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ACTEX

Exam IFM Study Manual

Spring 2019 Edition, Volume I

Johnny Li, Ph.D., FSA

ACTEX Learning | Learn Today. Lead Tomorrow.

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Preface

Thank you for choosing ACTEX Learning.

The Investment and Financial Markets (IFM) Exam is a new exam that is first launched by the Society of Actuaries (SoA) in July 2018. Although the IFM Exam draws heavily from the MFE Exam (which is no longer offered after July 1, 2018), it covers a lot of topics (including corporate finance and the interface between derivatives and insurance) which the MFE Exam does not cover. This brand new study manual is created to help you best prepare for the IFM Exam.

Given that the IFM Exam covers a very wide range of topics, it is crucial to learn them in a logical sequence and to see through the connections among them. We have meticulously categorized the exam topics into two broad themes: Quantitative and Qualitative.

The first part of this manual focuses on the Quantitative theme, which encompasses all of the topics covered in *Derivatives Markets* (the required text authored by R.L. McDonald) and the technical topics from *Corporate Finance* (the required text authored by J. Berk and others). To help you develop a strong foundation, we begin with the easiest calculations that are just straightforward extensions of what you have learnt in Exam FM. These are then followed by progressively harder calculations, ranging from the binomial model to various versions of the Black-Scholes formula.

The following features concerning the Quantitative part of the manual are noteworthy:

1. The connections between the option pricing models (from *Derivatives Markets*) and real options (from *Corporate Finance*) are clearly explained.

2. We do not want to overwhelm readers with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and/or integrated into the practice problems.

3. We provide sufficient practice problems (which are similar to the real exam problems in terms of format and level of difficulty), so that you do not have to go through the textbooks' end-of-chapter problems. We find that the end-of-chapter problems in *Derivatives Markets* are either too trivial (simple substitutions) or too computationally intensive (Excel may be required), compared to the real exam questions.

4. We do not follow the order in *Derivatives Markets*, because the focus of this textbook is somewhat different from what the SoA expects from the candidates. According to the SoA, the purpose of the exam is "to develop candidates' knowledge of the theoretical basis," but the book emphasizes more on applications.

We believe that the materials in the Quantitative theme should be studied first, as a lot of time has to be spent on the practice problems in order to develop a solid mastery of these materials.

The second part of the manual is devoted to the Qualitative theme, which encompasses a lot of definitions and hard facts that you have to memorize (unfortunately). There are some calculations in the Qualitative theme, but they are typically trivial. To help you breeze through this theme, the materials in this theme are presented in an easy-to-read point form, with the most important points being clearly highlighted. Of course, we have practice problems to test how well you can remember the materials.

We recommend that you go through the Qualitative theme after the Quantitative theme, simply because everyone's short-term memory is limited.

The manual is concluded with several mock exams, which are written in a similar format to the released exam and sample questions provided by the SoA. This will enable you to, for example, retrieve information more quickly in the real exam. Further, we have integrated all of the relevant released exam and sample questions into the examples, practice problems, mock exams in the manual. These exam/sample questions include:

- The released MFE sample and released exam questions that are still relevant to the IFM exam syllabus.
- The released FM sample and released exam questions that are relevant to the IFM exam syllabus.
- The sample questions on Finance and Investment (Corporate Finance, IFM-21-18, IFM-22-18).

This integration seems to be a better way to learn how to solve those questions, and of course, you will need no extra time to review those questions.

We recommend you to use of this study manual is as follows:

1. Read the lessons in order.
2. Immediately after reading a lesson, complete the practice problems for that lesson.
3. After studying all lessons, work on the mock exams.

If you find a possible error in this manual, please let us know at the "Customer Feedback" link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the "Errata & Updates" link.

A Note on Rounding and the Normal Distribution Calculator

To achieve the desired accuracy, we recommend that you store values in intermediate steps in your calculator. If you prefer not to, please keep at least six decimal places.

In this study guide, normal probability values and z-values are based on a normal distribution calculator instead of a normal table in other exams. In the actual examination you will be able to use the same normal distribution calculator.

The calculator is very easy to use. Simply go to

https://www.prometric.com/en-us/clients/SOA/Pages/calculator.aspx

Recall that $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z$ is the cumulative distribution function of a standard normal random variable. To find $N(x)$, you may use the first panel of the calculator. Type in the value of x and press "Normal CDF". Then you would get $N(x)$. For example, when $x = -1.282$, the calculator would report 0.09992.

To find the 100pth percentile of the standard normal random variable (i.e. to find the value of x such that $N(x) = p$), enter p into the cell adjacent to $N(x)$, and press "Inverse CDF". Then you would get x. For example, when $N(x) = 0.25$, the calculator would report –0.67449.

If you do not want to go online every time when you follow the examples and work on the practice problems, you can set up your own normal distribution calculator using Excel. Open a blank workbook, and set up the following

Cell A1: x
Cell A2: N(x)
Cell B1: -1.282
Cell B2: = round(norm.s.dist(B1, 1), 5)

Cell A5: N(x)
Cell A6: x
Cell B5: 0.25
Cell B6: = round(norm.s.inv(B5), 5)

Cell B2 would report 0.09992 and Cell B6 would report –0.67449 if you are using Excel 2010 or more recent versions of Excel. You can alter the values in B1 and B5 to calculate any probability and percentile. Save your workbook for later use.

Syllabus Reference

In what follows,

- BM stands for the textbook *Corporate Finance* (4^{th} Ed) coauthored by Berk and DeMarzo,
- McD stands for the textbook *Derivatives Markets* (3^{rd} Ed) authored by McDonald,
- SN1 stands for the SoA published Study Note IFM-21-18 *Measures of Investment Risk, Monte Carlo Simulation and Empirical Evidence on the Efficient Market Hypothesis*,
- SN2 stands for the SoA published Study Note IFM-22-18 *Actuarial Applications of Options and Other Financial Derivatives*.

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1.1.4	BM 11.3, 10.5, 10.6
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Mock Exams

Lesson 1 Risk and Return

OBJECTIVES

1. To calculate returns (arithmetic and geometric)
2. To calculate volatility and correlation
3. To calculate the return and volatility of a portfolio of assets
4. To understand the notion of diversification
5. To understand the concept of systematic and firm-specific risks

This module is about portfolio mathematics. You will be introduced the classical model for computing the required return on an asset / a portfolio / a real project given a specified level of risk. We will define how the risk can be measured, and discuss some simple ideas in asset allocation.

The first lesson is about how one can convert asset prices to returns and calculate summary statistics from a series of returns. We assume that you have taken Exam P and this lesson is a review of probability and statistics put into the context of portfolios.

1. 1. 1 Calculating Historical Returns and Volatility

One-period Realized Return

Suppose that you are a shareholder (i.e., you own shares of a certain firm). Over a certain period of time, you have *capital gain* or *capital loss* due to changes in share price:

$$\text{Capital gain} = \text{change in share price} \times \text{number of shares owned.}$$

For example, if you own two hundred shares of a stock whose current price is \$35.2 per share, and after one day the stock price jumps to \$36.4, your capital gain is $(36.4 - 35.2) \times 200 = \$240$.

If the share price drops so that the change in share price is negative, the capital gain would be negative too. This means you suffer a capital loss. In the following, we will refer to capital losses as negative capital gains.

You may wonder why capital gain is really a "gain." If you do not sell the stock then you cannot capitalize the gain of $240 because you do not receive any cash. But since you have the rights to sell the stock and realize that $240, we can still regard capital gain as a gain.

Sometimes, when the company is profitable, it may pay *dividends* to its shareholders. Therefore, when you calculate your total gain / loss, you must include dividends:

$$\text{Total dollar return} = \text{Dividend income} + \text{Capital gain.}$$

For example, if the stock pays a dividend of $1 per share after one day, then the total dollar return is

$$1 \times 200 + 240 = \$440.$$

The realized return is the rate of return that actually occurs over a particular time period. Let P_t and P_{t+1} be the per share price of the stock at time t and $t+1$, respectively, and Div_{t+1} be the dividend paid at time $t+1$ per share:

$$\begin{array}{cc} P_t & P_{t+1} + \text{Div}_{t+1} \\ \hline t & t+1 \end{array}$$

Then the realized return on the stock over $(t, t+1]$ is defined as

$$\text{Realized return} = \text{Dividend yield} + \text{Capital gain rate}$$

$$R_{t+1} = \frac{\text{Div}_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

In our example, the percentage change in share price is $\frac{36.4 - 35.2}{35.2} = 3.41\%$, and the dividend yield is $\frac{1}{35.2} = 2.84\%$, giving a realized return of 6.25%.

The concept of calculating realized return applies to all financial instruments, including bonds and preferred stocks. Any intermediate cash flows paid during a year can be treated as dividends.

Example **1.1.1**

Consider a 20-year corporate bond with a par value of 1,000. The bond, which pays no coupons, was issued one year ago, and the price was $140.50. The current effective annual market interest rate is 11%.

Compute the realized return on the bond over the last year.

Solution

The price of the zero-coupon bond is now

$$1{,}000 \times 1.11^{-19} = \$137.6776,$$

giving a capital gain of 137.6776 − 140.50 = −$2.82236. Since the zero-coupon bond does not pay an intermediate cash flow during the previous year, the realized return is

$$-\frac{2.82236}{140.5} \times 100\% = -2.009\%.$$

[END]

The realized return defined here should be distinguished from "logarithmic return" (also known as "continuously compounded return") which we will introduce in Module 3 when we study option pricing. For your information, the one-period logarithmic return is defined as

$$\ln \frac{P_{t+1} + \text{Div}_{t+1}}{P_t}.$$

In this module, we do not use logarithmic return.

