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CHAPTER FIVE

SURVIVAL MODELS (CONTINUOUS PARAMETRIC CONTEXT)

A *survival model* is simply a probability distribution for a particular type of random variable. Thus the general theory of probability, as reviewed in Chapter 2, is fully applicable here. However the particular history of the survival model random variable is such that specific terminology and notation has developed, particularly in an actuarial context. In this chapter (and the next) the reader will see this specialized terminology and notation, and recognize that it is *only* the terminology and notation that is new; the underlying probability theory is the same as that applying to any other continuous or discrete random variable and its distribution.

In actuarial science, the survival distribution is frequently summarized in tabular form, which is called a *life table*.[1] Because the life table form is so prevalent in actuarial work, we will devote a full chapter to it in this textbook (see Chapter 6).

5.1 THE AGE-AT-FAILURE RANDOM VARIABLE

We begin our study of survival distributions by defining the generic concept of *failure*. In any situation involving a survival model, there will be a defined entity and an associated concept of *survival*, and hence of failure, of that entity.[2] Here are some examples of entities and their associated random variables.

(1) The operating lifetime of a light bulb. The bulb is said to survive as long as it keeps burning, and fails at the instant it burns out.

(2) The duration of labor/management harmony. The state of harmony continues to survive as long as regular work schedules are met, and fails at the time a strike is called. (Conversely, we could model the duration of a strike, where the strike survives until it is settled and workers return to the job. The settlement event constitutes the failure of the strike status.)

(3) The lifetime of a new-born person. The person survives until death occurs, which constitutes the failure of the human entity. This will be the most common example considered in this text.

Let T_0 denote the continuous random variable for the age of the entity at the instant it fails.

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[1] Alternatively, the tabular model is also called a *mortality table*.

[2] Another term for failure is *decrement*. If an entity has a particular status, such as survival, then failure to retain that status is often described as being *decremented* from that status. This terminology is particularly useful in the context of multiple decrements, which we encounter in Chapters 13 and 14.

We assume that the entity exists at age 0, so the domain of the random variable T_0 is $T_0 > 0$. We refer to T_0 as the *age-at-failure* random variable. We will consider the terms "failure" and "death" to be synonymous, so we will also refer to T_0 as the *age-at-death* random variable.[3]

It is easy to see that the numerical value of the age at failure is the same as the *length of time* that survival lasts until failure occurs, since the variable begins at age 0, so we can also refer to T_0 as the *time-to-failure* random variable. (If failure occurs at exact *age t*, then *t* is also the time until failure occurs.)

Later (see Section 5.3) we will consider the case where the entity of interest is known to have survived to some age $x > 0$. Then the time-to-failure random variable, to be denoted by T_x, will not be identical to the age-at-failure random variable T_0, although they will be related to each other by $T_0 = x + T_x$. When dealing with this more general case we will do our thinking in terms of the time-to-failure random variable.

5.1.1 THE CUMULATIVE DISTRIBUTION FUNCTION OF T_0

For the age-at-failure random variable T_0, we denote its CDF by

$$F_0(t) = Pr(T_0 \leq t), \tag{5.1}$$

for $t \geq 0$.[4] We have already noted, however, that $T_0 = 0$ is not possible, so we will always consider that $F_0(0) = 0$. We observe that $F_0(t)$ gives the probability that failure will occur prior to (or at) precise age *t* for our entity known to exist at age 0. In standard actuarial notation,[5] this probability is denoted by ${}_tq_0$, so we have

$${}_tq_0 = F_0(t) = Pr(T_0 \leq t). \tag{5.2}$$

5.1.2 THE SURVIVAL DISTRIBUTION FUNCTION OF T_0

The *survival distribution function* (SDF) for the survival random variable T_0 is denoted by $S_0(t)$, and is defined by

$$S_0(t) = 1 - F_0(t) = Pr(T_0 > t), \tag{5.3}$$

for $t \geq 0$. Since we take $F_0(0) = 0$, it follows that we will always take $S_0(0) = 1$. The SDF gives the probability that the age at failure exceeds *t*, which is the same as the probability that the entity known to exist at age 0 will survive to age *t*. Since the notion of infinite survival is unrealistic, we consider that

[3] In practice, age-at-failure is often used for inanimate objects, such as light bulbs or labor strikes, and age-at-death is used for animate entities, such as laboratory animals or human persons under an insurance arrangement.

