ACTEX

PUBLICATIONS

The Experts In Actuarial Career Advancement

Product Preview

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BOND VALUATION

"Gentlemen prefer bonds."
– Andrew Mellon, 1855-1937

It is often necessary for corporations and governments to raise funds to cover planned expenditures. Corporations have two main ways of doing so; one is to issue **equity** by means of common (or preferred) shares of ownership (stocks) which usually give the shareholder a vote in deciding the way in which the corporation is managed. The other is to issue **debt**, which is to take out a loan requiring interest payments and repayment of principal. For borrowing in the short term, the corporation might obtain a **demand loan** (a loan that must be repaid at the lender's request with no notice) or a **line of credit** (an account which allows the borrower to maintain outstanding balances up to a specified maximum amount, with periodic interest payable). For longer term borrowing it is possible to take out a loan that is amortized in the standard way, but this would usually be done only for loans of a relatively small amount. To borrow large amounts over a longer term a corporation can issue a **bond**, also called a **debenture** (from the Latin word "debentur", meaning "they are due"). A bond is a debt that usually requires periodic interest payments called **coupons** (at a specified rate) for a stated term and also requires the return of the principal at the end of the term. It will often be the case that the amount borrowed is too large for a single lender or investor, and the bond is divided into smaller units to allow a variety of investors to participate in the issue.

Definition 4.1 – Bond

A bond is an interest-bearing certificate of public (government) or private (corporate) indebtedness.

Governments generally have the option of raising funds via taxes. Governments also raise funds by borrowing, in the short term by issuing Treasury bills, and in the longer term by issuing coupon bonds (called Treasury notes for maturities of 10 years or less). Government **savings bonds** pay periodic interest and might not have a fixed maturity date, and

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can usually be redeemed by the owner of the bond at any time for the return of principal and any accrued interest. Savings bonds would be purchased and held by individual investors, while government T-Bills and coupon bonds are held by individuals, financial institutions such as insurance companies and banks, and other investors. Bonds are crucial components in government and corporate financing.

The initial purchaser of a bond might not retain ownership for the full term to maturity, but might sell the bond to another party. Ownership of the bond refers to the right to receive the payments specified by the bond. There is a very active and liquid secondary market in which bonds are bought and sold. Through this market bonds also provide an important investment vehicle, and can make up large parts of pension funds and mutual funds. Bonds issued by corporations are usually backed by various corporate assets as collateral, although a type of bond called a **junk bond** has been used with little or no collateral, often to raise funds to finance the takeover of another company. Bonds issued by financially and politically stable governments are virtually risk-free and are a safe investment option. There are agencies that rate the risk of default on interest and principal payments associated with a bond issuer. The purchaser of a bond will take into account the level of risk associated with the bond when determining its value.

The risk of loss of principal or loss of a financial reward stemming from a borrower's failure to repay a loan or otherwise meet a contractual obligation is referred to as **credit risk**. Credit risk arises whenever a borrower is expecting to use future cash flows to pay a current debt, such as in the case of a bond. Investors are compensated for assuming credit risk by way of interest payments from the borrower or issuer of a debt obligation. Credit risk is closely tied to the potential return of an investment, the most notable being that the yields on bonds correlate strongly to their perceived credit risk.

4.1 Determination of Bond Prices

A bond is a contract that specifies a schedule of payments that will be made by the issuer to the bondholder (purchaser). The most common type of bond issue is the **straight-term bond**, for which the schedule of payments is similar to that of a loan with regular payments of interest plus a single payment of principal at the end of the term of the loan. A bond specifies a **face amount** and a **bond interest rate**, also called the **coupon rate**, which are analogous

to the principal amount of a loan and rate at which interest is paid. The bond also specifies the **maturity date** or **term to maturity** during which the **coupons** (bond interest payments) are to be paid, and the **redemption amount** that is to be repaid on the maturity date. It is generally the case that the face amount and the redemption amount on a bond are the same, and this will be assumed to be the case throughout this chapter unless specified otherwise.

For bonds issued in Canada and the United States, the coupons are nearly always paid semiannually, with the coupon rate quoted on a nominal annual basis. (Some bonds issued in some European countries have coupons payable annually). Unless specified otherwise, when coupon rates are quoted in this chapter they will refer to annual rates payable semiannually, with the first coupon payable one period after the bond is issued, and with coupons payable up to and including the time at which the redemption amount is paid. Note that bonds may be issued on a non-coupon date, in which case the first coupon is paid less than one coupon period after issue.

The website of the U.S. Bureau of the Public Debt has a great deal of information about bonds issued by the U.S. Government. At that website, a historical record of all notes, bonds (and T-Bills) issued by the U.S. Government is available. A **Treasury Note** is a bond with a term to maturity of less than 15 years when issued, and a **Treasury Bond** has a term to maturity of 15 or more years when issued. An excerpt from that website is in Table 4.1. In the table, "CUSIP" is a Treasury assigned identifier for the T-bill or bond.

TABLE 4.1

Historical Securities Search Results Treasury Notes and Bonds							
Security Term	Auction Date	Issue Date	Maturity Date	Interest Rate %	Yield %	Price Per $100	CUSIP
2-YR	03-24-2015	03-31-2015	03-31-2017	0.500	0.598	99.805456	912828J92
30-YR	08-14-2014	08-15-2014	08-15-2044	3.125	3.224	98.105640	912810RH3

The first bond in the website excerpt has the following characteristics:

Issue Date:	March 31, 2015
Maturity Date:	March 31, 2017
Interest Rate:	0.500%
Yield Rate:	0.598%
Price per $100:	99.805456

This describes a 2-year Treasury Note with a face amount of $100. The purchaser of this bond will receive payments of $\frac{1}{2} \times 0.500 = 0.250$ every September 30 and March 31, starting September 31, 2015 and ending March 31, 2017 (the maturity date of the bond). The purchaser will also receive $100 on March 31, 2017. This payment of $100 is the **redemption value** or **maturity value** of the bond.

The "interest rate" in the table is the coupon rate on the bond, quoted on an annual basis. It is understood in practice that coupons are paid semi-annually, so that the coupon rate is divided by 2 to calculate the actual coupon amount of 0.250. The purchaser will pay a price that is equal to the present value of the series of coupon and redemption payments based on a rate of return, or **yield rate** that is indicated by current financial market conditions. For this example, at the time of purchase, it is indicated that the yield rate is 0.598%. This is the rate used by the purchaser to calculate the present value of the series of bond payments for that 2-year Treasury Note. **It is a long established convention that bond yield rates, like coupon rates, are quoted as nominal annual interest rates compounded twice per year**. Therefore, the yield rate per half-year is 0.299%.

It should be emphasized again that the phrase "interest rate" in Table 4.1 is the rate used to determine the coupons that the bond will pay, and the phrase "yield rate" is the rate which is used to calculate the present value of the stream of coupons and redemption amount. During the lifetime of the bond, the coupon rate will not change; it is a part of the contract describing the stream of payments that the bond issuer will make to the bond purchaser. The yield rate is set by market conditions, and will fluctuate as time goes on and market conditions change.

The following time diagram describes the payments made by the first bond in Table 4.1.

3/31/15	9/30/15	3/31/16	9/30/16	3/31/17
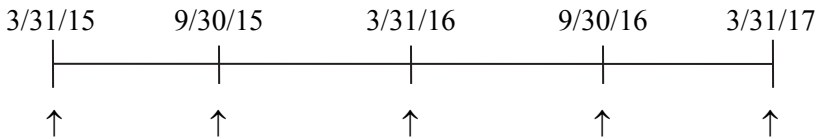
Purchase bond on issue date,

pay 99.805456

Receive   0.250   0.250   0.250   100.250

If we regard the period from the issue date to the first coupon (March 31, 2015 to September 30, 2015) and the time from one coupon period to the next each as exactly one-half year, then the present value of the bond payments is

$$100 \times \frac{1}{(1.00299)^4}$$

$$+\, 0.250 \times \left[\frac{1}{1.00299} + \frac{1}{(1.00299)^2} + \frac{1}{(1.00299)^3} + \frac{1}{(1.00299)^4} \right] = 99.805456.$$

This is the listed purchase price of the bond. One of the exercises at the end of this chapter asks to verify the price for the 30-year bond in Table 4.1.

4.1.1 THE PRICE OF A BOND ON A COUPON DATE

The following notation will be used to represent the various parameters associated with a bond:

     F – The face amount (also called the par value) of the bond

     r – The coupon rate per coupon period (six months unless otherwise specified)

     C – The redemption amount on the bond (equal to F unless otherwise noted)

     n – The number of coupon periods until maturity or **term** of the bond

The coupons are each of amount $F \times r$, and the sequence of payments associated with the bond is shown in the following time diagram

$$\begin{array}{ccccc} 0 & 1 & 2 & \cdots & n \end{array}$$

$$\begin{array}{cccc} Fr & Fr & \cdots & Fr + C \end{array}$$
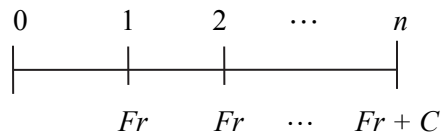
FIGURE 4.1

The purchase price of the bond is determined as the present value, on the purchase date, of that series of payments. There will be a number of factors that influence the yield rate used by the purchaser to find the price of the bond. We will not explore here the relationship between economic factors and interest rates on investments, but this is addressed briefly in Chapter 8. We will simply accept that "market forces" determine the interest rate used to value the bond and thus to determine the purchase price. We will use j to denote the six-month yield rate.

