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Fall 2017 Edition | **Volume I**

Johnny Li, P.h.D., FSA | Andrew Ng, Ph.D., FSA

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Fall 2017 Edition

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Preface

Thank you for choosing ACTEX.

A new version of Exam MLC is launched in Spring 2014. The new Exam MLC is significantly different from the old one, most notably in the following aspects:

(1) Written-answer questions are introduced and form a major part of the examination.

(2) The number of official textbooks is reduced from two to one. The new official textbook, *Actuarial Mathematics for Life Contingent Risks* 2nd edition (AMLCR), contains a lot more technical materials than other textbooks written on the same topic.

(3) The level of cognitive skills demanded from candidates is much higher. In particular, the new learning objectives require candidates to not only calculate numerical values but also, for example, interpret the results they obtain.

(4) Several new (and more advanced) topics, such as participating insurance, are added to the syllabus.

Because of these major changes, ACTEX have decided to bring you this new study manual, which is written to fit the new exam.

We know very well that you may be worried about written-answer questions. To help you best prepare for the new exam, this manual contains some 150 written-answer questions for you to practice. Eight full-length mock exams, written in exactly the same format as that announced in SoA's Exam MLC Introductory Note, are also provided. Many of the written-answer questions in our mock exams are highly challenging! We are sorry for giving you a hard time, but we do want you to succeed in the real exam.

The learning outcomes of the new exam syllabus require candidates to be able to interpret a lot of actuarial concepts. This skill is drilled extensively in our written-answer practice problems, which often ask you to interpret a certain actuarial formula or to explain your calculation. Also, as seen in SoA's Exam MLC Sample Written-Answer Questions (e.g., #9), you may be asked in the new exam to define or describe a certain insurance product or actuarial terminology. To help you prepare for this type of exam problems, we have prepared a special chapter (Chapter 0), which contains definitions and descriptions of various products and terminologies. The special chapter is written in a "fact sheet" style so that you can remember the key points more easily.

Proofs and derivations are another key challenge. In the new exam, you are highly likely to be asked to prove or derive something. This is evidenced by, for example, problem #4 in SoA's Exam MLC Sample Written-Answer Questions, which demands a mathematical derivation of the Kolmogorov forward differential equations for a certain transition probability. In this new study manual, we do teach (and drill) you how to prove or derive important formulas. This is in stark contrast to some other exam prep products in which proofs and derivations are downplayed, if not omitted.

We have paid special attention to the topics that are newly introduced in the recent two syllabus updates. Seven full-length chapters (Chapters 0, 10, 12 – 16) and two sections (amount to more than 300 pages) are especially devoted to these topics. Moreover, instead of treating the new topics as "orphans", we demonstrate, as far as possible, how they can be related to the old topics in an exam setting. This is very important for you, because multiple learning outcomes can be examined in one single exam question.

We have made our best effort to ensure that all topics in the syllabus are explained and practiced in sufficient depth. For your reference, a detailed mapping between this study manual and the official textbook is provided on pages P-10 to P-12.

Besides the topics specified in the exam syllabus, you also need to know a range of numerical techniques in order to succeed. These techniques include, for example, Euler's method, which is involved in SoA's Exam MLC Sample Multiple-Choice Question #299. We know that quite a few of you have not even heard of Euler's method before, so we have prepared a special chapter (Appendix 1, appended to the end of the study manual) to teach you all numerical techniques required for this exam. In addition, whenever a numerical technique is used, we clearly point out which technique it is, letting you follow our examples and exercises more easily.

Other distinguishing features of this study manual include:

- We use graphics extensively. Graphical illustrations are probably the most effective way to explain formulas involved in Exam MLC. The extensive use of graphics can also help you remember various concepts and equations.
- A sleek layout is used. The font size and spacing are chosen to let you feel more comfortable in reading. Important equations are displayed in eye-catching boxes.
- Rather than splitting the manual into tiny units, each of which tells you a couple of formulas only, we have carefully grouped the exam topics into 17 chapters. Such a grouping allows you to more easily identify the linkages between different concepts, which, as we mentioned earlier, are essential for your success.
- Instead of giving you a long list of formulas, we point out which formulas are the most important. Having read this study manual, you will be able to identify the formulas you must remember and the formulas that are just variants of the key ones.
- We do not want to overwhelm you with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.
- We write the practice problems and the mock exams in a similar format as the released exam and sample questions. This will enable you to comprehend questions more quickly in the real exam.

On page P-13, you will find a flow chart showing how different chapters of this manual are connected to one another. You should first study Chapters 0 to 10 <u>in order</u>. Chapter 0 will give you some background factual information; Chapters 1 to 4 will build you a solid foundation; and Chapters 5 to 11 will get you to the core of the exam. You should then study Chapters 12 to 16 in any order you wish. Immediately after reading a chapter, do <u>all</u> practice problems we provide for that chapter. Make sure that you understand every single practice problem. Finally, work on the mock exams.

Before you begin your study, please download the exam syllabus from SoA's website:

https://www.soa.org/education/exam-req/edu-exam-m-detail.aspx

On the last page of the exam syllabus, you will find a link to Exam MLC Tables, which are frequently used in the exam. You should keep a copy of the tables, as we are going to refer to them from time to time. You should also check the exam home page periodically for updates, corrections or notices.

If you find a possible error in this manual, please let us know at the "Customer Feedback" link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the "Errata & Updates" link.

Enjoy your study!

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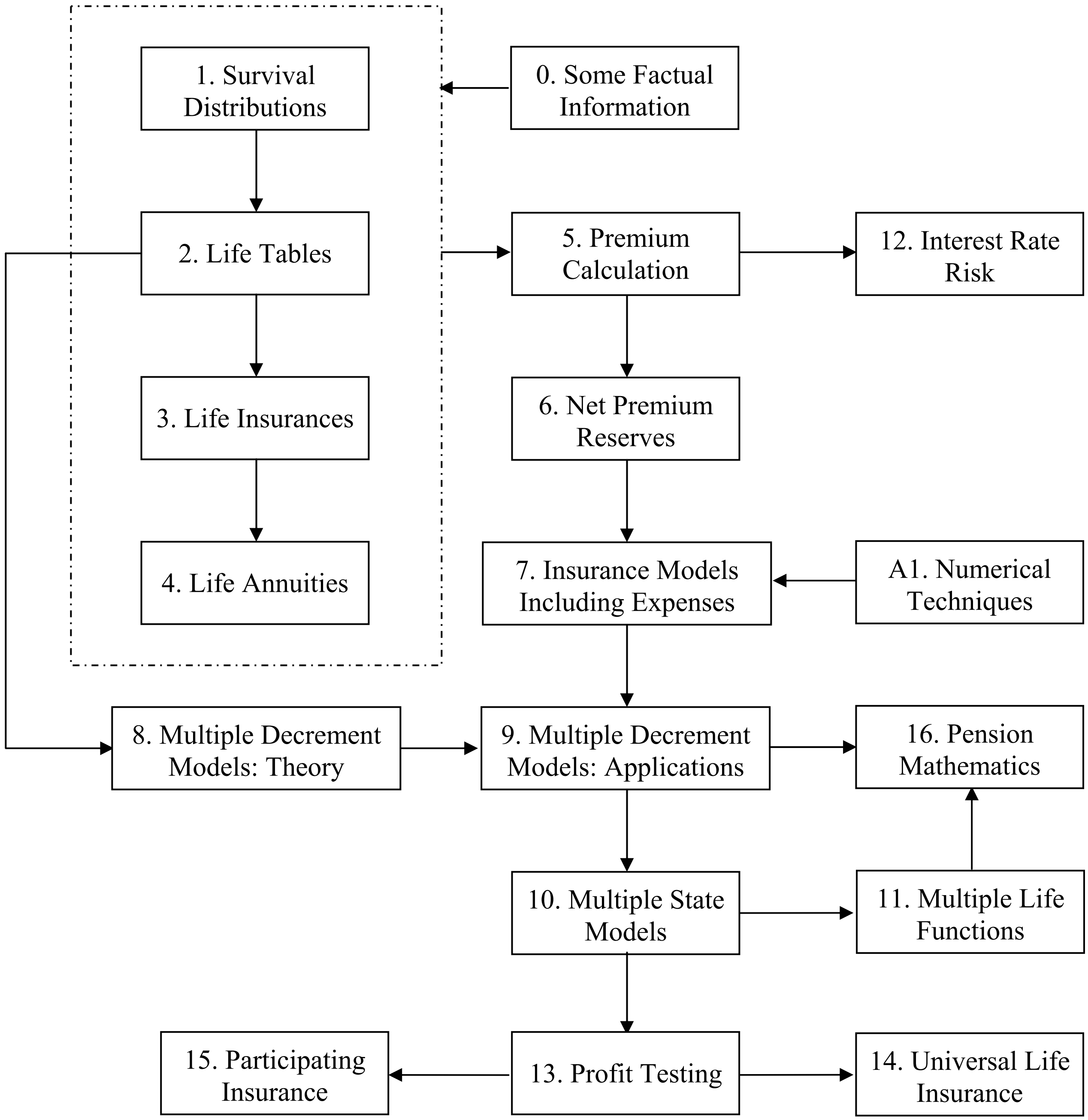
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13. Profit Testing
14. Universal Life Insurance

Chapter 0 Some Factual Information

This chapter serves as a summary of Chapter 1 in AMLCR. It contains descriptions of various life insurance products and pension plans. There is absolutely no mathematics in this chapter.

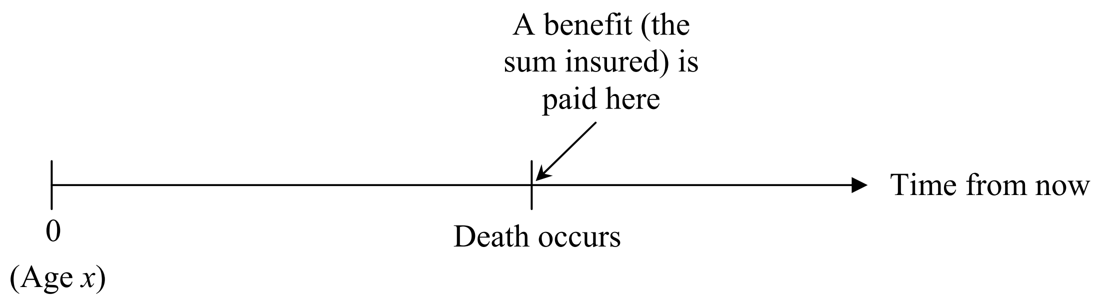
You should know (and remember) the information presented in this chapter, because in the written answer questions, you may be asked to define or describe a certain pension plan or life insurance policy. Most of the materials in this chapter are presented in a "fact sheet" style so that you can remember the key points more easily.