[4] In probability theory, it is customary to subscript the CDF symbol with the name of the random variable, which suggests the notation $F_{T_0}(t)$ in this case. With the name of the random variable understood to be T_0 in this section, we prefer the notation $F_0(t)$ to avoid the awkwardness of subscripting a subscript.

[5] As stated in the Preface, this text uses Standard International Actuarial Notation whenever possible.

$$\lim_{t \to \infty} S_0(t) = 0 \tag{5.4a}$$

and

$$\lim_{t \to \infty} F_0(t) = 1. \tag{5.4b}$$

In actuarial notation, the probability represented by $S_0(t)$ is denoted ${}_t p_0$, so we have

$${}_t p_0 = S_0(t) = Pr(T_0 > t). \tag{5.5}$$

In probability textbooks in general, the CDF is given greater emphasis than is the SDF. (Some textbooks do not even define the SDF at all.) But when we are dealing with an age-at-failure random variable, and its associated distribution, the SDF will receive greater attention.

EXAMPLE 5.1

Use both the CDF and the SDF to express the probability that an entity known to exist at age 0 will fail between the ages of 10 and 20.

SOLUTION

We seek the probability that T_0 will take on a value between 10 and 20. In terms of the CDF we have

$$Pr(10 < T_0 \leq 20) = F_0(20) - F_0(10).$$

Since $S_0(t) = 1 - F_0(t)$, then we also have

$$Pr(10 < T_0 \leq 20) = S_0(10) - S_0(20).$$ ❒

5.1.3 THE PROBABILITY DENSITY FUNCTION OF T_0

For a continuous random variable in general, the *probability density function* (PDF) is defined as the derivative of the CDF. Thus we have here

$$f_0(t) = \frac{d}{dt} F_0(t) = -\frac{d}{dt} S_0(t), \tag{5.6}$$

for $t > 0$. Consequently,

$$F_0(t) = \int_0^t f_0(y)\, dy \tag{5.7}$$

and

$$S_0(t) = \int_t^\infty f_0(y)\, dy. \tag{5.8}$$

Of course it must be true that

$$\int_0^\infty f_0(y)\,dy = 1. \tag{5.9}$$

Although we have given mathematical definitions of $f_0(t)$, it will be useful to describe $f_0(t)$ more fully in the context of the age-at-failure random variable. Whereas $F_0(t)$ and $S_0(t)$ are probabilities that relate to certain *time intervals*, $f_0(t)$ relates to a *point of time*, and is not a probability. It is the density of failure *at* age *t*, and is therefore an *instantaneous* measure, as opposed to an interval measure.

It is important to recognize that $f_0(t)$ is the *unconditional* density of failure at age *t*. By this we mean that it is the density of failure at age *t* given *only* that the entity existed at $t = 0$. The concept of *conditional* density is presented in the next subsection.

5.1.4 THE HAZARD RATE FUNCTION OF T_0

Recall that the PDF of T_0, $f_0(t)$, is the *unconditional* density of failure at age *t*. We now define a *conditional* density of failure at age *t*, with such density conditional on survival to age *t*. This conditional instantaneous measure of failure at age *t*, given survival to age *t*, is called the *hazard rate* at age *t*, or the *hazard rate function* (HRF) when viewed as a function of *t*. (In some textbooks the hazard rate is called the *failure rate*.) It will be denoted by $\lambda_0(t)$.

In general, if a conditional measure is multiplied by the probability of obtaining the conditioning event, then the corresponding unconditional measure will result. Specifically,

(Conditional density of failure at age *t*, given survival to age *t*)
× (Probability of survival to age *t*)
= (Unconditional density of failure at age *t*).

