac{i}{2}\right)^{16}$. The amount of interest earned by Eric in the period is

$$X\left(1+\frac{i}{2}\right)^{16} - X\left(1+\frac{i}{2}\right)^{15} = X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right).$$

The balance in Mike's account at the end of the 15^{th} half-year (7.5 years) is $2X(1+7.5i)$, and the balance at the end of the 16^{th} half-year (8 years) is $2X(1+8i)$.

The interest earned by Mike in that period is

$$2X(1+8i) - 2X(1+7.5i) = 2X(.5i) = 2X\left(\frac{i}{2}\right).$$

We are told that Eric and Mike earn the same amount of interest. Therefore, $X\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right) = 2X\left(\frac{i}{2}\right)$, so that $\left(1+\frac{i}{2}\right)^{15} = 2$.

Then $i = 2[2^{1/15} - 1] = .0946$.

1.4.5 Quarterly effective rate is $\frac{.0325}{4}=.008125$. Initial amount invested after commission is .99. At the end of 3 months, the accumulated value is .99(1.008125). This is then subject to the 1% commission for the rollover and then the 3-month interest rate of .008125. At the end of the year, the accumulated value is $[.99(1.008125)]^4 = .992198 = 1-.0078$. The effective after-commission return is $-.78\%$.

1.4.6
$$\begin{aligned} i^{(.5)} &= .5\left[(1.10)^{1/.5}-1\right] = .105 \\ i^{(.25)} &= .25\left[(1.1)^{1/.25}-1\right] = .116025 \\ i^{(.1)} &= .10\left[(1.10)^{1/.1}-1\right] = .159374 \\ i^{(.01)} &= .01\left[(1.10)^{1/.01}-1\right] = 137.796 \end{aligned}$$

1.4.7 From November 9 to January 1 (53 days) Smith earns (two full months) interest of $\frac{2}{12}(.1125)(1000)=18.75$. Thus, Smith earns a 53-day effective rate of interest of .01875. The equivalent effective annual rate of interest is $i=(1.01875)^{365/53}-1 = .1365$.

1.4.8 Left on deposit for a year at $i^{(12)}=.09$, X accumulates to $X(1.0075)^{12}$. If the monthly interest is reinvested at monthly rate .75%, the accumulated value at the end of the year is

$$X+X(.0075)\left[(1.0075)^{11}+(1.0075)^{10}+\cdots+(1.0075)+1\right].$$

Since $1+r+r^2+\cdots+r^k=\frac{r^{k+1}-1}{r-1}$, it follows that the total at the end of the year with reinvestment of interest is

$$X\left[1+(.0075)\cdot\frac{(1.0075)^{12}-1}{1.0075-1}\right] = X(1.0075)^{12}.$$

1.4.9 (a) We wish to show that

$$f'(m) = f(m)\cdot\left[\ln\left(1+\frac{j}{m}\right)-\frac{\frac{j}{m}}{1+\frac{j}{m}}\right] > 0.\text{ First, } f(m)>0,$$

since $j>0$. Also, if $x>0$ and $h(x) = \ln(1+x)-\frac{x}{1+x}$, then $h'(x) = \frac{x}{(1+x)^2} > 0$, and since $h(0)=0$, it follows that $h(x)>0$ for all $x>0$. Letting $x=\frac{j}{m}>0$, we see that $\ln\left(1+\frac{j}{m}\right)-\frac{\frac{j}{m}}{1+\frac{j}{m}} > 0$, which implies that $f'(m)>0$.

(b) $g'(m) = (1+j)^{1/m}-1-\frac{(1+j)^{1/m}\cdot\ln(1+j)}{m}$

$$= (1+j)^{1/m}\cdot\left[(1-\ln(1+j)^{1/m}\right]-1.$$

But $x[1-\ln x]$ has a maximum of 1 at $x=1$, so that with $(1+j)^{1/m}=x$, we see that $g'(m)<0$ for $m>1$.

(c) Consider $\ln[f(m)] = m\cdot\ln\left(1+\frac{j}{m}\right) = \frac{\ln\left(1+\frac{j}{m}\right)}{\frac{1}{m}}$. Then

$$\lim_{m\to\infty}\ln[f(m)] = \lim_{m\to\infty}\frac{\ln\left(1+\frac{j}{m}\right)}{\frac{1}{m}} = \lim_{m\to\infty}\frac{\frac{1}{1+(j/m)}\cdot\left(-\frac{j}{m^2}\right)}{-\frac{1}{m^2}} = j.$$

Thus $\lim_{m\to\infty} f(m)=e^j$.

