Mathematics of Investment & Credit

Seventh Edition

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PREFACE

While teaching an intermediate level university course in mathematics of investment over a number of years, I found an increasing need for a textbook that provided a thorough and modern treatment of the subject, while incorporating theory and applications. This book is an attempt (as a 7th edition, it must be a seventh attempt) to satisfy that need. It is based, to a large extent, on notes that I have developed while teaching and my use of a number of textbooks for the course. The university course for which this book was written has also been intended to help students prepare for the mathematics of investment topic that is covered on one of the professional examinations of the Society of Actuaries and the Casualty Actuarial Society. A number of the examples and exercises in this book are taken or adapted from questions on past SOA/CAS examinations.

As in many areas of mathematics, the subject of mathematics of investment has aspects that do not become outdated over time, but rather become the foundation upon which new developments are based. The traditional topics of compound interest and dated cashflow valuations, and their applications, are developed in the first five chapters of the book. In addition, in Chapters 6 to 10, a number of topics are introduced which have become of increasing importance in modern financial practice over the past number of years. The past three decades or so have seen a great increase in the use of derivative securities, particularly financial options. The subjects covered in Chapters 6 to 9, such as the term structure of interest rates, interest rate swaps and forward contracts, form the foundation for the mathematical models used to describe and value derivative securities, which are introduced in Chapter 10. This 7th edition expands on and updates the 6th edition's coverage of measures of duration and convexity, interest rate swaps and topics in fixed income securities.

The purpose of the methods developed in this book is to facilitate financial valuations. This book emphasizes a direct calculation approach, assuming that the reader has access to a financial calculator with standard financial functions.

The mathematical background required for the book is a course in calculus at the freshman level. Chapters 8 and 10 cover a couple of topics that involve the notion of probability, but mostly at an elementary level. A very basic understanding of probability concepts should be sufficient background for those topics.

The topics in the first five Chapters of this book are arranged in an order that is similar to traditional approaches to the subject, with Chapter 1 introducing the various measures of interest rates, Chapter 2 developing methods for valuing a series of payments, Chapter 3 considering amortization of loans, Chapter 4 covering bond valuation, and Chapter 5 introducing the various methods of measuring the rate of return earned by an investment.

The content of this book is probably more than can reasonably be covered in a one-semester course at an introductory or even intermediate level, but it might be possible for the FM Exam material to be covered in a one-semester course. At the University of Toronto, the contents of this books are covered in two consecutive one-semester courses at the Sophomore level.

Samuel A. Broverman, ASA, Ph.D.   University of Toronto, October 2017

CHAPTER 1

INTEREST RATE MEASUREMENT

"Money makes the world go round, the world go round, the world go round."
– Fred Ebb, lyricist for the 1966 Broadway musical "Cabaret"

1.0 INTRODUCTION

Almost everyone, at one time or another, will be a saver, borrower, or investor, and will have access to insurance, pension plans, or other financial benefits and liabilities. There is a wide variety of financial transactions in which individuals, corporations, or governments can become involved. The range of available investments is continually expanding, accompanied by an increase in the complexity of many of these investments.

Financial transactions involve numerical calculations, and, depending on their complexity, may require detailed mathematical formulations. It is therefore important to establish fundamental principles upon which these calculations and formulations are based. The objective of this book is to systematically develop insights and mathematical techniques which lead to these fundamental principles upon which financial transactions can be modeled and analyzed.

The initial step in the analysis of a financial transaction is to translate a verbal description of the transaction into a mathematical model. Unfortunately, in practice, a transaction may be described in language that is vague and which may result in disagreements regarding its interpretation. The need for precision in the mathematical model of a financial transaction requires that there be a correspondingly precise and unambiguous understanding of the verbal description before the translation to the model is made. To this end, terminology and notation, much of which is in standard use in financial and actuarial practice, will be introduced.

A component that is common to virtually all financial transactions is **interest**, the "time value of money." Most people are aware that interest rates play a central role in their own personal financial situations as well as in the economy as a whole. Many governments and private enterprises

employ economists and analysts who make forecasts regarding the level of interest rates.

The U.S. Federal Reserve Board sets the "federal funds discount rate," a target rate at which banks can borrow and invest funds with one another. This rate affects the more general cost of borrowing and also has an effect on the stock and bond markets. Bonds and stocks will be considered in more detail later in the book. For now, it is not unreasonable to accept the hypothesis that higher interest rates tend to reduce the value of other investments, if for no other reason than that the increased attraction of investing at a higher rate of interest makes another investment earning a lower rate relatively less attractive.

Irrational Exuberance

After the close of trading on North American financial markets on Thursday, December 5, 1996, U.S. Federal Reserve Board chairman Alan Greenspan delivered a lecture at The American Enterprise Institute for Public Policy Research.

In that speech, Mr. Greenspan commented on the possible negative consequences of "irrational exuberance" in the financial markets.

The speech was widely interpreted by investment traders as indicating that stocks in the U.S. market were overvalued, and that the Federal Reserve Board might increase U.S. interest rates, which might affect interest rates worldwide.

Although U.S. markets had already closed, those in the Far East were just opening for trading on December 6, 1996. Japan's main stock market index dropped 3.2%, the Hong Kong stock market dropped almost 3%. As the opening of trading in the various world markets moved westward throughout the day, market drops continued to occur. The German market fell 4% and the London market fell 2%. When the New York Stock Exchange opened at 9:30 AM EST on Friday, December 6, 1996, it dropped about 2% in the first 30 minutes of trading, although the market did recover later in the day.

Source:
http://www.federalreserve.gov/boarddocs/speeches/1996/19961205.htm

The variety of interest rates and the investments and transactions to which they relate is extensive. Figure 1.1 provides a snapshot of a variety of current and historic interest rates as of April 17, 2017 and is an illustration of just a few of the types of interest rates that arise in practice. **Libor** refers to the London Interbank Overnight Rate, which is an international rate charged by one bank to another for very short term loans denominated in U.S. dollars. The **prime rate** is the interest rate that banks charge their most creditworthy customers. An **ARM** is an Adjustable Rate Mortgage, which is a mortgage loan whose interest rate is periodically reset based on changes in market interest rates.

KEY RATES (in %)					
	April 2017	March 2017	April 2016	April 2007	April 1992
Fed Reserve Target Rate	0.91	0.91	0.37	5.20	3.75
3-Month Libor	1.15	1.15	0.63	5.35	4.06
Prime Rate	4.00	3.75	3.50	8.25	6.50
AAA Average 20-Year Corporate Bond Yields	3.54	3.73	3.62	5.47	8.33
High Yield Bonds	5.85	5.96	7.93	7.41	
MORTGAGE RATES provided by Bankrate.com					
	April 2017	March 2017	April 2016	April 2007	April 1992
15-Year Mortgage	3.36	3.50	2.86	5.89	8.38
30-Year Mortgage	4.08	4.30	3.58	6.17	8.76
1-Year ARM	3.18	3.28	2.84	5.92	6.15
U.S. TREASURIES					
Bills					
	Coupon	**Maturity Date**	**Current Discount Rate***		
3-Month	N.A.	07/20/2017	.20728		
12-Month	N.A.	03/29/2018	1.03639		

*This term will be defined in Section 1.5

FIGURE 1.1

To analyze financial transactions, a clear understanding of the concept of interest is required. Interest can be defined in a variety of contexts, and most people have at least a vague notion of what it is. In the most common context, interest refers to the consideration or rent paid by a borrower of money to a lender for the use of the money over a period of time.

This chapter provides a detailed development of the mechanics of interest rates: how they are measured and applied to amounts of principal over time to calculate amounts of interest. A standard measure of interest rates will be defined and two commonly used growth patterns for investment, simple and compound interest, will be described. Various alternative standard measures of interest, such as nominal annual rate of interest, rate of discount, and force of interest, are discussed. A general way in which a financial transaction is modeled in mathematical form will be presented using the notions of accumulated value, present value, and equation of value.