Many of the policies and plans mentioned in this chapter will be discussed in detail in later parts of this study guide.

0. 1 Traditional Life Insurance Contracts

Whole life insurance

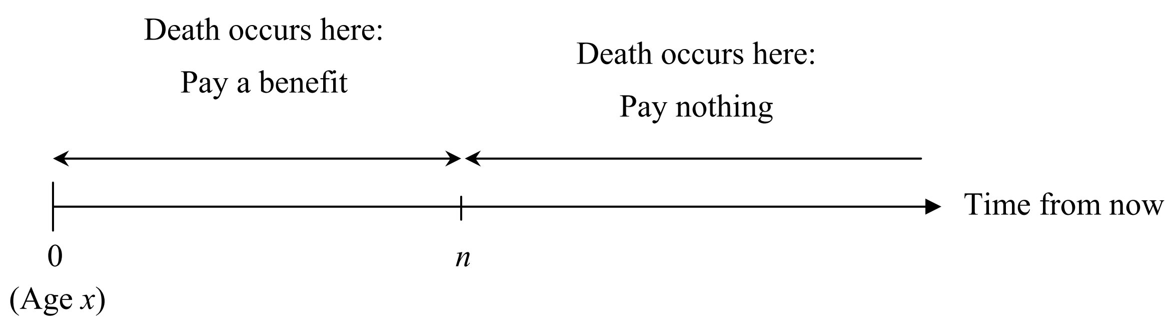
A whole life insurance pays a benefit on the death of the policyholder whenever it occurs. The following diagram illustrates a whole life insurance sold to a person age x.



The amount of benefit is often referred to as the *sum insured*. The policyholder, of course, has to pay the "price" of policy. In insurance context, the "price" of a policy is called the *premium*, which may be payable at the beginning of the policy, or periodically throughout the life time of the policy.

Term life insurance

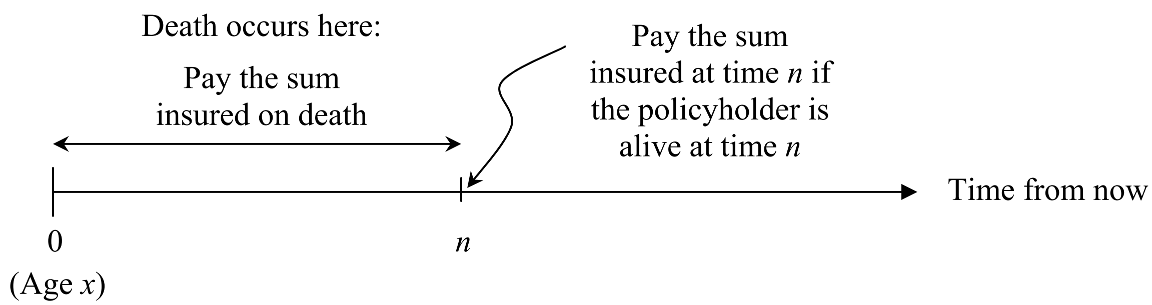
A term life insurance pays a benefit on the death of the policyholder, provided that death occurs before the end of a specified term.



The time point n in the diagram is called the *term* or the *maturity date* of the policy.

Endowment insurance

An endowment insurance offers a benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs earlier.



These three types of traditional life insurance will be discussed in Chapter 3 of this study guide.

Participating (with profit) insurance

Any premium collected from the policyholder will be invested, for example, in the bond market. In a participating insurance, the profits earned on the invested premiums are shared with the policyholder. The profit share can take different forms, for example, cash dividends, reduced premiums or increased sum insured. This product type will be discussed in detail in Chapter 15 of this study manual.

0. 2 Modern Life Insurance Contracts

Modern life insurance products are usually more flexible and often involve an investment component. The table below summarizes the features of several modern life insurance products.

Product	**Features**
Universal life insurance	– Combines investment and life insurance – Premiums are flexible, as long as the accumulated value of the premiums is enough to cover the cost of insurance
Unitized with-profit insurance	– Similar to traditional participating insurance – Premiums are used to purchase shares of an investment fund. – The income from the investment fund increases the sum insured.
Equity-linked insurance	– The benefit is linked to the performance of an investment fund. – Examples: equity-indexed annuities (EIA), unit-linked policies, segregated fund policies, variable annuity contracts – Usually, investment guarantees are provided.

In Chapter 14 of this study guide, we will discuss universal life insurance policies in detail.

0. 3 Underwriting

Underwriting refers to the process of collecting and evaluating information such as age, gender, smoking habits, occupation and health history. The purposes of this process are:

- To classify potential policyholders into broadly homogeneous risk categories
- To determine if additional premium has to be charged.

The following table summarizes a typical categorization of potential policyholders.

Category	**Characteristics**
Preferred lives	Have very low mortality risk
Normal lives	Have some risk but no additional premium has to be charged
Rated lives	Have more risk and additional premium has to be charged
Uninsurable lives	Have too much risk and therefore not insurable

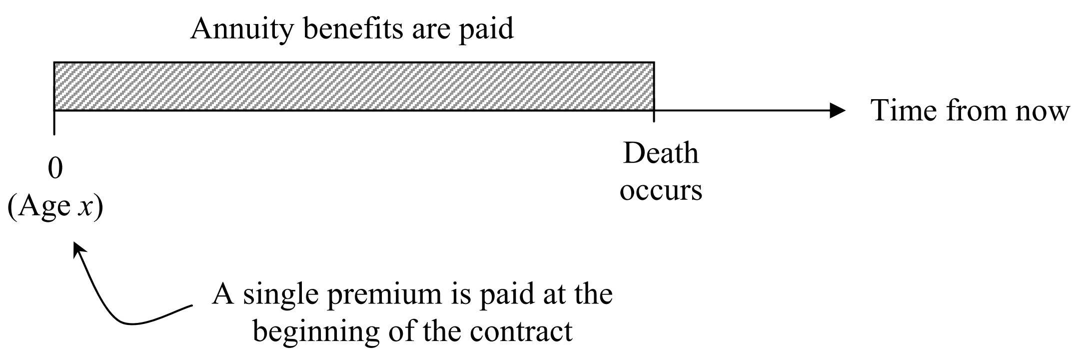
Underwriting is an important process, because with no (or insufficient) underwriting, there is a risk of adverse selection; that is, the insurance products tend to attract high risk individuals, leading to excessive claims. In Chapter 2, we will introduce the select-and-ultimate table, which is closely related to underwriting.

0. 4 Life Annuities

A life annuity is a benefit in the form of a regular series of payments, conditional on the survival of the policyholder. There are different types of life annuities.

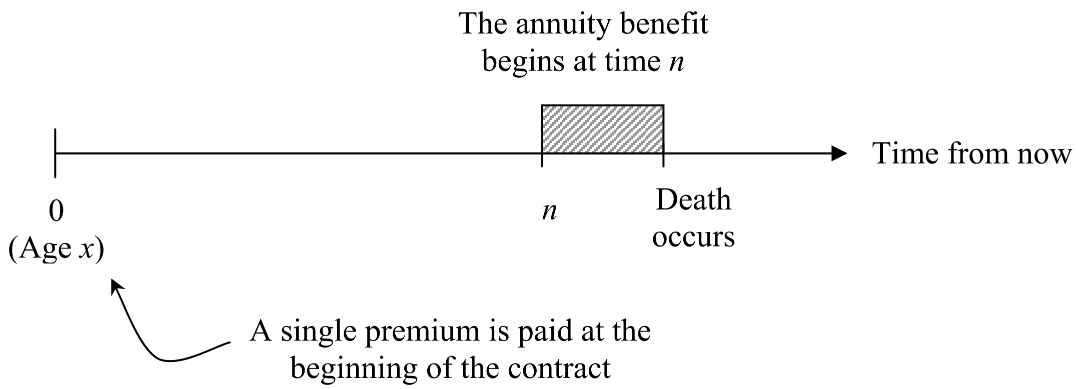
Single premium immediate annuity (SPIA)

The annuity benefit of a SPIA commences as soon as the contract is written. The policyholder pays a single premium at the beginning of the contract.



Single premium deferred annuity (SPDA)

The annuity benefit of a SPDA commences at some future specified date (say n years from now). The policyholder pays a single premium at the beginning of the contract.



Regular Premium Deferred Annuity (RPDA)

An RPDA is identical to a SPDA except that the premiums are paid periodically over the deferred period (i.e., before time n).

These three annuity types will be discussed in greater depth in Chapter 4 of this study guide.

Some life annuities are issued to two lives (a husband and wife). These life annuities can be classified as follows.

Joint life annuity	The annuity benefit ceases on the first death of the couple.
Last survivor annuity	The annuity benefit ceases on the second death of the couple.
Reversionary annuity	The annuity benefit begins on the first death of the couple, and ceases on the second death.

These annuities will be discussed in detail in Chapter 11 of this study guide.

0. 5 Pensions

A pension provides a lump sum and/or annuity benefit upon an employee's retirement. In the following table, we summarize a typical classification of pension plans:

Defined contribution (DC) plans	The retirement benefit from a DC plan depends on the accumulation of the deposits made by the employ and employee over the employee's working life time.
Defined benefit (DB) plans	The retirement benefit from a DB plan depends on the employee's service and salary. **Final salary plan:** the benefit is a function of the employee's final salary. **Career average plan:** the benefit is a function of the average salary over the employee's entire career in the company.

Pension plans will be discussed in detail in Chapter 16 of this study guide.

Chapter 1 Survival Distributions

OBJECTIVES

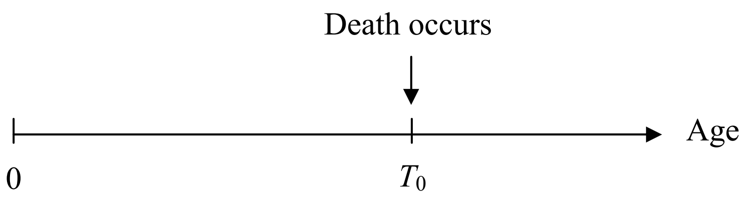
1. To define future lifetime random variables
2. To specify survival functions for future lifetime random variables
3. To define actuarial symbols for death and survival probabilities and develop relationships between them
4. To define the force of mortality

In Exam FM, you valued cash flows that are paid at some known future times. In Exam MLC, by contrast, you are going to value cash flows that are paid at some unknown future times. Specifically, the timings of the cash flows are dependent on the future lifetime of the underlying individual. These cash flows are called life contingent cash flows, and the study of these cash flows is called life contingencies.