It is now a straightforward matter to formulate the price of the bond on a coupon date using the notation defined above:

$$P \ = \ C \times \frac{1}{(1+j)^n} + Fr \times \left[\frac{1}{1+j} + \frac{1}{(1+j)^2} + \cdots + \frac{1}{(1+j)^n} \right] = Cv_j^n + Fra_{\overline{n}|j}. \quad (4.1)$$

The first term on the right hand side of the equation is the present value of the redemption amount to be received by the bondholder in n coupon periods. The second term is the present value of the annuity of coupons to be received until the bond matures. Note that it is being assumed that the next coupon is payable one full coupon period from the valuation point (hence the annuity-immediate symbol), and there are n coupon periods until the bond matures.

It is usually the case that the redemption amount and the face amount are the same $(C = F)$. If this is so, the bond price can be expressed as

$$P \ = \ Fv_j^n + Fr \, a_{\overline{n}|j}. \quad (4.2)$$

Using the identity $v_j^n = 1 - ja_{\overline{n}|j}$, Equation 4.1 becomes

$$P \ = \ C + (Fr - Cj) a_{\overline{n}|j}. \quad (4.3)$$

Alternatively, writing $a_{\overline{n}|j}$ in Equation (4.1) as $a_{\overline{n}|j} = \dfrac{1-v_j^n}{j}$, and letting the present value of the redemption amount be denoted by $K = Fv_j^n$, Equation (4.1) becomes

$$P \;=\; Fv_j^n + \frac{r}{j}\left(F - Fv_j^n\right) \;=\; K + \frac{r}{j}(F - K). \tag{4.4}$$

Equation (4.4) is known as **Makeham's Formula**, which is the same as Equation (3.22) with F replacing the loan amount L, and r replacing the interest rate i in Equation (3.22). Note that, as in Equation (3.22), Equation (4.4) requires r and j to be based on the same period.

Equation (4.2) was used in finding the price of the U.S. Treasury note in the previous section. For that note, the price was found on the issue date March 31, 2015 for a bond with a face and redemption amount of $F = C = 100$. The coupons were payable semiannually at rate $r = .00250$ (this is the six-month coupon rate found from the annual quoted coupon rate of 0.500%. The bond matures on March 31, 2017, which is $n = 4$ coupon periods after the issue date. The yield rate is quoted as 0.598% (compounded semi-annually), which corresponds to a yield rate of $j = .00299$ every six months. The price of the other bond in Table 4.1 can be found in the same way.

EXAMPLE 4.1   (*Price of a bond on a coupon date*)

A 10% bond with semiannual coupons has a face amount of 100,000,000 and was issued on June 18, 2000. The first coupon was paid on December 18, 2000, and the bond has a maturity date of June 18, 2020.

(a) Find the price of the bond on its issue date using $i^{(2)}$ equal to

   (i) 5%,   (ii) 10%,   and (iii) 15%.

(b) Find the price of the bond on June 18, 2010, just after the coupon is paid, using the yield rates of part (a).

SOLUTION

(a) We have $F = C = 100,000,000$, $r = .05$, and $n = 40$. Using Equation (4.2), we see that the price of the bond is

$$P \;=\; 100,000,000 v_j^{40} + (100,000,000)(.05)a_{\overline{40}|j}.$$

With (i) $j = .025$, (ii) $j = .05$, and (iii) $j = .075$, the bond prices are (i) $P = 162,756,938$, (ii) $P = 100,000,000$, and (iii) $P = 68,513,978$.

(b) We still have $F = C = 100,000,000$ and $r = .05$, but now $n = 20$ (since there are 10 years, or 20 coupons, remaining on the bond). Using Equation (4.2), we have

$$P = 100,000,000 + 100,000,000(.05 - j)a_{\overline{20}|j}.$$

This results in prices of (i) 138,972,906, (ii) 100,000,000, and (iii) 74,513,772. ☐

Note that as the yield rate is increased in Example 4.1 the bond price decreases. This is due to the inverse relationship between interest rate and present value; the bond price is the present value of a stream of payments valued at the yield rate.

In the Canadian and U.S. financial press bond prices are generally quoted as a value per 100 of face amount, to the nearest .001 and yield rates are quoted to the nearest .001% (one-thousandth of a percent). Thus the quoted prices in Example 4.1 would be (i) 162.757, (ii) 100.000, and (iii) 68.514 in part (a), and (i) 138.972, (ii) 100.000, and (iii) 74.514 in part (b).

For corporate bond quotations in the financial press, along with yield, price and coupon rate listed, another measure that is often listed is the **current yield**. The current yield is the coupon rate divided by the bond's price.

4.1.2 BONDS BOUGHT OR REDEEMED AT A PREMIUM OR DISCOUNT

In looking at the bond price formulation of Equation (4.3) it is clear that the relative sizes of the bond price and face amount are directly related to the relative sizes of the coupon rate and yield rate. We have the relationships

$$P > F \quad \leftrightarrow \quad r > j, \tag{4.5a}$$

$$P = F \quad \leftrightarrow \quad r = j, \tag{4.5b}$$

and

$$P < F \quad \leftrightarrow \quad r < j, \tag{4.5c}$$

These relationships are similar to the ones between the loan amount L and the price paid for the loan based on Makeham's Formula in Section 3.4. The terminology associated with these relationships between P and F is as follows.

Definition 4.2 – Bond Purchase Value

(a) If $P > C$, the bond is said to be bought **at a premium**.
(b) If $P = C$, the bond is said to be bought **at par**.
(c) If $P < C$, the bond is said to be bought **at a discount**.

Equation (4.3) can be rewritten as $P - F = F(r - j)a_{\overline{n}|j}$. Suppose that the bond is bought at a premium so that $P > F$. The rewritten version of Equation (4.3) indicates that the amount of premium in the purchase price $(P - F)$ can be regarded as a loan (from the buyer to the seller) re-paid at rate j by n payments of $F(r - j)$, the excess of coupon over yield.

We can also see from Equation (4.3) that if $r > j$ and the time n until maturity is increased, then the bond price P increases, but if $r < j$ then P decreases as n increases. This can be seen another way. If $r > j$ then $P > F$, so that the bondholder will realize a **capital loss** of $P - F$ at the time of redemption. Having the capital loss deferred would be of some value to the bondholder, so he would be willing to pay a larger P for such a bond with a later maturity date. The reverse of this argument applies if $r < j$. In any event, the level of bond yield rates would influence a bond issuer in setting the coupon rate and maturity date on a new issue, since both coupon rate and maturity date have an effect on the actual price received for the bond by the issuer.

In the case where F and C are not equal, an additional parameter can be defined, called the **modified coupon rate** and denoted by $g = \frac{F \cdot r}{C}$, so that $Cg = Fr$. Exercise 4.1.18 develops alternative formulations for bond prices when $C \neq F$, that are equivalent to Equations (4.2), (4.3) and (4.4). When $C = F$, the bond is said to be **redeemed at par**, when $C > F$ the bond is said to be **redeemed at a premium**, and when $C < F$ the bond is said to be **redeemed at a discount**.

**The U.S. Treasury Department "Halloween Surprise"
of October 31, 2001**

On October 31, 2001, the U.S. Department of the Treasury announced
that it would no longer be issuing 30-year treasury bonds. The yield on
a 30-year treasury bond had become a bellwether for long term interest
rates. The announcement resulted in a surge in demand for the long-
term bonds, prices rose and yield rates fell on 30-year bonds by about
33 basis points (.33%) that day (yields on 2-year Treasury bonds fell
only 4 basis points that day). Among the reasons given by the Treasury
Department to discontinue the 30-year bonds were

(i) they were expensive for the government (30 year yields at that
     time were about 5.25% and 10-year yields were below 4.25%;
     yield represents a measure of the cost to the bond issuer) and

(ii) the government was running a large budget surplus and govern-
     ment borrowing needs had diminished.

A significant deficit reappeared a few years later, resulting in in-
creased borrowing needs by the government. On February 15, 2006,
the Treasury Department reintroduced 30-year bonds with a $14 bil-
lion issue. The deficit is in the neighborhood of $500 billion at the
time this edition of the book is being prepared.

4.1.3 BOND PRICES BETWEEN COUPON DATES

We have thus far considered only the determination of a bond's price on its
issue date or at some later coupon date. In practice bonds are traded daily,
and we now consider the valuation of a bond at a time between coupon
dates. Let us regard the coupon period as the unit of time, and suppose that
we wish to find the purchase price P_t of a bond at time t, where $0 \le t \le 1$,
with t measured from the last coupon payment. The value of the bond is
still found as the present value at the yield rate of all future payments (cou-
pons plus redemption). Suppose that there are n coupons remaining on the
bond, including the next coupon due. At yield rate j per coupon period, the
value P_1 of the bond *just after* the next coupon could be found using one of
Equations (4.2), (4.3) or (4.4). Then the value of the bond at time t is the
present value of the amount $P_1 + Fr$ due at time 1 (the present value of
both the coupon due then and the future coupons and redemption), so that

$$P_t \;=\; v_j^{1-t}[P_1 + Fr]. \tag{4.6a}$$

$$\begin{array}{ccc}
0 & t & 1 \\
\uparrow & \uparrow & \uparrow \\
P_0 & P_t & Fr + P_1
\end{array}$$

FIGURE 4.2

Alternatively, if we define P_0 to be the value of the bond just after the last coupon, then (see Exercise 4.1.27) we also have

$$P_t = P_0(1+j)^t. \tag{4.6b}$$

The value P_t given by Equations (4.6a) and (4.6b) is the purchase price paid for the bond at time t, and it is also called the **price-plus-accrued** of the bond. Price-plus-accrued may also be referred to as the "full price," the "flat price" or the "dirty price" of the bond.