Historical Average Return and Return Variability

The previous formula on realized return tells you how the return over a single period $(t, t + 1]$ can be computed. If you gather year-by-year asset prices and dividends, then you can compute a series of realized returns R_t's as shown in the following table:

Time	Asset Price	Dividend	Return
0	P_0	–	–
1	P_1	Div_1	R_1
2	P_2	Div_2	R_2
⋮	⋮	⋮	⋮
t	P_t	Div_t	R_t
$t+1$	P_{t+1}	Div_{t+1}	R_{t+1}
⋮	⋮	⋮	⋮

The asset can be a bond, and in such a case the dividends should be replaced by coupon payments.

After we have calculated the returns over many periods, we can construct a frequency curve for the returns and calculate summary statistics. Suppose that we have returns $\{R_1, R_2, \ldots, R_T\}$, then the following can be calculated.

(1) The *average annual return* (or the mean return) on an asset during $(0, T]$ is

$$\overline{R} = \frac{R_1 + R_2 + \ldots + R_T}{T}.$$

This average annual return is also known as *arithmetic average return*.

(2) The (sample) *variance* of the realized returns is

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{i=1}^{T}(R_i - \bar{R})^2 = \frac{1}{T-1}\left[\sum_{i=1}^{T} R_i^2 - T\bar{R}^2\right].$$

Note the use of $T - 1$ in the denominator! The (sample) *standard deviation* of the returns is the square root of the variance of the returns:

$$\hat{\sigma} = \sqrt{\frac{1}{T-1}\sum_{i=1}^{T}(R_i - \bar{R})^2}\ .$$

We often refer $\hat{\sigma}$ to the volatility of the stock. If returns are reported as a percentage (e.g., x%) per year, then you may treat % as a "unit" for returns. You may then express the variance of returns as, for example, $y\%^2$ per year, and the volatility of return as, for example, z% per year.

(3) The average annual rate is just an estimate of the true (population) annual return rate on the asset. It is subject to estimation error. The standard error of this estimate, assuming that the annual returns are iid (independent and identically distributed), is

$$\hat{\sigma} / \sqrt{\text{Number of Observations}}\ .$$

(This follows from the fact that if $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ where X_i are iid with variance σ^2, then

$$\text{Var}(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.)$$

The textbook mentions that the 95% confidence interval for the true annual rate of return of the asset can be computed as

(Average annual rate − 1.96 × Standard error, Average annual rate + 1.96 × Standard error).

(The textbook uses 2 instead of 1.96 actually.) This confidence interval is an approximation that works well only when the number of observations is large.

Example 1.1.2

Standard & Poor's 500 is a portfolio constructed by Standard and Poor's of 500 US stocks. The stocks represented are large firms and are leaders in their respective industries. The annual returns of the S&P 500 index from 1991 to 1995 are as follows:

Year	1991	1992	1993	1994	1995
Return (in %)	30.55	7.67	9.99	1.31	37.43

Estimate the average return and the volatility of the S&P 500 index.

Solution

We have $\sum R_i = 0.3055 + 0.0767 + \ldots + 0.3743 = 0.8695$. The average is

$$\frac{0.3055+0.0767+0.0999+0.0131+0.03743}{5} = 17.39\%.$$

We also have $\sum R_i^2 = 0.3055^2 + 0.0767^2 + \ldots + 0.3743^2 = 0.24946525$.

The sample variance is

$$\frac{0.24946525-5\times 0.1739^2}{4} = 0.024565.$$

The volatility is $0.024565^{0.5} = 15.673\%$.

[END]

Example 1.1.3

Based on the mean and volatility obtained from the previous example and assuming normality, compute the probability that the return in year 1996 is in between –5% and 15%.

Solution

Let R_{96} be the return in 1996. The assumption is $R_{96} \sim \mathrm{N}(0.1739, 0.15673^2)$.

The probability required is

$$\Pr(-0.05 < R_{96} < 0.15) = \Pr\left(\frac{-0.05-0.1739}{0.15673} < Z < \frac{0.15-0.1739}{0.15673}\right)$$
$$= \Pr(-1.4286 < Z < -0.15249) = N(-0.15249) - N(-1.4286)$$
$$= 0.43940 - 0.07656 = 0.3628$$

[END]

Compound Annual Return

Suppose that a stock pays dividends at the end of each quarter, with realized returns R_{Q1}, R_{Q2}, R_{Q3} and R_{Q4} each quarter. Then the compound annual return of the stock is given by

$$R = (1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})(1 + R_{Q4}) - 1.$$

The idea of the formula above is that if you start with a dollar investment of the asset at time 0, then at the end of quarter 1 your investment would grow to $1 + R_{Q1}$. You reinvest all dividends immediately and use them to purchase additional shares of the same stock. At the end of quarter 2 your investment would grow by a rate of R_{Q2} and hence your investment would grow to

$$(1 + R_{Q1}) \times (1 + R_{Q2}).$$

You continue to reinvest all dividends by purchasing additional shares of the same stock. At the end of quarter 3 your investment would grow to $(1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})$. Repeating the same procedure, you would end up with

$$(1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})(1 + R_{Q4})$$

at the end of the year. The total return over the year is thus

$$R = (1 + R_{Q1})(1 + R_{Q2})(1 + R_{Q3})(1 + R_{Q4}) - 1.$$

To be slightly more general, let us suppose that you have an investment and the realized annual return over $(t, t + 1]$ is R_{t+1} for $t = 0, 1, \ldots, T - 1$. If you make a dollar of investment of the asset at time 0, then at the end of year T your investment would grow to

$$(1 + R_1)(1 + R_2) \ldots (1 + R_T).$$

The total return can be "annualized" into a **geometric average return**, as follows:

$$R = [(1 + R_1)(1 + R_2) \ldots (1 + R_T)]^{1/T} - 1$$

The geometric average return can be thought of as the average *compound return*. Recall that $(1 + R_1)(1 + R_2) \ldots (1 + R_T)$ is the amount resulting from 1 dollar of investment at time 0. Keep in mind that you have to reinvest any dividends. Assuming a constant return of R per year, then

$$(1 + R)^T = (1 + R_1)(1 + R_2) \ldots (1 + R_T)$$

and hence the value of R is the geometric return.

Let us use an example to clarify the difference between arithmetic (which is the average defined before) and geometric average returns. Suppose that $R_1 = -50\%$ and $R_2 = 100\%$.

(a) The arithmetic average is $(-0.5 + 1) / 2 = 25\%$.

(b) The geometric return is $[(1 - 0.5)(1 + 1)]^{1/2} - 1 = 0$. If you make a dollar investment in a stock at time 0 in the stock that produces R_1 and R_2, then at time 1 you have only \$0.5, and at time 2 you have $0.5 \times (1 + 100\%) = 1$. Since your investment does not grow or shrink, the average compounded return is 0.

In this example, the geometric return is less than the arithmetic return. In fact, unless all the returns are equal, the geometric return is always less than the arithmetic return. (This follows from the famous inequality of arithmetic and geometric means.)

Example **1.1.4**

Compute the realized geometric return for S&P 500 in Example 1.1.2.

Solution

For an investment of 1 at the beginning of the year 1991, the accumulated value in 5 years is

$$(1 + 0.3055) \times (1 + 0.0767) \times \ldots \times (1 + 0.3743) = 2.152577.$$

The geometric return is $2.152577^{1/5} - 1 = 16.5713\%$, which is slightly less than the average return.

[END]

Historical Trade-off between Risk and Return

Historically, we have observed that securities with large average return tend to (but not always) have large volatility. Volatility is a measure of risk and riskier investments need to reward their investors for taking greater risk. Since it is unlikely for a government to default on its debt, government-issued debt securities can be treated as riskless assets. The return on a government bond can thus be treated as the risk-free return. Among all government bonds, T-bills (3-month US Treasury bills, not to be confused with T-bonds, which have maturities of 1 year or longer) have the smallest maturity and are thus subject to the smallest default risk. Thus it is common to use T-bills to calculate the risk-free rate. Corporate bonds have a greater volatility and a higher historical average return, though they are not as volatile as stocks in general.

Risky assets normally have a return that is greater than the risk-free rate. Let us define the following:

Risk premium of an asset = Excess return = Return on that asset – Risk-free return

Sharpe ratio of an asset = Risk premium of that asset / volatility of return on that asset

Sharpe ratio is a measure of return to the level of risk taken (as measured by volatility).

The textbook reports the mean returns and volatilities of the following six portfolios over the period 1926 – 2014:

(1) S&P 500

(2) Small stocks (a portfolio of US stocks traded on the New York Stock Exchange with market capitalizations in the bottom 20%)

(3) World portfolio (a portfolio of international stocks from all of the world's major stock markets)

(4) Corporate bonds (a portfolio of long-term, AAA-rated US corporate bonds)

(5) Treasury bills

(6) Mid-cap stocks (a portfolio of US stocks with a market capitalization between 2 to 10 million)

It can be observed that average returns and volatilities follow the ordering below:

small stocks > mid-cap stocks > S&P 500 > world portfolio > corporate bonds > treasury bills.

While volatility is a measure of risk, you would see later that it is not a perfect measure. In the last section of this lesson, you will see that there are two kinds of risks and investors would only be compensated for taking one kind of risk. Volatility does not differentiate the two kinds of risks. This means that a stock with a high volatility may not give a higher expected return because the high volatility may be contributed from the non-compensated risk. Also, volatility (and also variance) do not differentiate between downside and upside risk. In Module 2 Lesson 3 we will introduce measures of downside risk.