It is obvious that an important part of life contingencies is the modeling of future lifetimes. In this chapter, we are going to study how we can model future lifetimes as random variables. A few simple probability concepts you learnt in Exam P will be used.

1. 1 Age-at-death Random Variable

Let us begin with the age-at-death random variable, which is denoted by T_0. The definition of T_0 can be easily seen from the diagram below.



The age-at-death random variable can take any value within $[0, \infty)$. Sometimes, we assume that no individual can live beyond a certain very high age. We call that age the limiting age, and denote it by ω. If a limiting age is assumed, then T_0 can only take a value within $[0, \omega]$.

We regard T_0 as a continuous random variable, because it can, in principle, take any value on the interval $[0, \infty)$ if there is no limiting age or $[0, \omega]$ if a limiting age is assumed. Of course, to model T_0, we need a probability distribution. The following notation is used throughout this study guide (and in the examination).

– $F_0(t) = \Pr(T_0 \le t)$ is the (cumulative) distribution function of T_0.

– $f_0(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} F_0(t)$ is the probability density function of T_0. For a small interval Δt, the product $f_0(t)\Delta t$ is the (approximate) probability that the age at death is in between t and $t + \Delta t$.

In life contingencies, we often need to calculate the probability that an individual will survive to a certain age. This motivates us to define the survival function:

$$S_0(t) = \Pr(T_0 > t) = 1 - F_0(t).$$

Note that the subscript "0" indicates that these functions are specified for the age-at-death random variable (or equivalently, the future lifetime of a person age 0 now).

Not all functions can be regarded as survival functions. A survival function must satisfy the following requirements:

1. $S_0(0) = 1$. This means every individual can live at least 0 years.
2. $S_0(\omega) = 0$ or $\lim_{t \to \infty} S_0(t) = 0$. This means that every individual must die eventually.
3. $S_0(t)$ is monotonically decreasing. This means that, for example, the probability of surviving to age 80 cannot be greater than that of surviving to age 70.

Summing up, $f_0(t)$, $F_0(t)$ and $S_0(t)$ are related to one another as follows.

FORMULA

Relations between $f_0(t)$, $F_0(t)$ and $S_0(t)$

$$f_0(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_0(t) = -\frac{\mathrm{d}}{\mathrm{d}t} S_0(t), \tag{1.1}$$

$$S_0(t) = \int_t^\infty f_0(u)\mathrm{d}u = 1 - \int_0^t f_0(u)\mathrm{d}u = 1 - F_0(t), \tag{1.2}$$

$$\Pr(a < T_0 \le b) = \int_a^b f_0(u)\mathrm{d}u = F_0(b) - F_0(a) = S_0(a) - S_0(b). \tag{1.3}$$

Note that because T_0 is a continuous random variable, $\Pr(T_0 = c) = 0$ for any constant c. Now, let us consider the following example.

Example **1.1** [Structural Question]

You are given that $S_0(t) = 1 - t/100$ for $0 \le t \le 100$.

(a) Verify that $S_0(t)$ is a valid survival function.

(b) Find expressions for $F_0(t)$ and $f_0(t)$.

(c) Calculate the probability that T_0 is greater than 30 and smaller than 60.

Solution

(a) First, we have $S_0(0) = 1 - 0/100 = 1$.

Second, we have $S_0(100) = 1 - 100/100 = 0$.

Third, the first derivative of $S_0(t)$ is $-1/100$, indicating that $S_0(t)$ is non-increasing.

Hence, $S_0(t)$ is a valid survival function.

(b) We have $F_0(t) = 1 - S_0(t) = t/100$, for $0 \le t \le 100$.

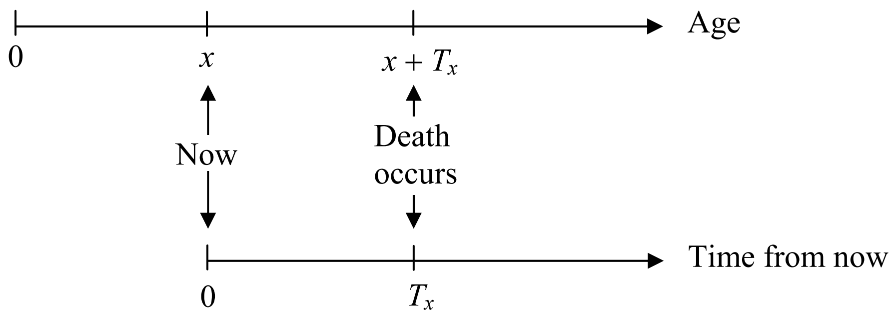
Also, we have and $f_0(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} F_0(t) = 1/100$, for $0 \le t \le 100$.

(c) $\Pr(30 < T_0 < 60) = S_0(30) - S_0(60) = (1 - 30/100) - (1 - 60/100) = 0.3$.

[END]

1.2 Future Lifetime Random Variable

Consider an individual who is age x now. Throughout this text, we use (x) to represent such an individual. Instead of the entire lifetime of (x), we are often more interested in the future lifetime of (x). We use T_x to denote the future lifetime random variable for (x). The definition of T_x can be easily seen from the diagram below.



[Note: For brevity, we may only display the portion starting from age x (i.e., time 0) in future illustrations.]

If there is no limiting age, T_x can take any value within $[0, \infty)$. If a limiting age is assumed, then T_x can only take a value within $[0, \omega - x]$. We have to subtract x because the individual has attained age x at time 0 already.

We let $S_x(t)$ be the survival function for the future lifetime random variable. The subscript "x" here indicates that the survival function is defined for a life who is age x now. It is important to understand that when modeling the future lifetime of (x), we always know that the individual is alive at age x. Thus, we may evaluate $S_x(t)$ as a conditional probability:

$$\begin{aligned} S_x(t) &= \Pr(T_x > t) = \Pr(T_0 > x + t \mid T_0 > x) \\ &= \frac{\Pr(T_0 > x + t \cap T_0 > x)}{\Pr(T_0 > x)} = \frac{\Pr(T_0 > x + t)}{\Pr(T_0 > x)} = \frac{S_0(x+t)}{S_0(x)}. \end{aligned}$$

The third step above follows from the equation $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$, which you learnt in Exam P.

FORMULA

Survival Function for the Future Lifetime Random Variable

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} \tag{1.4}$$

With $S_x(t)$, we can obtain $F_x(t)$ and $f_x(t)$ by using

$$F_x(t) = 1 - S_x(t) \quad \text{and} \quad f_x(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_x(t),$$

respectively.

Example 1.2 [Structural Question]

You are given that $S_0(t) = 1 - t/100$ for $0 \le t \le 100$.

(a) Find expressions for $S_{10}(t)$, $F_{10}(t)$ and $f_{10}(t)$.

(b) Calculate the probability that an individual age 10 now can survive to age 25.

(c) Calculate the probability that an individual age 10 now will die within 15 years.

Solution

(a) In this part, we are asked to calculate functions for an individual age 10 now (i.e., $x = 10$). Here, $\omega = 100$ and therefore these functions are defined for $0 \le t \le 90$ only.

First, we have $S_{10}(t) = \dfrac{S_0(10+t)}{S_0(10)} = \dfrac{1-(10+t)/100}{1-10/100} = 1 - \dfrac{t}{90}$, for $0 \le t \le 90$.

Second, we have $F_{10}(t) = 1 - S_{10}(t) = t/90$, for $0 \le t \le 90$.

Finally, we have $f_{10}(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} F_{10}(t) = \dfrac{1}{90}$.

(b) The probability that an individual age 10 now can survive to age 25 is given by

$$\Pr(T_{10} > 15) = S_{10}(15) = 1 - \frac{15}{90} = \frac{5}{6}.$$

(c) The probability that an individual age 10 now will die within 15 years is given by

$$\Pr(T_{10} \le 15) = F_{10}(15) = 1 - S_{10}(15) = \frac{1}{6}.$$

[END]

1. 3 Actuarial Notation

For convenience, we have designated actuarial notation for various types of death and survival probabilities.

Notation 1: ${}_tp_x$

We use ${}_tp_x$ to denote the probability that a life age x now survives to t years from now. By definition, we have

$$ {}_tp_x = \Pr(T_x > t) = S_x(t). $$

When $t = 1$, we can omit the subscript on the left-hand-side; that is, we write ${}_1p_x$ as p_x.

Notation 2: ${}_tq_x$

We use ${}_tq_x$ to denote the probability that a life age x now dies before attaining age $x + t$. By definition, we have

$$ {}_tq_x = \Pr(T_x \le t) = F_x(t). $$

When $t = 1$, we can omit the subscript on the left-hand-side; that is, we write ${}_1q_x$ as q_x.

Notation 3: ${}_{t|u}q_x$

We use ${}_{t|u}q_x$ to denote the probability that a life age x now dies between ages $x + t$ and $x + t + u$. By definition, we have

$$ {}_{t|u}q_x = \Pr(t < T_x \le t + u) = F_x(t + u) - F_x(t) = S_x(t) - S_x(t + u). $$

When $u = 1$, we can omit the subscript u; that is, we write ${}_{t|1}q_x$ as ${}_{t|}q_x$.

Note that when we describe survival distributions, "p" always means a survival probability, while "q" always means a death probability. The "|" between t and u means that the death probability is deferred by t years. We read "$t \mid u$" as "t deferred u". It is important to remember the meanings of these three actuarial symbols. Let us study the following example.

Example 1.3

Express the probabilities associated with the following events in actuarial notation.

(a) A new born infant dies no later than age 45.

(b) A person age 20 now survives to age 38.

(c) A person age 57 now survives to age 60 but dies before attaining age 65.

Assuming that $S_0(t) = e^{-0.0125t}$ for $t \geq 0$, evaluate the probabilities.

Solution

(a) The probability that a new born infant dies no later than age 45 can be expressed as ${}_{45}q_0$. [Here we have "q" for a death probability, $x = 0$ and $t = 45$.]

Further, ${}_{45}q_0 = F_0(45) = 1 - S_0(45) = 0.4302$.

