PENSION MATHEMATICS

for

ACTUARIES,

COMMENTARY AND SOLUTIONS

by

Keith P. Sharp, FSA, FCIA, Ph.D.
University of Waterloo

ACTEX Publications, Inc.
Winsted, CT
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Acknowledgements


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Keith P. Sharp, FSA, PhD.
April, 2006
CHAPTER 1: INTRODUCTION

It is clear that saving a portion of income to provide a future pension is an important goal. One can perform a simplistic calculation to determine a funding rate as a proportion of salary. Use the following “dream plan”:

<table>
<thead>
<tr>
<th>Pension:</th>
<th>100% of final salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Termination benefit:</td>
<td>Full accrued liability</td>
</tr>
<tr>
<td>Indexation:</td>
<td>Full CPI after retirement.</td>
</tr>
<tr>
<td>Working life</td>
<td>40 years</td>
</tr>
<tr>
<td>Retired life</td>
<td>18 years</td>
</tr>
</tbody>
</table>

Assume equality of future interest rate, salary increase rate and price inflation. Assume zero pre-retirement deaths, or equivalently assume the value of the pre-retirement death benefit to be the full accrued liability. Entry age is 25. Then the funding rate depends on the ratio of the number of years retired to the number of working years: 18/40 = 0.45. We can see this without using funding formulas. Each of the 40 years of working life needs to pay for 18/40 years of pension. If we include the contributions paid by or on behalf of the individual to government and employment pension plan we should, on this very rough basis, be saving 45 percent of our salaries.

Very few individuals are saving 45% of their income. And most employment pension plans have a total contribution rate much less one-third of the 45%. The above calculation has made several excessively simplifying assumptions. The biggest simplification is the assumption that investment returns will be no more than the rate of salary rise. Another significant point is that the after-retirement income should be compared with the pre-retirement income net of pension contributions. Also, in the context of a corporate pension plan, there are usually substantial cost-savings when employees resign or are terminated, since the termination benefit is typically much less in value than an accrued liability calculated assuming future salary increases. It is clearly important to make more accurate determinations of the rate at which millions of people should be saving. Pensions are important. Let’s start looking at them more closely.
CHAPTER 2: ACTUARIAL COST METHODS

2.2 Unit Credit

First we will set up the notation. The arrow ‘PV’ implies that present values under the unit credit method are calculated as at age x, the age at valuation. Strictly speaking we should use the notation x\(j\) for the age at valuation of person \(j\). However, the \(j\) is usually omitted.

\[
\begin{align*}
T & \quad \text{set of terminated pension plan members} \\
R & \quad \text{set of retired members} \\
A_t & \quad \text{set of actively working plan members} \\
w & \quad \text{entry age from which pensionable service is calculated} \\
u & \quad \text{age at plan inception} \\
x & \quad \text{attained age at valuation} \\
PVFB & \quad \text{present value of future benefits} \\
F & \quad \text{fund actuarial value} \\
NC & \quad \text{normal cost} \\
S_x & \quad \text{actual salary in the year from age } x \\
s_x & \quad \text{salary scale} \\
AL & \quad PVFB - PVFNC = \text{accrued (actuarial) liability} \\
& \quad = AL_A + AL_I \text{ where } AL_I \text{ is for inactives/pensioners.}
\end{align*}
\]

The above gives the most complicated case. It allows for the possibility of generosity at inception: the period \(w\) to plan inception is given ‘pensionable service’ status. For a person hired after inception we can drop plan inception from the diagram.
Pensions are assumed actually not paid until retirement age \( y \). But we allocate a portion \( \Delta B(x) \) to the year following the valuation date. For a United Auto Workers - type plan for example, pension = \( (y-w) \times 40 \times 12 \) per annum from age \( y \). So we have \( \Delta B(x) = 40 \times 12 \) per annum ‘accrued’ in the 12 months from \( x \).

We turn now to the derivation of accrued liability under unit credit:

\[
B(x) = \Delta B(w) + \Delta B(w+1) + \ldots + \Delta B(x-1)
\]

which, it should be noted, does not include \( \Delta B(x) \)

For Traditional Unit Credit (TUC), which is usually used for non-final salary plans, \( \Delta B \) and \( B(x) \) come straight from plan provisions. A career pay plan might define the retirement benefit as, for example, \( B(y) = 2\% \times \text{total career earnings} \), with no adjustment for interest or inflation:

\[
\Delta B(y) = 0.02 \times S_x
\]

\[
B(x) = 0.02 \times (S_w + S_{w+1} + \ldots + S_{x-1})
\]

Considering now Projected Unit Credit (PUC), usually used for final/final average salary/final average earnings plans, \( \Delta B(x) \) and \( B(x) \) use the salary projected using scale \( s_2 \). For example,

\[
B(y) = 2\% \times \text{final 12 months salary} \times (r-w) = 0.02 \times S_x \times (s_{y-1}/s_x) \times (y-w)
\]

\[
\Delta B(x) = B(r)/(y-w)
\]

\[
B(x) = B(r) \times (x-w)/(y-w)
\]

The normal cost is the payment for the next 12 months accrual

\[
NC^j = \Delta B^j \times v \times p_x \times v^{px} \times \dd_y^{12}
\]
The accrued liability is the value of the benefit ‘earned’ by age $x$:

$$AL^j = B^j(x) \times \ddot{p}_x \times v^{x} \times \ddot{a}_y^{(12)}$$

The normal cost is, under Unit Credit, the amount to pay for ‘this year’s benefit accruals, summed over all active plan members

$$NC = \sum_{j \in AL} NC^j(x)$$

Use of the Accrued Liability

A major use of the accrued liability is to assess the safety of the benefits. The accrued liability is compared with the value of the assets. The accrued liability should include all who are entitled to benefits that will deplete the assets. Inactives should be included, e.g. those receiving pension $AL = B(y) \ddot{a}_k^{(12)}$. The comparison of the plan $AL_t$ with assets $F_t$ gives the unfunded accrued liability:

$$UAL_t = AL_t - F_t = \sum_{j \in \text{actives}} AL^j_t + \sum_{k \in \text{inactives}} AL^k_t - F_t$$

$UAL_t$ indicates, roughly, the value of promises not yet paid for, a matter of particular interest to regulators.