1.1 INTEREST ACCUMULATION AND EFFECTIVE RATES OF INTEREST

An interest rate is most typically quoted as an annual percentage. If interest is credited or charged annually, the quoted annual rate, in decimal or fraction form, is multiplied by the amount invested or loaned to calculate the amount of interest that accrues over a one-year period. It is generally understood that as interest is credited or paid, it is reinvested. This reinvesting of interest leads to the process of compounding interest. The following example illustrates this process.

EXAMPLE 1.1   (*Compound interest calculation*)

The current rate of interest quoted by a bank on its savings account is 9% per annum (per year), with interest credited annually. Smith opens an account with a deposit of 1000. Assuming that there are no transactions on the account other than the annual crediting of interest, determine the account balance just after interest is credited at the end of 3 years.

SOLUTION

After one year the interest credited will be $1000 \times .09 = 90$, resulting in a balance (with interest) of $1000 + 1000 \times .09 = 1000(1.09) = 1090$. It is standard practice that this balance is reinvested and earns interest in the second year, producing an interest amount of $1090 \times .09 = 98.10$, and a total balance of

$$1090 + 1090 \times .09 \ = \ 1090(1.09) \ = \ 1000(1.09)^2 \ = \ 1188.10$$

at the end of the second year. The balance at the end of the third year will be

$$1188.10 + 1188.10 \times .09 = (1188.10)(1.09) = 1000(1.09)^3 = 1295.03.$$

The following time diagram illustrates this process.

0	1	2	3
↑	↑	↑	↑
1000	$1000 \times .09 = 90$	$1090 \times .09 = 98.10$	$1188.10 \times .09 = 106.93$
Deposit	Interest	Interest	Interest
Total	1000+90	1090+98.10	1188.10+106.93
	$=1090$	$=1188.10$	$=1295.03$
	$=1000 \times 1.09$	$=1090 \times 1.09$	$=1188.10 \times 1.09$
		$=1000(1.09)^2$	$=1000(1.09)^3$

FIGURE 1.2 ❑

It can be seen from Example 1.1 that with an interest rate of i per annum and interest credited annually, an initial deposit of C will earn interest of Ci for the following year. The accumulated value or future value at the end of the year will be $C + Ci = C(1+i)$. If this amount is reinvested and left on deposit for another year, the interest earned in the second year will be $C(1+i)i$, so that the accumulated balance is $C(1+i) + C(1+i)i = C(1+i)^2$ at the end of the second year. The account will continue to grow by a factor of $1+i$ per year, resulting in a balance of $C(1+i)^n$ at the end of n years. This is the pattern of accumulation that results from compounding, or reinvesting, the interest as it is credited.

0	1	2	n–1	n
↑	↑	↑	↑	↑
C ⟶	Ci	$C(1+i)i$	$C(1+i)^{n-2}i$	$C(1+i)^{n-1}i$
Deposit	Interest	Interest	Interest	Interest
	↑	↑	↑	↑
	$C + Ci$	$C(1+i)$	$C(1+i)^{n-2}$	$C(1+i)^{n-1}$
Total	$= C(1+i)$	$+C(1+i)i$	$+C(1+i)^{n-2}i$	$+C(1+i)^{n-1}i$
		$= C(1+i)^2$	$= C(1+i)^{n-1}$	$= C(1+i)^n$

FIGURE 1.3

In Example 1.1, if Smith were to observe the accumulating balance in the account by looking at regular bank statements, Smith would see only one entry of interest credited each year. If Smith made the initial deposit on January 1, 2017 then Smith would have interest added to his account on December 31 of 2017 and every December 31 after that for as long as the account remained open.

The rate of interest may change from one year to the next. If the interest rate is i_1 in the first year, i_2 in the second year, and so on, then after n years an initial amount C will accumulate to $C(1+i_1)(1+i_2)\cdots(1+i_n)$, where the growth factor for year t is $1+i_t$ and the interest rate for year t is i_t. Note that "year t" starts at time $t-1$ and ends at time t.

EXAMPLE 1.2 (*Average annual rate of return*)

The excerpts below are taken from the 2016 year-end report of National Bank Global Equity Fund, a fund managed by a Canadian mutual fund company. The excerpts below focus on the performance of the fund and the Dow Jones Industrial Average during the five year period ending December 31, 2016. The Dow Jones Industrial Average is a price-weighted average of stocks traded on major American stock exchanges.

	Annual Rate of Return				
	2016	2015	2014	2013	2012
NB Global Equity	0.00%	18.08%	13.38%	33.57%	14.14%
Dow Jones Ind. Avg.	13.42%	-2.23%	7.52%	26.50%	7.26%
	Average Annual Return				
		1 yr.%	2 yr.%	3 yr.%	5 yr.%
NB Global Equity		0.00%	8.66%	10.21%	15.34%
Dow Jones Ind. Avg.		13.42%	5.30%	6.04%	10.10%

FIGURE 1.4

For the five year period ending December 31, 2016, the total compound growth in the Global Equity Fund can be found by compounding the annual rates of return for the 5 years.

$$(1+0.00)(1+.1808)(1+.1338)(1+.3357)(1+.1414) = 2.0411$$

This would be the value on December 31, 2016 of an investment of 1 made into the fund on January 1, 2012.

This five year growth can be described by means of an average annual re-

turn per year for the five-year period. In practice the phrase "average an-nual return" refers to an annual compound rate of interest for the period of years being considered. The average annual return would be i, where $(1+i)^5 = 2.0411$. Solving for i results in a value of $i = .1534$. This is the average annual return for the five year period ending December 31, 2016. For the Dow Jones average, an investment of 1 made January 1, 2012 would have a value on December 31, 2016 of

$$(1+.1342)(1-.0223)(1+.0752)(1+.2650)(1+.0726) = 1.6178$$

Solving for i in the equation $(1+i)^5 = 1.6178$ results in $i = .1010$, or a 5-year average annual return of 10.10%.

The Global Equity Fund is described on the National Bank website as follows: "The fund's investment objective is to achieve long-term capital growth. It builds a diversified portfolio of common and preferred shares listed on recognized stock exchanges."

The Dow Jones Industrial Average is a stock index of 30 large publicly owned U.S. companies. ❒

1.1.1 EFFECTIVE RATES OF INTEREST

In practice, interest may be credited or charged more frequently than once per year. Many bank accounts pay interest monthly and credit cards gener-ally charge interest monthly on unpaid balances. If a deposit is allowed to accumulate in an account over time, the algebraic form of the accumulation will be similar to the one given earlier for annual interest. At interest rate j per compounding period, an initial deposit of amount C will accumulate to $C(1+j)^n$ after n compounding periods. (It is typical to use i to denote an annual rate of interest, and in this text j will often be used to denote an in-terest rate for a period of time other than a year.)

At an interest rate of .75% per month on a bank account, with interest cred-ited monthly, the growth factor for a one-year period at this rate would be $(1.0075)^{12} = 1.0938$. The account earns 9.38% over the full year and 9.38% is called the **annual effective rate of interest** earned on the account.

Definition 1.1 – Annual Effective Rate of Interest

The annual effective rate of interest earned by an investment during a one-year period is the percentage change in the value of the investment from the beginning to the end of the year, without regard to the investment behavior at intermediate points in the year.

In Example 1.2, the annual effective rates of return for the fund and the index are given for years 2012 through 2016. Comparisons of the performance of two or more investments are often done by comparing the respective annual effective interest rates earned by the investments over a particular year. The mutual fund earned an annual effective rate of interest of 13.38% for 2014, but the Dow index earned 7.52%. For the 5-year period from January 1, 2012 to December 31, 2016, the mutual fund earned an average annual effective rate of interest of 15.34%, but the Dow index average annual effective rate was 10.10%.