In the calculation involved in Equations (4.6a) and (4.6b), the value of t is between 0 and 1 and measures the time since the last coupon was paid as a fraction of a coupon period. Given the coupon dates and the date of time t, the numerical value of t is

$$t = \frac{number\ of\ days\ since\ last\ coupon\ paid}{number\ of\ days\ in\ the\ coupon\ period}. \tag{4.7}$$

The price given by Equations (4.6a) and (4.6b), with t defined by Equation (4.7), is not the price that would be quoted for the bond in a financial newspaper. This quote price is called the **market price** (or simply **price** or **clean price**), and is equal to the price-plus-accrued minus the fraction of the coupon accrued to time t (throughout this section, the terms market price and **purchase price** will have the specific meanings given here). This fractional coupon is proportional to the fractional part of the coupon period that has elapsed since the last coupon was paid, so the fractional coupon is $t \times F \times r$. The market price of the bond is then

$$Price_t = Price\text{-}plus\text{-}accrued_t - t \times F \times r = P_0(1+j)^t - t \times F \times r. \tag{4.8}$$

This is the market price of the bond as calculated in practice. It is sometimes called the "actual/actual" method of calculating the market price. There is a variation to this calculation in which $P_0(1+j)^t$ is replaced by $P_0(1+jt)$ (a simple interest approximation).

> **EXAMPLE 4.2**   (*Bond price between coupon dates*)

For each of the yield rates (i), (ii), and (iii) in Example 4.1, find both the purchase price and market price on August 1, 2010. Quote the prices (to the nearest .001) per 100 of face amount.

SOLUTION

Using the results in part (b) of Example 4.1, we see that on the last coupon date, June 18, 2010, the value of the bond was (i) 138,972,906, (ii) 100,000,000, and (iii) 74,513,772. The number of days from June 18 to August 1 is 44, and the number of days in the coupon period from June 18 to December 18 is 183. Using Equation (4.6b), with $t = \frac{44}{183}$, we have purchase prices of

(i) $138,972,906(1.025)^{44/183} = 139,800,445,$

(ii) $100,000,000(1.05)^{44/183} = 101,180,005,$ and

(iii) $74,513,772(1.075)^{44/183} = 75,820,791.$

Per 100 of face amount, the purchase prices, to the nearest .001, are

(i) 139.800,   (ii) 101.180,   and   (iii) 75.821.

The market prices are:

(i) $139,800,445 - (44/183)(.05)(100,000,000) = 138,598,259,$

(ii) $101,180,004 - (44/183)(.05)(100,000,000) = 99,977,818,$ and

(iii) $75,820,791 - 44/13)(.05)(100,000,000) = 74,618,605.$

Per 100 of face we have quoted prices of

(i)   138.598,

(ii)   99.978, and

(iii)   74.619.   ❏

4.1.4 BOOK VALUE OF A BOND

In reporting the value of assets at a particular time, a bondholder would have to assign a value to the bond at that time. This value is called the **book value** of the bond, and is usually taken as the current price of the bond valued at the original yield rate at which the bond was purchased. For accounting purposes the accrued interest since the last coupon would be considered as a separate item from the book value of the bond. The next section of this chapter looks at book value in more detail.

Figure 4.3 below displays the graph of the purchase price and the market price of a bond over several consecutive coupon periods (all prices are calculated at the original yield rate of the bond). Nodes in the graph indicate the price just after a coupon is paid. The upper line in the graph is the purchase price or book value, and the lower line is the price without the accrued coupon. This graph is for a bond bought at a premium. The graph for a bond bought at a discount would be rising, but otherwise would be similar. Note that the market price (without the accrued coupon) is approximately the linear interpolation between two successive coupon dates (see Exercise 4.1.26), and the purchase price readjusts back to the market price on the coupon dates.

Bond
Price

Purchase Price

Market Price

1   2   3

Time (coupon periods)
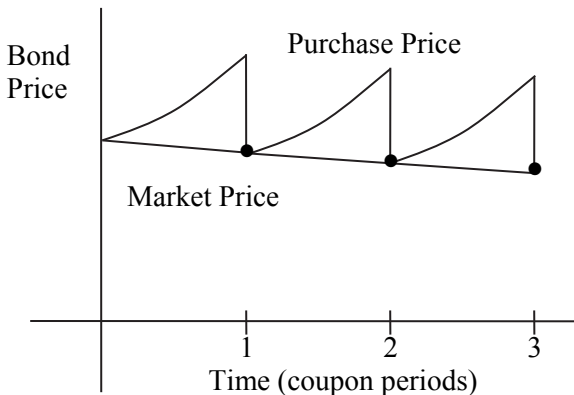
FIGURE 4.3

A bond trader would often be comparing the relative values of different bonds, and would want an equitable basis on which to compare, at a specific point in time, bonds with different calendar coupon dates. The price without accrued coupon (see Figure 4.3) provides a smooth progression of

bond values from one coupon date to the next, and it is this price that is used by bond traders to compare relative bond values. This can easily be seen if we consider a bond for which $r = j$. We see from Equation (4.3) that if $r = j$ (the coupon and yield rates are equal), then on a coupon date the price of the bond would be F (the bond is bought at par). However the purchase price of the bond grows from $P_0 = F$ at $t = 0$ to $P_0(1+j)$ just before the coupon is paid at $t = 1$, and just after that coupon is paid the price drops to $P_1 = F$ (see Exercise 4.1.28). For a bond trader comparing two bonds with different coupon dates, but both with $r = j$, it would be appropriate to compare using market price, since this price has eliminated a "distortion" caused by the accrued coupon included in the purchase price.

The notion, introduced earlier, of a bond bought at a premium, at par, or at a discount just after a coupon is paid was based on comparing the bond price with the face amount. To describe a bond as being bought at a premium, par or discount when bought at a time between coupon dates, the comparison is made between the market price and the face amount. It was pointed out that just after a coupon payment, a bond is priced at a premium, par or discount according to whether $r > j$, $r = j$, or $r < j$, respectively. This relationship remains valid when comparing the market price with the face amount at a time between coupon dates.

4.1.5 FINDING THE YIELD RATE FOR A BOND

When bonds are actually bought and sold on the bond market, the trading takes place with buyers and sellers offering **bid** and **ask** prices, respectively, with an intermediate settlement price eventually found. The amount by which the ask price exceeds the bid is the **spread**. This is the difference in price between the highest price that a buyer is willing to pay for an asset and the lowest price for which a seller is willing to sell it. This terminology applies to investments of all types, bonds (Chapter 8) and equities and derivative investments (Chapters 9 and 10). Some assets are more liquid than others and this may affect the size of the spread.

We have considered bond valuation mainly from the point of view of calculating the price of a bond when the coupon rate, time to maturity and yield rate are known. In practice, bond prices are settled first, and the corresponding yield rate is then determined and made part of the overall quotation describing the transaction. The determination of the yield rate from the price becomes an unknown interest rate problem which would

be solved using an unknown interest function, a bond yield function on a financial calculator, or a spreadsheet program.

For a bond with face amount F, coupon rate r, with n coupons remaining until maturity, and bought at a purchase price P one period before the next coupon is due, the yield rate j is the solution of the equation $P = Fv_j^n + Fr\,a_{\overline{n}|j}$. j is the yield rate, or **internal rate of return** for this transaction, and there is a unique positive solution, $j > 0$. If the bond is bought at time t, $0 < t < 1$, measured from the last coupon, and there are n coupons remaining, then j is the solution of the equation $P = [Fv_j^n + Fr\,a_{\overline{n}|j}](1+j)^t$, where P is the purchase price of the bond. Again, there will be a unique positive solution, $j > 0$.

EXAMPLE 4.3   (*Finding the yield rate from the price of a bond*)

A 20-year 8% bond has semi-annual coupons and a face amount of 100. It is quoted at a purchase price of 70.400.

(a) Find the yield rate.

(b) Suppose that the bond was issued January 15, 2010, and is bought by a new purchaser for a price of 112.225 on January 15, 2015 just after a coupon has been paid.

   (i) Find the yield rate for the new purchaser.

   (ii) Find the yield rate (internal rate of return) earned by the original bondholder.

   (iii) Suppose that the original bondholder was able to deposit coupons into an account earning an annual interest rate of 6% convertible semi-annually. Find the average effective annual rate of return earned by the original bond purchaser on his 5-year investment. Assume that interest on the deposit account is credited every January 15 and July 15.

(c) Suppose that the bond was issued January 15, 2010, and is bought by a new purchaser on April 1, 2015 for a market price of 112.225.

   (i) Find the yield rate for the new purchaser.

   (ii) Find the yield rate (internal rate of return) earned by the original bondholder.

SOLUTION

(a) We solve for j, the 6-month yield rate, in the equation

$$70.400 = 100v_j^{40} + 4a_{\overline{40}|j}$$

(40 coupon periods to maturity, each coupon amount is 4). Using a financial calculator returns a value of $j = .059565$. This would be a quoted yield rate of 11.913% (compounded semi-annually).