Symbolically this states that

$$\lambda_0(t) \cdot S_0(t) = f_0(t), \tag{5.10}$$

or

$$\lambda_0(t) = \frac{f_0(t)}{S_0(t)}. \tag{5.11}$$

Equations (5.11) and (5.6) give formal definitions of the HRF and the PDF, respectively, of the age-at-failure random variable. Along with the definitions it is also important to have a clear understanding of the *conceptual meanings* of $\lambda_0(t)$ and $f_0(t)$. They are both instantaneous measures of the density of failure at age *t*; they differ from each other in that $\lambda_0(t)$ is conditional on survival to age *t*, whereas $f_0(t)$ is unconditional (i.e., given only existence at age 0).

In the actuarial context of survival models for animate objects, including human persons, failure means death, or mortality, and the hazard rate is normally called the *force of mortality*. We will discuss the actuarial context further in Section 5.1.6 and in Chapter 6.

Some important mathematical consequences follow directly from Equation (5.11). Since $f_0(t) = -\frac{d}{dt} S_0(t)$, it follows that

$$\lambda_0(t) = \frac{-\frac{d}{dt} S_0(t)}{S_0(t)} = -\frac{d}{dt} \ln S_0(t). \tag{5.12}$$

Integrating, we have

$$\int_0^t \lambda_0(y)\, dy = -\ln S_0(t), \tag{5.13}$$

or

$$S_0(t) = \exp\left[-\int_0^t \lambda_0(y)\, dy\right]. \tag{5.14}$$

The *cumulative hazard function* (CHF) is defined to be

$$\Lambda_0(t) = \int_0^t \lambda_0(y)\, dy = -\ln S_0(t), \tag{5.15}$$

so that

$$S_0(t) = e^{-\Lambda_0(t)}. \tag{5.16}$$

EXAMPLE 5.2

An age-at-failure random variable has a distribution defined by

$$F_0(t) = 1 - .10(100-t)^{1/2},$$

for $0 \le t \le 100$. Find (a) the PDF and (b) the HRF for this random variable.

SOLUTION

(a) The PDF is given by

$$f_0(t) = \frac{d}{dt} F_0(t) = -(.10)(.50)(100-t)^{-1/2} \cdot (-1) = .05(100-t)^{-1/2}.$$

(b) The HRF is given by

$$\lambda_0(t) = \frac{f_0(t)}{S_0(t)} = \frac{.05(100-t)^{-1/2}}{.10(100-t)^{1/2}} = .50(100-t)^{-1}.$$ ❐

5.1.5 THE MOMENTS OF THE AGE-AT-FAILURE RANDOM VARIABLE T_0

The first moment, or expected value, of a continuous random variable defined on $[0,\infty)$ is given by

$$E[T_0] = \int_0^\infty t \cdot f_0(t)\, dt, \tag{5.17}$$

if the integral exists, and otherwise the first moment is undefined. Integration by parts yields the alternative formula

$$E[T_0] = \int_0^\infty S_0(t)\, dt, \tag{5.18}$$

provided $\lim_{t\to\infty} t \cdot S_0(t) = 0$. Equation (5.18) is frequently used to find the first moment of an age-at-failure random variable.

The second moment of T_0 is given by

$$E\left[T_0^{\,2}\right] = \int_0^\infty t^2 \cdot f_0(t)\; dt, \tag{5.19}$$

if the integral exists, so the variance of T_0 can be found from

$$Var(T_0) = E\left[T_0^{\,2}\right] - \left\{E\left[T_0\right]\right\}^2. \tag{5.20}$$

Specific expressions can be developed for the moments of T_0 for specific forms of $f_0(t)$. This will be pursued in the following section.