1. 1. 2. Calculating Covariance and Correlation

In the previous section we have discussed how we can use historical data to estimate the population mean return and standard deviation of an individual asset. We now expand our discussion to consider portfolios of assets. In this context, correlations play an important role.

Recall that for two random variables, X and Y, the covariance is defined as

$$\mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mathrm{E}X)(Y - \mathrm{E}Y)] = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y),$$

while the correlation of X and Y is defined as $\mathrm{Corr}(X, Y) = \dfrac{\mathrm{Cov}(X,Y)}{\mathrm{SD}(X)\mathrm{SD}(Y)}$.

Also, for a and b being numerical constants,

$$\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y) \text{ and } \mathrm{Var}(aX + bY) = a^2\,\mathrm{Var}(X) + 2ab\,\mathrm{Cov}(X, Y) + b^2\,\mathrm{Var}(Y).$$

Example **1.1.5**

The following table gives the joint probability mass function of two random returns, R_A and R_B, for two companies ACY and BOC.

State	ACY Returns R_A	BOC Returns R_B	Probability
Depression	–20%	5%	0.25
Recession	10%	20%	0.25
Normal	30%	–12%	0.25
Boom	50%	9%	0.25

Calculate the correlation of R_A and R_B.

Solution

First we calculate the mean returns:

$$\mathrm{E}(R_A) = 0.25 \times (-20) + 0.25 \times 10 + 0.25 \times 30 + 0.25 \times 50 = 17.5\%,$$
$$\mathrm{E}(R_B) = 0.25 \times 5 + 0.25 \times 20 + 0.25 \times (-12) + 0.25 \times 8 = 5.5\%.$$

Then we calculate the second moments of the returns:

$$\mathrm{E}(R_A^2) = 0.25 \times (-20)^2 + 0.25 \times 10^2 + 0.25 \times 30^2 + 0.25 \times 50^2 = 975\%^2,$$
$$\mathrm{E}(R_B^2) = 0.25 \times 5^2 + 0.25 \times 20^2 + 0.25 \times (-12)^2 + 0.25 \times 8^2 = 162.5\%^2.$$

Thus the variances of the returns are

$$\mathrm{Var}(R_A) = 975 - 17.5^2 = 668.75\%^2,\ \mathrm{Var}(R_B) = 162.5 - 5.5^2 = 132.25\%^2$$

giving $\mathrm{SD}(R_A) = 25.86\%$ and $\mathrm{SD}(R_B) = 11.5\%$.

Finally, we calculate $E(R_A R_B) = \frac{1}{4}[(-20) \times 5 + 10 \times 20 + 30 \times (-12) + 50 \times 9] = 47.5\%^2$

to get $\text{Cov}(R_A, R_B) = 47.5 - 17.5 \times 5.5 = -48.75\%^2$ and

$$\text{Corr}(R_A, R_B) = \frac{-48.75}{25.86 \times 11.5} = -0.1639.$$

[END]

Estimating Correlation from Realized Returns

Suppose that we have the return series for two stocks: R_{Ai} and R_{Bi}. We can then estimate the covariance and correlation of the returns on the two stocks as follows:

$$\text{Covariance estimate } = \frac{1}{T-1}\sum_{i=1}^{T}(R_{Ai} - \overline{R}_A)(R_{Bi} - \overline{R}_B) = \frac{1}{T-1}\left(\sum_{i=1}^{T} R_{Ai}R_{Bi} - T\overline{R}_A\overline{R}_B\right),$$

Correlation estimate = Covariance estimate / ($\hat{\sigma}_A \hat{\sigma}_B$).

Example 1.1.6

Use the data below to estimate the correlation between the returns on two stocks A and B.

Year	Return on A	Return on B
2008	−10%	11%
2009	−6%	−3%
2010	10%	25%

Solution

$$\overline{R}_A = \frac{-0.1 - 0.06 + 0.1}{3} = -0.02, \ \overline{R}_B = \frac{0.11 - 0.03 + 0.25}{3} = 0.11$$

$$\hat{\sigma}_A = \sqrt{\frac{(-0.08)^2 + (-0.04)^2 + 0.12^2}{2}} = \sqrt{0.0112}, \ \hat{\sigma}_B = \sqrt{\frac{0^2 + (-0.14)^2 + 0.14^2}{2}} = 0.14$$

$$\text{Covariance estimate} = \frac{(-0.08) \times 0 + (-0.04) \times (-0.14) + 0.12 \times 0.14}{2} = 0.0112$$

The estimated correlation is $\frac{0.0112}{\sqrt{0.0112} \times 0.14} = 0.755929.$

[END]

1. 1. 3 Portfolio Return and Volatility

Two Risky Assets

Suppose the current share price of ACY is 50, while the current share price of BOC is 40. You own a portfolio with 6 shares of ACY and 5 shares of BOC.

The current portfolio value is $50 \times 6 + 40 \times 5 = 500$.

The weight on ACY is $x_A = \dfrac{50 \times 6}{500} = 60\%$, while the weight on BOC is $x_B = \dfrac{40 \times 5}{500} = 40\%$.

Let R_P be the return on your portfolio. How can R_P be computed?

After one period, the portfolio becomes $50 \times 6(1 + R_A) + 40 \times 5(1 + R_B)$.

The dollar return is

$$50 \times 6(1 + R_A) + 40 \times 5(1 + R_B) - (50 \times 6 + 40 \times 5) = 50 \times 6R_A + 40 \times 5R_B .$$

The portfolio return is

$$\frac{50 \times 6R_A + 40 \times 5R_B}{50 \times 6 + 40 \times 5} = x_A R_A + x_B R_B .$$

In general, the return on a portfolio with portfolio weights x_A and x_B is

$$R_P = x_A R_A + x_B R_B \quad \text{(where } x_A + x_B = 1.)$$

Since R_A and R_B are both random variables, R_P is also a random variable.

Mean and Variance of Portfolio Returns

For $R_P = x_A R_A + x_B R_B$, where x_A and x_B are portfolio weights (with $x_A + x_B = 1$),

$$\mathrm{E}(R_P) = x_A \mathrm{E}(R_A) + x_B \mathrm{E}(R_B),$$

$$\mathrm{Var}(R_P) = x_A^2 \mathrm{Var}(R_A) + 2x_A x_B \mathrm{Cov}(R_A, R_B) + x_B^2 \mathrm{Var}(R_B).$$

A useful way to remember the formula for the calculation of variance is to first write down the covariance matrix of R_A and R_B,

$$\begin{bmatrix} \mathrm{Var}(R_A) & \mathrm{Cov}(R_A, R_B) \\ \mathrm{Cov}(R_B, R_A) & \mathrm{Var}(R_B) \end{bmatrix},$$

and then perform the following matrix multiplication:

$$\mathrm{Var}(R_P) = [x_A \quad x_B] \begin{bmatrix} \mathrm{Var}(R_A) & \mathrm{Cov}(R_A, R_B) \\ \mathrm{Cov}(R_B, R_A) & \mathrm{Var}(R_B) \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix}.$$

Example 1.1.7

Assume that $x_A = 60\%$. Calculate the mean and standard deviation of R_P in the previous example.

Solution

$E(R_P) = 0.6 \times 17.5\% + 0.4 \times 5.5\% = 12.7\%$.

$\text{Var}(R_P) = 0.6^2 \times 668.75\%^2 + 2 \times 0.6 \times 0.4 \times (-48.75\%^2) + 0.4^2 \times 132.25\%^2 = 238.51(\%^2)$

so that $SD(R_P) = 15.444\%$.

The shortcut calculation for the variance is

$$\text{Var}(R_P) = [0.6 \quad 0.4]\begin{bmatrix} 0.066875 & -0.004875 \\ -0.004875 & 0.013225 \end{bmatrix}\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = [0.6 \quad 0.4]\begin{bmatrix} 0.038175 \\ 0.002365 \end{bmatrix} = 0.023851.$$

[END]

Notice that the *weighted* average of the standard deviations of R_A and R_B is

$$0.6 \times 25.86\% + 0.4 \times 11.5\% = 20.12\%,$$

which is greater than 15.444%. This property leads to the so-called diversification effect.

Diversification Effect: The standard deviation of the portfolio is less than the weighted average of the standard deviations of the individual securities.

The above holds as long as both of the following conditions hold:
– The returns on the assets are not perfectly correlated.
– x_A and x_B are both positive.

If $\text{Corr}(R_A, R_B) < 1$, then $\text{Cov}(R_A, R_B) = \text{Corr}(R_A, R_B)\, SD(R_A)SD(R_B) < SD(R_A)SD(R_B)$, so that

$$\begin{aligned} \text{Var}(R_P) &= x_A^2 SD^2(R_A) + 2x_A x_B \text{Cov}(R_A, R_B) + x_B^2 SD^2(R_B) \\ &< x_A^2 SD^2(R_A) + 2x_A x_B SD(R_A)SD(R_B) + x_B^2 SD^2(R_B) \\ &= [x_A SD(R_A) + x_B SD(R_B)]^2 \end{aligned}$$

Taking square root on both sides of the equation (and keeping in mind that $x_A > 0$ and $x_B > 0$), we have

$$SD(R_P) < x_A SD(R_A) + x_B SD(R_B).$$

When the weight on an asset is negative, we say that we are "short selling" the asset. The act of short selling will be explained in Module 3. Suppose that one borrows shares of the second asset to finance the purchase of the first asset. Say, one short sells 3 shares of BOC to purchase 4 shares of ACY. The value of the portfolio would be $50 \times 4 - 40 \times 3 = 80$.