(b) (i) There are 30 coupons remaining when the new purchaser buys the bond. We solve for j, the 6-month yield rate, in the equation $112.225 = 100v_j^{30} + 4a_{\overline{30}|j}$ (30 coupon periods to maturity). Using a financial calculator returns a value of $j = .033479$. This would be a quoted yield rate of 6.696% (compounded semi-annually).

(ii) The original bondholder received 10 coupons plus the purchase price of 112.225 on January 15, 2015. The original bondholder's equation of value for that 5-year (10 half-years) period is $70.400 = 112.225v_j^{10} + 4a_{\overline{10}|j}$. A financial calculator gives a value of $j = .09500$ (6-month yield of 9.5%), which is a quoted nominal annual yield compounded semiannually of 19.0%.

(iii) On January 15, 2015, just after the coupon is deposited, the balance in the deposit account is $4s_{\overline{10}|.03} = 45.86$. Along with the sale of the bond, the original bondholder has a total of $45.86 + 112.225 = 158.08$. For the five year period, the annual effective return is i, found from the equation $158.08 = 70.40(1+i)^5$. Solving for i results in a value of $i = 17.6\%$.

(c) (i) There are 76 days from January 15, 2015 (the time of the most recent coupon) to April 1, 2015. The entire coupon period from January 15, 2015 to July 15, 2015 is 181 days. The purchase price of the bond on April 1, 2015 at yield rate j per coupon period is $\left[100v_j^{30} + 4a_{\overline{30}|j}\right](1+j)^{76/181}$. The market price is 112.225, so the purchase price is $112.225 + \frac{76}{181} \times 4 = 113.905$. The new purchaser's 6-month yield is j, the solution of the equation $113.905 = \left[100v_j^{30} + 4a_{\overline{30}|j}\right](1+j)^{76/181}$. This is a somewhat awkward equation. Using a financial calculator with a function for calculating the yield rate on a bond, we get $j = .033421$ (annual yield rate of 6.684%, compounded semi-annually).

(ii) The original purchaser will receive 10 coupons of amount 4 each, plus a payment of 113.905 on April 1, 2005. The payment of 113.905 is made $10\frac{76}{181} = 10.420$ coupon periods after the bond was originally purchased. The original purchaser's 6-month yield j, is the solution of the equation $70.400 = 113.905v_j^{10.420} + 4a_{\overline{10}|j}$.

Standard financial calculator functions will not be capable of solving for j, and some approximate computer routine such as EXCEL Solver would be needed. The resulting solution is $j = .093054$ (9.31% is the 6-month yield rate). ❒

Several exercises at the end of the chapter look at various methods for approximating the yield rate. These methods may provide some insight into the price-yield relationship, but they are no longer used in practice.

4.2   AMORTIZATION OF A BOND

For taxation and other accounting purposes, it may be necessary to determine the amount of interest received and principal returned in a bond coupon or redemption payment. This can be done by viewing the bond as a standard amortized loan.

The purchaser of the bond can be thought of as the lender and the issuer of the bond is the borrower. The purchase price of the bond, P, can be thought of as the loan amount, and the coupon and redemption payments are the loan payments made by the borrower (the bond issuer) to the lender (the bondholder). For an amortized loan, the loan amount is the present value of the loan payments using the loan interest rate for present value calculation. Since the bond purchase price is the present value of the coupon and redemption payments using the yield rate, the **yield rate is the loan rate** when thinking of the bond purchase as a loan. An amortization schedule for the bond would be constructed algebraically like the general amortization schedule in Table 3.2.

The payment amounts in the bond amortization schedule will be the coupon amounts Fr up to the $n-1^{st}$ coupon and the nth payment is a coupon plus the redemption amount, $Fr + C$. The prospective form of the outstanding balance just after a coupon payment is the present value of all future coupons plus redemption amount, valued at the "loan interest rate"

which is the yield rate on the bond from when it was originally purchased. This is the bond's **book value**.

The algebraic relationships for loan amortization also apply to bond amortization. One of the basic loan amortization relationships from Chapter 3 was Equation 3.4, $OB_{t+1} = OB_t(1+i) - K_{t+1}$. The corresponding bond amortization relationship up to the $n-1^{st}$ coupon is

$$BV_{t+1} = BV_t(1+j) - Fr \qquad (4.9)$$

BV denotes the book value (or amortized value) and j is the yield rate. We can also formulate the interest paid and principal repaid:

$$I_{t+1} = BV_t \times j \qquad (4.10)$$

$$PR_{t+1} = Fr - I_{t+1} \qquad (4.11)$$

The bond amortization schedule for a bond purchased on a coupon date (just after the coupon is paid) is given in the following table.

TABLE 4.2

K	Outstanding Balance (Book Value after Payment)	Payment	Interest Due	Principal Repaid
0	$P = F[1+(r-j)\cdot a_{\overline{n}}]$	–	–	–
1	$F[1+(r-j)\cdot a_{\overline{n-1}}]$	Fr	$F[j+(r-j)(1-v_j^n)]$	$F(r-j)\cdot v_j^n$
2	$F[1+(r-j)\cdot a_{\overline{n-2}}]$	Fr	$F[j+(r-j)(1-v_j^{n-1})]$	$F(r-j)\cdot v_j^{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$F[1+(r-j)\cdot a_{\overline{n-k}}]$	Fr	$F[j+(r-j)(1-v_j^{n-k+1})]$	$F(r-j)\cdot v_j^{n-k+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$F[1+(r-j)\cdot a_{\overline{1}}]$	Fr	$F[j+(r-j)(1-v_j^2)]$	$F(r-j)\cdot v_j^2$
n	0	$Fr+F$	$F[j+(r-j)(1-v_j)]$	$F[1+(r-j)\cdot v_j]$

Notice that since the payments are level throughout the term of the bond, except for the final payment, the principal repaid column forms a geo-

metric progression with ratio $1+j$. The following example illustrates a bond amortization.

EXAMPLE 4.4   (*Bond amortization*)

A 10% bond with face amount 10,000 matures 4 years after issue. Construct the amortization schedule for the bond over its term for nominal annual yield rates of (a) 8%, (b) 10%, and (c) 12%.

SOLUTION

(a) The entries in the amortization schedule are calculated as they were in Table 4.2, where k is the number of the coupon period. With a nominal yield rate of 8% the purchase price of the bond is 10,673.27. Then $I_1 = (10,673.27)(.04) = 426.93$, and so on. The complete schedule is shown in Table 4.3a.

TABLE 4.3a

k	Outstanding Balance	Payment	Interest Due	Principal Repaid
0	10,673.27	–	–	–
1	10,600.21	500	426.93	73.07
2	10,524.22	500	424.01	75.99
3	10,445.19	500	420.97	79.03
4	10,363.00	500	417.81	82.19
5	10,277.52	500	414.52	85.48
6	10,188.62	500	411.10	88.90
7	10,096.16	500	407.54	92.46
8	0	10,500	403.85	10,096.15

Note that since $OB_7 = 10,096.16$ and $PR_8 = 10.096.15$ we should have $OB_8 = .01$. Of course $OB_8 = 0$ and the one cent discrepancy is due to rounding. The values of OB_k decrease to the redemption value as k approaches the end of the term. The entry under Principal Repaid is called **the amount for amortization of premium** for that particular period. The amortization of a bond bought at a premium is also referred to as **writing down a bond**. If we sum the entries in the principal repaid column for k from 1 to 7 along with 96.15 at time 8 (the principal repaid before the redemption payment of 10,000), we get a total of 673.27

which is the total premium above redemption value at which the bond was originally purchased. The coupon payments are amortizing the premium, reducing the book value to 10,000 as of time 8 and then the redemption payment of 10,000 retires the bond debt.

(b) With a nominal yield rate of 10% the purchase price is 10,000. The schedule is shown in Table 4.3b.

TABLE 4.3b

k	Outstanding Balance	Payment	Interest Due	Principal Repaid
0	10,000.00	–	–	–
1	10,000.00	500	500.00	0
2	10,000.00	500	500.00	0
3	10,000.00	500	500.00	0
4	10,000.00	500	500.00	0
5	10,000.00	500	500.00	0
6	10,000.00	500	500.00	0
7	10,000.00	500	500.00	0
8	0	10,500	500.00	10,000

(c) With a nominal yield rate of 12% the purchase price is 9379.02; the schedule is shown in Table 4.3c.

TABLE 4.3c

k	Outstanding Balance	Payment	Interest Due	Principal Repaid
0	9379.02	—	—	—
1	9441.76	500	562.74	– 62.74
2	9508.27	500	566.51	– 66.51
3	9578.77	500	570.50	– 70.50
4	9653.50	500	574.73	– 74.73
5	9732.71	500	579.21	– 79.21
6	9816.67	500	583.96	– 83.96
7	9905.67	500	589.00	– 89.00
8	0	10,500	594.34	9905.66

Round-off error again gives a value of $OB_8 = .01,$ when it should be zero. Note that OB_k *increases* to the redemption amount as k approaches the end of the term. The principal repaid column entries are negative. This occurs because the coupon payment of 500 is not large enough to pay for the interest due, so the shortfall is added to the outstanding balance (amortized value). The negative of the principal repaid entry is called the **amount for accumulation of discount** when a bond is bought at a discount. This amortization is also referred to as **writing up a bond**. ☐

Since a bond amortization is algebraically the same as a loan amortization, calculator functions for amortization can be used to calculate bond amortization quantities.