(b) The probability that a person age 20 now survives to age 38 can be expressed as ${}_{18}p_{20}$. [Here we have "p" for a survival probability, $x = 20$ and $t = 38 - 20 = 18$.]

Further, we have ${}_{18}p_{20} = S_{20}(18) = \dfrac{S_0(38)}{S_0(20)} = 0.7985$.

(c) The probability that a person age 57 now survives to age 60 but dies before attaining age 65 can be expressed as ${}_{3|5}q_{57}$. [Here, we have "q" for a (deferred) death probability, $x = 57$, $t = 60 - 57 = 3$, and $u = 65 - 60 = 5$.]

Further, we have ${}_{3|5}q_{57} = S_{57}(3) - S_{57}(8) = \dfrac{S_0(60)}{S_0(57)} - \dfrac{S_0(65)}{S_0(57)} = 0.058357$.

[END]

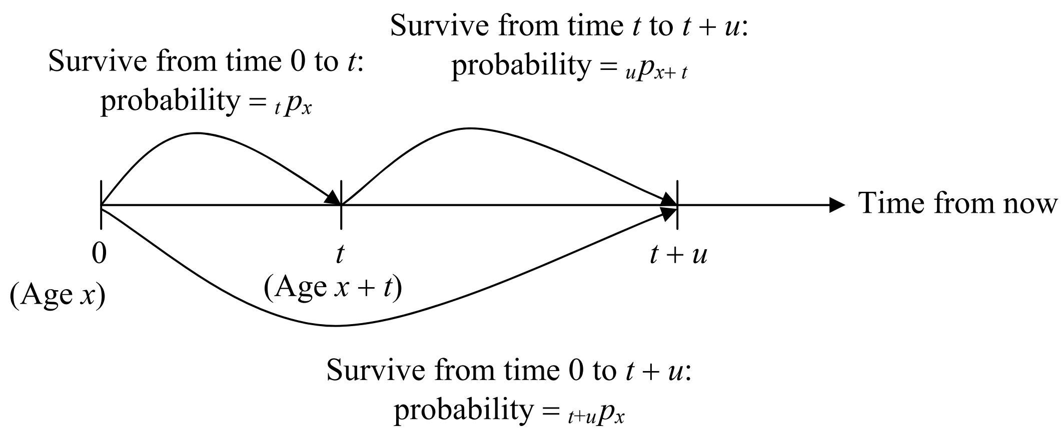
Other than their meanings, you also need to know how these symbols are related to one another. Here are four equations that you will find very useful.

Equation 1: ${}_tp_x + {}_tq_x = 1$

This equation arises from the fact that there are only two possible outcomes: dying within t years or surviving to t years from now.

Equation 2: ${}_{t+u}p_x = {}_tp_x \times {}_up_{x+t}$

The meaning of this equation can be seen from the following diagram.



Mathematically, we can prove this equation as follows:

$$ {}_{t+u}p_x = S_x(t+u) = \frac{S_0(x+t+u)}{S_0(x)} = \frac{S_0(x+t)}{S_0(x)}\frac{S_0(x+t+u)}{S_0(x+t)} = S_x(t)S_{x+t}(u) = {}_tp_x \times {}_up_{x+t}\,. $$

Equation 3: ${}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$

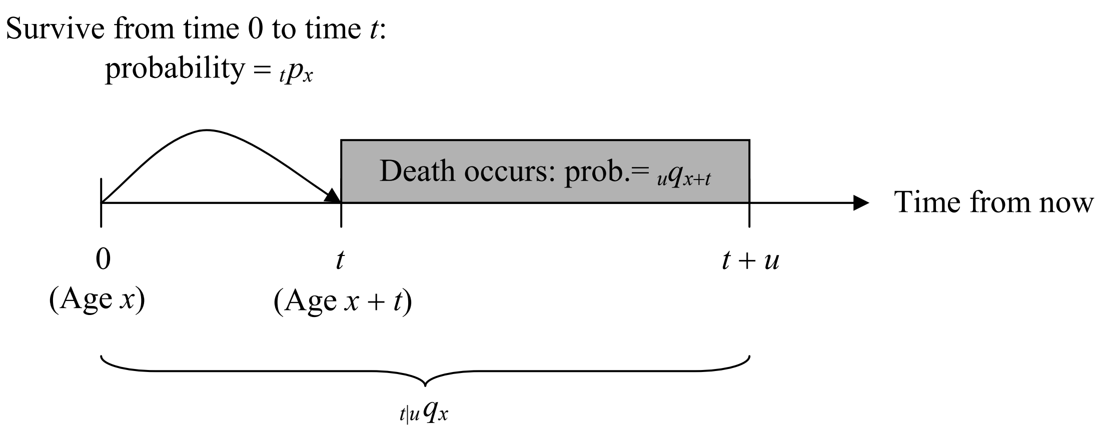
This equation arises naturally from the definition of ${}_{t|u}q_x$.

We have ${}_{t|u}q_x = \Pr(t < T_x \le t + u) = F_x(t + u) - F_x(t) = {}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x$.

Also, ${}_{t|u}q_x = \Pr(t < T_x \le t + u) = S_x(t) - S_x(t + u) = {}_tp_x - {}_{t+u}p_x$.

Equation 4: ${}_{t|u}q_x = {}_tp_x \times {}_uq_{x+t}$

The reasoning behind this equation can be understood from the following diagram:



Mathematically, we can prove this equation as follows:

$$
\begin{aligned}
{}_{t|u}q_x &= {}_t p_x - {}_{t+u}p_x && \text{(from Equation 3)} \\
&= {}_t p_x - {}_t p_x \times {}_u p_{x+t} && \text{(from Equation 2)} \\
&= {}_t p_x (1 - {}_u p_{x+t}) && \\
&= {}_t p_x \times {}_u q_{x+t} && \text{(from Equation 1)}
\end{aligned}
$$

Here is a summary of the equations that we just introduced.

FORMULA

Relations between ${}_t p_x$, ${}_t q_x$ and ${}_{t|u}q_x$

$$ {}_t p_x + {}_t q_x = 1, \tag{1.5} $$

$$ {}_{t+u}p_x = {}_t p_x \times {}_u p_{x+t}, \tag{1.6} $$

$$ {}_{t|u}q_x = {}_{t+u}q_x - {}_t q_x = {}_t p_x - {}_{t+u}p_x = {}_t p_x \times {}_u q_{x+t}. \tag{1.7} $$

Let us go through the following example to see how these equations are applied.

Example 1.4

You are given:

(i) $p_x = 0.99$

(ii) $p_{x+1} = 0.985$

(iii) ${}_3p_{x+1} = 0.95$

(iv) $q_{x+3} = 0.02$

Calculate the following:

(a) p_{x+3}

(b) ${}_2p_x$

(c) ${}_2p_{x+1}$

(d) ${}_3p_x$

(e) ${}_{1|2}q_x$

Solution

(a) $p_{x+3} = 1 - q_{x+3}$

$= 1 - 0.02 = 0.98$

(b) ${}_2p_x = p_x \times p_{x+1}$

$= 0.99 \times 0.985 = 0.97515$

(c) Consider ${}_3p_{x+1} = {}_2p_{x+1} \times p_{x+3}$

$\Rightarrow 0.95 = {}_2p_{x+1} \times 0.98$

$\Rightarrow {}_2p_{x+1} = 0.9694$

(d) ${}_3p_x = p_x \times {}_2p_{x+1}$

$= 0.99 \times 0.9694 = 0.9597$

(e) ${}_{1|2}q_x = p_x \times {}_2q_{x+1}$

$= p_x (1 - {}_2p_{x+1})$

$= 0.99 (1 - 0.9694) = 0.0303$

[END]

1.4 Curtate Future Lifetime Random Variable

In practice, actuaries use Excel extensively, so a discrete version of the future lifetime random variable would be easier to work with. We define

$$K_x = \lfloor T_x \rfloor,$$

where $\lfloor y \rfloor$ means the integral part of y. For example, $\lfloor 1 \rfloor = 1$, $\lfloor 4.3 \rfloor = 4$ and $\lfloor 10.99 \rfloor = 10$. We call K_x the curtate future lifetime random variable.

It is obvious that K_x is a discrete random variable, since it can only take non-negative integral values (i.e., 0, 1, 2, …). The probability mass function for K_x can be derived as follows:

$$\Pr(K_x = 0) = \Pr(0 \le T_x < 1) = q_x,$$
$$\Pr(K_x = 1) = \Pr(1 \le T_x < 2) = {}_{1|1}q_x,$$
$$\Pr(K_x = 2) = \Pr(2 \le T_x < 3) = {}_{2|1}q_x, \ldots$$

Inductively, we have

FORMULA

Probability Mass Function for K_x

$$\Pr(K_x = k) = {}_{k|1}q_x, \qquad k = 0, 1, 2, \ldots \tag{1.8}$$

The cumulative distribution function can be derived as follows:

$$\Pr(K_x \le k) = \Pr(T_x < k + 1) = {}_{k+1}q_x, \qquad \text{for } k = 0, 1, 2, \ldots .$$

It is just that simple! Now, let us study the following example, which is taken from a previous SoA Exam.

Example 1.5 [Course 3 Fall 2003 #28]

For (x):

(i) K is the curtate future lifetime random variable.

(ii) $q_{x+k} = 0.1(k + 1), \qquad k = 0, 1, 2, \ldots, 9$

Calculate $\text{Var}(K \wedge 3)$.

(A) 1.1 (B) 1.2 (C) 1.3 (D) 1.4 (E) 1.5

Solution

The notation $\wedge$ means "minimum". So here $K \wedge 3$ means $\min(K, 3)$. For convenience, we let $W = \min(K, 3)$. Our job is to calculate $\text{Var}(W)$. Note that the only possible values that W can take are 0, 1, 2, and 3.