Equivalent Rates of Interest

If the monthly compounding at .75% described earlier continued for another year, the accumulated or future value after two years would be $C(1.0075)^{24} = C(1.0938)^2$. We see that over an integral number of years a month-by-month accumulation at a monthly rate of .75% is *equivalent* to annual compounding at an annual rate of 9.38%; the word "equivalent" is used in the sense that they result in the same accumulated value.

Definition 1.2 - Equivalent Rates of Interest

Two rates of interest are said to be **equivalent** if they result in the same accumulated values at each point in time.

1.1.2 COMPOUND INTEREST

When compound interest is in effect, and deposits and withdrawals are occurring in an account, the resulting balance at some future point in time can be determined by accumulating all individual transactions to that future time point.

EXAMPLE 1.3 (*Compound interest calculation*)

Smith deposits 1000 into an account on January 1, 2011. The account credits interest at an annual effective interest rate of 5% every December 31. Smith withdraws 200 on January 1, 2013, deposits 100 on January 1, 2014, and withdraws 250 on January 1, 2016. What is the balance in the account just after interest is credited on December 31, 2017?

SOLUTION

One approach is to recalculate the balance after every transaction.

On December 31, 2012 the balance is $1000(1.05)^2 = 1102.50$;

on January 1, 2013 the balance is $1102.50 - 200 = 902.50$;

on December 31, 2013 the balance is $902.50(1.05) = 947.63$;

on January 1, 2014 the balance is $947.63 + 100 = 1047.63$;

on December 31, 2015 the balance is $1047.63(1.05)^2 = 1155.01$;

on January 1, 2016 the balance is $1155.01 - 250 = 905.01$; and

on December 31, 2017 the balance is $905.01(1.05)^2 = 997.77$.

An alternative approach is to accumulate each transaction to the December 31, 2017 date of valuation and combine all accumulated values, adding deposits and subtracting withdrawals. Then we have

$$1000(1.05)^7 - 200(1.05)^5 + 100(1.05)^4 - 250(1.05)^2 \quad = \quad 997.77$$

for the balance on December 31, 2017. This is illustrated in the following time line:

1/1/11 $\cdots$ 1/1/13	1/1/14 $\cdots$ 1/1/16 $\cdots$	12/31/17
+1000 (initial Deposit)		$\longrightarrow$ $1000(1.05)^7$
-200		$\longrightarrow$ $-200(1.05)^5$
	+100	$\longrightarrow$ $100(1.05)^4$
	-250	$\longrightarrow$ $\underline{-250(1.05)^2}$

Total $= 1000(1.05)^7 - 200(1.05)^5 + 100(1.05)^4 - 250(1.05)^2 = 997.77$.

FIGURE 1.5 ❑

The pattern for compound interest accumulation at rate i per period results in an accumulation factor of $(1+i)^n$ over n periods. The pattern of investment growth may take various forms, and we will use the general expression $a(n)$ to represent the accumulation (or growth) factor for an investment from time 0 to time n.

**Definition 1.3 – Accumulation Factor and
Accumulated Amount Function**

$a(t)$ is the accumulated value at time t of an investment of 1 made at time 0 and is defined as the accumulation factor from time 0 to time t. The notation $A(t)$ will be used to denote the accumulated amount of an investment at time t, so that if the initial investment amount is $A(0)$, then the accumulated value at time t is $A(t) = A(0) \times a(t)$. $A(t)$ is the accumulated amount function.

Compound interest accumulation at rate i per period can be defined with t as any positive real number.

Definition 1.4 – Compound Interest Accumulation

At effective rate of interest i per period, the accumulation factor from time 0 to time t (periods) is

$$a(t) \;=\; (1+i)^t \qquad\qquad (1.1)$$

The graph of compound interest accumulation is given in Figure 1.6.
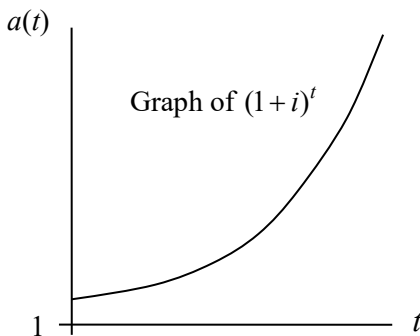
FIGURE 1.6

In Example 1.1, if Smith closed the account in the middle of the fourth year (3.5 years after the account was opened), the accumulated or future value at time $t = 3.5$ would be $1000(1.09)^{3.50} = 1000(1.09)^3(1.09)^{.50} = 1352.05$, which is the balance at the end of the third year followed by accumulation for one-half more year to the middle of the fourth year.

In practice, financial transactions can take place at any point in time, and it may be necessary to represent a period which is a fractional part of a year. A fraction of a year is generally described in terms of either an integral number of m months, or an exact number of d days. In the case that time is measured in months, it is common in practice to formulate the fraction of the year t in the form $t = \frac{m}{12}$, even though not all months are exactly $\frac{1}{12}$ of a year. In the case that time is measured in days, t is often formulated as $t = \frac{d}{365}$ (some investments use a denominator of 360 days instead of 365 days, in which case $t = \frac{d}{360}$).

The Magic of Compounding

Investment advice newsletters and websites often refer to the "magic" of compounding when describing the potential for investment accumulation. A phenomenon is magical only until it is understood. Then it's just an expected occurrence, and it loses its mystery.

A value of 10% is often quoted as the long-term historical average return on equity investments in the U.S. stock market. Based on the historical data, the 30-year average return (excluding dividends) was 8.0% on the Dow Jones index from the start of 1980 to the end of 2010. During that period, the average annual return in the 1980s was 10.3%, in the 1990s it was 15.4%, and in the 2000's it was -1.0%. The 2000s were not as magical a time for investors as the 1990s. The average annual return on the index for the seven year period from the start of 2010 to the end of 2016 was 9.56%.

In the 1980s heyday of multi-level marketing schemes, one such scheme promoted the potential riches that could be realized by marketing "gourmet" coffee in the following way. A participant had merely to recruit 6 sub-agents who could sell 2 pounds of coffee per week. Those sub-agents would then recruit 6 sub-agents of their own. This would continue to an ever increasing number of levels. The promotional literature stated the expected net profit earned by the "top" agent based on each number of levels of 6-fold sub-agents that could be recruited. The expected profit based on 9 levels of sub-agents was of the order of several hundred thousand dollars per week. There was no indication in the brochure that to reach this level would require over 10,000,000 (6^9) sub-agents. Reaching that level would definitely require some compounding magic.

When considering the equation $X(1+i)^t = Y$, given any three of the four variables X, Y, i, t, it is possible to find the fourth. If the unknown variable is t, then solving for this time factor results in $t = \frac{\ln(Y/X)}{\ln(1+i)}$ (*ln* is the natural log function). If the unknown variable is the interest rate i, then solving for i results in $i = \left(\frac{Y}{X}\right)^{1/t} - 1$. Financial calculators have functions that allow you to enter three of the variables and calculate the fourth.

1.1.3 SIMPLE INTEREST

When calculating interest accumulation over a fraction of a year or when executing short term financial transactions, a variation on compound interest commonly known as **simple interest** is often used. At an interest rate of i per year, an amount of 1 invested at the start of the year grows to $1 + i$ at the end of the year. If t represents a fraction of a year, then under the application of simple interest, the accumulated value at time t of the initial invested amount of 1 is as follows.