4.3 APPLICATIONS AND ILLUSTRATIONS

4.3.1 CALLABLE BONDS: OPTIONAL REDEMPTION DATES

A bond issuer may wish to add flexibility to a bond issue by specifying a range of dates during which redemption can occur, at the issuer's option. Such a bond is referred to as a **callable bond**. For a specific yield rate, the price of the bond will depend on the time until maturity, and for a specific bond price, the yield rate on the bond will depend on the time until maturity. The following example illustrates the way in which a callable bond is priced.

Definition 4.3 – Callable Bond

A callable bond is one for which there is a range of possible redemption dates, not known in advance to the bond purchaser. The redemption date is chosen by the bond issuer.

EXAMPLE 4.5 *(Finding the price of a callable bond)*

(a) A 10% bond with semi-annual coupons and with face and redemption amount 1,000,000 is issued with the condition that redemption can take place on any coupon date between 12 and 15 years from the issue date. At the time the bond is purchased, the purchaser does not know the actual redemption. Find the price paid by an investor wishing a minimum yield of (i) $i^{(2)} = .12,$ and (ii) $i^{(2)} = .08.$

(b) Suppose the purchaser pays the maximum of all prices for the range of redemption dates. Find the yield rate if the issuer chooses a redemption date corresponding to the minimum price in each of cases (i) and (ii) of part (a).

(c) Suppose the purchaser pays the minimum of all prices for the range of redemption dates. Find the yield rate if the issuer chooses a redemption date corresponding to the maximum price in each of cases (i) and (ii) of part (a).

(d) Suppose the purchaser pays 850,000 for the bond and holds the bond until it is called. Find the minimum yield that the investor will obtain.

SOLUTION

(a) (i) From Equation (4.3) $P = 1,000,000 \times \left[1 + (.05 - .06) \cdot a_{\overline{n}|.06}\right]$, where n is the number of coupons until redemption. At the time the bond is purchased, the purchaser knows only that n will be one of the following possible values: $n = 24, 25, \ldots, 30$. The price range for these redemption dates is from 874,496 for redemption at $n = 24$ to 862,352 for redemption at $n = 30$. It is most prudent for the purchaser to offer a price of 862,352.

(ii) The range of prices is from 1,152,470 if redemption occurs at 12 years, to 1,172,920 if redemption is at 15 years The prudent purchaser would pay 1,152,470.

(b) If the purchaser in (i) pays the maximum price of 874,496 (based on redemption at $n = 24$), and the bond is redeemed at the end of 15 years, the actual nominal yield is 11.80%. If the purchaser in (ii) pays 1,172,920 (based on redemption at $n = 30$), and the bond is redeemed at the end of 12 years, the actual nominal yield is 7.76%.

(c) If the purchaser in (i) pays 862,352 (based on 15 year redemption) and the bond is redeemed after 12 years, the actual nominal yield is 12.22%, and if the purchaser in (ii) pays 1,152,470 (based on 12 year redemption) and the bond is redeemed after 15 years, the actual nominal yield is 8.21%.

(d) We use the equation $850,000 = 1,000,000 v_j^n + 50,000 a_{\overline{n}|j}$.

For each n from 24 to 30, we can find the corresponding 6-month yield rate j. For $n = 24$, the 6-month yield rate is $j = .0622$, which corresponds to a nominal annual yield of 12.44%. For $n = 30$, the 6-month yield rate is .0610 (nominal annual 12.20%). If we check the

yield rate for $n = 25, 26, 27, 28, 29$, we will see that the minimum yield occurs if redemption is at $n = 30$, and this yield is 12.20%.

◻

For a callable bond, if a minimum yield j is desired, there are some general rules that can be established for finding the purchase price that will result in a yield rate of at least j. Using the bond price formula $P = C\left[1 + (r-j) \cdot a_{\overline{n}|j}\right]$, we have seen that for a given yield rate j, if $r < j$, then $P < C$ (the bond is bought at a discount). Also, in this case, since $r - j < 0$, the minimum price will occur at the maximum value of n (latest redemption date), because $(r-j) \times a_{\overline{n}|j}$ is most negative when n is largest. This can also be seen from general reasoning, because if a bond is bought at a discount, then it is most advantageous to the bond purchaser to receive the large redemption amount C as soon as possible, so the bond has greatest value if redemption occurs at the earliest redemption date and the bond has least value if redemption occurs at the latest date. Using similar reasoning, if $r > j$ (bond bought at a premium), the minimum price will occur at the minimum value of n. This is illustrated in part (a) of Example 4.5.

Similar reasoning to that in the previous paragraph provides a general rule for determining the minimum yield that will be obtained on a callable bond if the price is given. If a bond is bought at a discount for price P, then there will be a "capital gain" of amount $C - P$ when the bond is redeemed. The sooner the bondholder receives this capital gain, the greater will be the yield (return) on the bond, and therefore, the minimum possible yield would occur if the bond is redeemed on the latest possible call date. This is illustrated in part (d) of Example 4.5. In a similar way, we see that for a bond bought at a premium, there will be a "capital loss" of amount $P - C$ when the bond is redeemed. The earlier this loss occurs, the lower the return received by the bondholder, so the minimum yield occurs at the earliest redemption date.

If a purchaser pricing the bond desires a minimum yield rate of j, the purchaser will calculate the price of the bond at rate j for each of the redemption dates in the specified range. The minimum of those prices will be the purchase price. If the purchaser pays more than that minimum price, and if the issuer redeems at a point such that the price is the minimum price, then the purchaser has "overpaid" and will earn a yield less then the minimum yield originally desired. This is illustrated in parts (b) and (c) of Example 4.5.

When the first optional call date arrives, the bond issuer, based on market

conditions and its own financial situation, will make a decision on whether or not to call (redeem) the bond prior to the latest possible redemption date. If the issuer is not in a position to redeem at an early date, under appropriate market conditions, it still might be to the issuer's advantage to redeem the bond and issue a new bond for the remaining term. As a simple illustration of this point, suppose in Example 4.5(a) that 12 years after the issue date, the yield rate on a new 3-year bond is 9%. If the issuer redeems the bond and immediately issues a new 3-year bond with the same coupon and face amount, the issuer must pay 1,000,000 to the bondholder, but then receives 1,025,789 for the new 3-year bond, which is bought at a yield rate of 9%.

A callable bond might have different redemption amounts at the various optional redemption dates. It might still be possible to use some of the reasoning described above to find the minimum price for all possible redemption dates. In general, however, it may be necessary to calculate the price at several (or all) of the optional dates to find the minimum price.

Example 4.6 (*Varying redemption amounts for a callable bond*)

A 15-year 8% bond with face amount 100 is callable (at the option of the issuer) on a coupon date in the 10th to 15th years. In the 10th year the bond is callable at par, in the 11th or 12th years at redemption amount 115, or in the 13th, 14th or 15th years at redemption amount 135.

(a) What price should a purchaser pay in order to ensure a minimum nominal annual yield to maturity of (i) 12%, and (ii) 6%?

(b) Find the purchaser's minimum yield if the purchase price is (i) 80, and (ii) 120.

Solution

(a) (i) Since the yield rate is larger than the coupon rate (or modified coupon rate for any of the redemption dates), the bond will be bought at a discount. Using Equation (4.3E) from Exercise 4.1.18, we see that during any interval for which the redemption amount is level, the lowest price will occur at the latest redemption date. Thus we must compute the price at the end of 10 years, 12 years and 15 years. The corresponding prices are 77.06, 78.60 and 78.56. The lowest price corresponds to a redemption date of 10 years, which is near the earliest possible redemption date. This ex-

ample indicates that the principal of pricing a bond bought at a discount by using the latest redemption date may fail when the redemption amounts are not level.

(ii) For redemption in the 10th year and the 11th or 12th years, the yield rate of 3% every six months is smaller than the modified coupon rate of 4% (for redemption in year 10) or $\frac{100(.04)}{115} = .0348$ (for redemption in years 11 or 12). The modified coupon rate is $.0296 < 3\%$ for redemption in the 13th to 15th years. Thus the minimum price for redemption in the 10th year occurs at the earliest redemption date, which is at $9\frac{1}{2}$ years, and the minimum price for redemption in the 11th or 12th years also occurs at the earliest date, which is at $10\frac{1}{2}$ years. Since $g < j$ in the 13th to 15th years, the minimum price occurs at the latest date, which is at 15 years. Thus we must calculate the price of the bond for redemption at $9\frac{1}{2}$ years, $10\frac{1}{2}$ years and 15 years. The prices are 114.32, 123.48, and 134.02. The price paid will be 114.32, which corresponds to the earliest possible redemption date.

(b) (i) Since the bond is bought at a discount (to the redemption value), it is to the purchaser's disadvantage to have the redemption at the latest date. Thus we find the yield based on redemption dates of 10 years, 12 years and 15 years. These nominal yield rates are 11.40%, 11.75% and 11.77% The minimum yield is 11.40%.