To accomplish our goal, we need the probability function of W, which is related to that of K. The probability function of W is derived as follows:

$\Pr(W = 0) = \Pr(K = 0) = q_x = 0.1$

$\Pr(W = 1) = \Pr(K = 1) = {}_{1|}q_x$

$\qquad = p_x \times q_{x+1}$

$= (1 - q_x)q_{x+1}$

$= (1 - 0.1) \times 0.2 = 0.18$

$\Pr(W = 2) = \Pr(K = 2) = {}_{2|}q_x$

$= {}_2p_x \times q_{x+2} = p_x \times p_{x+1} \times q_{x+2}$

$= (1 - q_x)(1 - q_{x+1})\, q_{x+2}$

$= 0.9 \times 0.8 \times 0.3 = 0.216$

$\Pr(W = 3) = \Pr(K \geq 3) = 1 - \Pr(K = 0) - \Pr(K = 1) - \Pr(K = 2) = 0.504.$

From the probability function for W, we obtain $\mathrm{E}(W)$ and $\mathrm{E}(W^2)$ as follows:

$\mathrm{E}(W) = 0 \times 0.1 + 1 \times 0.18 + 2 \times 0.216 + 3 \times 0.504 = 2.124$

$\mathrm{E}(W^2) = 0^2 \times 0.1 + 1^2 \times 0.18 + 2^2 \times 0.216 + 3^2 \times 0.504 = 5.58$

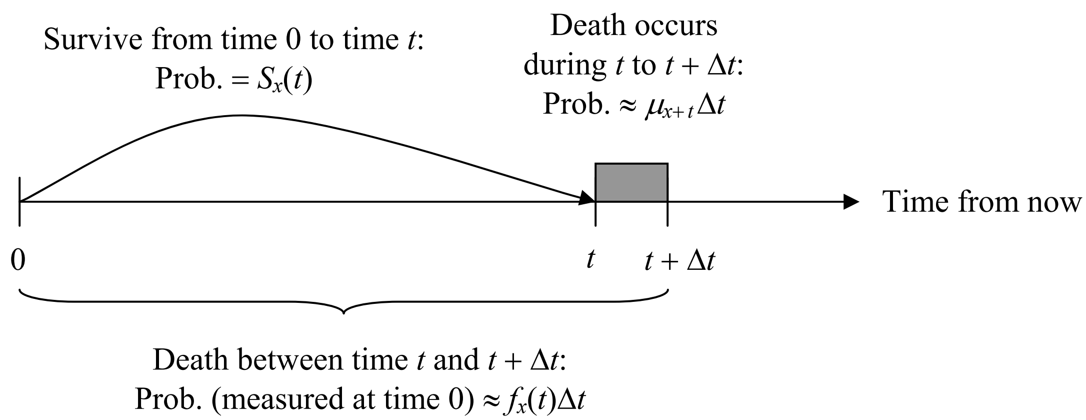
This gives $\mathrm{Var}(W) = \mathrm{E}(W^2) - [\mathrm{E}(W)]^2 = 5.58 - 2.124^2 = 1.07$. Hence, the answer is (A).

[END]

1. 5 Force of Mortality

In Exam FM, you learnt a concept called the force of interest, which measures the amount of interest credited in a very small time interval. By using this concept, you valued, for example, annuities that make payouts continuously. In this exam, you will encounter continuous life contingent cash flows. To value such cash flows, you need a function that measures the probability of death over a very small time interval. This function is called the force of mortality.

Consider an individual age x now. The force of mortality for this individual t years from now is denoted by μ_{x+t} or $\mu_x(t)$. At time t, the (approximate) probability that this individual dies within a very small period of time Δt is $\mu_{x+t}\Delta t$. The definition of μ_{x+t} can be seen from the following diagram.



From the diagram, we can also tell that $f_x(t)\Delta t = S_x(t)\mu_{x+t}\Delta t$. It follows that

$$f_x(t) = S_x(t)\mu_{x+t} = {}_tp_x\,\mu_{x+t}\,.$$

This is an extremely important relation, which will be used throughout this study manual.

Recall that $f_x(t) = F_x'(t) = -S_x'(t)$. This yields the following equation:

$$\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)},$$

which allows us to find the force of mortality when the survival function is known.

Recall that $\dfrac{\mathrm{d}\ln x}{\mathrm{d}x} = \dfrac{1}{x}$, and that by chain rule, $\dfrac{\mathrm{d}\ln g(x)}{\mathrm{d}x} = \dfrac{g'(x)}{g(x)}$ for a real-valued function g.

We can rewrite the previous equation as follows:

$$\begin{aligned}
\mu_{x+t} &= -\frac{S_x'(t)}{S_x(t)} \\
\mu_{x+t} &= -\frac{\mathrm{d}[\ln S_x(t)]}{\mathrm{d}t} \\
-\mu_{x+t}\mathrm{d}t &= \mathrm{d}[\ln S_x(t)].
\end{aligned}$$

Replacing t by u,

$$\begin{aligned}
-\mu_{x+u}\mathrm{d}u &= \mathrm{d}[\ln S_x(u)] \\
-\int_0^t \mu_{x+u}\mathrm{d}u &= \int_0^t \mathrm{d}[\ln S_x(u)] \\
-\int_0^t \mu_{x+u}\mathrm{d}u &= \ln S_x(t) - \ln S_x(0) \\
S_x(t) &= \exp\left(-\int_0^t \mu_{x+u}\mathrm{d}u\right).
\end{aligned}$$

This allows us to find the survival function when the force of mortality is known.

F O R M U L A

Relations between μ_{x+t}, $f_x(t)$ and $S_x(t)$

$$f_x(t) = S_x(t)\mu_{x+t} = {}_tp_x\mu_{x+t}, \tag{1.9}$$

$$\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)}, \tag{1.10}$$

$$S_x(t) = \exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right). \tag{1.11}$$

Not all functions can be used for the force of mortality. We require the force of mortality to satisfy the following two criteria:

(i) $\mu_{x+t} \geq 0$ for all $x \geq 0$ and $t \geq 0$.

(ii) $\int_0^\infty \mu_{x+u}\,\mathrm{d}u = \infty$.

Criterion (i) follows from the fact that $\mu_{x+t}\,\Delta t$ is a measure of probability, while Criterion (ii) follows from the fact that $\lim_{t\to\infty} S_x(t) = 0$.

Note that the subscript $x + t$ indicates the age at which death occurs. So you may use μ_x to denote the force of mortality at age x. For example, μ_{20} refers to the force of mortality at age 20. The two criteria above can then be written alternatively as follows:

(i) $\mu_x \geq 0$ for all $x \geq 0$.

(ii) $\int_0^\infty \mu_x\,\mathrm{d}x = \infty$.

The following two specifications of the force of mortality are often used in practice.

Gompertz' law

$$\mu_x = Bc^x$$

Makeham's law

$$\mu_x = A + Bc^x$$

In the above, A, B and c are constants such that $A \geq -B$, $B > 0$ and $c > 1$.

Let us study a few examples now.

Example 1.6 [Structural Question]

For a life age x now, you are given:

$$S_x(t) = \frac{(10-t)^2}{100}, \qquad 0 \le t < 10.$$

(a) Find μ_{x+t}.

(b) Find $f_x(t)$.

Solution

(a) $$\mu_{x+t} = -\frac{S'_x(t)}{S_x(t)} = -\frac{-\dfrac{2(10-t)}{100}}{\dfrac{(10-t)^2}{100}} = \frac{2}{10-t}.$$

(b) You may work directly from $S_x(t)$, but since we have found μ_{x+t} already, it would be quicker to find $f_x(t)$ as follows:

$$f_x(t) = S_x(t)\mu_{x+t} = \frac{(10-t)^2}{100} \times \frac{2}{10-t} = \frac{10-t}{50}.$$

[END]

Example 1.7 [Structural Question]

For a life age x now, you are given

$$\mu_{x+t} = 0.002t, \qquad t \ge 0.$$

(a) Is μ_{x+t} a valid function for the force of mortality of (x)?

(b) Find $S_x(t)$.

(c) Find $f_x(t)$.

Solution

(a) First, it is obvious that $\mu_{x+t} \ge 0$ for all x and t.

Second, $\int_0^\infty \mu_{x+u} \mathrm{d}u = \int_0^\infty 0.002u \mathrm{d}u = 0.001u^2\Big|_0^\infty = \infty$.

Hence, it is a valid function for the force of mortality of (x).

(b) $S_x(t) = \exp\left(-\int_0^t \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^t 0.002u\mathrm{d}u\right) = \exp(-0.001t^2)$.

(c) $f_x(t) = S_x(t)\mu_{x+t} = 0.002t\exp(-0.001t^2)$.

[END]

Example 1.8 [Course 3 Fall 2002 #35]

You are given:

(i) $R = 1 - \exp\left(-\int_0^1 \mu_{x+t}\mathrm{d}t\right)$

(ii) $S = 1 - \exp\left(-\int_0^1 (\mu_{x+t} + k)\mathrm{d}t\right)$

(iii) k is a constant such that $S = 0.75R$.

Determine an expression for k.

(A) $\ln((1-q_x)/(1-0.75q_x))$

(B) $\ln((1-0.75q_x)/(1-p_x))$

(C) $\ln((1-0.75p_x)/(1-p_x))$

(D) $\ln((1-p_x)/(1-0.75q_x))$

(E) $\ln((1-0.75q_x)/(1-q_x))$

Solution

First, $R = 1 - S_x(1) = 1 - p_x = q_x$.

Second,

$$S = 1 - \exp\left(-\int_0^t (\mu_{x+t} + k)\mathrm{d}u\right) = 1 - e^{-k}\exp\left(-\int_0^t \mu_{x+t}\mathrm{d}u\right) = 1 - e^{-k}S_x(1) = 1 - e^{-k}p_x.$$

Since $S = 0.75R$, we have

$$\begin{aligned}
1-e^{-k}p_x &= 0.75q_x \\
e^{-k} &= \frac{1-0.75q_x}{p_x} \\
e^{k} &= \frac{p_x}{1-0.75q_x}. \\
k &= \ln\left(\frac{p_x}{1-0.75q_x}\right) = \ln\left(\frac{1-q_x}{1-0.75q_x}\right)
\end{aligned}$$

Hence, the answer is (A).

[END]

Example 1.9 [Structural Question]

(a) Show that when $\mu_x = Bc^x$, we have

$$_t p_x = g^{c^x (c^t - 1)} ,$$

where g is a constant that you should identify.

(b) For a mortality table constructed using the above force of mortality, you are given that $_{10}p_{50}$ = 0.861716 and $_{20}p_{50}$ = 0.718743. Calculate the values of B and c.

Solution

(a) To prove the equation, we should make use of the relationship between the force of mortality and $_tp_x$.

$$_t p_x = \exp\left(-\int_0^t \mu_{x+s}\,\mathrm{d}s\right) = \exp\left(-\int_0^t Bc^{x+s}\,\mathrm{d}s\right) = \exp\left(\frac{-B}{\ln c}c^x(c^t-1)\right).$$

This gives $g = \exp(-B/\ln c)$.