The Projected Unit Credit method apportions projected the benefit $B^j(y)$ proportional to service:

$$\Delta B^j(x) = B^j(y)/(y - w)$$

$$B^j(x) = B^j(y) \times (x-w)/(y-w)$$

and the normal cost and accrued liability are found from

$$NC^j = \Delta B^j(x) \times \ddot{p}_x \times v^{x} \times \ddot{a}_y^{(12)}$$

$$AL^j = B^j(x) \times \ddot{p}_x \times v^{x} \times \ddot{a}_y^{(12)}$$
2.3 Entry Age Normal, Level Dollar

Compared with benefit allocation methods such as Unit Credit, this is a whole new way of thinking about pension cost. We know that we need a lump sum $B(y) \ddot{a}_y$ at retirement. Now we arrange to spread cost (not benefit) level over service. A defining feature of Entry Age Normal is that we spread the pension cost from entry age $w$ to retirement age $y$. Here it is possible that entry age equals a hire age at a time before the plan had been created, with past service to hire granted:

As for a mortgage, we have that the initial $PV$ (payments) equals the value of debt when payments start. Hence we have for one particular member $j$:

$$p_{w}^j = p_{w}B^j$$

and more specifically:

$$NC_{x_{\text{EAN}}}^j \times \ddot{a}_{y-w}^{(T)} w = y-w p^{(T)} w \times v^{w} \times B(y) \times \ddot{a}_y$$

The total plan normal cost is given by:

$$NC_t = \sum_{j \in \text{actives}} NC_{x_{j, t}}$$

and the accrued liability for member $j$ is given by:

$$Al_{x_{\text{EAN}}}^j (\text{Prospective}) = pv_x B^r - pv_x NC^j$$

If there are zero decrements we can prove that the same $AL$ is given by

$$Al_{x}^j (\text{Retrospective}) = \text{Accumulated value of past } NC^j$$

$$= (1+i) NC^j s_{x-w}$$
This is much the same as the equivalence of retrospective and prospective reserves in life insurance, using the valuation net premium.

However, in pensions the ‘past NC’ is a notional NC using the age x plan document and the current age x valuation assumptions. Use of the prospective formula

$$A_l^{EAN}(Prospective) = pv_x B'_y - pv_x NC'$$

for $AL$ is safer and less confusing since it is clear that the most recent assumptions, data and benefit definitions are being used throughout.

2.4 Entry Age Normal, Level %

As at entry we set the payments so that they will fund the pension:

$$U^l \times (S_x \times s_w/s_x) \times s\tilde{a}_{w-y-w} = y-w p^{(7)}_w \times \varphi^{x-w} \times B(y)^l \times \tilde{a}^{(12)}_y$$

There is a subtlety in getting NC as percentage $U$ of salary. We use $S_x \times s_w/s_x$, not the actual historical $S_w$ even if $S_w$ is known. This is consistent with using projected $S_x$ to get $B(y)$

Then we can calculate the prospective accrued liability:

$$A_l^l(Prospective) = pv_x B'_y - pv_x NC'$$

$$= y-x p^{(7)}_x \times \varphi^{x} \times B'_y \times \tilde{a}^{(12)}_y - U^l \times S_x \times s\tilde{a}_{x-y-w}$$

Not that under EAN at plan inception $AL_{u^l} \neq 0$. If it is desired to avoid making supplemental cost payments, and regulations permit, then ILP might be used instead of EAN.
Amortization of UAL: e.g. Under EAN

\[ SC = UAL_{1.1.01}/\bar{a}_{1.0} \]

1.1.01 plan inception with back service

We assume that at plan inception, service before 1.1.01 is made ‘pensionable’. Rarely, a pension is awarded to those who retired before the plan existed, which also boosts the plan accrued liability. No normal cost NC was actually paid before 1.1.01 since the plan didn’t exist. Probably there is no fund F at the 1.1.01 plan inception so \( F_{1.1.01} = 0 \). Hence

\[ UAL_{1.1.01} = AL - F_{1.1.01} = \sum_j (pv_x B_j - pv_x NC) - 0 > 0 \]

Because, recalling at entry age \( w \) (not \( x \)) we have:

\[ pv_w NC_j = pv_w B_j \]

So at inception, we have \( UAL_{1.1.01} > 0 \). We pay off the \( UAL_{1.1.01} \), typically by making level dollar supplemental cost payments. The maximum period for the amortization is set by applicable regulation (e.g. ERISA). Assume here that 10 years is chosen. So the plan is to pay for 10 years a total cost

\[ NC + SC = \sum_{j \in \text{actives}} NC_{EAN,j} + UAL_{1.1.01}/\bar{a}_{1.0} \]

2.5 Individual Level Premium

\[ NC^{ILP} \]

Under ILP we adopt the philosophy that at retirement we need a lump sum \( B(y) \times \bar{a}_y^{(12)} \): If plan inception is at age \( u \), after pensionable service commences at \( w \), then the obvious way of
accumulating the lump sum is over the period \( u \) (not \( w \)) to \( y \). Assume in this discussion \( u > w \), otherwise ILP is the same as EAN.