Another property of the age-at-failure random variable that is of interest is its *median* value. We recall that the median of a continuous random variable is the value for which there is a 50% chance that the random variable will exceed (and thus also not exceed) that value. Mathematically, y is the median of T_0 if

$$Pr(T_0 > y) = Pr(T_0 \le y) = \frac{1}{2}, \tag{5.21}$$

so that $S_0(y) = F_0(y) = \frac{1}{2}$.

5.1.6 ACTUARIAL SURVIVAL MODELS

When the age-at-failure random variable is considered in an actuarial context, special symbols are used for some of the concepts defined in this section. The hazard rate, now called the force of mortality, is denoted by μ_t, rather than $\lambda_0(t)$. Thus we have

$$\mu_t = \frac{-\frac{d}{dt} S_0(t)}{S_0(t)} = -\frac{d}{dt} \ln S_0(t). \tag{5.22}$$

It is customary to denote the first moment of T_0 by $\overset{\circ}{e}_0$. Thus we have

$$\overset{\circ}{e}_0 = E[T_0] = \int_0^{\infty} t \cdot f_0(t)\, dt. \tag{5.23}$$

Since $\overset{\circ}{e}_0$ is the unconditional expected value of T_0, given only alive at $t = 0$, it is called the *complete expectation of life at birth*.[6]

We recognize that the moments of T_0 given above are all unconditional. Conditional moments, and other conditional measures, are defined in Section 5.3, and the standard actuarial notation for them is reviewed in Chapter 6.

EXAMPLE 5.3

For the distribution of Example 5.2, find (a) $E[T_0]$ and (b) the median of the distribution.

SOLUTION

(a) The expected value is given by Equation (5.18) as

$$\begin{aligned} E[T_0] &= \int_0^{100} .10(100-t)^{1/2}\, dt \\ &= -\left(\frac{2}{3}\right)(.10)(100-t)^{3/2}\Big|_0^{100} = \left(\frac{2}{3}\right)(.10)(100)^{3/2} = \frac{200}{3}. \end{aligned}$$

(b) The median is the value of y satisfying $S_0(y) = .10(100-y)^{1/2} = .50$, which solves for $y = 75$. ❐

5.2 EXAMPLES OF PARAMETRIC SURVIVAL MODELS

In this section we explore several non-negative continuous probability distributions that are candidates for serving as survival models. In practice, some distributions fit better than others to the empirical evidence of the shape of a survival distribution, so we will comment on each distribution we present regarding its suitability as a survival model.

5.2.1 THE UNIFORM DISTRIBUTION

The continuous uniform distribution, defined in Section 2.3.1, is a simple two-parameter distribution with a constant PDF. The parameters of the distribution are the limits of the interval

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[6] The significance of the adjective "complete" will become clearer when we consider an alternative measure of the expectation of life in Sections 5.3.6 and 6.3.4.

on the real number axis over which it is defined, and its PDF is the reciprocal of that interval length. Thus if a generic random variable X is defined over the interval $[a,b]$, then $f_X(x) = \frac{1}{b-a}$, for $a \le x \le b$, and $f_X(x) = 0$ elsewhere.

For the special case of the age-at-failure random variable, $a = 0$ so b is the length of the interval, as well as the greatest value of t for which $f_0(t) > 0$. When the uniform distribution is used as a survival model, the Greek ω is frequently used for this parameter (which then represents the maximum survival age), so the distribution is defined by

$$f_0(t) = \frac{1}{\omega}, \tag{5.24}$$

for $0 < t \le \omega$. The following properties of the uniform distribution easily follow, and should be verified by the reader:

$$F_0(t) = \int_0^t f_0(y)\, dy = \frac{t}{\omega} \tag{5.25}$$

$$S_0(t) = 1 - F_0(t) = \int_t^\omega f_0(y)\, dy = \frac{\omega - t}{\omega} \tag{5.26}$$

$$\lambda_0(t) = \frac{f_0(t)}{S_0(t)} = \frac{1}{\omega - t} \tag{5.27}$$

$$E[T_0] = \int_0^\omega t \cdot f_0(t)\, dt = \frac{\omega}{2} \tag{5.28}$$

$$Var(T_0) = E\left[T_0^2\right] - \left\{E\left[T_0\right]\right\}^2 = \frac{\omega^2}{12} \tag{5.29}$$

The uniform distribution, as a survival model, is not appropriate over a broad range of age, at least as a model for *human* survival. It is of historical interest, however, to note that it was the first continuous probability distribution to be suggested for that purpose, in 1724, by Abraham de Moivre. As a result, actuarial literature and exams often refer to the uniform distribution as "de Moivre's law."