(b) From (a), we have $0.861786 = g^{c^{50}(c^{10}-1)}$ and $0.718743 = g^{c^{50}(c^{20t}-1)}$. This gives

$$\frac{c^{20}-1}{c^{10}-1} = \frac{\ln(0.718743)}{\ln(0.861716)}.$$

Solving this equation, we obtain c = 1.02000. Substituting back, we obtain $g = 0.776856$ and $B = 0.00500$.

[END]

Now, let us study a longer structural question that integrates different concepts in this chapter.

Example 1.10 [Structural Question]

The function

$$\frac{18000 - 110x - x^2}{18000}$$

has been proposed for the survival function for a mortality model.

(a) State the implied limiting age ω.

(b) Verify that the function satisfies the conditions for the survival function $S_0(x)$.

(c) Calculate ${}_{20}p_0$.

(d) Calculate the survival function for a life age 20.

(e) Calculate the probability that a life aged 20 will die between ages 30 and 40.

(f) Calculate the force of mortality at age 50.

Solution

(a) Since

$$S_0(\omega) = \frac{18000 - 110\omega - \omega^2}{18000} = 0,$$

We have $\omega^2 + 110\omega - 18000 = 0 \Rightarrow (\omega - 90)(\omega + 200) = 0 \Rightarrow \omega = 90$ or $\omega = -200$ (rejected).

Hence, the implied limiting age is 90.

(b) We need to check the following three conditions:

(i) $$S_0(0) = \frac{18000 - 110 \times 0 - 0^2}{18000} = 1$$

(ii) $$S_0(\omega) = \frac{18000 - 110\omega - \omega^2}{18000} = 0$$

(iii) $$\frac{\mathrm{d}}{\mathrm{d}x} S_0(x) = -\frac{2x + 110}{18000} < 0$$

Therefore, the function satisfies the conditions for the survival function $S_0(x)$.

(c)
$$ {}_{20}p_0 = S_0(20) = \frac{18000 - 110\times 20 - 20^2}{18000} = 0.85556 $$

(d)
$$ S_{20}(x) = \frac{S_0(20+x)}{S_0(20)} = \frac{\dfrac{(90-20-x)(20+x+200)}{18000}}{\dfrac{(90-20)(20+200)}{18000}} $$
$$ = \frac{(70-x)(x+220)}{15400} = \frac{15400 - 150x - x^2}{15400}. $$

(e) The required probability is

$$ {}_{10|10}q_{20} = {}_{10}p_{20} - {}_{20}p_{20} $$
$$ = \frac{(70-10)(10+220)}{15400} - \frac{(70-20)(20+220)}{15400} = 0.89610 - 0.77922 = 0.11688 $$

(f) First, we find an expression for μ_x.

$$ \mu_x = -\frac{S_0'(x)}{S_0(x)} = -\frac{\dfrac{-110-2x}{18000}}{\dfrac{(90-x)(x+200)}{18000}} = \frac{2x+110}{(90-x)(x+200)}. $$

Hence, $\mu_{50} = \dfrac{2\times 50 + 110}{(90-50)(50+200)} = 0.021.$

[END]

You may be asked to prove some formulas in the structural questions of Exam MLC. Please study the following example, which involves several proofs.

Example 1.11 [Structural Question]

Prove the following equations:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\,{}_t p_x = -{}_t p_x \mu_{x+t}$

(b) ${}_t q_x = \int_0^t {}_s p_x \mu_{x+s}\,\mathrm{d}s$

(c) $\int_0^{\omega-x} {}_t p_x \mu_{x+t}\,\mathrm{d}t = 1$

Solution

(a) LHS $= \dfrac{\mathrm{d}}{\mathrm{d}t}\,{}_t p_x = \dfrac{\mathrm{d}}{\mathrm{d}t}\exp(-\int_0^t \mu_{x+s}\,\mathrm{d}s) = \exp(-\int_0^t \mu_{x+s}\,\mathrm{d}s)\left(-\dfrac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mu_{x+s}\,\mathrm{d}s\right) = {}_t p_x(-\mu_{x+t}) =$ RHS

(b) LHS $= {}_t q_x = \Pr(T_x \le t) = \int_0^t f_x(s)\,\mathrm{d}s = \int_0^t {}_s p_x \mu_{x+s}\,\mathrm{d}s =$ RHS

(c) LHS $= \int_0^{\omega-x} {}_t p_x \mu_{x+t}\,\mathrm{d}t = \int_0^{\omega-x} f_x(t)\,\mathrm{d}t = {}_{\omega-x} q_x = 1 =$ RHS

[END]

Exercise 1

1. [**Structural Question**] You are given:

$$S_0(t) = \frac{1}{1+t}, \quad t \geq 0.$$

   (a) Find $F_0(t)$.

   (b) Find $f_0(t)$.

   (c) Find $S_x(t)$.

   (d) Calculate p_{20}.

   (e) Calculate ${}_{10|5}q_{30}$.

2. You are given:

$$f_0(t) = \frac{(30-t)^2}{9000}, \quad \text{for } 0 \leq t < 30$$

   Find an expression for ${}_tp_5$.

3. You are given:

$$f_0(t) = \frac{20-t}{200}, \quad 0 \leq t < 20.$$

   Find μ_{10}.

4. [**Structural Question**] You are given:

$$\mu_x = \frac{1}{100-x}, \quad 0 \leq x < 100.$$

   (a) Find $S_{20}(t)$ for $0 \leq t < 80$.

   (b) Compute ${}_{40}p_{20}$.

   (c) Find $f_{20}(t)$ for $0 \leq t < 80$.

5. You are given:

$$\mu_x = \frac{2}{100-x}, \quad \text{for } 0 \leq x < 100.$$

   Find the probability that the age at death is in between 20 and 50.

6. You are given:

   (i) $S_0(t) = \left(1 - \dfrac{t}{\omega}\right)^{\alpha} \qquad 0 \leq t < \omega, \qquad \alpha > 0.$

   (ii) $\mu_{40} = 2\mu_{20}$.

   Find ω.

7. Express the probabilities associated with the following events in actuarial notation.
   (a) A new born infant dies no later than age 35.
   (b) A person age 10 now survives to age 25.
   (c) A person age 40 now survives to age 50 but dies before attaining age 55.

   Assuming that $S_0(t) = e^{-0.005t}$ for $t \geq 0$, evaluate the probabilities.

8. You are given:

$$S_0(t) = \left(1 - \frac{t}{100}\right)^2, \qquad 0 \leq t < 100.$$

   Find the probability that a person aged 20 will die between the ages of 50 and 60.

9. You are given:
   (i) ${}_2p_x = 0.98$
   (ii) $p_{x+2} = 0.985$
   (iii) ${}_5q_x = 0.0775$

   Calculate the following:
   (a) ${}_3p_x$
   (b) ${}_2p_{x+3}$
   (c) ${}_{2|3}q_x$

10. You are given:

$$q_{x+k} = 0.1(k+1), \qquad k = 0, 1, 2, \ldots, 9.$$

   Calculate the following:
   (a) $\Pr(K_x = 1)$
   (b) $\Pr(K_x \leq 2)$

11. [**Structural Question**] You are given $\mu_x = \mu$ for all $x \geq 0$.
   (a) Find an expression for $\Pr(K_x = k)$, for $k = 0, 1, 2, \ldots$, in terms of μ and k.
   (b) Find an expression for $\Pr(K_x \leq k)$, for $k = 0, 1, 2, \ldots$, in terms of μ and k.

   Suppose that $\mu = 0.01$.
   (c) Find $\Pr(K_x = 10)$.
   (d) Find $\Pr(K_x \leq 10)$.

12. Which of the following is equivalent to $\int_0^t {}_u p_x \mu_{x+u} \mathrm{d}u$?

    (A) ${}_t p_x$

    (B) ${}_t q_x$

    (C) $f_x(t)$

    (D) $-f_x(t)$

    (E) $f_x(t)\mu_{x+t}$

13. Which of the following is equivalent to $\dfrac{\mathrm{d}}{\mathrm{d}t}\,{}_t p_x$?

    (A) $-{}_t p_x \mu_{x+t}$

    (B) μ_{x+t}

    (C) $f_x(t)$

    (D) $-\mu_{x+t}$

    (E) $f_x(t)\mu_{x+t}$

14. (2000 Nov #36) Given:

    (i) $\mu_x = F + e^{2x},\ x \ge 0$

    (ii) ${}_{0.4}p_0 = 0.50$

    Calculate F.

    (A) –0.20

    (B) –0.09

    (C) 0.00

    (D) 0.09

    (E) 0.20

15. (CAS 2004 Fall #7) Which of the following formulas could serve as a force of mortality?

    (I) $\mu_x = Bc^x, \quad B > 0, C > 1$

    (II) $\mu_x = a(b + x)^{-1}, \quad a > 0, b > 0$

    (III) $\mu_x = (1 + x)^{-3}, \quad x \ge 0$

    (A) (I) only

    (B) (II) only

    (C) (III) only

    (D) (I) and (II) only

    (E) (I) and (III) only

16 (2002 Nov #1) You are given the survival function $S_0(t)$, where

(i) $S_0(t) = 1, \quad 0 \le t \le 1$

(ii) $S_0(t) = 1 - \dfrac{e^t}{100}, \; 1 \le t \le 4.5$

(iii) $S_0(t) = 0, \quad 4.5 \le t$

Calculate μ_4.

(A) 0.45

(B) 0.55

(C) 0.80

(D) 1.00

(E) 1.20

17. (CAS 2004 Fall #8) Given $S_0(t) = \left(1 - \dfrac{t}{100}\right)^{1/2}$, for $0 \le t \le 100$, calculate the probability that a life age 36 will die between ages 51 and 64.

(A) Less than 0.15

(B) At least 0.15, but less than 0.20

(C) At least 0.20, but less than 0.25

(D) At least 0.25, but less than 0.30

(E) At least 0.30

18. (2007 May #1) You are given:

(i) ${}_3p_{70} = 0.95$

(ii) ${}_2p_{71} = 0.96$

(iii) $\int_{71}^{75} \mu_x \, dx = 0.107$

Calculate ${}_5p_{70}$.

(A) 0.85

(B) 0.86

(C) 0.87

(D) 0.88

(E) 0.89

19. (2005 May #33) You are given:

$$\mu_x = \begin{cases} 0.05 & 50 \le x < 60 \\ 0.04 & 60 \le x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38
(B) 0.39
(C) 0.41
(D) 0.43
(E) 0.44

20. (2004 Nov #4) For a population which contains equal numbers of males and females at birth:

(i) For males, $\mu_x^m = 0.10, x \ge 0$

(ii) For females, $\mu_x^f = 0.08, x \ge 0$

Calculate q_{60} for this population.