We set up the accumulation by taking an equation of value at \( u \). We require \( PV_u \) (payments) to equal the present value of the required lump sum. So PVs and annuities now start at inception \( u \):

\[
NC_x^{ILP} \times \bar{a}_{y-u|u} = y-u p^{(T)}_u \times v^{y-u} \times B(y) \times \bar{a}_y^{(12)} \quad (ILP.01)
\]

For the whole plan we add the normal cost of each individual:

\[
NC_t = \sum_x NC_x
\]

The accrued liability found prospectively is:

\[
Al_x^I(Prospective) = pv_x B'_y - pv_x NC_y
\]

\[
= y-x p^{(T)}_x \times v^{y-x} \times B'_y \times \bar{a}_y^{(12)} - NC_x \times \bar{a}_{y-x|y}
\]

Note that under ILP at plan inception we have a zero accrued liability:

\[
AL_u^I = y-u p^{(T)}_u \times v^{y-u} \times B'_y \times \bar{a}_y^{(12)} - NC_u \times \bar{a}_{y-u|y}
\]

\[
= 0 \quad (using \ ILP.01)
\]

ILP is virtually always used in its level dollar form. For both \( EA \) and \( ILP \), the accrued liability can be calculated prospectively \( (AL = PVFB - PVFNC) \) or retrospectively as an accumulation of normal costs.

### 2.6-2.9 FIL, AAN and Aggregate Methods

One formula can be used for the Frozen Initial Liability, Attained age Normal and Aggregate methods in their level percent form. We use the notation \( U \) for the normal cost as a fraction of payroll. All quantities are at time \( t \), where ages are \( x'(t) \), abbreviated to \( x \):

\[
U(plan) = (\sum_{plan} pv_x B - UAL_t - F) / \sum_x pv_x S \quad (1.01)
\]

We multiply by \( \sum_x S_x \) to get plan dollar TNC.

The level dollar form for these Frozen Initial Liability, Attained age Normal and Aggregate methods is, per person ('Joe')

\[
NC(Joe) = (\sum_{plan} pv_x B - UAL_t - F) / \sum_x \bar{a}^{(T)}_{y-u} \quad (1.02)
\]
We multiply by the number of actives to get the plan total dollar normal cost.

We choose $UAL_t$ appropriate to the cost method through considering the gain formula (see for example Anderson’s Equation (2.6.5)): 

$$Gain_t = (UAL_{t-1} + NC_{t-1}) (1+i) - C - IC - UAL_t$$  \hspace{1cm} (II.01)

Under FIL and FIL(AAN) we define $Gain_t = 0$.

So we can now get $UAL_t$ from $UAL_{t-1}$ (see Anderson’s (2.6.11)):

$$UAL_t = (UAL_{t-1} + NC_{t-1}) (1+i) - C - IC$$  \hspace{1cm} (II.02)

Equation (II.02) says for FIL and AAN the UAL is frozen. We can use (II.02) to iterate $UAL_t$ from year to year.

The starting values of the iteration at plan inception 00 for the two methods are defined as:

$$UAL_{00}(FIL) = UAL_{00}(Individual-EAN)$$

$$UAL_{00}(AAN) = UAL_{00}(Unit Credit, usually TUC)$$

The phrase ‘Frozen Initial Liability’ implies that the UAL decreases only by the effect of the amortization payments. Items such as good investment performance do not reduce the UAL for these methods. Let’s examine this more closely in the light of our UAL iteration formula:

$$UAL_t = (UAL_{t-1} + NC_{t-1}) (1+i) - C - IC$$  \hspace{1cm} (II.02)

Let’s say we are amortizing level dollar over n years, so SC is chosen as one would calculate a periodic mortgage payment. The SC is the portion of cost paid which does not represent normal cost:

$$SC = (C + IC) / (1+i) - NC_{t-1} = UAL_{00} / \bar{a}_{n}$$

Hence the iteration is:

$$UAL_t = (UAL_{t-1} - UAL_{00} / \bar{a}_{n}) (1+i)$$

This is a standard level dollar loan amortization with fully predictable (frozen) progression of the outstanding principal:

$$UAL_t = UAL_{00} \times \bar{a}_{n-1} / \bar{a}_{n}$$
It may seem as if defining the gain to be zero is a dramatic step, which we consider here. Notice that the pension methods take the $PVFB$ and say that this is paid for in three ways:

1. the fund $F$
2. the present value of future normal costs $PVFNC$
3. the present value of future amortization payments (supplemental costs) $UAL = PVFSC$

So we have the very general pension formula (see e.g. Anderson’s equation (2.6.12))

$$PVFB = F + PVFNC + UAL = AL + PVFNC$$

This formula, suitably specialized, applies to all pension methods. There is an analogy to buying a house with a combination of down payment, mortgage and second mortgage: there are two payment streams, one for each mortgage.

In pension funding, the two payment streams must sum to $PVFB - F$. But the choice of method is really the actuary’s decision about the load taken by each payment stream (normal cost or supplemental cost for amortization). Our ‘gain’ is really ‘unfunded liability gain’. We could (but usually we don’t) define also a normal cost gain, an unexpected decrease in the present value of necessary future normal cost payments. By saying ‘gain=0’ for the FIL and AAN method we are really saying that any good or bad experience is all taken into the normal cost. For example, (1.01) above leads to a decrease in $U$ if the fund $F$ is increased by good investment performance.