Then $x_A = \dfrac{50\times 4}{80} = 250\%$ (positive) and $x_B = \dfrac{-40\times 3}{80} = -150\%$ (negative). You can check that the portfolio standard deviation is 69.59%, while $x_A\text{SD}(R_A) + x_B\text{SD}(R_B) = 47.40\%$. In this case we don't have diversification!

The General Case: *n* Risky Assets

Consider a portfolio of n (> 2) risky assets with random returns $R_1, R_2, \ldots, R_n$. If portfolio weights are $x_1, x_2, \ldots, x_n$ (with $x_1 + x_2 + \ldots + x_n = 1$), then the portfolio return can be expressed as

$$R_P = x_1R_1 + x_2R_2 + \ldots + x_nR_n.$$

The mean and variance of R can be computed from

$$\text{E}(R_P) = x_1\,\text{E}(R_1) + x_2\,\text{E}(R_2) + \ldots + x_n\,\text{E}(R_n),$$

and

$$\text{Var}(R_P) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \text{Cov}(R_i, R_j) = \sum_{i=1}^{n} x_i^2 \text{Var}(R_i) + 2\sum_{i<j} x_i x_j \text{Cov}(R_i, R_j),$$

respectively.

A good way to remember the formula for the variance is to use the following table. The value of $\text{Var}(R_P)$ is the sum of all values in the following array:

Stock	1	2	3	…	n
1	$x_1^2\text{Var}(R_1)$	$x_1x_2\text{Cov}(R_1, R_2)$	$x_1x_3\text{Cov}(R_1, R_3)$		$x_1x_n\text{Cov}(R_1, R_n)$
2	$x_2x_1\text{Cov}(R_2, R_1)$	$x_2^2\text{Var}(R_2)$	$x_2x_1\text{Cov}(R_2, R_3)$		$x_2x_n\text{Cov}(R_2, R_n)$
3	$x_3x_1\text{Cov}(R_3, R_1)$	$x_3x_2\text{Cov}(R_3, R_2)$	$x_3^2\text{Var}(R_3)$		$x_3x_n\text{Cov}(R_3, R_n)$
⋮					⋮
n	$x_nx_1\text{Cov}(R_n, R_1)$	$x_nx_2\text{Cov}(R_n, R_2)$	$x_nx_3\text{Cov}(R_n, R_3)$	…	$x_n^2\text{Var}(R_n)$

A more elegant way to write the above formula is to first form the covariance matrix of the returns,

$$\begin{bmatrix} \text{Var}(R_1) & \text{Cov}(R_1, R_2) & \cdots & \text{Cov}(R_1, R_n) \\ \text{Cov}(R_2, R_1) & \text{Var}(R_2) & \cdots & \text{Cov}(R_2, R_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(R_n, R_1) & \text{Cov}(R_n, R_2) & \cdots & \text{Var}(R_n) \end{bmatrix}$$

and then perform the following matrix multiplication:

$$\text{Var}(R_P) = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} \text{Var}(R_1) & \text{Cov}(R_1, R_2) & \cdots & \text{Cov}(R_1, R_n) \\ \text{Cov}(R_2, R_1) & \text{Var}(R_2) & \cdots & \text{Cov}(R_2, R_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(R_n, R_1) & \text{Cov}(R_n, R_2) & \cdots & \text{Var}(R_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If all weights are non-negative and the risky assets do not have a perfect positive correlation with one another, diversification effect exists. To prove this, we first express the portfolio variance as

$$\text{Var}(R_P) = \text{Cov}(R_P, R_P) = \sum_{i=1}^{n} x_i \text{Cov}(R_i,\ R_P).$$

Dividing both sides of this equation by the standard deviation of the portfolio, we get:

$$\text{SD}(R_P) = \sum_{i=1}^{n} x_i \frac{\text{Cov}(R_i,\ R_P)}{\text{SD}(R_P)} = \sum_{i=1}^{n} x_i \text{SD}(R_i)\text{Corr}(R_i,\ R_P) < \sum_{i=1}^{n} x_i \text{SD}(R_i).$$

Example **1.1.8**

Consider the following information:

State of Economy	Probability	Rate of return on Stock A	Rate of return on Stock B	Rate of return on Stock C
Boom	0.6	0.07	0.15	0.33
Bust	0.4	0.13	0.00	–0.06

(a) Calculate the mean return and variance for each stock.

(b) Find the three covariances for the returns.

(c) Find the volatility of a portfolio invested 20% each in A and B, and 60% in C.

Solution

(a) In what follows, $\%^2$ means 0.01^2.

Stock	Mean	Variance
A	$0.6 \times 0.07 + 0.4 \times 0.13 = 0.094$	$0.6 \times 0.07^2 + 0.4 \times 0.13^2 - 0.094^2 = 8.64\%^2$
B	$0.6 \times 0.15 = 0.09$	$0.6 \times 0.15^2 - 0.09^2 = 54\%^2$
C	$0.6 \times 0.33 - 0.4 \times 0.06 = 0.174$	$0.6 \times 0.33^2 + 0.4 \times 0.06^2 - 0.174^2 = 365.04\%^2$

(b) Recall that $\text{Cov}(X, Y) = \text{E}(XY) - \text{E}(X)\,\text{E}(Y)$.
$\text{Cov}(R_A, R_B) = 0.6 \times 0.07 \times 0.15 - 0.094 \times 0.09 = -21.6\%^2$
$\text{Cov}(R_A, R_C) = (0.6 \times 0.07 \times 0.33 - 0.4 \times 0.13 \times 0.06) - 0.094 \times 0.174 = -56.16\%^2$
$\text{Cov}(R_B, R_C) = 0.6 \times 0.15 \times 0.33 - 0.09 \times 0.174 = 140.4\%^2$

(c) The variance of the portfolio return is

$$0.2^2 \times 8.64 + 0.2^2 \times 54 + 0.6^2 \times 365.04 + 2(0.2 \times 0.2) \times (-21.6) + 2(0.2 \times 0.6) \times (-51.16)$$
$$+ 2(0.2 \times 0.6) \times (140.4)$$
$$= 152.4096\ \%^2$$

The portfolio volatility is $152.4096^{1/2}\% = 12.34543\%$.

You may also use the matrix formula

$$[0.2 \quad 0.2 \quad 0.6]\begin{bmatrix} 8.64 & -21.6 & -56.16 \\ -21.6 & 54 & 140.4 \\ -56.16 & 140.4 & 365.04 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} = [-36.288 \quad 90.72 \quad 235.872]\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$$

$$= 152.4096\%^2$$

[END]

1. 1. 4 More on Diversification of Risk

Out of the $n \times n$ elements in the covariance matrix of the returns, n of them are variance terms, and the remaining $n^2 - n$ are covariance terms. How do they affect Var(R)?

A Simple Model for Portfolio Risk

To study the effect of diversification in the case of n risky assets, we make the following assumptions:

(a) the returns on the n assets have the same variance: $\text{Var}(R_1) = \text{Var}(R_2) = \ldots = \text{Var}(R_n) = \text{var}$;

(b) the covariances between all possible pairs of assets are identical: $\text{Cov}(R_i, R_j) = \text{cov}$ for all $i \neq j$ (note that cov = corr × sd × sd = corr × var ≤ var);

(c) the portfolio is equally weighted: $x_1 = x_2 = \ldots = x_n = \dfrac{1}{n}$.

The assumptions lead to the following consequences:

$$\begin{aligned} \text{Var}(R_P) &= n\left(\frac{1}{n}\right)^2 \text{var} + (n^2 - n)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\text{cov} \\ &= \frac{1}{n}\text{var} + \left(1 - \frac{1}{n}\right)\text{cov} \\ &= \frac{1}{n}(\text{var} - \text{cov}) + \text{cov} \end{aligned}$$

The second equality says that Var(R) is a weighted average of var and cov. The weights are $\dfrac{1}{n}$ and $\left(1 - \dfrac{1}{n}\right)$, respectively. The last equality shows that when $n \to \infty$, $\text{Var}(R_P) \to \text{cov}$.

The essence of the equation above can be summarized as follows:

(1) When n increases, Var(R_P) decreases, and hence diversification effect increases with n.

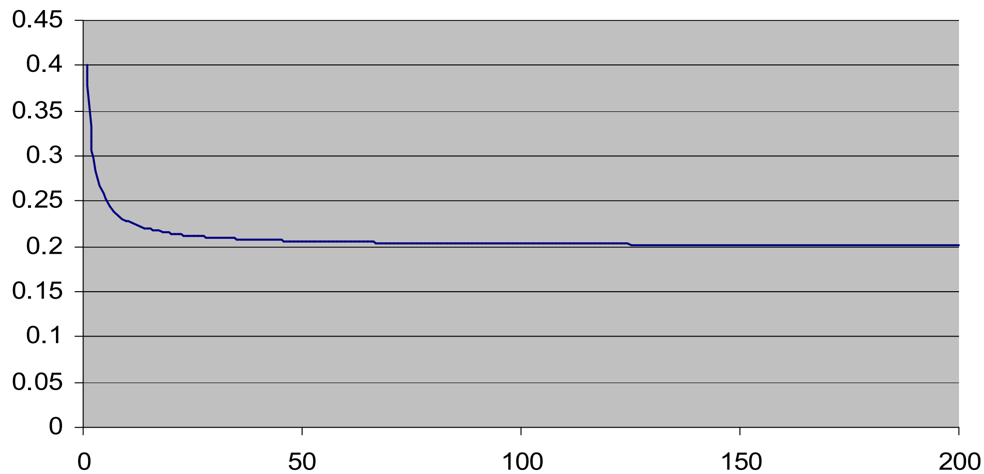
(2) Suppose that cov > 0. Even if n tends to infinity, Var(R_P) does not drop to 0. There is a limit to diversification effect: A diversified portfolio can eliminate some, but not all, of the risk of the individual securities.