(A) 0.076
(B) 0.081
(C) 0.086
(D) 0.091
(E) 0.096

21. (2001 May #28) For a population of individuals, you are given:

(i) Each individual has a constant force of mortality.

(ii) The forces of mortality are uniformly distributed over the interval (0, 2).

Calculate the probability that an individual drawn at random from this population dies within one year.

(A) 0.37
(B) 0.43
(C) 0.50
(D) 0.57
(E) 0.63

22. [**Structural Question**] The mortality of a certain population follows the De Moivre's Law; that is

$$\mu_x = \frac{1}{\omega - x}, \quad x < \omega.$$

(a) Show that the survival function for the age-at-death random variable T_0 is

$$S_0(x) = 1 - \frac{x}{\omega}, \quad 0 \le x < \omega.$$

(b) Verify that the function in (a) is a valid survival function.

(c) Show that

$$_t p_x = 1 - \frac{1}{\omega - x}, \quad 0 \le t < \omega - x, \quad x < \omega.$$

23. [**Structural Question**] The probability density function for the future lifetime of a life age 0 is given by

$$f_0(x) = \frac{\alpha \lambda^\alpha}{(\lambda + x)^{\alpha+1}}, \qquad \alpha, \lambda > 0$$

(a) Show that the survival function for a life age 0, $S_0(x)$, is $S_0(x) = \left(\dfrac{\lambda}{\lambda + x}\right)^\alpha$.

(b) Derive an expression for μ_x.

(c) Derive an expression for $S_x(t)$.

(d) Using (b) and (c), or otherwise, find an expression for $f_x(t)$.

24. [**Structural Question**] For each of the following equations, determine if it is correct or not. If it is correct, prove it.

(a) $_{t|u}q_x = {}_tp_x + {}_uq_{x+t}$

(b) $_{t+u}q_x = {}_tq_x \times {}_uq_{x+t}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, {}_tp_x = {}_tp_x(\mu_x - \mu_{x+t})$

Solutions to Exercise 1

1. (a) $F_0(t) = 1 - \dfrac{1}{1+t} = \dfrac{t}{1+t}$.

   (b) $f_0(t) = \dfrac{\mathrm{d}}{\mathrm{d}t}F_0(t) = \dfrac{1+t-t}{(1+t)^2} = \dfrac{1}{(1+t)^2}$.

   (c) $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{\dfrac{1}{1+x+t}}{\dfrac{1}{1+x}} = \dfrac{1+x}{1+x+t}$.

   (d) $p_{20} = S_{20}(1) = 21/22$.

   (e) ${}_{10|5}q_{30} = {}_{10}p_{30} - {}_{15}p_{30} = S_{30}(10) - S_{30}(15) = \dfrac{1+30}{1+30+10} - \dfrac{1+30}{1+30+15} = \dfrac{31}{41} - \dfrac{31}{46} = 0.0822$.

2. $S_0(t) = \displaystyle\int_t^{30} f_0(u)\mathrm{d}u = \dfrac{\int_t^{30}(30-u)^2\,\mathrm{d}u}{9000} = -\dfrac{\left[(30-u)^3\right]_t^{30}}{27000} = \dfrac{(30-t)^3}{27000}$.

   If follows that ${}_tp_5 = S_5(t) = \dfrac{S_0(5+t)}{S_0(5)} = \dfrac{(30-5-t)^3}{(30-5)^3} = \left(1-\dfrac{t}{25}\right)^3$.

3. $S_0(t) = \displaystyle\int_t^{20} f_0(u)\mathrm{d}u = \dfrac{\int_t^{20}(20-u)\mathrm{d}u}{200} = -\dfrac{\left[(20-u)^2\right]_t^{20}}{400} = \dfrac{(20-t)^2}{400}$.

   $\mu_t = \dfrac{f_0(t)}{S_0(t)} = \dfrac{\dfrac{20-t}{200}}{\dfrac{(20-t)^2}{400}} = \dfrac{2}{20-t}$.

   Hence, $\mu_{10} = 2/(20-10) = 0.2$.

4. (a) First, note that $\mu_{20+t} = \dfrac{1}{100-20-t} = \dfrac{1}{80-t}$. We have

   $$S_{200}(t) = \exp\left(-\int_0^t \mu_{20+u}\mathrm{d}u\right) = \exp\left(-\int_0^t \frac{1}{80-u}\mathrm{d}u\right)$$
   $$= \exp([\ln(80-u)]_0^t) = \exp(\ln\frac{80-t}{80}) = 1-\frac{t}{80}.$$

   (b) ${}_{40}p_{20} = S_{20}(40) = 1 - 40/80 = 1/2$.

   (c) $f_{20}(t) = S_{20}(t)\mu_{20+t} = \left(1-\dfrac{t}{80}\right)\left(\dfrac{1}{80-t}\right) = \dfrac{1}{80}$.

5. Our goal is to find $\Pr(20 < T_0 < 50) = S_0(20) - S_0(50)$.

   Given the force of mortality, we can find the survival function as follows:

$$S_0(t) = \exp\left(-\int_0^t \mu_u \mathrm{d}u\right) = \exp\left(-\int_0^t \frac{2}{100-u}\mathrm{d}u\right)$$

$$= \exp(2[\ln(100-u)]_0^t) = \exp(2\ln\frac{100-t}{100}) = \left(1-\frac{t}{100}\right)^2$$

   So, the required probability is $(1 - 20/100)^2 - (1 - 50/100)^2 = 0.8^2 - 0.5^2 = 0.39$.

6. $$\mu_x = -\frac{S_0'(x)}{S_0(x)} = -\frac{-\alpha\left(-\frac{1}{\omega}\right)\left(1-\frac{x}{\omega}\right)^{\alpha-1}}{\left(1-\frac{x}{\omega}\right)^{\alpha}} = \frac{\alpha}{\omega-x}.$$

   We are given that $\mu_{40} = 2\mu_{20}$. This implies $\dfrac{\alpha}{\omega-40} = \dfrac{2\alpha}{\omega-20}$, which gives $\omega = 60$.

7. (a) The probability that a new born infant dies no later than age 35 can be expressed as ${}_{35}q_0$. [Here we have "q" for a death probability, $x = 0$ and $t = 35$.]

   Further, ${}_{35}q_0 = F_0(35) = 1 - S_0(35) = 0.1605$.

   (b) The probability that a person age 10 now survives to age 25 can be expressed as ${}_{15}p_{10}$. [Here we have "p" for a survival probability, $x = 10$ and $t = 25 - 10 = 15$.]

   Further, we have ${}_{15}p_{10} = S_{10}(15) = \dfrac{S_0(25)}{S_0(15)} = 0.9277$.

   (c) The probability that a person age 40 now survives to age 50 but dies before attaining age 55 can be expressed as ${}_{10|5}q_{40}$. [Here, we have "q" for a (deferred) death probability, $x = 40$, $t = 50 - 40 = 10$, and $u = 55 - 50 = 5$.]

   Further, we have ${}_{10|5}q_{40} = S_{40}(10) - S_{40}(15) = \dfrac{S_0(50)}{S_0(40)} - \dfrac{S_0(55)}{S_0(40)} = 0.0235$.

8. The probability that a person aged 20 will die between the ages of 50 and 60 is given by

$${}_{30|10}q_{20} = {}_{30}p_{20} - {}_{40}p_{20} = S_{20}(30) - S_{20}(40).$$

$$S_{20}(t) = \frac{S_0(20+t)}{S_0(20)} = \frac{\left(1-\frac{20+t}{100}\right)^2}{\left(1-\frac{20}{100}\right)^2} = \left(1-\frac{t}{80}\right)^2.$$

   So, $S_{20}(30) = \left(1-\dfrac{30}{80}\right)^2 = \dfrac{25}{64}$, $S_{20}(40) = \left(1-\dfrac{40}{80}\right)^2 = \dfrac{16}{64}$. As a result, ${}_{30|10}q_{20} = 9/64$.

9. (a) ${}_3p_x = {}_2p_x \times p_{x+2} = 0.98 \times 0.985 = 0.9653$.

(b)
$$ {}_3p_x \times {}_2p_{x+3} = {}_5p_x = 1 - {}_5q_x $$
$$ \Rightarrow {}_2p_{x+3} = \frac{1 - {}_5q_x}{{}_3p_x} = \frac{1 - 0.0775}{0.9653} = 0.95566 $$

(c) ${}_{2|3}q_x = {}_2p_x - {}_5p_x = 0.98 - (1 - 0.0775) = 0.0575$.

10. (a) $\Pr(K_x = 1) = {}_{1|}q_x = p_x \times q_{x+1} = (1 - q_x)q_{x+1} = (1 - 0.1) \times 0.2 = 0.18$

(b) $\Pr(K_x = 0) = q_x = 0.1$

$\Pr(K_x = 2) = {}_{2|}q_x = {}_2p_x \times q_{x+2} = p_x \times p_{x+1} \times q_{x+2} = (1 - q_x)(1 - q_{x+1})q_{x+2}$
$= 0.9 \times 0.8 \times 0.3 = 0.216$.

Hence, $\Pr(K_x \le 2) = 0.1 + 0.18 + 0.216 = 0.496$.

11. (a) Given that $\mu_x = \mu$ for all $x \ge 0$, we have ${}_tp_x = e^{-\mu t}$, $p_x = e^{-\mu}$ and $q_x = 1 - e^{-\mu}$.
$$ \Pr(K_x = k) = {}_{k|}q_x = {}_kp_x\, q_{x+k} = e^{-k\mu}(1 - e^{-\mu}). $$

(b) $\Pr(K_x \le k) = {}_{k+1}q_x = 1 - {}_{k+1}p_x = 1 - e^{-(k+1)\mu}$.

(c) When $\mu = 0.01$, $\Pr(K_x = 10) = e^{-10 \times 0.01}(1 - e^{-0.01}) = 0.0090$.

(d) When $\mu = 0.01$, $\Pr(K_x \le 10) = 1 - e^{-(10+1) \times 0.01} = 0.1042$.