For the aggregate method we define $UAL_t = 0$ at all times. We throw away the option of having the second payment stream and just use one payment stream, usually calling it a normal cost. For the level percent of payroll version, which is usually used (1.01) becomes

$$U(plan) = (\sum_{plan} pv_B - UAL_t - F_t)/\sum_A pv_S$$

$$= (\sum_{plan} pv_B - F_t)/\sum_A pv_S$$
Gains and Losses:

The above $Gain_I$ given by II.01 is the total gain. The interest portion of the total gain is found from the excess of interest earned over that which would have been earned at the assumed rate $i$:

$$G_I^t = I - (iF_t + I_C - I_B)$$

$$= F_t (I+i) - (C + I_C + (B + I_B)$$

For decrements such as mortality and termination, the gain in general is given as the difference between the actual and expected release of liabilities; if a lump sum benefit $L_{x+1}$ is given and interest at the assumed rate to year-end is $I_L$ then:

$$G_{Dec} = \sum_T (AL_{x+1}^t - L_{x+1}^t - I_L) - \sum_T q_x (AL_{x+1}^t - L_{x+1}^t - I_L)$$
SOLUTIONS

2.2.1

\[ \text{LHS} = (p_x + q_x) \frac{D_y}{D_{y+1}} = \frac{\ell_{x+1}}{\ell_x} \frac{D_y}{v^{x+1}e^{x+1}} + q_x \frac{D_y}{D_{y+1}} \]
\[ = \frac{D_y}{v^{x+1}e^{x}} + q_x \frac{D_y}{D_{y+1}} = (1 + i) \frac{D_y}{D_x} + q_x \frac{D_y}{D_{y+1}} \]

2.2.2 (a) We recall that \( \mu_x = -D \log \ell_x \),
\[ 1 + i = e^\delta, \log (1 + i) = \delta, D = d/dx. \]
\[ \text{LHS} = D \log NC^x \]
\[ = D \log \Delta B_x + D \log a_x + D \log D_y - D \log D_x \]
\[ = D \log \Delta B_x + 0 + 0 - D \log v_x - D \log \ell_x \]
\[ = D \log \Delta B_x + D_x \log (1 + i) + \mu_x \]
\[ = D \log \Delta B_x + \delta + \mu_x \]

(b)
\[ \Delta B_x = .01 S_x = .01 S_w (1 + r)^{e - w} \]
\[ \log \Delta B_x = \log .01 + \log S_w + x \log (1 + r) - w \log (1 + r) \]
\[ D \log \Delta B_x = \log (1 + r) \]
\[ \therefore D \log NC_x = \mu_x + \delta + \log (1 + r) \]

(c)
\[ D \log S_x = D \log 100 \Delta B_x = D \log 100 + D \log \Delta B_x \]
\[ = D \log \Delta B_x \]
\[ D \log \frac{NC_x}{S_x} = D \log NC_x - D \log S_x \]
\[ = \mu_x + \delta + \log (1 + r) - \log(1 + r) \]
\[ = \mu_x + \delta \]

(d)
\[ D \log \frac{NC_x}{S_x} = \mu_x + \delta = ae^{bx} + \delta \]
Under the credit method, costs generally rise more rapidly than covered payroll.

2.2.3

\[
\begin{align*}
AL_j^i &= B(x) \cdot a_y \cdot \frac{D_y}{D_x} \\
&= \sum_{z=w}^{x-1} \Delta B(z) \cdot a_y \frac{D_y}{D_x} \frac{D_z}{D_x} \\
&= \sum NC^j \cdot \frac{D_z}{D_x} \\
&= \text{accumulated past normal costs.}
\end{align*}
\]

2.2.4

\[
\begin{align*}
\Delta AL_x^j &= B(x) \cdot a_y \cdot \frac{D_y}{D_x} \\
\frac{1}{AL_x^j} \Delta AL_x^j &= \Delta \log AL_x^j \\
\therefore \Delta AL_x^j &= AL_x^j \left( \Delta \log B(x) + 0 + 0 - \Delta \log (\nu^y \ell_x) \right) \\
&= AL_x^j \left( \frac{1}{B(x)} \Delta B(x) + \mu_x + \delta \right) \\
&= a_y \nu^y \ell_x \frac{dB(x)}{d_x} + (\mu_x + \delta) AL_x^j \\
\therefore \frac{d}{dx} AL_x^j &= NC_x^j + (\mu_x + \delta) AL_x^j
\end{align*}
\]

Equation (2.2.16) says that the accrued liability increases because of the increasing accrued benefit, survivorship and interest. Equation (2.2.17) says that the normal cost pays for the increase in the accrued benefit.
2.2.5 Equation (2.2.3) is:

\[ AL_{t+1} = \left[ AL_t + \sum_{\alpha_t} \Delta B^i \hat{\alpha}^{(12)} \frac{D_z}{D_x} \right] (1 + i) \]

\[ - \left[ \sum_{i} B^i(x + 1) \hat{\alpha}_y^{(12)} \frac{D_x}{D_{x+1}} - \sum_{\alpha_x} q_x B^i(x + 1) \hat{\alpha}_y^{(12)} \frac{D_y}{D_{x+1}} \right] \]

\[ = 0 \text{ if assumptions OK} \]

\[ - \sum_R B^j_{x+1} \hat{\alpha}^{(12)} \frac{D_y}{D_{x+1}} \]

\[ = (P + I_p) \text{ if correctly guess amounts for retirements} \]

\[ = [AL_t - NC_t] (1 + i) - P - I_p \]  

(2.2.18)

\[ NC_{t+1} = \sum_{\alpha_{t+1}} \Delta B^i \hat{\alpha}_y^{(12)} \frac{D_y}{D_{x+1}} \]

\[ = \sum_{\alpha_{t} - R-T} \Delta B^i \hat{\alpha}_y^{(12)} \left[ \frac{D_x}{D_z} (1 + i) + q_x \frac{D_y}{D_{x+1}} \right] \]

\[ = NC_t(1 + i) \]

\[ + \sum_{\alpha_{t}} \Delta B^i \hat{\alpha}_y^{(12)} q_x \frac{D_y}{D_{x+1}} - \sum_{i} B^i \hat{\alpha}_y^{(12)} \frac{D_y}{D_{x+1}} \]

\[ = 0 \text{ if assumptions OK} \]

\[ - \sum_R \Delta B^j \hat{\alpha}_y^{(12)} \frac{D_y}{D_{x+1}} \]

\[ = 0 \text{ since } \Delta B^j = 0 \text{ for rets} \]

\[ = NC_t \]  

(2.2.19)