(d) $g(m) = \frac{(1+j)^{1/m}-1}{\frac{1}{m}}$, $\lim_{m\to\infty} g(m) = \lim_{m\to\infty}\frac{(1+j)^{1/m}\cdot\left(-\frac{\ln(1+j)}{m^2}\right)}{-\frac{1}{m^2}}$

$= \ln(1+j)$, since $\lim_{m\to\infty}(1+j)^{1/m} = 1$.

1.4.10 We want to find the smallest integer m so that

$f(m) = \left[1+\frac{.17}{m}\right]^m \geq 1.18$, $f(2) = 1.1772$, $f(3) = 1.1798$, $f(4) = 1.1811 \to m = 4$.

With 16%, we see that $\lim_{m\to\infty}\left[1+\frac{.16}{m}\right]^m = e^{.16} = 1.1735$, so that no matter how many times per year compounding takes place, a nominal rate of interest of 16% cannot accumulate to an effective rate of more than 17.35%.

SECTION 1.5

1.5.1 (a) $4992 = \frac{X}{(1.08)^{1/2}} \to X = 5187.84$

(b) $4992 = \frac{X}{\left[1+(.08)\left(\frac{1}{2}\right)\right]} \to X = 5191.68$

(c) $4992 = X(1-.08)^{1/2} \to X = 5204.52$

(d) $4992 = X\left[1-(.08)\left(\frac{1}{2}\right)\right] \to X = 5200$

1.5.2 With a quoted discount rate of .940, the price of a 91-day T-Bill should be $100(1-\frac{28}{360}\times.00050) = 99.996111$ as quoted.

The investment rate is found as
$\left(\frac{100}{99.996111}-1\right)\times\frac{365}{28} = .00051,$ as quoted.

1.5.3 $d_j = \frac{A(r+T)-A(r)}{A(r+T)} = 1-\frac{A(r)}{A(r+T)}$, and

$j = \frac{A(r+T)-A(r)}{A(r)} = \frac{A(r+T)}{A(r)}-1$

$\to 1-d_j = \frac{A(r)}{A(r+T)} = \frac{1}{1+j} \to$

(a) $d_j = \frac{j}{1+j}$ and (b) $j = \frac{d_j}{1-d_j}$.

1.5.4 $1.15 = (1-d)(1.3) \to d = .1154$

1.5.5 Bruce’s interest in year 11: $100(1-d)^{-10}\cdot\left[(1-d)^{-1}-1\right] = X$.
Robbie’s interest in year 17:

$$50(1-d)^{-16}\cdot\left[(1-d)^{-1}-1\right] = X = 100(1-d)^{-10}\cdot\left[(1-d)^{-1}-1\right]$$
$$\to 50(1-d)^{-16} = 100(1-d)^{-10} \to (1-d)^6 = .5 \to d = .1091$$
$$\to X = 38.9$$

1.5.6 The present value of 1 due in n years is $(1-d)^n$, so the accumulated value after n years of an initial investment of 1 is

$$\frac{1}{(1-d)^n} = (1-d)^{-n}.$$

1.5.7 The initial deposit of 10 grows to $10\left(1-\frac{d}{4}\right)^{-40}$ at the end of 10 years (40 quarters), and then continues to grow at 3% per half year after that. The accumulated value of the initial deposit of 10 at the end of 30 years is $10\left(1-\frac{d}{4}\right)^{-40} \times (1.03)^{40}$ (20 more years, 40 more half-years at 3% per half-year).

The second deposit is 20 made at time 15. The accumulated value of the second deposit at time 30 (15 years after the second deposit) is $20(1.03)^{30}$ (15 years is 30 half-years).

The total accumulated value at the end of 30 years is

$$10\left(1-\frac{d}{4}\right)^{-40} \times (1.03)^{40} + 20(1.03)^{30} = 100.$$

Solving for d results in $d = .0453$.