(ii) Since the bond is bought at a premium to the redemption value in the 10th year and in the 11th and 12th years, the minimum yield to maturity occurs at the earliest redemption date for those periods. This is $9\frac{1}{2}$ years for the 10th year and $10\frac{1}{2}$ years for the 11th and 12th years. The bond is bought at a discount to the redemption amount in the 13th to 15th years, so the minimum yield occurs at the latest redemption date, which is 15 years. We find the yield based on redemption at $9\frac{1}{2}$ years, $10\frac{1}{2}$ years and 15 years. These nominal yield rates are 5.29%, 6.38% and 7.15% The minimum is 5.29%. ❑

Through the latter part of the 1980s, bonds callable at the option of the issuer became less common in the marketplace. The U.S. and Canadian governments no longer issue callable bonds. The increased competition for funds by governments and corporations over time has produced various incentives that are occasionally added to a bond issue. One such incentive is a **retractable-extendible** feature, which gives the bondholder the option of having the bond redeemed (retracted) on a specified date, or having the redemption date extended to a specified later date. This is similar to a callable bond with the option in the hands of the bondholder rather than the bond issuer. Another incentive for the bondholder is to provide **warrants** with the bond. A warrant gives the bondholder the option to purchase additional amounts of the bond at a later date at a guaranteed price.

4.3.2 SERIAL BONDS AND MAKEHAM'S FORMULA

A bond issue may consist of a collection of bonds with a variety of redemption dates, or redemption in installments. This might be done so that the bond issuer can stagger the redemption payments instead of having a single redemption date with one large redemption amount. Such an issue can be treated as a series of separate bonds, each with its own redemption date, and it is possible that the coupon rate differs for the various redemption dates. It may also be the case that purchasers will want different yield rates for the different maturity dates. This bond is called a **serial bond** since redemption occurs with a series of redemption payments.

Suppose that a serial bond has redemption amounts $F_1, F_2, \ldots, F_m$, to be redeemed in $n_1, n_2, \ldots, n_m$ coupon periods, respectively, and pays coupons at rates $r_1, r_2, \ldots, r_m$, respectively, Suppose also that this serial bond is purchased to yield $j_1, j_2, \ldots, j_m$, respectively, on the m pieces. Then the price of the tth piece can be formulated using any one of Equation (4.2), (4.3) or (4.4). Using Makeham's bond price formula given by Equation (4.4), the price of the tth piece is

$$P_t = K_t + \frac{r_t}{j_t} \times (F_t - K_t), \tag{4.12}$$

where $K_t = F_t \times v_{j_t}^{n_t}$. The price of the total serial issue would be

$P = \sum_{t=1}^{m} P_t.$ In the special case where the coupon rates and yield rates on

CHAPTER 7

CASHFLOW DURATION AND IMMUNIZATION

"I have enough money to last me the rest of my life, unless I buy something.
Jackie Mason – American Comedian (1934 -)

An investor who holds a fixed-income investment such as a bond will see the value of the bond change over time for a number of reasons. It was seen in the discussion of bond amortization in Chapter 4 that there is a natural progression of the amortized value of a bond toward the maturity value as the bond approaches its maturity date. Over the lifetime of the bond, the market value of the bond will also converge to the maturity amount as well. These longer term changes in bond values are somewhat predictable, although the market value will fluctuate to a large extent as a result of changing market conditions, such as changing interest rates and the perception of the chance of the bond defaulting on some of its scheduled payments.

The market value of a bond at a particular time is directly related to the yield to maturity that prevails in the bond market at that time. Changes in market yield rates can have a sudden and significant impact on the market value of a bond. There is no guarantee that when an investor buys a bond, the yield rate at which the bond was bought will continue to be the bond's yield rate for the entire term of the bond. In fact, the yield rate will almost surely not stay constant as time goes on. The next example illustrates one of the reasons why yield rates on a particular bond might change over time.

EXAMPLE 7.1   (*Yield curve slide*)

Suppose that the current term structure of interest rates has the following schedule of spot rates for maturities of 1, 2, 3 and 4 years:

Maturity	1-year	2-year	3-year	4-year
Spot Rate	$s_0(1) = .05$	$s_0(2) = .10$	$s_0(3) = .15$	$s_0(4) = .20$

Suppose a 4-year zero coupon bond with maturity amount 100 is purchased and that as time goes on, the term structure does not change, so that at any time the spot rates are the same as they are now for any time to maturity. This means that at time 1, $s_1(1) = .05$ (yield on a one year zero coupon bond issued at time 1 is 5%), $s_1(2) = .10$, etc. Find the book value and market value of the bond in one, two and three years.

SOLUTION

The purchase price is $100(1.2)^{-4} = 48.23$. In one year the amortized (book) value will be $100(1.2)^{-3} = 57.87$. At that time the bond will be three years from maturity, so the market yield will be 15% (we are assuming that the term structure doesn't change, so three-year zero coupon bonds will always have a yield rate of 15%). In one year the market value of the bond will be $100(1.15)^{-3} = 65.75$. Note that after one year the holder of the bond will have had a one-year **holding period return** of $\frac{65.87}{48.23} - 1 = .3657$ (36.57%).

At the end of the second year the book value will be $100(1.2)^{-2} = 69.44$ and the market value will be $100(1.1)^{-2} = 82.64$. At the end of the third year the book value will be $100(1.2)^{-1} = 83.33$ and the market value will be $100(1.05)^{-1} = 95.24$. At the end of the fourth year the bond matures and its value (book and market) is 100. These book and market values are summarized in the following table.

Time	Book Value	Market Value
0	48.23	48.23
1	57.87	65.75
2	69.44	82.64
3	83.33	95.24
4	100.00	100.00

The phrase "yield curve slide" in the title of this example refers to the move away from book value that the market value takes as time goes on. This can occur not because there is any change in the term structure, but because the term structure may not evolve in a way that is consistent with the original forward rate structure. For instance, at time 0, the one-

year forward effective annual rate of interest is $i_0(1,2) = 15.24\%$, but when time 1 arrives, the actual one year zero coupon bond yield is 5%.

Rates of return demanded by investors may also change, which also affects the market value of the bond. If the market yield rate for the bond is higher than the bondholder's original (book) yield, then the present value of the payments represented by the bond will be less at the higher market yield rate than at the original yield rate (with the reverse occurring if the market yield is below the book yield).

There is no requirement that the bondholder must sell the bond before maturity, and if the bondholder keeps the bond until maturity he will realize the original book yield-to-maturity as the internal rate of return on his investment no matter what changes in interest rates occur during the term of the bond. If the bondholder sells the bond before maturity at a price other than the book value, the bondholder's return for the period that the bond was held will most likely not be the original book yield, but will be related to the market yield at the time of sale.

Suppose that in Example 7.1 above, the investor purchases the 4-year zero coupon bond at the yield rate of 20% per year for a price of 48.23. Suppose that later that same day some significant event occurs that changes the economic and financial outlook of investors and the 4-year spot rate suddenly changes to 22%. The value of the 4-year zero coupon bond becomes $100(1.22)^{-4} = 45.14$, an almost immediate loss of 3.09. The numerical values used in this illustration are not likely to occur in the world's major financial markets, and they exaggerate the possible consequences of short term changes in interest rates. The following section presents a systematic analysis of the sensitivity of a bond's price to changes in the yield rate.

7.1 DURATION OF A SET OF CASHFLOWS AND BOND DURATION

In 1938, economist Frederick Macaulay published a comprehensive study of interest rate behavior from 1856 to 1938 in the book *Some Theoretical Problems Suggested by the Movement of Interest Rates*. As part of that study, Macaulay addressed the notion of the security of bond and loan

values and yields. Macaulay commented that "only short term loans can be imagined to be 'absolutely secure'". He created a measure of the "length" of a loan by which the "security" of loans of various terms to maturity and yield rates could be compared.

"Length" in the definition that Macaulay introduced did not mean the term of the loan (time until the final payment). As an exaggerated example of comparing the lengths of two loans, consider the following two 10 year loans. Both loan amounts are $100, and both loans are at annual effective interest rate 10%.

Loan 1: single payment of 259.37 in 10 years ($100 = 259.37 \times v_{.1}^{10}$)

Loan 2: payment of 100 in 1 year and payment of 23.60 in 10 years ($100 = 100 \times v_{.1} + 23.60 \times v_{.1}^{10}$).

Macaulay wanted to define a measure that would classify Loan 1 as "longer" than Loan 2 in the sense that "bulk" of payment comes later for Loan 1 than for Loan 2. Macaulay called his measure of "length" of a loan the **duration** of the loan, and it is often referred to as **Macaulay duration**. The duration measure he developed took into account the amount and time of each payment and resulted in a weighted average of the time until the payments take place.

Macaulay duration was originally defined assuming a flat term structure applied to a fixed set of cashflows such as an annuity or a coupon bond. The original definition is easily adaptable to a non-flat term structure.

Definition 7.1 – Macaulay Duration

Flat Term Structure

Suppose that the term structure is flat at annual effective rate i.

Suppose that a fixed income set of cashflows consists of payments $K_1, K_2, \cdots, K_n$ at times (years) $1, 2, \cdots, n$. The **Macaulay Duration** of the set of cashflows is denoted D and is defined to be

$$D = \frac{\sum_{t=1}^{n} t K_t (1+i)^{-t}}{\sum_{t=1}^{n} K_t (1+i)^{-t}} \tag{7.1}$$

General Term Structure

Suppose that the term structure is $\{s_0(t)\}_{t>0}$, where $s_0(t)$ is the annual effective yield rate as of time 0 for a zero coupon bond maturing at time t. The **Macaulay Duration** is

$$D = \frac{\sum_{t=1}^{n} t K_t (1+s_0(t))^{-t}}{\sum_{t=1}^{n} K_t (1+s_0(t))^{-t}} \tag{7.2}$$

That duration is a weighted average of time until payment can be seen as follows. Suppose that in Equation 7.2 (or 7.1) we define the factor w_t as

$w_t = \dfrac{K_t (1+s_0(t))^{-t}}{\sum_{t=1}^{n} K_t (1+s_0(t))^{-t}}$. We see that $\sum_{t=1}^{n} w_t = 1$ and $D = \sum_{t=1}^{n} w_t \times t$.