The major use of this distribution is over short ranges of time (or age). We will explore this use of the uniform distribution quite thoroughly in Section 6.5.1.

5.2.2 THE EXPONENTIAL DISTRIBUTION

This very popular one-parameter distribution (see Section 2.3.3) is defined by its SDF to be

$$S_0(t) = e^{-\lambda t}, \tag{5.30}$$

for $t > 0$ and $\lambda > 0$. It then follows that the PDF is

$$f_0(t) = -\frac{d}{dt} S_0(t) = \lambda \cdot e^{-\lambda t}, \tag{5.31}$$

so that the HRF is

$$\lambda_0(t) = \frac{f_0(t)}{S_0(t)} = \lambda, \tag{5.32}$$

a constant. In the actuarial context, where the hazard rate is generally called the force of mortality, the exponential distribution is referred to as the *constant force distribution*.

The exponential distribution, with its property of a constant hazard rate, is frequently used in reliability engineering as a survival model for inanimate objects such as machine parts. Like the uniform distribution, however, it is not appropriate as a model for human survival over a broad range, but might be used over short intervals, such as one year, due to its mathematical simplicity. This will be explored in Section 6.5.2.

5.2.3 THE GOMPERTZ DISTRIBUTION

This distribution was suggested as a model for human survival by Gompertz [9] in 1825. The distribution is usually defined by its force of mortality as

$$\mu_t = Bc^t, \tag{5.33}$$

for $t > 0, B > 0,$ and $c > 1.$ Then the SDF is given by

$$S_0(t) = \exp\left[-\int_0^t Bc^y \, dy\right] = \exp\left[\frac{B}{\ln c}(1-c^t)\right]. \tag{5.34}$$

The PDF is given by $\mu_t \cdot S_0(t),$ and is clearly not a very convenient mathematical form. A closed-form expression for the mean of the distribution, $E[T_0],$ does not exist, but the mean can be approximated by numerical integration with a large finite upper limit replacing the actual upper limit of infinity.

5.2.4 THE MAKEHAM DISTRIBUTION

In 1860 Makeham [19] modified the Gompertz distribution by taking the force of mortality to be

$$\mu_t = A + Bc^t, \tag{5.35}$$

for $t > 0, B > 0, c > 1,$ and $A > -B.$ Makeham was suggesting that part of the hazard at any age is independent of the age itself, due, for example, to the risk of accident, so a constant was added to the Gompertz force of mortality.

The SDF for this distribution is given by

$$S_0(t) \;=\; \exp\left[-\int_0^t (A+Bc^y)\,dy\right] \;=\; \exp\left[\frac{B}{\ln c}(1-c^t)-At\right]. \tag{5.36}$$

Again it is clear that the PDF for this distribution is not mathematically tractable. As with the Gompertz distribution, there is no closed-form expression for $E[T_0]$, although it can also be approximated by numerical integration.[7]

5.2.5 SUMMARY OF PARAMETRIC SURVIVAL MODELS

We have briefly explored four distributions here: two (uniform and exponential) which are mathematically simple, and two (Gompertz and Makeham) which are not. For many illustrations, where we wish to avoid mathematical complexity, we will use the uniform or the exponential for illustrative purposes only, not necessarily suggesting that they are applicable in practice. The exponential distribution has been applied in many situations not involving healthy human lives, and has been widely used in those situations.