(3) Only in the extreme case when all stocks are uncorrelated (such that cov = 0) or independent would Var(R_P) tend to 0 as n tends to infinity.

(4) If cov = var, that is, when the stocks are all perfectly correlated, then Var(R_P) = cov for any n. That is, there is no diversification of risk.

The following figure shows how volatility varies with n when var = 0.16 and corr = 0.25 (such that cov = 0.28 × 0.16 = 0.04). The volatility is computed from

$$\text{SD}(R) = \sqrt{\frac{1}{n}\times 0.16 + \left(1-\frac{1}{n}\right)\times 0.04}$$



It is evident that the volatility decreases very rapidly towards 20% as n increases.

Firm-Specific Versus Systematic Risk

Correlation is one of the keys in understanding stock price risk and the risks in insurance. The textbook defines risks that are perfectly correlated as common risk, and risks that share no correlation independent risks.

When risks are independent, some would be unlucky and others are lucky, and the famous central limit theorem says that the average of the risk would be quite predictable because the variance of the average is inversely proportional to the number of risks. For a portfolio of insurance policies, independence is usually not too bad an assumption, unless you are talking about some very rare events such as large scale earthquakes that can potentially affect millions of lives.

For a stock portfolio, stock prices and dividends fluctuate owing to two types of information in the market:

(1) Firm-specific news that only affects one particular firm (e.g. the firm announces that it is going to launch a unique product in the market, or the death of its CEO)

(2) Market-wide news that affects the economy as a whole and therefore affects all stocks (e.g. the discovery of a large energy source would harm the profitability of oil firms and electricity firms, but other firms may benefit)

The risk of (1) is called **firm-specific**, **idiosyncratic** or **unique** risk. It is also called **diversifiable** risk because it can be diversified away by holding a large stock portfolio. The risk of (2) is called **systematic**, **market** or **undiversifiable** risk.

Going back to our simple model for portfolio risk at the beginning of this section, (var − cov) represents diversifiable risk because it can be diversified away by increasing the number of risky assets. On the other hand, cov represents undiversifiable risk because it still remains even if n tends to infinity.

Risk Premium for Diversifiable and Undiversifiable Risk

As diversifiable risk can be eliminated by diversification, investors are not compensated for taking diversifiable risk. The risk premium for diversifiable risk is zero.

Because undiversifiable risk cannot be eliminated by diversification, investors are compensated for taking undiversifiable risk. The risk premium for a risky investment depends on the amount of its undiversifiable risk.

Exercise 1.1

1. Consider a 10-year corporate bond with a par value of 1,000. The bond, which pays no coupons, was issued one year ago, and the price at issue was \$352.2. The current effective annual market interest rate is 7%.

   Compute the realized return on the bond over the last year.

2. Consider a 10-year corporate bond with a par value of 1,000. The bond, which pays 40 worth of coupons at the end of each year, was issued one year ago, and the price at issue was \$731.6. The current effective annual market interest rate is 10%.

   Compute the realized return on the bond over the last year.

3. Which of the following has the greatest volatility over the past eighty years?

   (A) Mid-cap stocks

   (B) S&P 500

   (C) Small stocks

   (D) Corporate bonds

   (E) Treasury bills

4. You invest \$4000 in a mutual fund for 1 year and earn a return of 100%. You then reinvest the proceeds again and earn a return of –50% over the next year.

   (a) Calculate the arithmetic average return over the two-year period.

   (b) Calculate the geometric average return over the two-year period.

   (c) Which of the answer in (a) and (b) would describe your average annual return better?

5. Which of the following statements is/are true?

   I. The volatility of a stock is the square root of the variance of the return on that stock.

   II. Volatility increases when the size of a portfolio increases.

   III. Volatility differentiates between upside and downside risk.

   (A) I only

   (B) III only

   (C) I and II only

   (D) II and III only

   (E) I, II and III

For Questions 6 to 7, consider the following data for Telford:

Date	Price	Dividend per share
End of 2013	36.1	
End of 1st quarter of 2014	25.9	0
End of 2nd quarter of 2014	22.4	0.1
End of 3rd quarter of 2014	30.6	0
End of 2014	32.4	0.1

6. Compute the realized return for each of the quarters of 2014.

7. Compute the realized annual return for the year 2014.

For Questions 8 to 12, use the following realized annual returns on two stocks:

Year End	ACY Returns R_A	BOC Returns R_B
2013	–20%	25%
2014	10%	–2.5%
2015	–5%	5%
2016	50%	–10%

8. Calculate the historical average annual returns on the two stocks. Repeat for geometric average annual return.

9. Estimate the stocks' volatilities.

10. Estimate the correlation between the two stocks.

11. Construct a 95% confidence interval for the annual return on ACY.

12. The annual return on Treasury bills is 1.2%. Find the Sharpe ratio of BOC.

For Questions 13 to 15, use the following distribution for two stocks:

State	ACY Returns R_A	BOC Returns R_B	Probability
Depression	–20%	5%	0.2
Recession	10%	20%	0.3
Normal	30%	–12%	0.4
Boom	50%	9%	0.1

13. Calculate the expected annual returns on the two stocks.

14. Estimate the stocks' volatilities.

15. Calculate the correlation between the two stocks.

16. Which of the following statements is/are true?

    I. The volatility of an investment portfolio that is balanced evenly between two stocks is not greater than the average volatility of the two individual stocks.

    II. Full diversification of an investment portfolio eliminates market risk.

    III. The total risk of an individual stock held in isolation determines its contribution to the risk of a well-diversified portfolio.

    (A) I only

    (B) III only

    (C) I and II only

    (D) II and III only

    (E) I, II and III

For Questions 17 and 18, consider two banks A and B.

Both banks have 100 loans outstanding. The principal on each of the loan is 10000, due today. Each loan has a default probability of 0.1. If defaults happen, the recovery value is only 4000.

For Bank A, all loans concentrate in one industrial sector and the loans either all default or all not default. For Bank B, the 100 loans are independent.

17. Calculate the standard deviation of the overall payoff to Bank A.

18. Calculate the standard deviation of the overall payoff to Bank B.

19. Which of the following risk is diversifiable?

    (A) The risk that the government raises corporate tax rate

    (B) The risk that a key official in the government of the United States is kidnapped

    (C) The risk that a key employee in a start-up computer software firm would be hired away by a another similar start-up company

    (D) The risk that an asteroid would hit the earth

    (E) The risk that the oil reserve would be depleted in 20 years

20. Albert wants to invest in two different stocks, A and B.

    (i) Stock A has an expected return of 10% and a volatility of σ.

    (ii) Stock B has an expected return of 20% and a volatility of 1.5σ.

    After investing in both stocks, the expected return on Albert's portfolio is 14% and the volatility is σ.

    Find the correlation between the returns on A and B.

21. You are given the following information about the weekly returns on two stocks:

Stock 1: weekly returns follow a normal distribution with mean 0.12% and standard deviation of 0.20%

Stock 2: weekly returns follow a normal distribution with mean 0.15% and standard deviation of 0.18%

The covariance between the two weekly returns is 0.0001%.

(a) An investor holds a portfolio with 25% of the assets in stock 1 and 75% of the assets in stock 2. What are the mean and standard deviation of the weekly returns on the portfolio?

(b) Suppose that the weekly portfolio returns follow a normal distribution. Calculate the probability that the weekly return is greater than 0.3%.

(c) Suppose further that weekly portfolio returns are independent across different weeks. Calculate the probability that the monthly return is greater than 0.3%. Assume for simplicity that there are 4 weeks every month, and that the monthly return is the sum of four independently distributed weekly returns.

22. Which of the following statement is/are false?

I. Another name of undiversifiable risk is idiosyncratic risk

II. Investors would be rewarded for holding firm-specific risk

III. The risk premium of an asset is related to its unsystematic risk

(A) I only

(B) III only

(C) I and II only

(D) II and III only

(E) I, II and III

23. There are n stocks in the market. All stocks have the same volatility of 25%, and correlation of any stock to any other stock is 30%.

For each of the following value of n, calculate the volatility of a portfolio that equally weighted in the n stocks.

(a) $n = 10$

(b) $n = 50$

(c) $n = 100$

(d) n tends to infinity

24. Consider the two stocks A and B. You are given:

    (i) The expected return on A and B are 14% and 10%, respectively.

    (ii) The following variance covariance matrix:

	Stock A	Stock B
Stock A	0.30	0.12
Stock B	0.12	0.15

    You want to purchase (but not borrow any of) A and B to form a portfolio with a volatility of 40%. What is the expected return on your portfolio?

For Questions 25 to 27, consider three firms A, B and C. You purchase 200 shares of A at 10 per share, 10 shares of B at 175 per share, and finally 30 shares of C at 75 per share.

The volatility of A, B and C are 18%, 12% and 20%. The correlation between A and B is 35%, the correlation between A and C is –5%, while the correlation between B and C is 20%.