2.3.1

\[ AL^{PROSPECTIVE} = \sum_{\alpha_t} B^j(y) \hat{\alpha}_y^{(12)} \frac{D_y}{D_x} - \sum_{\alpha_t} NC^j \frac{N_x - N_y}{D_x} \]

\[ = \sum_{\alpha_t} NC^j \frac{N_x - N_y}{D_x} - \sum_{\alpha_t} NC^j \frac{N_x - N_y}{D_x} \]

\[ = \sum_{\alpha_t} NC^j \frac{N_x - N_y}{D_x} \]

\[ = AL^{RETROSPECTIVE} \]
2.3.2

\[ LHS = \frac{N_w - N_{x+1}}{D_{x+1}} \left( p_x + q_x \right) \]
\[ = \frac{N_w - N_{x+1}}{\frac{1}{1+r} + \frac{1}{1+r} \cdot N_w - N_x} \cdot \left( p_x + q_x \cdot \frac{N_w - N_{x+1}}{D_{x+1}} \right) \]
\[ = \frac{N_w - N_{x+1}}{D_x} \left( 1 + i \right) + q_x \cdot \frac{N_w - N_{x+1}}{D_{x+1}} \]
\[ = RHS \]

2.3.3 (a)

\[ \bar{AL}_x = \bar{NC}_x \cdot \frac{N_w - N_x}{D_x} \]
\[ = B \cdot a_y \cdot \frac{D_x}{N_w - N_y} \cdot \frac{N_w - N_x}{D_x} \]
\[ = B \cdot a_y \cdot \frac{D_x}{N_w - N_y} \cdot \frac{N_w - N_x}{D_x} \]

(b)

\[ \frac{d}{dx} \bar{AL}_x = \bar{NC} + (\mu_x + \delta) \bar{AL}_x \quad (2.2.16) \]
\[ = \bar{NC} \text{ at age } w \text{ because } \bar{AL}_w = 0 \]

(c) Rate of increase in \( \bar{AL}_x^{EAN} \) at age \( w \) is greater because \( \bar{NC}^{EAN} > \bar{NC}^{UC} \).

(d) They intersect at retirement age. The \( AL \) is greater under \( EAN \), so terminations greater than expected give a greater release under \( EAN \), i.e. greater gain. Early year \( NC \) is larger under \( EAN \), hence a larger fund.

2.4.1

\[ s_{N_x} = s_{D_x} + s_{D_{x+1}} + \ldots \]
\[ = s_x \cdot D_x + s_{x+1} \cdot D_{x+1} + \ldots \]
\[ = (1 + r)^x \cdot D_x + (1 + r)^{x+1} \cdot D_{x+1} + \ldots \]
\[ = N'_x \text{ iff } (1 + r)^x \cdot D_x = D'_x \text{ etc.} \]
which implies that \((1 + r)^x v^x \ell_x = v^x \ell_x\)

\[
\text{or } v' = (1 + r)v = \frac{1 + r}{1 + i}
\]

\[
\text{or } \frac{1}{1 + r'} = \frac{1 + r}{1 + i}
\]

\[
\text{or } i' = \frac{1 + i}{1 + r} - 1
\]

2.4.2 \(B\) in formulas (2.3.1) and (2.4.1) are the same if \(B\) not dependent on salary.

(2.3.1) implies that 
\[
NC_j = B\bar{a}_y \frac{D_y}{D_w} \left( \frac{N_w - N_y}{D_w} \right)
\]

(2.4.1) implies 
\[
NC_i = B\bar{a}_y \frac{D_y}{D_w} \left( \frac{s_w^{-1} \sum s_x D_x}{s_w \sum s_x D_x} \right)
\]

\[
\frac{s_z}{s_w} = 1
\]

\[
\frac{s_z}{s_w} > 1 \text{ for } z \geq 1 \text{ if salaries increase}
\]

\[
\therefore \sum_w \frac{y-1}{s_w} D_z > \sum_w \frac{y-1}{D_w} = \frac{N_w - N_y}{D_w}
\]

\[
\therefore \text{NC under (2.4.1) is lowered}
\]

2.4.3 

\[
B_y = C S_y , \text{ say}
\]

\[
\frac{s_{NC_y}}{NC_y} = \frac{C S_w (1 + r)^{y-w} a_y^{(12)} D_y^{D_w} D_x^{D_w + D_y + \ldots + D_y-1}}{C S_w a_y^{(12)} D_y^{D_w} N_w - N_y}
\]

\[
= (1 + r)^{y-w} \left( \frac{D_y + D_y + \ldots + D_y-1}{D_w + D_y + \ldots + D_y-1} \right)
\]

\[
= (1 + r)^{y-w} \left( 1 + \frac{D_y + D_y + \ldots + D_y-1}{(1 + r)^{y-w} D_w + (1 + r)^{y-w+1} D_y + \ldots + (1 + r)^{y-1} D_y-1} \right)
\]

\[
> 1
\]
because \((1 + r)^{w-y+i} < 1\) \(\forall i \leq i < w - y\)

No. To end up with same fund, the "no salary increase" \(NC\) must overtake at some time.