This question is from the May 2003 Course 2 exam that was conducted jointly by the Society of Actuaries and the Casualty Actuarial Society. It should be noted that the nominal interest rate notation $i^{(m)}$ and nominal discount rate notation $d^{(m)}$ is not always specifically used on the professional actuarial exams. In this example, the notation d was a nominal annual rate of discount compounded quarterly.

1.5.8 (a) Bank pays

$$1-d\cdot\frac{n}{365} = \frac{1}{1+i\cdot\frac{n}{365}} \rightarrow i = \frac{365}{n}\left[\frac{1}{1-d\cdot\frac{n}{365}}-1\right] = \frac{d}{1-d\cdot\frac{n}{365}}.$$

As n increases, i increases.

(b) From (a) $1-dt = \frac{1}{1+it} \rightarrow d = \frac{i}{1+it}$. If $i = .11$ then

$t = 1 \rightarrow d = .099099$, $t = .50 \rightarrow d = .104265$,

$t = \frac{1}{12} \rightarrow d = .109001$.

1.5.9 Suppose that the T-Bill's face amount is \$100. Then Smith purchases the bill for $100\left[1-\frac{182}{360}(.10)\right] = 94.94$ (nearest .01). 91 days later, the value of the T-Bill is

$$100\left[1-\frac{91}{360}(.10)\right] = 97.47.$$

Smith's return for the 91 days is $\frac{97.47}{94.94}-1 = .0266$ (2.66%).

1.5.10 From Exercise 1.5.3, we have $\frac{d^{(m)}}{m}=d_j$, and $\frac{i^{(m)}}{m}=j$, so

(a) $\frac{d^{(m)}}{m}=d_j=\frac{j}{1+j}=\frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}} \rightarrow d^{(m)}=\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}$, and

(b) $i^{(m)}=\frac{d^{(m)}}{1-\frac{d^{(m)}}{m}}$.

1.5.11 $1000 = 1200\left(\frac{1}{1+i}\right)(1-i) \rightarrow i = .0909$

1.5.12 $1000(1+j) = 1000+40(1+j)^2 \rightarrow (1+j)^2-25(1+j)+25 = 0$ $\rightarrow 1+j = 1.043561$ or 23.9564. Since $j<.10$, it follows that $j=.0436$.

SECTION 1.6

1.6.1 Accumulated value at time 1 is

$$10,000\times e^{\int_0^1 .05\,dt} = 10,000\times e^{.05} = 10,512.71.$$

Accumulated value at time 2 is

$$10,000\times e^{\int_0^1 .05\,dt+\int_1^2[.05+.02(t-1)]\,dt} = 10,000\times e^{.05+.06} = 11,162.78$$

1.6.2 $\left(1+\frac{i^{(4)}}{4}\right)^{16} = e^{\int_0^3 .02t\,dt+\int_3^4 .045dt} = e^{.09+.045} \rightarrow i^{(4)} = .0339.$

1.6.3 $\exp(\int_0^5 \frac{t^2}{k}\,dt) = (1-.04)^{-10} \rightarrow e^{125/3k} = 1.50414$

$\rightarrow \frac{125}{3k} = \ln(1.50414) \quad \rightarrow k = 102.$

1.6.4 Effective annual rate for Tawny is $i = (1.05)^2 - 1 = .1025.$
Tawny: $\delta = \ln(1.1025) = .09758.$

Fabio: Simple interest rate $j \rightarrow \delta_t = \frac{j}{1+tj}.$

At time 5,
$.09758 = \frac{j}{1+5j} \rightarrow j = .1906 \rightarrow Z = 1000[1+5j] = 1953.$

1.6.5 $100\left[e^{\int_0^6 .01t^2\,dt} - e^{\int_0^3 .01t^2\,dt}\right] + X\left[e^{\int_3^6 .01t^2\,dt} - 1\right] = X$

$\rightarrow 100(e^{.72} - e^{.09}) = X(2 - e^{.63}) \rightarrow X = 784.6.$

1.6.6 Bruce's 6-month rate of interest is $\frac{i}{2}$, and 7.25 years is 14.5 6-month periods. Bruce's accumulated value after 7.25 years is $100\left(1+\frac{i}{2}\right)^{14.5} = 200.$ Solving for *i*, we get

$$\left(1+\frac{i}{2}\right) = 2^{1/14.5} \rightarrow i = .0979.$$

Peter's account grows to $100e^{7.25\delta} = 200$, so that
$\delta = \frac{1}{7.25}\ln 2 = .0956.$ Then $i - \delta = .0023 = .23\%.$