5.3 THE TIME-TO-FAILURE RANDOM VARIABLE

In Section 5.1 we defined a continuous random variable, denoted T_0, which measured the length of time from age 0 until failure occurs. Now we turn to the case where our entity of interest is known to have survived to age x, where $x>0$, and we wish to consider the random variable for the *additional* time that the entity might survive beyond age x. We denote this random variable by T_x, and note that its domain is $T_x > 0$. We define the random variable T_x to be the *time-to-failure* random variable for an entity known to be alive (i.e., known to have not yet failed) at age x. We will use the notation (x) to denote the entity known to be alive at age x.[8]

If T_x is the random time-to-failure for an entity alive at age x, it follows that the age-at-failure will be T_x more than age x, so we have the relationship $T_0 = x + T_x$ between our two basic random variables. This is illustrated in the following figure.

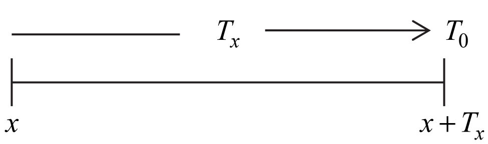

FIGURE 5.1

Rather than develop separate distributions for T_x for each different value of x, we will simply calculate probability values for T_x from the distribution of T_0. (An exception to this will be explored in Section 5.4.)

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[7] A generalization of the Makeham distribution is presented in Exercise 5-10.

[8] The time until failure of (x) can also be called the future lifetime of (x), so T_x is therefore often called the *future lifetime* random variable for the entity (x).

CHAPTER SIX

THE LIFE TABLE (DISCRETE TABULAR CONTEXT)

In this chapter we describe the nature of the traditional life table, showing that it can have all the properties of the survival models described in Chapter 5. When a survival model is presented in the life table format, it is customary to use notation and terminology which differ somewhat from that presented in Chapter 5. A major objective of this chapter will be to show clearly the correspondence between notation used in the probability model and that used in the life table model.

The reader should realize that life tables were developed by actuaries independently from (and a century earlier than) the development of the statistical theory of survival models as probability distributions[1]. For this reason, traditional life table notation and terminology will not tend to reveal the stochastic nature of the model as clearly as is done by the probability model in Chapter 5. By showing the correspondence of the life table symbols to those of the probability model, we intend to correct this.

6.1 DEFINITION OF THE LIFE TABLE

The life table can be defined as a table of numerical values of $S_0(x)$ for certain values of x (which we now prefer to use instead of *t*). Table 6.1 illustrates such a table.

TABLE 6.1

$\boldsymbol{x}$	0	1	2	3	4	$\cdots$	109	110
$\boldsymbol{S_0(x)}$	1.00000	.97408	.97259	.97160	.97082	$\cdots$	.00001	.00000

Typically a complete life table shows values of $S_0(x)$ for all integral values of $x, x=0,1,\ldots$. Since $S_0(x)$ is represented by these values, it is clear that a practical upper limit on x must be adopted beyond which values of $S_0(x)$ are taken to be zero. Traditionally, ω is used for the smallest value of x for which $S_0(x)=0$. Then $S_0(\omega-1)>0$, but $S_0(\omega)=0$. In Table 6.1, $\omega=110$.

From Table 6.1, we can calculate the conditional probabilities represented by ${}_np_x$ and ${}_nq_x$ for integral x and n. However, these are the *only* functions that can be determined from the tabular model. Functions such as $f_0(x)$, $\lambda_0(x)$, and $\mathring{e}_x$ cannot be determined from the tabular model unless we expand the model by adopting assumed values for $S_0(x)$ between adjacent integers. We will pursue this in Section 6.6.

[1] The first modern life table, called the Breslau Table, dates from 1693 and is attributed to Edmund Halley [10] of Halley's Comet fame.

EXAMPLE 6.1

From Table 6.1, calculate (a) the probability that a life age 0 will fail before age 3; (b) the probability that a life age 1 will survive to age 4.