Definition 1.5 – Simple Interest Accumulation

The accumulation function from time 0 to time t at annual simple interest rate i, where t is measured in years is

$$a(t) \;\; = \;\; 1 + it. \tag{1.2}$$

As in the case of compound interest, for a fraction of a year, t is usually either $m/12$ or $d/365$. The following example refers to a **promissory note**, which is a short-term contract (generally less than one year) requiring the issuer of the note (the borrower) to pay the holder of the note (the lender) a principal amount plus interest on that principal at a specified annual interest rate for a specified length of time. At the end of the time period the payment (principal and interest) is due. Promissory note interest is calculated on the basis of simple interest. The interest rate earned by the lender is sometimes referred to as the **yield rate** earned on the investment. As concepts are introduced throughout this text, we will see the expression "yield rate" used in a number of different investment contexts with differing meanings. In each case it will be important to relate the meaning of the yield rate to the context in which it is being used.

EXAMPLE 1.4　(*Promissory note and simple interest*)

On January 31, Smith borrows 5000 from Brown and gives Brown a promissory note. The note states that the loan will be repaid on April 30 of the same year, with interest at 12% per annum. On March 1, Brown sells the promissory note to Jones, who pays Brown a sum of money in return for the right to collect the payment from Smith on April 30. Jones pays Brown an amount such that Jones' yield (interest rate earned) from March 1 to the

maturity date can be stated as an annual rate of interest of 15%.

(a) Determine the amount Smith was to have paid Brown on April 30.

(b) Determine the amount that Jones paid to Brown and the yield rate (interest rate) Brown earned, quoted on an annual basis. Assume all calculations are based on simple interest and a 365 day year.

(c) Suppose instead that Jones pays Brown an amount such that Jones' yield is 12%. Determine the amount that Jones paid.

SOLUTION

(a) The payment required on the maturity date April 30 is

$5000\left[1+(.12)\left(\frac{89}{365}\right)\right] = 5146.30$ (there are 89 days from January 31 to April 30 in a non-leap year; financial calculators often have a function that calculates the number of days between two dates).

(b) Let X denote the amount Jones pays Brown on March 1. We will denote by j_1 the annual yield rate earned by Brown based on simple interest for the period of $t_1 = \frac{29}{365}$ years from January 31 to March 1, and we will denote by j_2 the annual yield rate earned by Jones for the period of $t_2 = \frac{60}{365}$ years from March 1 to April 30. Then $X = 5000(1 + t_1 \times j_1)$ and the amount paid on April 30 by Smith is $X(1 + t_2 \times j_2) = 5146.30$. The following time-line diagram indicates the sequence of events.

January 31	March 1	April 30
Smith borrows 5000 from Brown	Brown receives X from Jones	Jones receives 5146.30 from Smith
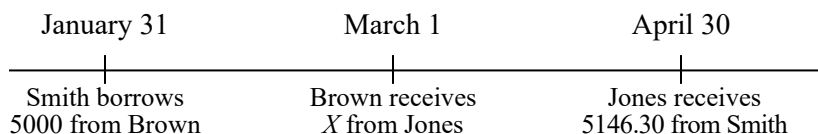
FIGURE 1.7

We are given $j_2 = .15$ (the annualized yield rate earned by Jones) and we can solve for X from $X = \frac{5146.30}{1 + t_2 \times j_2} = \frac{5146.30}{1 + \left(\frac{60}{365}\right)(.15)} = 5022.46$. Now that X is known, we can solve for j_1 from

$$X \; = \; 5022.46 \; = \; 5000(1+t_1 \times j_1) \; = \; 5000\left(1+\tfrac{29}{365} \times j_1\right)$$

to find that Brown's annualized yield is $j_1 = .0565$.

(c) If Jones' yield is 12%, then Jones paid

$$X \; = \; \frac{5146.30}{1+t_2 \times j_2} \; = \; \frac{5146.30}{1+\left(\tfrac{60}{365}\right)(.12)} \; = \; 5046.75. \qquad \square$$

In the previous example, we see that to achieve a yield rate of 15% Jones pays 5022.46 and to achieve a yield rate of 12%, Jones pays 5046.75. This inverse relationship between yield and price is typical of a **fixed-income** investment. A fixed-income investment is one for which the future payments are predetermined (unlike an investment in, say, a stock, which involves some risk, and for which the return cannot be predetermined). Jones is investing in a loan which will pay him 5146.30 at the end of 60 days. If the desired interest rate for an investment with fixed future payments increases, the price that Jones is willing to pay for the investment decreases (the less paid, the better the return on the investment). An alternative way of describing the inverse relationship between yield and price on fixed-income investments is to say that the holder of a fixed income investment (Brown) will see the market value of the investment decrease if the yield rate to maturity demanded by a buyer (Jones) increases. This can be explained by noting that a higher yield rate (earned by Jones) requires a smaller investment amount (made by Jones to Brown) to achieve the same dollar level of interest payments. This will be seen again when the notion of **present value** is discussed later in this chapter. Note that most banks use a 365 day count even in the case of a leap year.

1.1.4 COMPARISON OF COMPOUND INTEREST AND SIMPLE INTEREST

From Equations 1.1 and 1.2, it is clear that accumulation under simple interest forms a linear function whereas compound interest accumulation forms an exponential function. This is illustrated in Figure 1.8 with a graph of the accumulation of an initial investment of 1 at both simple and compound interest.
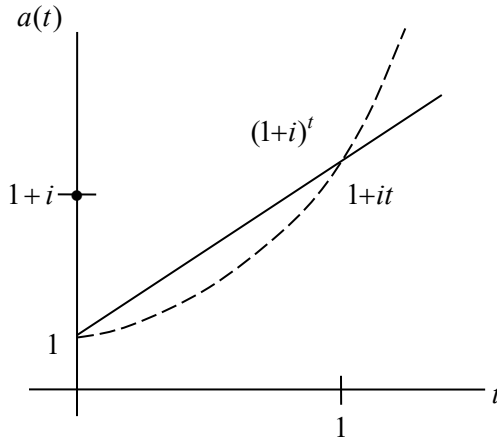
FIGURE 1.8

From Figure 1.8, it appears that simple interest accumulation is larger than compound interest accumulation for values of t between 0 and 1, but compound interest accumulation is greater than simple interest accumulation for values of t greater than 1. Using an annual interest rate of $i = .08$, we have, for example, at time $t = .25$,
$$1 + it = 1 + (.08)(.25) = 1.02 > 1.0194 = (1.08)^{.25} = (1+i)^t,$$
and at $t = 2$ we have
$$1 + it = 1 + (.08)(2) = 1.16 < 1.1664 = (1.08)^2 = (1+i)^t.$$
The relationship between simple and compound interest is verified algebraically in an exercise at the end of this chapter.

In practice, interest accumulation is often based on a combination of simple and compound interest. Compound interest would be applied over the completed (integer) number of interest compounding periods, and simple interest would be applied from then to the fractional point in the current interest period. For instance, under this approach, at annual rate 9%, over a period of 4 years and 5 months, an investment would grow by a factor of $(1.09)^4 \times \left[1 + \frac{5}{12}(.09)\right] = 1.4645$.