At age \(x_1\)

\[
\frac{\text{\(NC\)}}{\text{\(NC\)}} = \frac{C \cdot S \cdot (1+r)^{y-x} \delta_y^{(12)} \frac{D_y}{D_y} \frac{\delta_y}{\delta_y} \frac{D_y}{D_y} \frac{N_y-N_y}{N_y-N_y}}{C \cdot S \cdot \delta_y^{(12)} \frac{D_y}{D_y} \frac{E_y}{E_y} \frac{N_y-N_y}{N_y-N_y}}
\]

\[
= \frac{(1 + r)^{y-x} (1 + r)^w \frac{D_w}{D_w} + \frac{D_{w+1}}{D_w} + \cdots + \frac{D_{y-1}}{D_w}}{1 + (1+r)^{w-x} \frac{D_w}{D_w} + \frac{D_{w+1}}{D_w} + \cdots + \frac{D_{y-1}}{D_w}} \quad (> 1 \text{ for } x = w)
\]

\[
= \frac{(1 + r)^{y-x} \frac{D_w}{D_w} + \frac{D_{w+1}}{D_w} + \cdots + \frac{D_{y-1}}{D_w}}{1 + (1+r)^{y-x} \frac{D_w}{D_w} + \frac{D_{w+1}}{D_w} + \cdots + \frac{D_{y-1}}{D_w}} \quad (< 1 \text{ for } x = y)
\]

(bad for small \(x\), \(< 1\) for \(x\) close to \(y\) \(< 1\) for \(x = y\))
\[ NC_x^j = B_x^j \tilde{a}_y^{(12)} \frac{D_y}{D_w} \frac{D_{w}}{N_w-N_y} \]

\[ = c S_x \tilde{a}_y^{(12)} \frac{D_y}{D_w} \frac{D_{w}}{N_w-N_y} \text{ for some } c \]

\[ = c S_w (1 + r)^{x-w} \tilde{a}_y^{(12)} \frac{D_y}{D_w} \frac{D_{w}}{N_w-N_y} \]

So normal cost increases by ratio \((1 + r)\) each year

So constant as a fraction of salary

\[ AL_x^j = c S_x \tilde{a}_y^{(12)} \frac{D_y}{D_w} \frac{N_w-N_x}{N_x-N_y} \]

\[ = c S_w (1 + r)^{x-w} \tilde{a}_y^{(12)} \frac{D_y}{D_w} \frac{N_w-N_x}{N_x-N_y} \]

So have the usual increase in \(AL_x^j\) (proportional to \(\frac{N_x-N_w}{D_x} = \tilde{s}_{w, x-w}\)) accelerated by the \((1 + r)^{x-w}\) factor.

\[ \frac{s_x+1}{s_x} \frac{s_x+1-N_y}{iD_{x+1}} = \frac{s_x-N_y}{iD_x} (1 + i) \left( \frac{1}{1-q_x} \right) \]

\[ = \left( \frac{s_x-N_y}{iD_x} - 1 \right) (1 + i) \left( 1 + \frac{q_x}{1-q_x} \right) \]

\[ = \left( \frac{s_x-N_y}{iD_x} - 1 \right) (1 + i) + q_x \frac{s_x+1}{s_x} \frac{s_x+1-N_y}{iD_{x+1}} \]

(2.4.9)
2.4.6

Define $\tilde{A}L_{i+1} = B_i^j \tilde{a}_{\tilde{y}}^{(12)} D_y \left( \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right)$ (from 2.4.6)

$AL_{i+1}^j = (B_i^j + \Delta B_i^j) \tilde{a}_{\tilde{y}}^{(12)} D_y \left( \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right)$ (from (2.4.6))

$= B_i^j \tilde{a}_{\tilde{y}}^{(12)} D_y \left( \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right) + \Delta B_i^j \tilde{a}_{\tilde{y}}^{(12)} D_y \left( \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right)

+ \frac{D_y}{D_{x+1}} \left[ \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right] (1 + i) + q_x \frac{s_{x+1}^{N_y-^sN_{x+1}}}{s^{N_y-^sN_{x+1}}}

= (NC_i^j + AL_{i+1}^j) (1 + i) + q_x \tilde{A}L_{i+1}^j + \Delta B_i^j \tilde{a}_{\tilde{y}}^{(12)} D_y \left( \frac{s_{x+1}^{N_x-^sN_{x+1}}}{s^{N_x-^sN_{x+1}}} \right)

Recall $F_{i+1} = F_i + I + C - P$ (2.2.5)
\[ UAL_{t+1} = AL_{t+1} - F_{t+1} = \sum_{a_1} AL^j_{t+1} - \sum_{T+R} AL^j_{t+1} - F_{t+1} \]

\[ = \left( AL_t + NC_t \right)(1 + i) + \sum q_x \tilde{AL}^j_{t+1} + \sum \Delta B^j \tilde{a}^{(12)}_{y} \frac{D_y}{D_{x+1}} \]

\[ - \sum_{a_t} \Delta B^j \tilde{a}^{(12)}_{y} \frac{D_y}{D_{w}} \frac{s_{N_{w}}}{s_{N_y}} \frac{s_{x+1}}{s_{N_y}} \frac{s_{N_{x+1}}}{s_{D_{x+1}}} \]

\[ - \sum_{T+R} \left( B^j_t + \Delta B^j \right) \tilde{a}^{(12)}_{y} \left( \frac{D_y}{D_{x+1}} \frac{s_{N_{w}}}{s_{N_y}} \frac{s_{x+1}}{s_{N_y}} \frac{s_{N_{x+1}}}{s_{D_{x+1}}} \right) - F_{t+1} \]

(Note \( \Delta B = 0 \) on \( T, R \), so:

\[ = \left( AL_t + NC_t \right)(1 + i) + \sum q_x \tilde{AL}^j_{t+1} - \sum_{T+R} \tilde{AL}^j_{t+1} \]

\[ + \sum_{a_{t+1}} \Delta B^j \tilde{a}^{(12)}_{y} \left( \frac{D_y}{D_{x+1}} - \frac{D_y}{D_{w}} \frac{s_{N_{w}}}{s_{N_y}} \frac{s_{x+1}}{s_{N_y}} \frac{s_{N_{x+1}}}{s_{D_{x+1}}} \right) + P \]

\[ = UAL_t(1 + i) \]

\[ - \left[ C + I_c - NC_t(1 + i) \right] \]

\[ - \left[ I - IF_t - I_c + I_p \right] \]

\[ + \sum_{a_{t+1}} \Delta B^j \tilde{a}^{(12)}_{y} \left( \frac{D_y}{D_{x+1}} - \frac{D_y}{D_{w}} \frac{s_{N_{w}}}{s_{N_y}} \frac{s_{x+1}}{s_{N_y}} \frac{s_{N_{x+1}}}{s_{D_{x+1}}} \right) \]

\[ - \left[ \sum_{T} \tilde{AL}^j_{t+1} - \sum q_x \tilde{AL}^j_{t+1} \right] \]

\[ - \left[ \sum_{R} \tilde{AL}^j_{t+1} - P - I_p \right] \]  