SOLUTION

(a) This is given directly by $F_0(3) = 1 - S_0(3) = .02840.$

(b) This conditional probability is given by ${}_3p_1 = \frac{S_0(4)}{S_0(1)} = .99665.$ ❒

6.2 THE TRADITIONAL FORM OF THE LIFE TABLE

The tabular survival model was developed by the early actuaries many years ago. The history of this model is reported throughout actuarial literature, and a brief summary of this history is presented by Dobson [9].

Traditionally, the tabular survival model differs from Table 6.1 in two respects. Rather than presenting decimal values of $S_0(x)$, it is usual to multiply these values by, say, 100,000, and thereby present the $S_0(x)$ values as integers. Secondly, since these integers are not probabilities (which $S_0(x)$ values are), the column heading is changed from $S_0(x)$ to l_x, where l stands for *number living*, or *number of lives*. In this way the tabular survival model became known as the *life table*.

Since $S_0(0) = 1$, then l_0 is the same as the constant multiple which transforms all $S_0(x)$ into l_x. This constant is called the *radix* of the table. Formally,

$$l_x = l_0 \cdot S_0(x). \tag{6.1}$$

Using a radix of 100,000, we transform Table 6.1 into Table 6.1a.

TABLE 6.1a

x	0	1	2	3	4	$\cdots$	109	110
l_x	100,000	97,408	97,259	97,160	97,082	$\cdots$	1	0

The basic advantage of the traditional form of the life table is its susceptibility to interpretation. If we view $l_0 = 100{,}000$ as a hypothetical cohort group of newborn lives, or other new entities such as lightbulbs, electronic devices, or laboratory animals, then each value of l_x represents the survivors of that group to age x, according to the model. This is a convenient, deterministic, interpretation of the model. Of course, since $l_x = l_0 \cdot S_0(x)$, and $S_0(x)$ is a probability, then l_x is really the *expected number* of survivors to age x out of an original group of l_0 new entities. This connection between $S_0(x)$ and l_x is also given in Chapter 1 of Jordan [14].

Although the basic representation of the tabular survival model is in terms of the values of l_x, it is customary for the table to also show the value of several other functions derived from l_x. We define

$$d_x = l_x - l_{x+1}, \tag{6.2}$$

or, more generally,

$${}_nd_x = l_x - l_{x+n}. \tag{6.3}$$

Since l_x represents the size of the cohort at age x, and l_{x+n} is the number of them still surviving at age $x+n$, then clearly ${}_nd_x$ gives the number who fail (or die) between ages x and $x+n$. (This portrayal of number dying explains the frequent historical reference to these models as *mortality tables*.) Furthermore,

$$q_x = \frac{d_x}{l_x}, \tag{6.4}$$

or, more generally,

$${}_nq_x = \frac{{}_nd_x}{l_x} \tag{6.5}$$

gives the conditional probability of failure, given alive at age x. Finally, we have

$${}_np_x = 1 - {}_nq_x = \frac{l_x - {}_nd_x}{l_x} = \frac{l_{x+n}}{l_x} \tag{6.6}$$

as the conditional probability of surviving to age $x+n$, given alive at age x. With $n=1$, we have the special case

$$p_x = \frac{l_{x+1}}{l_x}. \tag{6.7}$$

Recall that the conditional probabilities ${}_np_x$ and ${}_nq_x$ were defined in Section 5.3 in terms of $S_0(x)$. The consistency of those definitions with the ones presented in this section is easily seen since l_x is simply $l_0 \cdot S_0(x)$. We redefined ${}_np_x$ and ${}_nq_x$ in terms of l_x here simply to complete our description of the life table form of the survival model.

EXAMPLE 6.2

From Table 6.1a, find (a) the number who fail between ages 2 and 4; (b) the probability that a life age 1 will survive to age 4.