1.6.7 $e^{\delta} \cdot (e^{1.5\delta})^4 = 1.36086 \rightarrow e^{7\delta} = 1.36086 \rightarrow 1+i = e^{\delta} = 1.045$

1.6.8 (a) $(1+i)^5 = \exp\left[\int_0^5 \left(.08+\frac{.025t}{t+1}\right) dt\right] = 1.616407 \rightarrow i = .1008$

(b) $1+i_1 = \exp\left[\int_0^1 \left(.08+\frac{.025t}{t+1}\right) dt\right] = 1.091629 \rightarrow i_1 = .091629$

$1+i_2 \;=\; \exp\left[\int_1^2 \left(.08+\frac{.025t}{t+1}\right) dt\right] = 1.099509 \rightarrow i_2 = .099509$

$i_3 = .102751, \quad i_4 = .104532, \quad i_5 = .105659$

(c) $1000 \cdot \exp\left[-\int_2^4 \left(.08+\frac{.025t}{t+1}\right) dt\right] \;=\; 821.00$

1.6.9 $Ke^{-2\delta} = 960,\ Ke^{-\delta} = 1200 \;\rightarrow\; e^{-\delta} = 1-d = .80 \;\rightarrow\; K = 1500$ and $d = .20$. If d changes to .10, then the present value becomes $1500(1-.10)^2 = 1215$.

1.6.10 $i \;= e^{\delta} - 1,\ \delta' = 2\delta \;\rightarrow$

$i' = e^{\delta'} - 1 = e^{2\delta} - 1 = (1+i)^2 - 1 = 2i + i^2 > 2i,$

$d' = 1 - e^{\delta'} = 1 - e^{-2\delta} = 1 - (1-d)^2 = 2d - d^2 < 2d$

1.6.11 (a) $1000(1.02)^2\left[1+(.08)\left(\frac{19}{365}\right)\right] \;=\; 1044.73$

(b) For $0 < t \le \frac{1}{4},\ A(t) \;=\; 1000[1+(.08)t]$

for $\frac{1}{4} \le t \le \frac{1}{2},\ A(t) = 1000(1.02)\left[1+(.08)\left(t-\frac{1}{4}\right)\right]$

for $\frac{1}{2} \le t \le \frac{3}{4},\ A(t) = 1000(1.02)^2\left[1+(.08)\left(t-\frac{1}{2}\right)\right]$

for $\frac{3}{4} \le t \le 1,\ A(t) = 1000(1.02)^3\left[1+(.08)\left(t-\frac{3}{4}\right)\right]$

(c) For $0 < t = \frac{1}{4}$, $\delta_t = \frac{S'(t)}{S(t)} = \frac{.08}{1+(.08)t}$.

To find $\delta_{t+1/4}$, let $r = t+\frac{1}{4}$, or $t = r-\frac{1}{4}$.

Then

$$\delta_{t+1/4,} = \delta_r = \frac{S'\left(t+\frac{1}{4}\right)}{S\left(t+\frac{1}{4}\right)}$$

$$= \frac{S'(r)}{S(r)} = \frac{1000(1.02)(.08)}{1000(1.02)\left[1+(.08)\left(r-\frac{1}{4}\right)\right]} = \frac{.08}{1+(.08)t}.$$

The same occurs for $t+\frac{1}{2}$ and $t+\frac{3}{4}$.

1.6.12 (a) $\frac{A\left(t+\frac{1}{m}\right)-A(t)}{A\left(t+\frac{1}{m}\right)}$

(b) $d^{(m)} = m\cdot\frac{A\left(t+\frac{1}{m}\right)-A(t)}{A\left(t+\frac{1}{m}\right)} = \frac{A\left(t+\frac{1}{m}\right)-A(t)}{\frac{1}{m}\cdot A\left(t+\frac{1}{m}\right)}$

(c) Let $h=\frac{1}{m}$. Then $\lim\limits_{m\to\infty} d^{(m)} = \lim\limits_{h\to 0}\frac{A(t+h)-A(t)}{h\cdot A(t+h)} = \frac{A'(t)}{A(t)}$.