1.1.5 Accumulated Amount Function

When analyzing the accumulation of a single invested amount, the value of the investment is generally regarded as a function of time. For example,

$A(t)$ is the value of the investment at time t, with t usually measured in years. Time $t = 0$ usually corresponds to the time at which the original investment was made. The amount by which the investment grows from time t_1 to time t_2 is often regarded as the amount of interest earned over that period, and this can be written as $A(t_2) - A(t_1)$. Also, with this notation, the annual effective interest rate for the one-year period from time u to time $u+1$ would be i_{u+1}, where $A(u+1) = A(u)(1+i_{u+1})$, or equivalently,

$$i_{u+1} = \frac{A(u+1) - A(u)}{A(u)}. \tag{1.3}$$

The subscript "$u+1$" indicates that we are measuring the interest rate in year $u+1$. Accumulation can have any sort of pattern, and, as illustrated in Figure 1.9, the accumulated value might not always be increasing. The Dow Jones Average in Example 1.2 had a negative growth rate in 2015 and the graph drops from 2014 to 2015. The graph indicates the Dow Jones index value on December 31 of each year based on an investment into the index of amount 1 on December 31, 2011. For the one-year period from December 31, 2014 to December 31, 2015 the annual effective rate of interest is

$$i_{2015} = \frac{1.4276 - 1.4602}{1.4602} = -.0223.$$
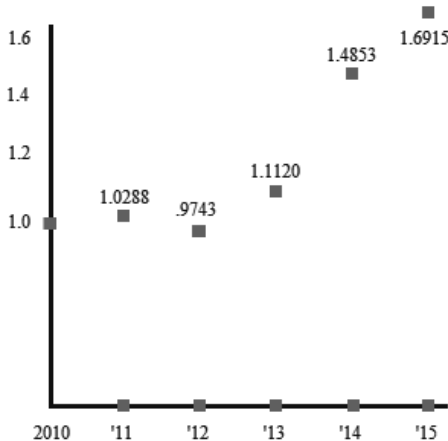
FIGURE 1.9

A more "continuous" picture of the progression of the fund's value over time could be obtained if month-end, or even daily fund values were plotted in Figure 1.9.

The relationship for i_{u+1} shows that the annual effective rate of interest for a particular one-year period is the amount of interest for the year as a proportion of the value of the investment at the start of the year, or equivalently, the rate of investment growth per dollar invested. In other words:

annual effective rate of interest for a specified one-year period
$$= \frac{\textit{amount of interest earned for the one-year period}}{\textit{value (or amount invested) at the start of the year}}.$$

The accumulated amount function can be used to find an effective interest rate for any time interval. For example, the three-month effective interest rate for the three months from time $3\frac{1}{4}$ to time $3\frac{1}{2}$ would be

$$\frac{A\left(3\frac{1}{2}\right)-A\left(3\frac{1}{4}\right)}{A\left(3\frac{1}{4}\right)}.$$

From a practical point of view, the accumulated amount function $A(t)$ would be a step function, changing by discrete increments at each interest credit time point, since interest is credited (or investment value is updated) at discrete points of time. For more theoretical analysis of investment behavior, it may be useful to assume that $A(t)$ is a continuous, or differentiable, function, such as in the case of compound interest growth on an initial investment of amount $A(0)$ at time $t = 0$, where $A(t) = A(0)(1+i)^t$ for any non-negative real number t.

1.2   PRESENT VALUE

If we let X be the amount that must be invested at the start of a year to accumulate to 1 at the end of the year at annual effective interest rate i, then $X(1+i) = 1$, or equivalently, $X = \frac{1}{1+i}$. The amount $\frac{1}{1+i}$ is the **present value of an amount of 1 due in one year**.

Definition 1.6 – One Period Present Value Factor

If the rate of interest for a period is i, the present value of an amount of 1 due one period from now is $\frac{1}{1+i}$. The factor $\frac{1}{1+i}$ is often denoted v in actuarial notation and is called a **present value factor** or discount factor.

When a situation involves more than one interest rate, the symbol v_i may be used to identify the interest rate i on which the present value factor is based.

The present value factor is particularly important in the context of compound interest. Accumulation under compound interest has the form $A(t) = A(0)(1+i)^t$. This expression can be rewritten as

$$A(0) \;=\; \frac{A(t)}{(1+i)^t} \;=\; A(t)(1+i)^{-t} \;=\; A(t)v^t.$$

Thus Kv^t is the present value at time 0 of an amount K due at time t when investment growth occurs according to compound interest. This means that Kv^t is the amount that must be invested at time 0 to grow to K at time t, and the present value factor v acts as a "compound present value" factor in determining the present value. Accumulation and present value are inverse processes of one another.
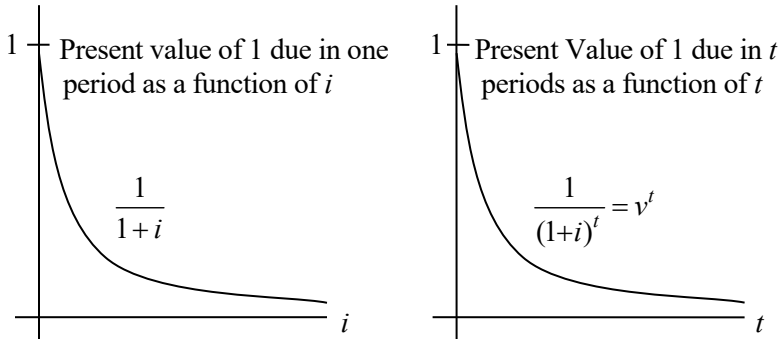
FIGURE 1.10

The right graph in Figure 1.10 illustrates that as the time horizon t increases, the present value of 1 due at time t decreases (if the interest rate is positive). The left graph of Figure 1.10 illustrates the classical "inverse

yield-price relationship," which states that at a higher rate of interest, a smaller amount invested is needed to reach a target accumulated value.

> **EXAMPLE 1.5** (*Present value calculation*)

Ted wants to invest a sufficient amount in a fund in order that the accumulated value will be one million dollars on his retirement date in 25 years. Ted considers two options. He can invest in Equity Mutual Fund, which invests in the stock market. E.M. Fund has averaged an annual compound rate of return of 19.5% since its inception 30 years ago, although its annual growth has been as low as 2% and as high as 38%. The E.M. Fund provides no guarantees as to its future performance. Ted's other option is to invest in a **zero-coupon bond** or **stripped bond** (this is a bond with no coupons, only a payment on the maturity date; this concept will be covered in detail later in the book), with a guaranteed annual effective rate of interest of 11.5% until its maturity date in 25 years.

(a) What amount must Ted invest if he chooses E.M. Fund and assumes that the average annual growth rate will continue for another 25 years?

(b) What amount must he invest if he opts for the stripped bond investment?

(c) What minimum annual effective rate is needed over the 25 years in order for an investment of $25,000 to accumulate to Ted's target of one million?

(d) How many years are needed for Ted to reach $1,000,000 if he invests the amount found in part (a) in the stripped bond?

> **SOLUTION**

(a) If Ted invests X at $t=0$, then $X(1.195)^{25} = 1,000,000$, so that the present value of 1,000,000 due in 25 years at an annual effective rate of 19.5% is $1,000,000v^{25} = 1,000,000(1.195)^{-25} = 11,635.96$.

(b) The present value of 1,000,000 due in 25 years at $i = .115$ is $1,000,000v^{25} = 1,000,000(1.115)^{-25} = 65,785.22$. Note that no subscript was used on v in part (a) or (b) since it was clear from the context as to the interest rate being used.

(c) We wish to solve for i in the equation $25,000(1+i)^{25} = 1,000,000$.

The solution for i is $i = \left(\frac{1,000,000}{25,000}\right)^{1/25} - 1 = .1590$.

(d) In t years Ted will have $11,635.96(1.115)^t = 1,000,000$. Solving for t

results in $t = \dfrac{\ln\left(\frac{1,000,000}{11,635}\right)}{\ln(1.115)} = 40.9$ years. ❐

If simple interest is being used for investment accumulation, then $A(t) = A(0)(1+it)$ and the present value at time 0 of amount $A(t)$ due at time t is $A(0) = \frac{A(t)}{1+it}$. It is important to note that implicit in this expression is the fact that simple interest accrual begins at the time specified as $t = 0$. The present value based on simple interest accumulation assumes that interest begins accruing at the time the present value is being found. There is no standard symbol representing present value under simple interest that corresponds to v under compound interest.