(2.4.8)

If experience follows assumptions, the last five terms, which comprise the gain, are zero

\[ \text{Gain} = (UAL_t + NC_t)(1 + i) - (C + I_c) - UAL_{t+1} \]  

(2.3.12)
2.4.7 No: \( AL_t = PVFB_t - PVFNC_t \) For UC & EAN and other cost methods:

**Entry Age**

\[
AL_t = \sum_{a_t} NC^j(x) \frac{N_x - N_x}{D_x} = \sum_{a_t} B^j(y) \ddot{a}_y^{(12)} \frac{D_x}{D_x} \frac{N_u - N_u + N_y - N_y}{N_u - N_y} \quad (2.3.2)
\]

\[
= \sum_{a_t} B^j(y) \ddot{a}_y^{(12)} \frac{D_x}{D_x} - \sum_{a_t} NC^j(x) \frac{N_x - N_x}{D_x} \quad (2.3.3)
\]

\[
= PVFB_t - PVFNC_t
\]

**Unit Credit**

\[
AL_t = \sum_{a_t} B^j(x) \ddot{a}_y^{(12)} \frac{D_x}{D_x} = \sum_{a_t} = \sum_{a_t} \sum_{x=0}^{t-1} NC^j(x) \frac{D_u + x}{D_x} \quad (2.2.13)
\]

\[
= AVPNC_t
\]

\[
= PVFB_t - PVFNC_t \quad \text{(Eqn of value at } t)\]

**What is Difference** UC vs. EAN?

**Unit Credit** \( AL^j_t = \sum B^j(x) \ddot{a}_y^{(12)} \frac{D_x}{D_x} = PV \) Benefit accrued to date.

Normal cost is “cost of benefit accruing this year”

**Entry Age** “Benefit accrued to date” not defined.

No simple interpretation of \( AL_t \).

Cost is spread over pensionable service.
2.4.8 Interpretation:

Increase in present value at \( x + 1 \) of retirement benefits because of change \( \Delta B \) in estimated benefit

LESS

value at \((x + 1)\) of increase in future normal costs

\[
\sum_{a_{t+1}} \Delta B^j \tilde{a}_{y}^{(12)} \left( \frac{D_y}{D_{x+1}} - \frac{D_y}{D_w} \right) \frac{s_{x+1} N_{x+1} - s_{N_y}}{s_{N_w} - s_{N_y}}
\]

\[
= \sum_{a_{t+1}} \Delta B^j \tilde{a}_{y}^{(12)} \frac{D_y}{D_{x+1}} \left( 1 - \frac{s_{N_{x+1}} - s_{N_k}}{s_{N_w} - s_{N_y}} \right)
\]

\[
= \sum_{a_{t+1}} \Delta B^j \tilde{a}_{y}^{(12)} \frac{D_y}{D_{x+1}} \left( \frac{s_{N_w} - s_{N_{x+1}}}{s_{N_w} - s_{N_y}} \right)
\]

\[
= \sum_{a_{t+1}} B^j_{t+1} \tilde{a}_{y}^{(12)} \frac{D_y}{D_{x+1}} \left( \frac{s_{N_w} - s_{N_{x+1}}}{s_{N_w} - s_{N_y}} \right) - \sum_{a_{t+1}} B^j_{t+1} \tilde{a}_{y}^{(12)} \frac{D_y}{D_{x+1}} \left( \frac{s_{N_w} - s_{N_{x+1}}}{s_{N_w} - s_{N_y}} \right)
\]

\[
= \sum_{a_{t+1}} \left( A L^j_{t+1} - \tilde{A} L^j_{t+1} \right)
\]
2.5.1 \( AL_{z+1}^i = B_o \frac{D_y}{N_z-N_y} \left( \frac{N_z-N_y}{D_{z+1}} - \frac{N_z+1-N_y}{D_{z+1}} \right) + \Delta B o_y \frac{D_y}{D_{z+1}} \)

\[ = \Delta B o_y \frac{D_y}{N_{z+1} - N_y} \frac{N_{z+1} - N_y}{D_{z+1}} \]

\[ = B_o \frac{D_y}{N_z - N_y} \frac{D_z}{D_{z+1}} + 0 \]

\[ = B_o \frac{D_y}{N_z - N_y} \frac{D_z}{D_{z+1}} = NC \frac{D_z}{D_{z+1}} \]

2.5.2 \( AL_{z+1}^i = \sum_{A_t} (AL_{z+1}^i + NC_{t}) \frac{D_{z+1}}{D_{z+1}} \)

\[ = \sum_{A_t} (AL_{z+1}^i + NC_{t}) (1+i) \left( p_{z+1} + q_{z+1} \right) \]

\[ = \sum_{A_t} (AL_{z+1}^i + NC_{t}) (1+i) + \sum_{A_t} q_{z+1} (AL_{z+1}^i + NC_{t}) (1+i) \]

\[ = \sum_{A_t} (AL_{z+1}^i + NC_{t}) (1+i) + \sum_{A_t} q_{z+1} AL_{z+1}^i + NC_{t} \]
2.5.3 Left side of (2.5.17) = \( AL_i \)