12. First of all, note that ${}_up_x\mu_{x+u}$ in the integral is simply $f_x(u)$.
$$ \int_0^t {}_up_x\mu_{x+u}\,\mathrm{d}u = \int_0^t f_x(u)\,\mathrm{d}u = \Pr(T_x \le t) = F_x(t) = {}_tq_x. $$

Hence, the answer is (B).

13. Method I: We use ${}_tp_x = 1 - {}_tq_x$. Differentiating both sides with respect to t,
$$ \frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = -\frac{\mathrm{d}}{\mathrm{d}t}{}_tq_x = -\frac{\mathrm{d}}{\mathrm{d}t}F_x(t) = -f_x(t). $$

Noting that $f_x(t) = {}_tp_x\,\mu_{x+t}$, the answer is (A).

Method II: We differentiate ${}_tp_x$ with respect to t as follows:
$$ \frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = \frac{\mathrm{d}}{\mathrm{d}t}S_x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right) $$
$$ = \exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right). $$

Recall the fundamental theorem of calculus, which says that $\frac{\mathrm{d}}{\mathrm{d}t}\int_c^t g(u)\mathrm{d}u = g(t)$. Thus
$$ \frac{\mathrm{d}}{\mathrm{d}t}{}_tp_x = \exp\left(-\int_0^t \mu_{x+u}\,\mathrm{d}u\right)(-\mu_{x+t}) = -{}_tp_x\mu_{x+t}. $$

Hence, the answer is (A).

14. First, note that

$$ {}_{0.4}p_0 = 0.5 = e^{-\int_0^{0.4}\mu_u \mathrm{d}u} = e^{-\int_0^{0.4}(F+e^{2u})\mathrm{d}u} . $$

The exponent in the above is

$$ \begin{aligned} -\int_0^{0.4}(F+e^{2u})\mathrm{d}u &= -\left(Fu + \frac{1}{2}e^{2u} \right)\Bigg|_0^{0.4} \\ &= -0.4F - 1.11277 + 0.5 \\ &= -0.4F - 0.61277 \end{aligned} $$

As a result, $0.5 = e^{-0.4F-0.61277}$, which gives $F = 0.2$. Hence, the answer is (E).

15. Recall that we require the force of mortality to satisfy the following two criteria:

(i) $\mu_x \geq 0$ for all $x \geq 0$, (ii) $\int_0^{\infty} \mu_x \mathrm{d}x = \infty$.

All three specifications of μ_x satisfy Criterion (i). We need to check Criterion (ii).

We have

$$ \int_0^{\infty} Bc^x \mathrm{d}x = \frac{Bc^x}{\ln c}\Bigg|_0^{\infty} = \infty , $$

$$ \int_0^{\infty} \frac{a}{b+x}\mathrm{d}x = a\ln(b+x)\Big|_0^{\infty} = \infty , $$

and

$$ \int_0^{\infty} \frac{1}{(1+x)^3}\mathrm{d}x = \frac{-1}{2(1+x)^2}\Bigg|_0^{\infty} = \frac{1}{2} . $$

Only the first two specifications can satisfy Criterion (ii). Hence, the answer is (D).

[Note: $\mu_x = Bc^x$ is actually the Gompertz' law. If you knew that you could have identified that $\mu_x = Bc^x$ can serve as a force of mortality without doing the integration.]

16. Recall that $\mu_{x+t} = -\dfrac{S_x'(t)}{S_x(t)}$.

Since we need μ_4, we use the definition of $S_0(t)$ for $1 \leq t \leq 4.5$:

$$ S_0(t) = 1 - \frac{e^t}{100}, \qquad -S_0'(t) = \frac{e^t}{100} . $$

As a result, $\mu_4 = \dfrac{\frac{e^4}{100}}{1-\frac{e^4}{100}} = \dfrac{e^4}{100-e^4} = 1.203$. Hence, the answer is (E).

17. The probability that a life age 36 will die between ages 51 and 64 is given by

$$S_{36}(15) - S_{36}(28).$$

We have $S_{36}(t) = \dfrac{S_0(36+t)}{S_0(36)} = \dfrac{\left(1-\dfrac{36+t}{100}\right)^{1/2}}{\left(1-\dfrac{36}{100}\right)^{1/2}} = \left(\dfrac{64-t}{64}\right)^{1/2} = \dfrac{\sqrt{64-t}}{8}$.

This gives $S_{36}(15) = \dfrac{7}{8}$ and $S_{36}(28) = \dfrac{6}{8}$. As a result, the required probability is

$$S_{36}(15) - S_{36}(28) = 1/8 = 0.125.$$

Hence, the answer is (A).

18. The computation of ${}_5p_{70}$ involves three steps.

First, $p_{70} = \dfrac{{}_3p_{70}}{{}_2p_{71}} = \dfrac{0.95}{0.96} = 0.9896$.

Second, ${}_4p_{71} = e^{-\int_{71}^{75}\mu_x \mathrm{d}x} = e^{-0.107} = 0.8985$.

Finally, ${}_5p_{70} = 0.9896 \times 0.8985 = 0.889$. Hence, the answer is (E).

19. ${}_4p_{50} = e^{-0.05 \times 4} = 0.8187$

${}_{10}p_{50} = e^{-0.05 \times 10} = 0.6065$

${}_8p_{60} = e^{-0.04 \times 8} = 0.7261$

${}_{18}p_{50} = {}_{10}p_{50} \times {}_8p_{60} = 0.6065 \times 0.7261 = 0.4404$

Finally, ${}_{4|14}q_{50} = {}_4p_{50} - {}_{18}p_{50} = 0.8187 - 0.4404 = 0.3783$. Hence, the answer is (A).

20. For males, we have

$$S_0^m(t) = e^{-\int_0^t \mu_u^m \mathrm{d}u} = e^{-\int_0^t 0.10 \mathrm{d}u} = e^{-0.10t}.$$

For females, we have

$$S_0^f(t) = e^{-\int_0^t \mu_u^f \mathrm{d}u} = e^{-\int_0^t 0.08 \mathrm{d}u} = e^{-0.08t}.$$

For the overall population,

$$S_0(60) = \frac{e^{-0.1\times 60} + e^{-0.08\times 60}}{2} = 0.005354,$$

and

$$S_0(61) = \frac{e^{-0.1\times 61} + e^{-0.08\times 61}}{2} = 0.00492.$$

Finally, $q_{60} = 1 - p_{60} = 1 - \dfrac{S_0(61)}{S_0(60)} = 0.081$. Hence, the answer is (B).

21. Let M be the force of mortality of an individual drawn at random, and T be the future lifetime of the individual. We are given that M is uniformly distributed over (0, 2). So the density function for M is $f_M(\mu) = 1/2$ for $0 < \mu < 2$ and 0 otherwise.

    This gives

$$\begin{aligned}
&\Pr(T \le 1) \\
&= \mathrm{E}[\Pr(T \le 1 \mid M)] \\
&= \int_0^\infty \Pr(T \le 1 \mid M = \mu) f_M(\mu)\mathrm{d}\mu \\
&= \int_0^2 (1 - e^{-\mu})\frac{1}{2}\mathrm{d}\mu \\
&= \frac{1}{2}(2 + e^{-2} - 1) \\
&= \frac{1}{2}(1 + e^{-2}) \\
&= 0.56767.
\end{aligned}$$

    Hence, the answer is (D).

22. (a) We have, for $0 \le x < \omega$,

$$S_0(x) = \exp(-\int_0^x \mu_s \mathrm{d}s) = \exp(-\int_0^x \frac{1}{\omega - s}\mathrm{d}s) = \exp([\ln(\omega - s)]_0^x = e^{\ln(1-\frac{x}{\omega})} = 1 - \frac{x}{\omega}.$$

    (b) We need to check the following three conditions:

    (i) $S_0(0) = 1 - 0/\omega = 1$

    (ii) $S_0(\omega) = 1 - \omega/\omega = 0$

    (iii) $S'_0(\omega) = -1/\omega < 0$ for all $0 \le x < \omega$, which implies $S_0(x)$ is non-increasing.

    Hence, the function in (a) is a valid survival function.

    (c) $_tp_x = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{1 - \frac{x+t}{\omega}}{1 - \frac{x}{\omega}} = \dfrac{\omega - x - t}{\omega - x} = 1 - \dfrac{t}{\omega - x}$, for $0 \le t < \omega - x,\ x < \omega$.

23. (a) $S_0(x) = 1 - F_0(x) = 1 - \int_0^x f_0(s)\mathrm{d}s = 1 - \int_0^x \dfrac{\alpha\lambda^\alpha}{(\lambda + s)^{\alpha+1}}\mathrm{d}s = \dfrac{\lambda^\alpha}{(\lambda + x)^\alpha}$.

    (b) $\mu_x = \dfrac{f_0(x)}{S_0(x)} = \dfrac{\alpha}{\lambda + x}$.

(c) $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{\left(\dfrac{\lambda}{\lambda+x+t}\right)^{\alpha}}{\left(\dfrac{\lambda}{\lambda+x}\right)^{\alpha}} \left(\dfrac{\lambda+x}{\lambda+x+t}\right)^{\alpha}.$

(d) $f_x(t) = S_x(t)\mu_{x+t} = \left(\dfrac{\lambda+x}{\lambda+x+t}\right)^{\alpha} \dfrac{\alpha}{\lambda+x+t}.$

24. (a) No, the equation is not correct. The correct equation should be ${}_{t|u}q_x = {}_tp_x \times {}_uq_{x+t}$.

(b) No, the equation is not correct. The correct equation should be ${}_{t+u}p_x = {}_tp_x \times {}_up_{x+t}$.

(c) Yes, the equation is correct. The proof is as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\,{}_tp_x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{S_0(x+t)}{S_0(x)} \\
&= \frac{S_0(x)S'_0(x+t) - S_0(x+t)S'_0(x)}{[S_0(x)]^2} \\
&= \frac{S_0(x)(-f_0(x+t)) - S_0(x+t)(-f_0(x))}{[S_0(x)]^2} \\
&= \frac{-f_0(x+t)}{S_0(x)} + \frac{S_0(x+t)}{S_0(x)}\frac{f_0(x)}{S_0(x)} \\
&= \frac{-f_0(x+t)}{S_0(x+t)}\frac{S_0(x+t)}{S_0(x)} + \frac{S_0(x+t)}{S_0(x)}\frac{f_0(x)}{S_0(x)} \\
&= -\mu_{x+t}\,{}_tp_x + {}_tp_x\mu_x \\
&= {}_tp_x(\mu_x - \mu_{x+t})
\end{aligned}$$