The w_t factor can be thought of as a "weight" applied to t.

The weight applied to time t is $w_t = \dfrac{K_t (1+s_0(t))^{-t}}{\sum_{t=1}^{n} K_t (1+s_0(t))^{-t}}$, which is the fraction of the overall present value of the series that is represented by that particular payment at time t.

It does not seem that at the time Macaulay introduced his concept duration that he meant for it to be applied to calculate changes in bond prices based on changes in yield rates. For many years prior to 1938, interest rates tended to be quite stable and "interest rate risk" may not have been perceived as being as important as it has been in recent years (since the 1970s) which have seen considerable volatility in interest rates. The following graph shows US Treasury bond rates since 1900.
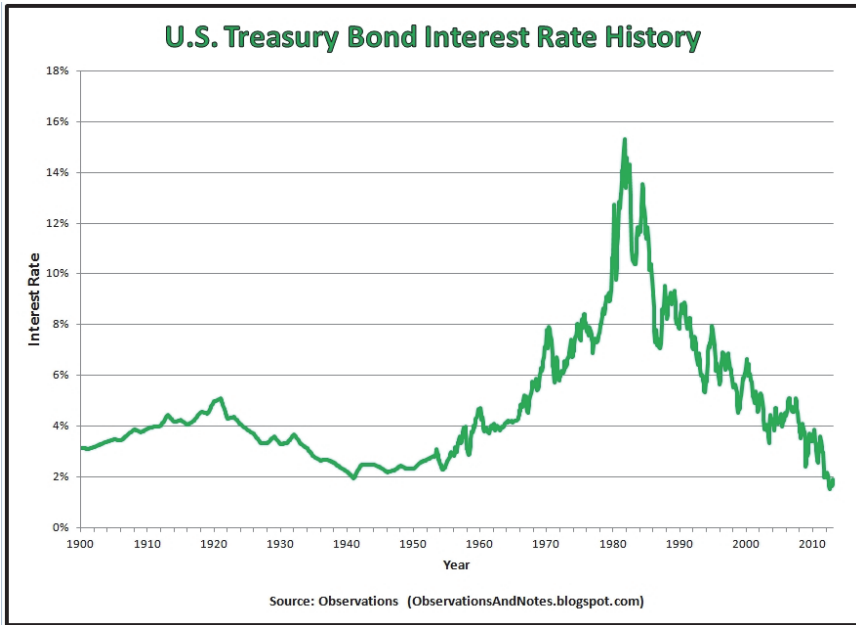
Figure 7.1

The graph indicates that interest rates did not become very volatile until the 1970s. As interest rate volatility and risk grew, the finance industry developed an alternative to Macaulay duration to measure the sensitivity of bond prices to changes in yield rates. The concept of **modified duration** was introduced, defined in terms of the derivative (rate of change) of bond price with respect to change in yield. Modified duration is defined in the case of a flat term structure.

Definition 7.2 – Modified Duration

Suppose that the term structure is flat at annual effective rate i.

Suppose that a fixed income set of cashflows consists of payments $K_1, K_2, \cdots, K_n$ at times (years) $1, 2, \cdots, n$, with a present value (at time 0) of $P(i) = \sum_{t=1}^{n} K_t (1+i)^{-t}$. The **Modified Duration** of the series of cashflows is denoted **MD** and is defined to be

---

$$MD = -\frac{\frac{d}{di}P(i)}{P(i)} = -\frac{d}{di}\ln P(i) \qquad (7.3)$$

Present values decrease when interest rates increase, so in the case of a series of positive cashflow amounts, the derivative $\frac{d}{di}P(i)$ would be negative. In order to have a positive number to measure the sensitivity of present value to changes in interest rate, the negative of the derivative is used.

The units of measure for Macaulay duration is years, since it measures a weighted average of times to payments. The units of measure for modified duration are change in present value per dollar of present value for a change in interest rate. The modified duration measure was developed, in part, to apply a commonly used method to obtain an approximation to changes in bond price when there is a small change in the yield to maturity. It turns out that Macaulay duration can be applied to provide a more accurate approximation for this. These applications will be considered in the following section of this chapter.

Another point to note is the following. From $P(i) = \sum\limits_{t=1}^{n} K_t(1+i)^{-t}$ we get $\frac{d}{di}P(i) = -\sum\limits_{t=1}^{n} t \times K_t(1+i)^{-t-1}$, and MD can be formulated as

$MD = \dfrac{\sum\limits_{t=1}^{n} t K_t(1+i)^{-t-1}}{\sum\limits_{t=1}^{n} K_t(1+i)^{-t}}$. Comparing this to the expression for D in

Equation 7.1, we see that, in the case of a flat term structure D and MD are linked by the relationship $MD = \dfrac{D}{1+i}$.

$$i_A + \text{prime} - \left[i_A - \left(f_A - \frac{c}{2} + \frac{d}{2} \right) \right] = \text{prime} + f_A - \frac{c}{2} + \frac{d}{2};$$

the net effect for borrower B is that B pays

$$\text{prime} + f_B + i_B - f_B - \frac{c}{2} + \frac{d}{2} - \text{prime} = i_B - \frac{c}{2} + \frac{d}{2};$$

the net effect for the intermediary is that the intermediary receives a fee of

$$\text{prime} - \left[i_A - \left(f_A - \frac{c}{2} + \frac{d}{2} \right) \right] + i_B - f_B - \frac{c}{2} + \frac{d}{2} - \text{prime}$$

$$= (i_B - i_A) - (f_B - f_A) - c + d = d.$$

8.2.2 Swapping a Floating Rate Loan for a Fixed Rate Loan

Another scenario in which an interest rate swap may be arranged is one in which a borrower has a floating rate loan that will take place for a certain period of time, and would like to convert it to a fixed rate loan for that period of time (this will be the **swap term**). One way to deal with this is to arrange a FRA for each particular interest payment.

We have seen in Section 6.4.2 that a FRA can be used to change a borrower's future floating interest rate payment to a fixed interest rate payment. In order that there are no arbitrage opportunities, the FRA will have a fixed rate equal to the forward rate of interest implied by the current term structure. An FRA can also be regarded as an interest rate swap, because the borrower's future interest payment based on a floating rate is being swapped for a future interest payment based on a fixed rate. No money exchanges hands at the time the swap (or FRA) is arranged, but there may be a payment from the borrower to the financial intermediary arranging the swap (or vice-versa) when the future interest period ends (or begins).

EXAMPLE 8.3   (*Forward rate agreement as an interest rate swap*)

A borrower has a floating rate loan that has payments of interest only at the end of each year for the next three years. Suppose that the current term structure has effective annual zero coupon bond yields for one, two and three years of 8%, 9% and 9.5%, respectively. The borrower arranges an FRA for each year's interest to exchange floating interest payments for fixed interest payments. Find the fixed interest rates that will be arranged

under each FRA, assuming that no arbitrage opportunities are created by any of the FRAs.

SOLUTION

The fixed interest rate for a particular year's FRA will be the forward rate for that year implied by the current term structure. Note that the one-year yield of 8% will be the floating rate for the first year. There is no FRA for the first year, because the first year's interest rate is known. The FRA for the second year's interest payment will have a fixed rate of

$i_0(1,2) = \frac{(1.09)^2}{1.08} - 1 = .1001$ (the one-year forward effective annual interest

rate). The FRA for the third year's interest payment will have a fixed

rate of $i_0(2,3) = \frac{(1.095)^3}{(1.09)^2} - 1 = .1051$ (the two-year forward effective annual

interest rate). The financial intermediary will pay the difference between $u_{1,2}$ (the floating rate for the second year) and 10.01% at the end of the second year. For the third year, the payment will be the difference between $u_{2,3}$ and 10.51%. ❏

Suppose that the notional amount of the loan in Example 8.3 is $10,000,000. The interest payment at the end of the first year will be $800,000. As a result of the FRAs for the second and third year interest, the borrower will have a net fixed interest payment at the end of the second year of $1,001,000, and $1,051,000 at the end of the third year.

8.2.3 The Swap Rate

A variation on the interest rate swap in Example 8.3 is for the borrower to arrange with the financial intermediary that the borrower's net interest rate is level for each of the three years. Assuming no arbitrage opportunities are created by the arrangement, we can find a fixed level effective annual interest rate j for the three-year period that is an appropriate (or equivalent) substitute for the rates found in the separate FRAs. With a notional amount of $10,000,000, we saw the previous paragraph the resulting fixed interest payments for the three years in the previous paragraph. We use the term structure to find the present value at time 0 of those fixed interest payments:

$$PV_0 = \frac{800,000}{1.08} + \frac{1,001,000}{(1.09)^2} + \frac{1,051,000}{(1.095)^2} = 2,383,761.$$

If these interest payments are swapped for level interest payments of $10,000,000j$ at the end of each year for three years, the present value at time 0 of the level payments is

$$PV_0 = 10,000,000\,j \times \left[\frac{1}{1.08} + \frac{1}{(1.09)^2} + \frac{1}{(1.095)^2} \right].$$

The appropriate rate j should result in the same present value. Therefore,

$$10,000,000\,j \times \left[\frac{1}{1.08} + \frac{1}{(1.09)^2} + \frac{1}{(1.095)^2} \right] = 2,383,761,$$

and solving for j results in $j = 9.42\%$. This 9.42% rate of interest is referred to as the **swap rate**.