1.6.13 (a) $\delta_t = \frac{A'(t)}{A(t)} = \frac{a_1+2a_2t+\cdots+na_nt^{n-1}}{a_0+a_1t+\cdots+a_nt^n} \to \lim\limits_{t\to\infty}\delta_t = 0$

(apply l'Hospital's rule)

(b) $A(t) = \exp\left[\int_0^t \delta_s\, ds\right] = \exp[k\cdot 2\cdot t^{1/2}]$.

$$\lim_{t\to\infty}\frac{A(t)}{1+it} = \lim_{t\to\infty}\frac{e^{2kt^{1/2}}\cdot\frac{k}{t^{1/2}}}{i} = \infty$$

$$\lim_{t\to\infty}\frac{A(t)}{(1+i)^t} = \lim_{t\to\infty}\frac{e^{2kt^{1/2}}\cdot\frac{k}{t^{1/2}}}{(1+i)^t\cdot\ln(1+i)}$$

$$= \lim_{t\to\infty}\frac{1}{t^{1/2}\ln(1+i)}\cdot\frac{1}{\exp\left[t\cdot\ln(1+i)-2kt^{1/2}\right]} = 0$$

SECTION 1.7

1.7.1 $i_{real} = \frac{i-r}{1+r} = \frac{.10-.15}{1+.15} = -.043478$

1.7.2 After-tax return is $\frac{(.12)(.55)-.10}{1.10} = -.0309.$

1.7.3 (a) Smith's ATI this year will be

$$21{,}000(.75)+21{,}000(.50) = 26{,}250$$

and taxes paid will be 15,750. The real growth from last year to this year in Smith's ATI is $\frac{26{,}250/25{,}000}{1.05} = 1.00$, and the real growth in taxes paid is $\frac{15{,}750/15{,}000}{1.05} = 1.00.$

(b) Continuing the old taxation scheme, Smith's taxes paid this year will be $(.25)(20{,}000)+(.50)(22{,}000) = 16{,}000$, and his ATI will be 26,000. The real growth in taxes paid will be $\frac{16{,}000/15{,}000}{1.05} = 1.015873\,(1.59\%)$ and the real growth in ATI is $\frac{26{,}000/25{,}000}{1.05} = .990476 = 1-.009524(-.95\%).$

1.7.4 Smith sells the items for $100{,}000\times 1.15 = 115{,}000$ at the end of the year and must pay back $100{,}000\times 1.10 = 110{,}000.$ Net gain is 5,000 (in year-end dollars).

1.7.5 $e^{\delta_{real}} = 1+i_{real} = \frac{1+i}{1+r} = \frac{e^{\delta}}{e^{\delta_r}} = e^{\delta-\delta_r}$

1.7.6 $\frac{.18-.14}{1.14} = \frac{i-1}{2} \rightarrow i = 1.070175\,(107\%)$

1.7.7 Smith needs $\frac{1000}{1.09}=917.4312$ US now if he invests in the US account. This is equivalent to $\frac{917.4312}{.73}=1256.7551$ Cdn., which grows to 1382.4306 in one year in a Canadian dollar account earning 10%. The implication is that one year from now, 1000 US $\equiv$ 1382.4306 Cdn., or, equivalently, .723364 US $\equiv$ 1 Cdn.

1.7.8 $$(1+r)^n \cdot v^n = \left(\frac{1+r}{1+i}\right)^n = \frac{1}{\left(\frac{1+i}{1+r}\right)^n} = \frac{1}{(1+i')^n}$$

$$\rightarrow 1+i' = \frac{1+i}{1+r} \rightarrow i' = \frac{i-r}{1+r}$$

1.7.9 (a) Real after-tax rate of return on standard term deposit is $\frac{i(1-t_x)-r}{1+r}$, and on the indexed term deposit is

$$\frac{r+i'(1+r)(1-t_x)-r}{1+r} = i'(1-t_x).$$

(b) Setting the two expressions in part (a) equal and solving for i, we have $i=i'(1+r)+\frac{r}{1-t_x}$.

If $i'=.02$ and $r=.12$, then $i=$ (i) .1424, (ii) .1824, (iii) .2224, (iv) .3224.