1.2.1 CANADIAN TREASURY BILLS

EXAMPLE 1.6

(*Canadian Treasury bills – present value based on simple interest*)
The figure below is an excerpt from the website of the Bank of Canada describing a sale of **Treasury Bills** by the Canadian federal government on Thursday, March 10, 2015 (www.bankofcanada.ca). A **T-Bill** is a debt obligation that requires the issuer to pay the owner a specified sum (the **face amount** or **amount**) on a specified date (the **maturity date**). The issuer of the T-Bill is the borrower, the Canadian government in this case. The purchaser of the T-Bill would be an investment company or an individual. Canadian T-Bills are issued to mature in a number of days that is a multiple of 7. Canadian T-Bills are generally issued on a Thursday, and mature on a Thursday, mostly for periods of (approximately) 3 months, 6 months or 1 year.

BANK OF CANADA					BANQUE DE CANADA	

OTTAWA
2015.03.10

Treasury Bills – Regular Auction Results	Bons du Trésor réguliers Résultats de l'adjudication
On behalf of the Minister of Finance, it was announced today that tenders for Government of Canada treasury bills have been accepted as follows:	On vient d'annoncer aujourd'hui, au nom du ministre des Finances, que les soumissions suivantes ont été acceptées pour les bons du Trésor du gouvernement du Canada:

Auction Date **2015.03.10** Date d'adjudication
Bidding Deadline **10:30:00** Heure limite de soumission
Total Amount **$7,000,000,000** Montant total

Multiple Price / Prix multiple

Amount Montant	Issue Émission	Maturity Échéance	Outstanding after Auction Encours après l'adjudication	(%) Yield and Equivalent Price Taux de rendement et prix correspondant		(%) Allotment Ratio Ratio de répartition	Bank of Canada Purchase Achat de la Banque du Canada
5,000,000,000	**2015.03.12**	**2015.06.18**	**$9,700,000,000**	Avg/Moy: **.593** Low/Bas: **.590** High/Haut: **.596**	**99.84104** **99.84184** **99.84023**	**50.66667**	**$300,000,000**
ISIN: **CA1350Z7WA84**							
2,000,000,000	**2015.03.12**	**2015.09.10**	**2,000,000,000**	Avg/Moy: **.624** Low/Bas: **.626** High/Haut: .627	**99.68982** **99.69180** **99.68853**	**80.42195**	**$150,000,000**
ISIN: **CA1350ZWR10**							

FIGURE 1.11

Two T-Bills are described in Figure 1.11, both issued March 12, 2015. The first one is set to mature June 18, 2015, which is 98 days (14 weeks) after issue. The *yield* is quoted as .593% and the price (per face amount The first one is set to mature June 18, 2015, which is 98 days (14 weeks) after issue. The *yield* is quoted as .593% and the price (per face amount of 100) is 99.84194. The price is the present value, on the issue date, of 100 due on the maturity date, and present value is calculated on the basis of simple interest and a 365-day year. The quoted price based on the quoted average yield rate of .593% can be calculated as follows:

$$Price = 100 \times \frac{1}{1 + (.00593)\left(\frac{98}{365}\right)} = 99.84104.$$

The price of the second T-Bill can be found in a similar way. It matures September 10, 2015, which is 182 days (26 weeks) after the issue date.

The quoted average yield is .624%. The price is

$$Price = 100 \times \frac{1}{1 + (.00624)\left(\frac{182}{365}\right)} = 99.68982.$$

Valuation of Canadian T-Bills is algebraically identical to valuation of promissory notes as described in Example 1.4. ❑

We can generalize the notion of present value based on any accumulation function $A(t)$. The investment grows from amount $A(t_1)$ at time t_1 to amount $A(t_2)$ at time $t_2 > t_1$. Therefore an amount of $\frac{A(t_1)}{A(t_2)}$ invested at time t_1 will grow to amount 1 at time t_2. In other words, $\frac{A(t_1)}{A(t_2)}$ is a generalized present value factor from time t_2 back to time t_1.

1.3 EQUATION OF VALUE

When a financial transaction is represented algebraically it is usually formulated by means of one or more equations that represent the values of the various components of the transaction and their interrelationships. Along with the interest rate, the other components of the transaction are the amounts disbursed and the amounts received. These amounts can be thought of as dated cash flows, in that both the amount and time of occurrence of each cash flow must be taken into account in an algebraic formulation. A mathematical representation of the transaction will be an equation that balances the dated cash outflows and inflows, according to the particulars of the transaction. The equation balancing these cash flows must take into account the "time values" of these payments, which are the accumulated and present values of the payments made at the various time points. Such a balancing equation is called an **equation of value** for the transaction, and its formulation is a central element in the process of analyzing a financial transaction.

In order to formulate an equation of value for a transaction, it is first necessary to choose a reference time point or valuation date. At the reference time point the equation of value balances, or equates, the following

two factors:

(1) the accumulated value of all payments already disbursed plus the present value of all payments yet to be disbursed, and

(2) the accumulated value of all payments already received plus the present value of all payments yet to be received.

EXAMPLE 1.7 (*Choice of valuation point for an equation of value*)

Every Friday in February (the 7, 14, 21, and 28) Walt places a $1000 bet, on credit, with his off-track bookmaking service. The betting service charges a weekly effective interest rate of 8% on all credit extended. Unfortunately for Walt, he loses each bet and agrees to repay his debt to the bookmaking service in four installments, to be made on March 7, 14, 21, and 28. Walt pays $1100 on each of March 7, 14, and 21. How much must Walt pay on March 28 to completely repay his debt?

SOLUTION

The payments in the transaction are represented in Figure 1.12. We must choose a reference time point at which to formulate the equation of value. If we choose February 7 ($t = 0$ in Figure 1.12), then Walt receives 1000 right "now". All other amounts received and paid are in the future, so we find their present values. The value at time 0 of what Walt will receive (on credit) is $1000(1 + v + v^2 + v^3)$, representing the four weekly credit amounts received in February, where $v = \frac{1}{1.08}$ is the weekly present value factor and t is measured in weeks. The value at $t = 0$ of what Walt must pay is $1100(v^4 + v^5 + v^6) + Xv^7$, representing the three payments of 1100 and the fourth payment of X.

Received				Paid			
1000	1000	1000	1000	1100	1100	1100	X
2/7	2/14	2/21	2/28	3/7	3/14	3/21	3/28
0	1	2	3	4	5	6	7
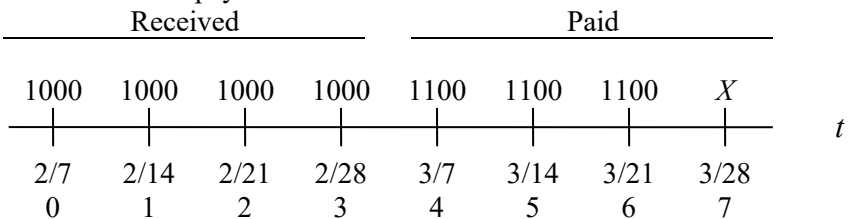
FIGURE 1.12

Equating the value at time 0 of what Walt will receive with the value of what he will pay results in the equation

$$1000(1+v+v^2+v^3) = 1100(v^4+v^5+v^6) + Xv^7. \quad\quad \text{(A)}$$

Solving for X results in

$$X = \frac{1000(1+v+v^2+v^3)-1100(v^4+v^5+v^6)}{v^7} = 2273.79. \quad \text{(B)}$$

If we choose March 28 $(t=7)$ as the reference time point for valuation, then we accumulate all amounts received and paid to time 7. The value of what Walt has received is $1000\left[(1+j)^7+(1+j)^6+(1+j)^5+(1+j)^4\right]$, and the value of what he has repaid is $1100\left[(1+j)^3+(1+j)^2+(1+j)\right]+X$, where again $j=.08$ is the effective rate of interest per week.