25. Find the weights on A, B and C in the portfolio.

26. Find the volatility of the portfolio.

27. If the realized capital gains on A, B and C are 10%, −10% and 20%, respectively, and the three firms pays no dividends, find the portfolio weights after 1 year. What do you notice?

Solutions to Exercise 1.1

1. The price of the zero-coupon bond is now

$$1000 \times 1.07^{-9} = 543.9337,$$

giving a capital gain of 543.9337 – 352.2 = 191.7337. Since the zero-coupon bond does not pay an intermediate cash flow during the previous year, the realized return is

$$\frac{191.7337}{352.2} \times 100\% = 54.44\%.$$

2. The price of the zero-coupon bond is now

$$1000 \times 1.1^{-9} + 40 \times \frac{1-1.1^{-9}}{0.1} = 654.4586,$$

giving a capital gain of 654.4586 – 731.6 = –77.1414. Since the zero-coupon bond pays 40 coupon at the end of year 1, the realized return is

$$\frac{-77.1414+40}{731.6} \times 100\% = -5.08\%.$$

3. (C)

4. (a) (100% – 50%) / 2 = 25%

(b) $\sqrt{(1+100\%)\times(1-50\%)} - 1 = 0\%$

(c) The answer in (b) is more appropriate. To see this, let us trace the amount of investment. The fund grows to 8000 (= 4000 × (1 + 100%)) at time 1, and then becomes 4000 (= 8000 × (1 – 50%)) at time 2. So, the investor actually earns nothing.

The arithmetic average is the appropriate way to calculate return that involves **reinvestment** over more than 1 period <u>only if you start with the same amount of money at the beginning of each reinvestment cycle</u>. In this question, the –50% is applied to a bigger amount of investment capital. If you withdraw 4000 at time 1 and only reinvest 4000, then at the end of year 2, you would have 4000 + 4000 × (1 – 50%) = 6000. The two-year average return would then be (6000 – 4000) / 4000 ÷ 2 = 25%, which is the same as the arithmetic average return. The geometric return automatically adjusts for the change in the amount of capital.

5. (A)

6. End of quarter 1: $\frac{25.9-36.1}{36.1} = -28.255\%$, End of quarter 2: $\frac{22.4-25.9+0.1}{25.9} = -13.127\%$

End of quarter 3: $\frac{30.6-22.4}{22.4} = 36.607\%$, End of quarter 4: $\frac{32.4-30.6+0.1}{30.6} = 6.209\%$

7. $(1 - 0.28255) \times (1 - 13.127) \times (1 + 0.366071) \times (1 + 0.062092) = 0.904296$
   $0.904296 - 1 = -9.57\%$

   Note that since we are computing realized annual return but not realized quarterly return, we do not use $0.904296^{0.25} - 1$.

8. Historical average annual returns:
   ACY: $0.25 \times (-0.2 + 0.1 - 0.05 + 0.5) = 8.75\%$
   BOC: $0.25 \times (0.25 - 0.025 + 0.05 - 0.1) = 4.375\%$

   Geometric average annual returns:
   ACY: $(0.8 \times 1.1 \times 0.95 \times 1.5)^{1/4} - 1 = 1.254^{1/4} - 1 = 5.82\%$
   BOC: $(1.25 \times 0.975 \times 1.05 \times 0.9)^{1/4} - 1 = 1.15171875^{1/4} - 1 = 3.59\%$

9. Variance:
   ACY: $\dfrac{0.2^2 + 0.1^2 + 0.05^2 + 0.5^2 - 4 \times 0.0875^2}{3} = 0.090625$
   BOC: $\dfrac{0.25^2 + 0.025^2 + 0.05^2 + 0.1^2 - 4 \times 0.04375^2}{3} = 0.02265625$

   Volatility
   ACY: $0.090625^{0.5} = 30.104\%$, BOC: $0.02265625^{0.5} = 15.052\%$

10. Covariance estimate
    $= \dfrac{-0.2 \times 0.25 - 0.1 \times 0.025 - 0.05^2 - 0.5 \times 0.1 - 4 \times 0.0875 \times 0.04375}{3} = -0.0401042$
    Correlation estimate $= -0.0401042 / 0.30104 / 0.15052 = -0.8851$

11. $(0.0875 - 1.96 \times \dfrac{0.30104}{\sqrt{4}}, 0.0875 + 1.96 \times \dfrac{0.30104}{\sqrt{4}}) = (-20.75\%, 38.25\%)$

12. Sharpe ratio = Excess return / Volatility = $(0.04375 - 0.012) / 0.15052 = 0.2109$

13. First we calculate the mean of the returns:

    $$E(R_A) = 0.2 \times 20\% + 0.3 \times 10\% + 0.4 \times 30\% + 0.1 \times 50\% = 16\%,$$
    $$E(R_B) = 0.2 \times 5\% + 0.3 \times 20\% + 0.4 \times (-12\%) + 0.1 \times 9\% = 3.1\%.$$

14. We calculate the second moment of the returns:

    $$E(R_A^2) = 0.2 \times (20\%)^2 + 0.3 \times (10\%)^2 + 0.4 \times (30\%)^2 + 0.1 \times (50\%)^2 = 0.072$$
    $$E(R_B^2) = 0.2 \times (5\%)^2 + 0.3 \times (20\%)^2 + 0.4 \times (12\%)^2 + 0.1 \times (9\%)^2 = 0.01907.$$

    Thus the variances of the returns are

    $$\text{Var}(R_A) = 0.072 - 0.16^2 = 0.0464, \text{Var}(R_B) = 0.01907 - 0.031^2 = 0.018109$$

    giving $SD(R_A) = 21.541\%$ and $SD(R_B) = 13.457\%$.

15. Finally, we calculate
$E(R_A R_B) = 0.2 \times (-0.2) \times 0.05 + 0.3 \times 0.1 \times 0.2 + 0.4 \times 0.3 \times (-0.12) + 0.1 \times 0.5 \times 0.09$
$= -0.0059$
to get $\text{Cov}(R_A, R_B) = -0.0059 - 0.16 \times 0.031 = -0.01086$ and
$$\text{Corr}(R_A, R_B) = \frac{-0.01086}{0.21541 \times 0.13457} = -37.47\%$$

16. (A)

I. Correct. For $X_A = X_B = 0.5$, the volatility of the portfolio would not be greater than the average of the volatility of the two stocks.

II. Full diversification of a portfolio can eliminate the risk that is specific to A and hence investor would not be compensated for taking such risk. Market risk affects all securities and it cannot be eliminated by diversification.

III. The individual stock's volatility can be split into two parts. The part contributed to market risk would contribute to the risk of a well-diversified portfolio.

17. There are only two scenarios. If default does not happen, then

$$\text{payoff} = 10000 \times 100 = 1000000.$$

If default happens, then payoff reduces to $4000 \times 100 = 400000$.

Mean $= 0.9 \times 1000000 + 0.1 \times 400000 = 940000$
SD $= (0.9 \times 60000^2 + 0.1 \times 540000^2)^{0.5} = 180000$

18. For one single loan, the mean payoff is 9400,
variance of payoff $= 0.9 \times 600^2 + 0.1 \times 5400^2 = 3240000$
Recall that for 100 independent random variables,
$\text{Var}(X_1 + \ldots + X_{100}) = \text{Var}(X_1) + \text{Var}(X_2) + \ldots + \text{Var}(X_{100})$.
Applying this result on the 100 independent payoffs,
Variance of 100 payoffs $= 100 \times 3240000 = 324000000$
Standard deviation of 100 payoffs $= 324000000^{0.5} = 18000$
Due to effect of diversification, the variance of the loan of B is much lower than that of A.

19. (C)
All other risks have worldwide consequence.

20. We have $E(R_P) = 0.14 = 0.1X_A + 0.2(1 - X_A)$, and hence $X_A = 0.6$.
$\text{Var}(R_P) = \sigma^2 = 0.6^2\sigma^2 + 2(0.6)(0.4)\rho\sigma(1.5\sigma) + 0.4^2(1.5\sigma^2) = 0.72\sigma^2 + 0.72\rho\,\sigma^2$
On solving, we get $\rho = 0.3889$.

21. (a) The mean return is $0.25 \times 0.12 + 0.75 \times 0.15 = 0.1425\%$.
The variance is
$0.25^2 \times 0.002^2 + 2 \times 0.25 \times 0.75 \times 0.000001 + 0.75^2 \times 0.0018^2 = 2.4475 \times 10^{-6}$.
So the standard deviation is 0.15644%.

(b) Let Z follow a standard normal distribution.

$$\Pr(R > 0.3\%) = \Pr(Z > \frac{0.3 - 0.1425}{0.15644}) = \Pr(Z > 1) = 0.1587.$$

(c) For X and Y being normally distributed and independent, $W = X + Y$ would be normally distributed, and the mean and variance of W would simply be the sum of means and variances of X and Y.

This means the monthly return follows a normal distribution with a mean of $0.1425 \times 4 = 0.57\%$, and a variance of $0.024475 \times 4 = 0.0979(\%^2)$.

So, the probability required is

$$\Pr(W > 0.3\%) = \Pr(Z > \frac{0.3 - 0.57}{\sqrt{0.0979}}) = \Pr(Z > -0.86) = 0.8051.$$

22. (E)

I. Idiosyncratic risk is the same as firm-specific or diversifiable risk

II. Investors would be rewarded for holding undiversifiable risk

III. The risk premium of an asset is related to its systematic risk

23. $\text{SD}(R_P) = \sqrt{\frac{1}{n}(\text{var} - \text{cov}) + \text{cov}}$ where var $= 0.25^2 = 0.0625$ and cov $= 0.3 \times 0.0625 = 0.01875$

Hence, $\text{SD}(R_P) = \sqrt{\frac{0.04375}{n} + 0.01875}$.