\[
\begin{align*}
(B+\Delta B)\bar{y} & \frac{D_y}{D_{z+1}} - (B\bar{y} \frac{D_z}{D_{z+1}} \frac{\delta D_z}{\delta z} \frac{\delta z+1}{\delta z} + \Delta B\bar{y} \frac{D_y}{D_{z+1}} \frac{\delta D_{z+1}}{\delta z} \frac{\delta z+1}{\delta D_{z+1}}) \cdot \frac{\delta N_{z+1} - \delta N_y}{\delta D_{z+1}} \\
&= \bar{y} \frac{D_y}{D_{z+1}} - \bar{y} \frac{D_y}{D_{z}} \frac{\delta D_z}{\delta z} \frac{\delta z+1}{\delta z} \frac{\delta N_{z+1} - \delta N_y}{\delta D_{z+1}} + 0 \\
&= \bar{y} \frac{D_y}{D_z} \frac{\delta D_z}{\delta z} \frac{\delta z+1}{\delta D_{z+1}} \left( \frac{D_z}{D_{z+1}} \frac{\delta N_{z+1} - \delta N_y}{\delta D_{z+1}} - \frac{\delta N_z - \delta N_y}{\delta D_z} \frac{D_z}{D_{z+1}} \right) \\
&= NC_i \left( \frac{D_z}{D_{z+1}} \frac{\delta N_z - \delta N_y}{\delta D_z} - \frac{\delta N_{z+1} - \delta N_y}{\delta D_{z+1}} \frac{D_z}{D_{z+1}} \right) \\
&= NC_i \frac{D_z}{D_{z+1}} \frac{\delta N_z - \delta N_y}{\delta D_z} - \frac{\delta N_{z+1} - \delta N_y}{\delta D_{z+1}} \frac{D_z}{D_{z+1}} \\
&= NC_i \frac{D_z}{D_{z+1}} \frac{\delta N_y - (\delta N_{z+1} - \delta N_y)}{\delta D_z} \\
&= NC_i \frac{D_z}{D_{z+1}} \frac{\delta D_z}{\delta D_z} = NC_i \frac{D_z}{D_{z+1}} \\
\end{align*}
\]

2.5.4 \( U_i \) = normal cost as a constant fraction of salary

\[
\text{PFVFNC}_i = U_i \text{PVFS}_i \quad \text{and} \quad U_i = \frac{NC_i}{S_i}
\]

\[
\text{PFVFBL}_i = \frac{\text{PFVFNC}_i}{S_i} = \frac{\text{PFVFBL}_i}{S_i} = \frac{AL_i}{S_i}
\]

(2.5.18)

Easier to use than 2.5.15 and 2.5.14 partly because there's no need to go back and examine last year's information.
2.6.1 Given $P\overline{VFB}_t = B_t(y)\frac{D_y}{D_z}$, $P\overline{VFB}_t = \sum_{A_t} P\overline{VFB}_t$

$$\sum_{A_{t+1}} P\overline{VFB}_{t+1} = \sum_{A_{t+1}} B_{t+1}(y)\frac{D_y}{D_x} - \sum_{T+R} P\overline{VFB}_t = \sum_{A_t} P\overline{VFB}_t$$

$$= \sum_{A_t} B_t(y)\frac{D_y}{D_x} + \sum_{A_t} \Delta B\frac{D_y}{D_x} - \sum_{T+R} B_t(y)\frac{D_y}{D_x} \Delta B = 0 \text{ for } T+R$$

$$\frac{D_y}{D_x} = \frac{D_y}{D_z}(1+i) + q_z \frac{D_y}{D_{z+1}}$$

$$= \sum_{A_t} B_t(y)\frac{D_y}{D_z}(1+i) + q_z \frac{D_y}{D_{z+1}} + \sum_{A_t} \Delta B\frac{D_y}{D_x} - \sum_{T+R} B_t(y)\frac{D_y}{D_x} \Delta B$$

$$= P\overline{VFB}_t(1+i) + \sum_{A_t} q_z P\overline{VFB}_{t+1} + \sum_{A_t} \Delta B\frac{D_y}{D_x} - \sum_{T+R} P\overline{VFB}_t$$

$$= P\overline{VFB}_t(1+i) - (\sum_{T} P\overline{VFB}_{t+1} - \sum_{R} q_z P\overline{VFB}_t) - \sum_{T+R} P\overline{VFB}_t + \sum_{A_t} \Delta B\frac{D_y}{D_x}$$

$\sim$ implies values computed as though the increase $\Delta B$ in the projected benefit between times $t$ and $t+1 = 0$.

2.6.2 \[
\frac{N_{z+1}-N_y}{D_{z+1}} = \frac{N_{z+1}-N_z}{D_{z+1}} (p_z + q_z) = \frac{N_{z+1}-N_y}{D_{z+1}} \frac{l_{z+1}}{l_z} + q_z \frac{N_{z+1}-N_y}{D_{z+1}}
\]

$$= \frac{N_z - N_y}{D_z}(1+i) + q_z \frac{N_y + N_{y+1}-N_y}{D_{z+1}} = \left(\frac{N_z - N_y}{D_z} - 1\right)(1+i) + q_z \frac{N_{z+1}-N_y}{D_{z+1}}$$

$$\sum_{A_{z+1}} \frac{N_{z+1}-N_y}{D_{z+1}} = \sum_{A_t} - \sum_{T} - \sum_{R}

= \sum_{A_t} \left(\frac{N_z - N_y}{D_z} - 1\right)(1+i) - \left[\sum_{T} \frac{N_{z+1}-N_y}{D_{z+1}} - \sum_{A_t} q_z \frac{N_{z+1}-N_y}{D_{z+1}} - \sum_{R} \frac{N_{z+1}-N_y}{D_{z+1}}\right]$$

2.6.3 \[D + \Delta D = \sqrt{N + \Delta N}
\]

$$\frac{D^2 N}{D} + \frac{N}{D} \Delta D
\]

$$\frac{\Delta N}{N} \Delta D = \frac{N}{D} + \frac{\Delta N}{D} \frac{\Delta D}{\Delta + \Delta D}$$