The equation of value formulated at $t=7$ can be written as

$$1000\left[(1+j)^7+(1+j)^6+(1+j)^5+(1+j)^4\right]$$
$$= 1100\left[(1+j)^3+(1+j)^2+(1+j)\right]+X. \quad \text{(C)}$$

Solving for X results in

$$1000\left[(1+j)^7+(1+j)^6+(1+j)^5+(1+j)^4\right]$$
$$-1100\left[(1+j)^3+(1+j)^2+(1+j)\right] = 2273.79. \quad \text{(D)}$$

Note that most financial transactions will have interest rates quoted as annual rates, but in the weekly context of this example it was unnecessary to indicate an annual rate of interest. (The equivalent annual effective rate would be quite high). ❑

We see from Example 1.7 that an equation of value for a transaction involving compound interest may be formulated at more than one reference time point with the same ultimate solution. Notice that Equation C can be

obtained from Equation A by multiplying Equation A by $(1+j)^7$. This corresponds to a change in the reference point upon which the equations are based, Equation A being based on $t = 0$ and Equation C being based on $t = 7$. In general, when a transaction involves only compound interest, an equation of value formulated at time t_1 can be translated into an equation of value formulated at time t_2 simply by multiplying the first equation by $(1+i)^{t_2 - t_1}$. In Example 1.7, when $t = 7$ was chosen as the reference point, the solution was slightly simpler than that required for the equation of value at $t = 0$, in that no division was necessary. For most transactions there will often be one reference time point that allows a more efficient solution of the equation of value than any other reference time point.

1.4   NOMINAL RATES OF INTEREST

Quoted annual rates of interest frequently do not refer to the annual effective rate. Consider the following example.

EXAMPLE 1.8   (*Monthly compounding of interest*)

Sam has just received a credit card with a credit limit of $1000. The card issuer quotes an annual charge on unpaid balances of 24%, payable monthly. Sam immediately uses his card to its limit. The first statement Sam receives indicates that his balance is $1000 but no interest has yet been charged. Each subsequent statement includes interest on the unpaid part of his previous month's statement. Smith ignores the statements for a year, and makes no payments toward the balance owed. What amount does Sam owe according to his thirteenth statement?

SOLUTION

Sam's first statement will have a balance of 1000 outstanding, with no interest charge. Subsequent monthly statements will apply a monthly interest charge of $\left(\frac{1}{12}\right)(24\%) = 2\%$ on the unpaid balance from the previous month. Thus Sam's unpaid balance is compounding monthly at a rate of 2% per month; the interpretation of the phrase "payable monthly" is that the

quoted annual interest rate is to be divided by 12 to calculate the one-month interest rate that will be applied. The balance on statement 13 (12 months after statement 1) will have compounded for 12 months to $1000(1.02)^{12} = 1268.23$ (this value is based on rounding to the nearest penny each month; the exact value is 1268.24). The annual effective interest rate charged on the account in the 12 months following the first statement is 26.82%. The quoted rate of 24% is a **nominal annual rate of interest, not an annual effective rate of interest**. This example shows that a **nominal annual interest rate of 24% compounded monthly** is equivalent to an annual effective rate of 26.82%.   ❐

Definition 1.7 – Nominal Annual Rate of Interest

A nominal annual rate of interest compounded or convertible m times per year refers to an interest compounding period of $\frac{1}{m}$ years.

$$\text{interest rate for } \frac{1}{m} \text{ year period} = \frac{\text{quoted nominal annual interest rate}}{m}$$

Nominal rates of interest occur frequently in practice. They are used in situations in which interest is credited or compounded more often than once per year. A nominal annual rate can be associated with any interest compounding period, such as six months, one month, or one week. In order to apply a quoted nominal annual rate, it is necessary to know the number of interest conversion periods in a year. In Example 1.8 the associated interest compounding period is indicated by the phrase "payable monthly," and this tells us that the interest compounding period is one month. This could also be stated in any of the following ways:

(i)   annual interest rate of 24%, compounded monthly,

(ii)   annual interest rate of 24%, convertible monthly, or

(iii) annual interest rate of 24%, convertible 12 times per year.

All of these phrases mean that the 24% quoted annual rate is to be transformed to a one-month effective rate that is one-twelfth of the quoted annual rate, $\left(\frac{1}{12}\right)(.24) = .02$. The effective interest rate per interest compound-

ing period is a fraction of the quoted annual rate corresponding to the fraction of a year represented by the interest compounding period.

The notion of equivalence of two rates was introduced in Section 1.1, where it was stated that rates are equivalent if they result in the same pattern of compound accumulation over any equal periods of time. This can be seen in Example 1.8. The nominal annual 24% refers to a compound monthly rate of 2%. Then in t years ($12t$ months) the growth of an initial investment of amount 1 will be

$$(1.02)^{12t} = \left[(1.02)^{12}\right]^{t} = (1.2682)^{t}.$$

Since $(1.2682)^{t}$ is the growth in t years at annual effective rate 26.82%, this verifies the equivalence of the two rates. The typical way to verify equivalence of rates is to convert one rate to the compounding period of the other rate, using compound interest. In the case just considered, the compound monthly rate of 2% can be converted to an equivalent annual effective growth factor of $(1.02)^{12} = 1.2682$. Alternatively, an annual effective rate of 26.82% can be converted to a compound monthly growth factor of $(1.2682)^{1/12} = 1.02$.

The 24% rate quoted in Example 1.8 is sometimes called an **annual percentage rate**, and the rate of 2% per month is the **periodic rate**. In practice, a credit card issuer will usually quote an "APR" (annual percentage rate), and may also quote a daily percentage rate which is $\frac{APR}{365}$. When a monthly billing cycle ends, an "average daily balance" is calculated, usually by taking the average of the account balances at the start of each day during the billing cycle. This is multiplied by the daily percentage rate, and this is multiplied by the number of days in the billing cycle. Under this approach, the monthly interest rates compounded in Example 1.8 would not be exactly 2% per month, but would be $\frac{.24}{365} \times 31 = .02038356$ for a 31 day billing cycle, $\frac{.24}{365} \times 30 = .01972603$ for a 30 day billing cycle, etc.

Once the nominal annual interest rate and compounding interest period are known, the corresponding compound interest rate for the interest conversion period can be found. The accumulation function then follows a compound interest pattern, with time usually measured in units of effective interest conversion periods. When comparing nominal annual

interest rates with differing interest compounding periods, it is necessary to convert the rates to equivalent rates with a common effective interest period. The following example illustrates this.

EXAMPLE 1.9 (*Comparison of nominal annual rates of interest*)

Tom is trying to decide between two banks in which to open an account. Bank A offers an annual rate of 15.25% with interest compounded semi-annually, and Bank B offers an annual rate of 15% with interest compounded monthly. Which bank will give Tom higher annual effective growth?

SOLUTION

Bank A pays an effective 6-month interest rate of $\frac{1}{2}(15.25\%) = 7.625\%$. In one year (two effective interest periods) a deposit of amount 1 will grow to $(1.07625)^2 = 1.158314$ in Bank A.

Bank B pays an effective monthly interest rate of $\frac{1}{12}(15\%) = 1.25\%$. In one year (12 effective interest periods) a deposit of amount 1 in Bank B will grow to $(1.0125)^{12} = 1.160755$. Bank B has an equivalent annual effective rate that is almost .25% higher than that of Bank A. ❐

In order to make a fair comparison of quoted nominal annual rates with differing interest conversion periods, it is necessary to transform them to a common interest conversion period, such as an annual effective period as in Example 1.9.

Payday Loans

As long as there have been people who run short of money before their next paycheck, there have been lenders who will provide short term loans to be repaid at the next payday, usually within a few weeks of the loan. Providers of these loans became more visible with storefront operations in the 1980s, and in recent years with internet based lending operations. Interest rates charged by some lenders for these loans can be surprisingly high. Regulation and interest rate restriction on this industry has grown in recent years and a number of states do not allow payday lenders to operate.

The U.S. Truth in Lending Act requires that, for consumer loans, the APR (annual percentage rate) associated with the loan must be disclosed to the borrower. The APR is generally disclosed as a nominal annual rate of interest whose conversion period is the payment period for the loan.