(a) 15.21%
(b) 14.01%
(c) 13.85%
(d) $0.01875^{0.5} = 13.69\%$

24. $\text{Var}(R_P) = x_A^2 \times 0.3 + 2x_A(1 - x_A) \times 0.12 + (1 - x_A)^2 \times 0.15 = 0.4^2$

This means $0.21x_A^2 - 0.06x_A - 0.01 = 0$, and hence $x_A = 0.403677$ or -0.11796. Since the question specifies that the portfolio has positive weights on both A and B (more explanation would be given in the next lesson), we have $x_A = 0.403677$.
The mean return is $0.403677 \times 0.14 + (1 - 0.403677) \times 0.1 = 11.61\%$.

25. The total value of the portfolio is $200 \times 10 + 10 \times 175 + 30 \times 75 = 6000$.
The weight on A is $200 \times 10 / 6000 = 33.33\%$.
The weight on B is $10 \times 175 / 6000 = 29.17\%$.
The weight on C is $30 \times 75 / 6000 = 37.5\%$.

26. The variance-covariance matrix is

$$\begin{bmatrix} 0.18^2 & 0.35\times0.18\times0.12 & -0.05\times0.18\times0.2 \\ 0.35\times0.18\times0.12 & 0.12^2 & 0.2\times0.12\times0.2 \\ -0.05\times0.18\times0.2 & 0.2\times0.12\times0.2 & 0.2^2 \end{bmatrix} = \begin{bmatrix} 0.0324 & 0.00756 & -0.0018 \\ 0.00756 & 0.0144 & 0.0048 \\ -0.0018 & 0.0048 & 0.04 \end{bmatrix}.$$

The variance is

$$[0.3333 \quad 0.2917 \quad 0.375]\begin{bmatrix} 0.0324 & 0.00756 & -0.0018 \\ 0.00756 & 0.0144 & 0.0048 \\ -0.0018 & 0.0048 & 0.04 \end{bmatrix}\begin{bmatrix} 0.3333 \\ 0.2917 \\ 0.375 \end{bmatrix}$$

$$= [0.01233 \quad 0.00852 \quad 0.0158]\begin{bmatrix} 0.3333 \\ 0.2917 \\ 0.375 \end{bmatrix} = 0.01252$$

The volatility is $0.01252^{0.5} = 11.19\%$.

27. The value of A becomes $2000 \times 1.1 = 2200$,
the value of B becomes $1750 \times 0.9 = 1575$,
and the value of C becomes $30 \times 75 \times 1.2 = 2700$.

The total value is 6475. The weights become

$x_A = 2200 / 6475 = 33.977\%$, $x_B = 1575 / 6475 = 24.324\%$, $x_C = 2700 / 6475 = 41.699\%$.

It can be observed that the weights would be different from those in Question 25. The weights on A and C increase, while the weight in C decreases.

Lesson 2 Portfolio Theory

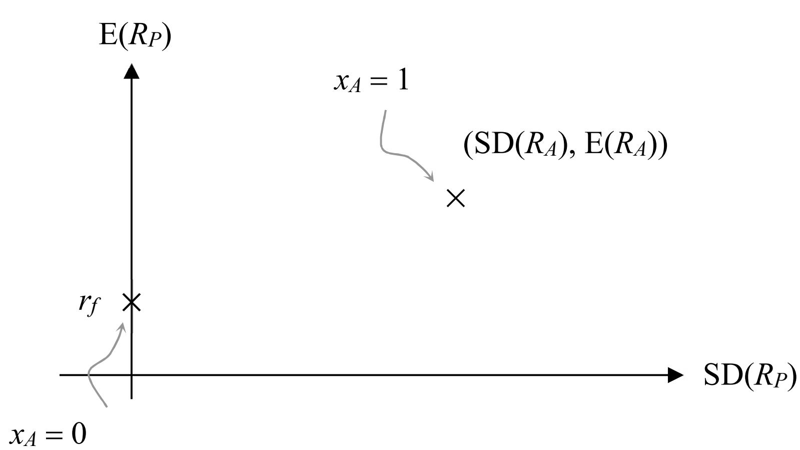
OBJECTIVES

1. To understand mean-standard deviation diagram
2. To calculate the optimal portfolio
3. To locate the capital market line
4. To understand market portfolio

In this lesson, we introduce the celebrated mean-variance portfolio analysis.

1. 2. 1 The Efficient Set for One Risky Asset and One Risk-free Asset

In the first section, we consider portfolios constructed from risk-free Treasury bill F and one risky asset A. The return on F is r_f, while the standard deviation of the return on F is 0. The mean return on A is $E(R_A)$ and the volatility is $SD(R_A)$. Suppose that the weight on A is x_A. If $x_A = 0$, then one only holds F. If $x_A = 1$, then one only holds A. These two situations correspond to the two points in the following **mean-standard deviation diagram**:



We can express $E(R_P)$ and $Var(R_P)$ as functions of x_A as follows:

Mean and Variance of Portfolio Return when One Asset is Risk-free

For $R_P = x_A R_A + x_F r_f$, where x_A and x_F are portfolio weights (with $x_A + x_F = 1$),

$$E(R_P) = x_A\, E(R_A) + x_F r_f,$$

$$Var(R_P) = x_A^2\, Var(R_A).$$

From the first equation, we have

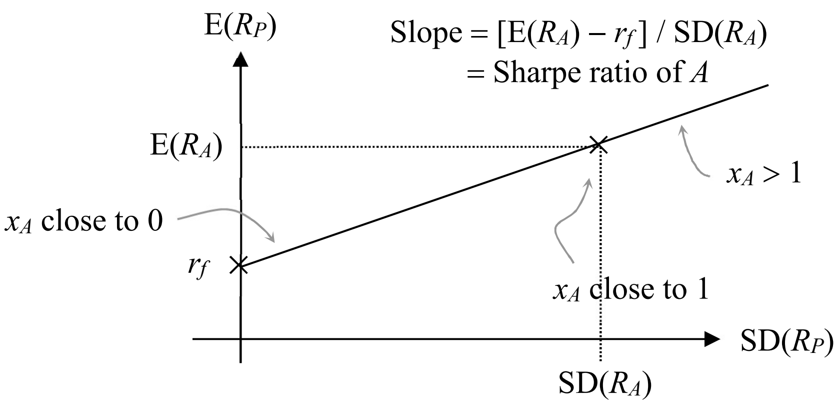
$$x_A = \frac{E(R_P) - r_f}{E(R_A) - r_f}.$$

Taking square root on both sides of the second equation,

$$SD(R_P) = x_A SD(R_A) = \frac{E(R_P) - r_f}{E(R_A) - r_f} SD(R_A)$$

and hence

$$E(R_P) = r_f + \frac{E(R_A) - r_f}{SD(R_A)} SD(R_P)$$



Opportunity set

The possible pairs of $E(R_P)$ and $SD(R_P)$ that can be formed by varying x_A (and hence x_F) is called the **opportunity set** or feasible set.

The opportunity set is a straight line that starts from the point that corresponds to $x_A = 0$ (100% Treasury bills) and passes through the point that corresponds to $x_A = 1$ (100% A). The minimum variance portfolio is the one with $x_A = 0$. Which point on the opportunity set an investor would pick depends on the **risk preference** of the investor. If he is more risk-averse, he would choose a point with a lower SD. When $x_A > 1$, we are in the situation when one is borrowing money at the risk-free rate to purchase A (i.e. buying on margin).

Example 1.2.1

There is a stock with an expected return of 8% and a volatility of 12%. The annual effective risk-free rate is 4%. You have 500 of cash, and you borrow 1000 at the risk-free rate to invest 1500 in the stock.

Calculate the expected rate of return and volatility of the portfolio. If the stock's realized return over the course of the year is –20%, what would the realized return on your portfolio? Repeat if the stock's realized return is +20%.

Solution

The weight on A is $x_A = 1500 / 500 = 3$, and the weight on risk-free asset is –2.
The expected return is $E(R_P) = 3 \times 0.08 - 2 \times 0.04 = 16\%$.
The variance of return is $Var(R_P) = 3^2 \times 0.12^2$. The volatility is $3 \times 0.12 = 36\%$.

If the realized return on the stock is R_A, then the realized return on the portfolio is

$$R_P = \frac{1500(1+R_A) - 1000 \times 1.04 - 500}{500} = 3R_A - 2 \times 0.04.$$

For $R_A = -0.2$, $R_P = -68\%$.
For $R_A = 0.2$, $R_P = 52\%$.

So we can see that buying the stock by borrowing money can greatly magnify profit and loss.

[END]

1. 2. 2 The Efficient Set for Many Risky Assets

What we have presented is a special case of the famous mean-variance portfolio analysis introduced by Harry Markowitz in 1952. He was awarded a Nobel Prize in Economics in 1990. Let us continue with his work by replacing the risk-free asset with a risky asset.

Two Risky Assets

Then we consider portfolios constructed from two risky assets A and B. If x_A and x_B change (in a way such that $x_A + x_B = 1$), then $E(R_P)$ and $SD(R_P)$ would also change. We will show that

> The opportunity set formed by two risky assets is hyperbolic unless the two assets are perfectly positively or negatively correlated.