The following three sets of cash flows can be swapped for one another:

(i)   floating rate interest payments,

(ii) varying fixed rate interest payments based on the forward rates of interest implied by the current term structure, and

(iii) level fixed interest payments based on the swap rate.

We have seen that the swap rate is a level interest rate on an interest only loan that results in level interest payments that have the same present value as interest payments based on the forward rates of interest. Suppose that the current term structure implies the following set of effective annual forward rates of interest for the next n years: $i_0(0,1), i_0(1,2), \ldots, i_0(n-1,n)$. A loan with floating rate interest can be swapped for a loan with fixed interest based on the forward rates, which, in turn, can be swapped for a loan with level interest at the swap rate j. The swap rate is found by setting equal the present values at time 0 of the floating rate interest payments and the swap rate interest payments. This can be expressed in the equation

$$j \times \left[\frac{1}{1+s_0(1)} + \frac{1}{\left(1+s_0(2)\right)^2} + \cdots + \frac{1}{\left(1+s_0(n)\right)^n} \right]$$

$$= \frac{i_0(0,1)}{1+s_0(1)} + \frac{i_0(1,2)}{\left(1+s_0(2)\right)^2} + \cdots + \frac{i_0(n-1,n)}{\left(1+s_0(n)\right)^n}. \quad (8.3)$$

The swap rate j can be interpreted as a weighted average of the forward rates. This can be seen by reformulating Equation 8.3:

$$j = w_1 i_0(0,1) + w_2 i_0(1,2) + \cdots + w_n i_0(n-1,n),$$

where

$$w_t = \frac{\dfrac{1}{\left(1+s_0(t)\right)^t}}{\dfrac{1}{1+s_0(1)} + \dfrac{1}{\left(1+s_0(2)\right)^2} + \cdots + \dfrac{1}{\left(1+s_0(n)\right)^n}} \quad (8.4)$$

We can follow the progress of the level payment swap from the point of view of the financial intermediary using this example. The first year interest rate is 8%, so the financial intermediary receives a net amount of $\$942,473.77 - \$800,000 = \$142,473.77$ at the end of the first year (the level interest payment received from the borrower minus the floating interest payment paid to the borrower). This amount is not necessarily a gain made by the intermediary. In fact, if interest rates evolve over time in a way that is consistent with the original term structure at time 0, this amount will disappear over the three-year term of the swap.

When the swap was initiated at time 0, the forward rates were $i_0(0,1) = .1001$ and $i_0(2,3) = .1051$. When the first year is done, there will be a term structure as of that time. Suppose that the term structure at the end of the first year has one-year zero yield of $s_1(2) = 10.01\%$ and two-year zero yield at $s_1(3) = 10.26\%$. This implies a one-year forward rate of $i_1(2,3) = \frac{(1.1026)^2}{1.1001} - 1 = .1051$. If a new two-year swap was initiated at time 1, it would call for fixed payments of $\$1,001,000$ at time 2 and $\$1,051,000$ at time 3. The original level payment swap has payments of $\$942,473.77$ at time 2 and time 3. The present value at time

1 of the difference between what the intermediary will receive and what a new swap would have the intermediary receive at times 2 and 3 is

$$\frac{942,473.77 - 1,001,000}{1.1001} + \frac{942,473.77 - 1,051,000}{(1.1026)^2} = -142,469.$$

This cancels out the $142,473 balance after the first-year swap payment.

This suggests a way in which the value to the intermediary of the swap can be found some time after the swap is made. Using the example above, suppose that the term structure at the end of the first year is $s_1(2) = 9\%$ and $s_1(3) = 10\%$. The forward rates as of time 1 implied by this term structure are $i_1(1,2) = .09$ and $i_1(2,3) = \frac{(1.10)^2}{1.09} - 1 = .1101.$

A two-year swap initiated at time 1 would have fixed payments of $900,000 at the end of the second year, and $1,101,000 at the end of the third year. The value of the original swap at time 1 is equal to the $142,473 plus the present value of the difference between the actual swap payments that will be received and the swap payments that are implied by the new term structure:

$$142,473 + \frac{942,473 - 900,000}{1.09} + \frac{942,473 - 1,101,000}{(1.10)^2} = 50,425.$$

If interest rates had risen significantly as of time 1, the swap would have a smaller value, possibly becoming negative.

It is also interesting to note the relationship between the at-par yield and the swap rate on a interest rate swap. At-par yield was introduced in Chapter 6. From Equation 8.3, the swap rate is j,

$$j \times \left[\frac{1}{1 + s_0(1)} + \frac{1}{(1 + s_0(2))^2} + \cdots + \frac{1}{(1 + s_0(n))^n} \right]$$

$$= \frac{i_0(0,1)}{1 + s_0(1)} + \frac{i_0(1,2)}{(1 + s_0(2))^2} + \cdots + \frac{i_0(n-1,n)}{(1 + s_0(n))^n}. \qquad (8.3)$$

The right side of the equation can be written as

$$\frac{1+i_0(0,1)-1}{1+s_0(1)}+\frac{1+i_0(1,2)-1}{\left(1+s_0(2)\right)^2}+\cdots+\frac{1+i_0(n-1,n)-1}{\left(1+s_0(n)\right)^n}$$

$$= 1-\frac{1}{1+s_0(1)}+\frac{1+i_0(1,2)}{\left(1+s_0(2)\right)^2}-\frac{1}{\left(1+s_0(2)\right)^2}$$

$$+\cdots+\frac{1+i_0(n-1,n)}{\left(1+s_0(n)\right)^n}-\frac{1}{\left(1+s_0(n)\right)^n}$$

$$= 1-\frac{1}{1+s_0(1)}+\frac{1}{1+s_0(1)}-\frac{1}{\left(1+s_0(2)\right)^2}$$

$$+\cdots+\frac{1}{\left(1+s_0(n-1)\right)^{n-1}}-\frac{1}{\left(1+s_0(n)\right)^n}$$

$$= 1-\frac{1}{\left(1+s_0(n)\right)^n}$$

Then solving for j results in the at-par yield rate. **The swap rate and the at-par yield rate are the same.**

The Growing Swap Market

The practice of interest rate swapping began in the late 1970s. It has grown incredibly quickly since that time. An estimate of 1992 activity in interest rate and currency swaps is 5 trillion dollars (U.S.). An estimate of the same activity in 2005 was over 60 trillion dollars, and in 2014, the average notional amount outstanding on interest rate derivatives was about 600 trillion dollars.

8.3 DETERMINANTS OF INTEREST RATES

There are a number of factors that influence interest rates. Certain types of interest rates are set by government agencies such as the Federal Reserve Bank in the U.S. and the Bank of Canada in Canada. Government policies are influenced by factors such as inflation, unemployment and the behavior of securities markets. Other interest rates may not be so closely related to central bank rates, such as interest

3.4.9   Merchant's Rule: $X = 211.54$,   US Rule: $X = 212.16$

3.4.11 Straight-Line: 41,078.46 each year
        Actuarial: 16,058.78 in 1st year, 82,568.81 in 20th year

3.4.12 US Rule payment is 328,
        Merchant's Rule payment is 325

CHAPTER 4

SECTION 4.1

4.1.1   (a) 84.5069   (b) 84.8501   (c) 82.5199   (d) 82.9678

4.1.2   115

4.1.3   12,229

4.1.4 $21\frac{1}{2}$ years

4.1.5   .0852

4.1.6   .1264

4.1.7   109.03

4.1.9   (b) 102.79 and 102.74   (c) 3.692% and 3.690%

4.1.10 $i^{(2)} = .0525$

4.1.11 I. False   II. True   III. False

4.1.12 Coupon rates are .0225 and .045 every 6 months

4.1.13 12 years

4.1.14 908.78

4.1.15 875.38

4.1.17 97.896

4.1.19 860

4.1.20 $r = .0354$ every 6 months for 10 year issue and $r = .0417$ every 6 months for 20 year issue

4.1.21 1055

4.1.22 $i^{(2)} = \dfrac{4r_2 - 2r_1}{1 + r_1 - r_2}$

4.1.23 $H = .6446$

4.1.24 2000

4.1.25 $X = 114.28$

SECTION 4.2

4.2.3 $n = 5, \ j = .025$

T	K_t	I_t	PR_t	OB_t
1	500	279.04	220.96	10,940.49
2	500	273.51	226.49	10,714.01
3	500	267.85	232.15	10,481.86
4	500	262.05	237.95	10,243.90
5	10,500	256.10	10,243.90	0

$n = 5, \ j = .075$

T	K_t	I_t	PR_t	OB_t
1	500	674.14	−174.14	9,162.67
2	500	687.20	−187.20	9,349.89
3	500	701.24	−201.24	9,551.10
4	500	716.33	−216.33	9,767.44
5	10,500	732.56	9,767.44	0

4.2.4   90.47

4.2.5   13 years or 26 coupon periods

4.2.6   (a) 8764

4.2.7   48,739

4.2.9   (a) $P = 8117.73, \ r = .06336$

       (b) $P = 29,039.25, \ r = 2483$

       (c) $r = .60, P = 68,821.07$

All values of r are nominal annual rates.