A review of online payday loan websites in 2017 turned up a lender charging a fee of $30 for a one week loan of $100. This one week interest rate of 30% is quoted as an APR of 1564.29% (this is $30 \times \frac{365}{7}$), which is the corresponding nominal annual rate convertible every 7 days. The equivalent annual effective growth of an investment that accumulates at a rate of 30% per week with weekly compounding is $(1.3)^{365/7} = 873,639$, which represents an equivalent annual effective rate of interest of a little more than 87,000,000% (a little more, meaning 363,786 percentage points more)! The lender also allows the loan to be repaid in up to 18 days for the same $30 fee for the 18 days. In this case, the APR is only 608.33%, and the equivalent annual effective rate of interest is a mere 20,342%.

1.4.1 ACTUARIAL NOTATION FOR NOMINAL RATES OF INTEREST

There is **standard actuarial notation** for denoting nominal annual rates of interest, although this notation is not generally seen outside of actuarial practice. In actuarial notation, the symbol i is generally reserved for an annual effective rate, and the symbol $i^{(m)}$ is reserved for a nominal annual rate

with interest compounded m times per year. Note that the superscript is for identification purposes and is <u>not an exponent</u>. The notation $i^{(m)}$ is taken to mean that interest will have a compounding period of $\frac{1}{m}$ years and compound rate per period of $\frac{1}{m} \times i^{(m)} = \frac{i^{(m)}}{m}$.

In Example 1.8, $m = 12$, so the nominal annual rate would be denoted as $i^{(12)} = .24$. The information indicated by the superscript "(12)" in this notation is that there are 12 interest conversion periods per year, and that the effective rate of 2% per month is $\frac{1}{12}$ of the quoted rate of 24%. Similarly, in Example 1.9 the nominal annual rates would be $i_A^{(2)} = .1525$ and $i_B^{(12)} = .15$ for Banks A and B, respectively.

In Example 1.8 the equivalent annual effective growth factor is $1 + i = \left(1 + \frac{.24}{12}\right)^{12} = 1.2682$. In Example 1.9 the equivalent annual effective growth factors for Banks A and B, respectively, are

$$1 + i_A = \left(1 + \frac{.1525}{2}\right)^2 = 1.158314$$

and

$$1 + i_B = \left(1 + \frac{.15}{12}\right)^{12} = 1.160755.$$

The general relationship linking equivalent nominal annual interest rate $i^{(m)}$ and annual effective interest rate i is

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m. \tag{1.4}$$

The comparable relationships linking i and $i^{(m)}$ can be summarized in the following two equations

$$i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1 \quad \text{and} \quad i^{(m)} = m\left[(1+i)^{1/m} - 1\right]. \tag{1.5}$$

Note that $(1+i)^{1/m}$ is the $\frac{1}{m}$-year growth factor, and $(1+i)^{1/m}-1$ is the equivalent effective $\frac{1}{m}$-year compound interest rate.

It should be clear from general reasoning that with a given nominal annual rate of interest, the more often compounding takes place during the year, the larger the year-end accumulated value will be, so the larger the equivalent annual effective rate will be as well. This is verified algebraically in an exercise at the end of the chapter. The following example considers the relationship between equivalent i and $i^{(m)}$ as m changes.

EXAMPLE 1.10

(*Equivalent effective and nominal annual rates of interest*)

Suppose the annual effective rate of interest is 12%. Find the equivalent nominal annual rates for $m = 1, 2, 3, 4, 6, 8, 12, 52, 365, \infty$.

SOLUTION

$m = 1$ implies interest is convertible annually ($m = 1$ time per year), which implies the annual effective interest rate is $i^{(1)} = i = .12$. We use Equation (1.5) to solve for $i^{(m)}$ for the other values of m. The results are given in Table 1.1.

TABLE 1.1

M	$(1+i)^{1/m} - 1$	$i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$
1	.1200	.12
2	.0583	.1166
3	.0385	.1155
4	.0287	.1149
6	.0191	.1144
8	.0143	.1141
12	.0095	.1139
52	.00218	.1135
365	.000311	.113346
∞		$\displaystyle\lim_{m\to\infty} m\left[(1+i)^{1/m} - 1\right] = \ln(1+i) = .113329$

❑

Note that $(1.12)^{1/2} - 1 = .0583$ is the 6-month effective rate of interest that is equivalent to an annual effective rate of interest of 12% (two successive 6-month periods of compounding at effective 6-month rate 5.83% results in one year growth of $(1.0583)^2 = 1.12$). The limit in the final line of Table 1.1 can be found using l'Hospital's Rule. It can also be seen from Table 1.1 that the more frequently compounding takes place (i.e., as m increases), the smaller is the equivalent nominal annual rate. The change is less significant, however, in going from monthly to weekly or even daily compounding, so we see that there is a limit to the benefit of compounding. With an annual effective rate of 12%, the minimum equivalent nominal annual rate is never less than 11.333% no matter how often compounding takes place. The limiting case $(m \rightarrow \infty)$ in Example 1.10 is called **continuous compounding** and is related to the notions of **force of interest** and instantaneous growth rate of an investment. This is discussed in detail in Section 1.6.

A nominal rate, although quoted on an annual basis, might refer to only the immediately following fraction of a year. For instance, in Example 1.9 Bank B's quoted nominal annual rate of 15% with interest credited monthly might apply only to the coming month, after which the quoted rate (still credited monthly) might change to something else, say 13.5%. Thus when interest is quoted on a nominal annual basis, the actual rate may change during the course of the year, from one interest period to the next.

1.5   EFFECTIVE AND NOMINAL RATES OF DISCOUNT

1.5.1 ANNUAL EFFECTIVE RATE OF DISCOUNT

In previous sections of this chapter, interest amounts have been regarded as paid or charged at the end of an interest compounding period, and the corresponding interest rate is the ratio of the amount of interest paid at the end of the period to the amount of principal at the start of the period. Interest rates and amounts viewed in this way are sometimes referred to as **interest payable in arrears** (payable at the end of an interest period). This is the standard way in which interest rates are quoted, and it is the standard way by which interest amounts are calculated. In many situations it is the method required by law.

Occasionally a transaction calls for **interest payable in advance**. In this case the quoted interest rate is applied to obtain an amount of interest which is payable at the start of the interest period. For example, if Smith borrows 1000 for one year at a quoted rate of 10% with interest payable in advance, the 10% is applied to the loan amount of 1000, resulting in an amount of interest of 100 for the year. The interest is paid at the time the loan is made. Smith receives the loan amount of 1000 and must immediately pay the lender 100, the amount of interest on the loan. One year later Smith must repay the loan amount of 1000. The net effect is that Smith receives 900 and repays 1000 one year later. The annual effective rate of interest on this transaction is $\frac{100}{900} = .1111$, or 11.11%. This rate of 10% payable in advance is called the **rate of discount** for the transaction. The rate of discount is the rate used to calculate the amount by which the year end value is reduced to determine the present value.

The annual effective rate of discount can be another way of describing investment growth in a financial transaction. In the example just considered, we see that an annual effective interest rate of 11.11% is equivalent to an annual effective discount rate of 10%, since both describe the same transaction.

Definition 1.8 – Annual effective Rate of Discount

In terms of an accumulated amount function $A(t)$, the general definition of the annual effective rate of discount from time $t = 0$ to time $t = 1$ is

$$d = \frac{A(1) - A(0)}{A(1)}. \qquad (1.6)$$

This definition is in contrast with the definition for the annual effective rate of interest, which has the same numerator but has a denominator of $A(0)$. Annual effective interest measures growth on the basis of the initially invested amount, whereas annual effective discount measures growth on the basis of the year-end accumulated amount. Either measure can be used in the analysis of a financial transaction.