While the textbook does not provided the algebraic proof of this statement, we provide it here for your reference. To derive the opportunity set, we express $\text{Var}(R_P)$ as a function of $\text{E}(R_P)$. Firstly, from

$$\text{E}(R_P) = x_A\, \text{E}(R_A) + x_B\, \text{E}(R_B) = x_A\, \text{E}(R_A) + (1 - x_A)\text{E}(R_B),$$

we get

$$x_A = \frac{\text{E}(R_P) - \text{E}(R_B)}{\text{E}(R_A) - \text{E}(R_B)}.$$

Then we substitute the formula above into

$$\text{Var}(R_P) = x_A^2\, \text{Var}(R_A) + 2x_A(1 - x_A)\text{Cov}(R_A, R_B) + (1 - x_A)^2\, \text{Var}(R_B) \qquad (1.2.1)$$

to obtain the relationship between $\text{Var}(R_P)$ and $\text{E}(R_P)$. The derivation can be summarized as follows:

Case 1: R_A and R_B are perfectly positively correlated

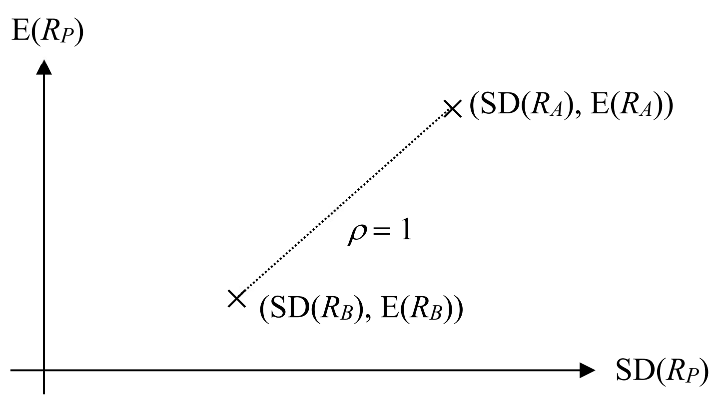
When $\text{Corr}(R_A, R_B) = 1$, we can simplify (1.2.1) as

$$\text{SD}(R_P) = x_A\text{SD}(R_A) + (1 - x_A)\text{SD}(R_B) = x_A[\text{SD}(R_A) - \text{SD}(R_B)] + \text{SD}(R_B)$$

and hence

$$\begin{aligned} \text{SD}(R_P) &= \frac{\text{E}(R_P) - \text{E}(R_B)}{\text{E}(R_A) - \text{E}(R_B)}[\text{SD}(R_A) - \text{SD}(R_B)] + \text{SD}(R_B) \\ &= \frac{\text{SD}(R_A) - \text{SD}(R_B)}{\text{E}(R_A) - \text{E}(R_B)}\text{E}(R_P) - \frac{\text{SD}(R_A)\text{E}(R_B) - \text{SD}(R_B)\text{E}(R_A)}{\text{E}(R_A) - \text{E}(R_B)} \end{aligned}$$

is linear in $\text{E}(R_P)$. This means that the opportunity set is a straight line joining A and B:



Case 2: R_A and R_B are perfectly negatively correlated (i.e. $\rho = -1$)

When $\text{Corr}(R_A, R_B) = -1$, we can simplify (1.2.1) (noting that $\text{SD}(R_P) \geq 0$) as

$$\mathrm{SD}(R_P) = |\, x_A\mathrm{SD}(R_A) - (1 - x_A)\mathrm{SD}(R_B) \,| = |\, x_A[\mathrm{SD}(R_A) + \mathrm{SD}(R_B)] - \mathrm{SD}(R_B) \,|$$

and hence

$$\begin{aligned}\mathrm{SD}(R_P) &= \left| \frac{\mathrm{E}(R_P) - \mathrm{E}(R_B)}{\mathrm{E}(R_A) - \mathrm{E}(R_B)}[\mathrm{SD}(R_A) + \mathrm{SD}(R_B)] - \mathrm{SD}(R_B) \right| \\ &= \left| \frac{\mathrm{SD}(R_A) + \mathrm{SD}(R_B)}{\mathrm{E}(R_A) - \mathrm{E}(R_B)}\mathrm{E}(R_P) - \frac{\mathrm{SD}(R_A)\mathrm{E}(R_B) + \mathrm{SD}(R_B)\mathrm{E}(R_A)}{\mathrm{E}(R_A) - \mathrm{E}(R_B)} \right|,\end{aligned}$$

which gives <u>two straight lines</u> with slopes having opposite signs.

Case 3: R_A and R_B are not perfectly positively or negatively correlated

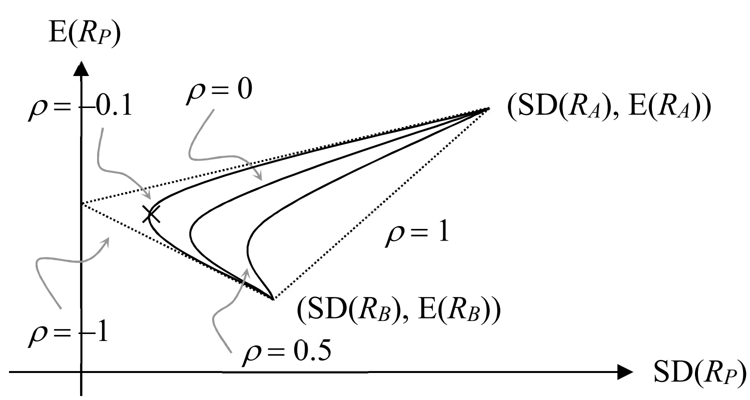
In this case we have to expand (1.2.1). Let $g = \dfrac{1}{\mathrm{E}(R_A) - \mathrm{E}(R_B)}$. After some very tedious calculation (you do not need to bother with the details!), we get

$$\begin{aligned}\mathrm{Var}(R_P) &= g^2\mathrm{Var}(R_A - R_B)\mathrm{E}^2(R_P) - 2g^2\mathrm{E}(R_A)\mathrm{E}(R_B)\mathrm{Cov}(R_A - R_B, \frac{R_A}{\mathrm{E}(R_A)} - \frac{R_B}{\mathrm{E}(R_B)})\mathrm{E}(R_P) \\ &\quad + g^2\mathrm{E}^2(R_A)\mathrm{E}^2(R_B)\mathrm{Var}(\frac{R_A}{\mathrm{E}(R_A)} - \frac{R_B}{\mathrm{E}(R_B)}).\end{aligned}$$

It can be shown (by completing square) that

$$\mathrm{Var}(R_P) = a[\mathrm{E}(R_P) - b]^2 + c$$

for $a > 0$ (because $\mathrm{Var}(R_A - R_B) > 0$) and some constants b and c. Since $\mathrm{Var}(R_P)$ is quadratic function of $\mathrm{E}(R_P)$, $\mathrm{SD}(R_P)$ is <u>**hyperbolic**</u> in $\mathrm{E}(R_P)$. As illustrated in the diagram below, the shape of the opportunity set depends crucially on the value of $\rho = \mathrm{Corr}(R_A, R_B)$.



Here we summarize the findings on the opportunity set for two risky assets.

(a) The effect of **diversification** (the curvature of the opportunity set)

The straight line that corresponds to $\rho = 1$ represents the points that would have been generated had the two assets been perfectly positively correlated. When $\rho \neq 1$, we see that the resulting opportunity set is always located on the left of the straight line and hence the SD of the portfolio is less than that of the case when $\rho = 1$ for the same value of $E(R_P)$. This illustrates the diversification effect. Since the curve bends towards the left more significantly as ρ decreases, the effect of diversification increases with decreasing ρ.

(b) **Minimum variance portfolio** (the point MV),

When $\rho \neq 1$, there is a point on the opportunity set so that $SD(R_P)$ is a minimum. This point corresponds to a portfolio that can be formed from mixing assets A and B such that the variance of the resulting portfolio is a minimum. For $\rho = -0.1$, the minimum variance portfolio is marked by a cross in the diagram. By (1.2.1),

$$\begin{aligned}\text{Var}(R_P) &= x_A^2 \text{Var}(R_A) + 2x_A(1-x_A)\text{Cov}(R_A, R_B) + (1-x_A)^2 \text{Var}(R_B) \\ &= [\text{Var}(R_A) + \text{Var}(R_B) - 2\text{Cov}(R_A, R_B)]x_A^2 - 2[\text{Var}(R_B) - \text{Cov}(R_A, R_B)]x_A \\ &\quad + \text{constant},\end{aligned}$$

and the value of x_A that minimizes $\text{Var}(R_P)$ is

$$x_A = \frac{\text{Var}(R_B) - \text{Cov}(R_A, R_B)}{\text{Var}(R_A) + \text{Var}(R_B) - 2\text{Cov}(R_A, R_B)} = \frac{\text{Var}(R_B) - \text{Cov}(R_A, R_B)}{\text{Var}(R_A - R_B)}.$$

(c) The **efficient frontier** (the part of curve above MV),

Suppose that you want to create a portfolio by mixing assets A and B such that the standard deviation of the resulting portfolio is greater than the minimum SD that can be achieved. From the figure, we find two possible portfolios (1) and (2) with this SD. But portfolio (1) dominates because it has a greater expected return. Using the same argument, the opportunity set below the point MV would never be selected. The opportunity set above the point MV is called the efficient frontier. The portfolios below MV are inefficient.

In general, a portfolio is inefficient if there is a portfolio lying to the northwest of it.