SOLUTION

(a) This is given by ${}_2d_2 = l_2 - l_4 = 177.$

(b) This is given by ${}_3p_1 = \frac{l_4}{l_1} = .99665.$ (Compare with part (b) of Example 6.1.) ❐

6.3 OTHER FUNCTIONS DERIVED FROM l_x

Although a life table only presents values of l_x for certain (say, integral) values of x, we wish to adopt the view that the l_x function which produces these values is a continuous and differentiable function. In other words, we assume that a continuous and differentiable l_x function *exists*, but only certain values of it are presented in the survival model. The reason we make this assumption is that there are several other important functions that can be derived from l_x if l_x is continuous and differentiable.

If values of l_x are known only at integral x, the question of how to *evaluate* these additional functions then arises, and the usual way to accomplish this evaluation is to make an assumption about the form of l_x between adjacent integral values of x.

In this section we will derive these several new functions from l_x symbolically, assuming l_x to be continuous and differentiable. In Section 6.6 we will discuss three common distribution assumptions, and show how they allow us to evaluate the functions of this section from a table of l_x values at integral x only. We will also interpret these distribution assumptions in terms of both l_x and $S_0(x)$.

6.3.1 THE FORCE OF FAILURE

The derivative of l_x can be interpreted as the absolute instantaneous annual rate of change of l_x. Since l_x represents the number of survivors at age x, then the derivative, which is the annual rate at which l_x is changing, gives the annual rate at which failures are occurring at age x. This derivative is negative since l_x is a decreasing function. To obtain the absolute magnitude of this instantaneous rate of failure, we will use the negative of the derivative. Finally, since the magnitude of the derivative depends on the size of l_x itself, we obtain the *relative instantaneous rate of failure* by dividing the negative derivative of l_x by l_x itself. Thus we have

$$\mu_x = \frac{-\frac{d}{dx}l_x}{l_x}, \tag{6.8}$$

which we call the *force of failure* (or *force of mortality*) at age x. Since $l_x = l_0 \cdot S_0(x)$, we see that Equation (6.8) is the same as

$$\lambda(x) = \frac{-\frac{d}{dx}S_0(x)}{S_0(x)} = \frac{f_0(x)}{S_0(x)}. \tag{6.9}$$

Thus the hazard rate and the force of failure are identical.

If we multiply both sides of Equation (5.14) by l_0 and substitute μ_y for $\lambda_0(y)$, we obtain

$$l_x = l_0 \cdot S_0(x) = l_0 \cdot \exp\left[-\int_0^x \mu_y\, dy\right]. \tag{6.10}$$

In the life table context, $S_0(x) = {}_xp_0 = \exp[-\int_0^x \mu_y\, dy]$ can be interpreted as a decremental factor that reduces the initial cohort of size l_0 to size l_x at age x.

By a simple variable change we can write Equation (6.8) as

$$\mu_{x+t} = \frac{-\frac{d}{dt}l_{x+t}}{l_{x+t}}, \tag{6.8a}$$

a form in which the force of failure will frequently be expressed.

EXAMPLE 6.3

Show that the force of failure, μ_x, is the limiting value of the probability of failure over an interval divided by the interval length (in years), as the interval length approaches zero.

SOLUTION

Consider first a one-year interval, with $q_x = \frac{d_x}{l_x}$. Then consider a half-year interval with $\frac{_{1/2}q_x}{1/2} = \frac{l_x - l_{x+1/2}}{1/2 \cdot l_x}$. Now, in general, consider $\frac{_{\Delta x}q_x}{\Delta x} = \frac{l_x - l_{x+\Delta x}}{\Delta x \cdot l_x}$, and show that $\lim_{\Delta x \to 0} \frac{_{\Delta x}q_x}{\Delta x} = \mu_x$. We have

$$\lim_{\Delta x \to 0}\left[\frac{l_x - l_{x+\Delta x}}{\Delta x \cdot l_x}\right] = \frac{1}{l_x} \cdot \lim_{\Delta x \to 0}\left[\frac{l_x - l_{x+\Delta x}}{\Delta x}\right] = \frac{1}{l_x}\left[-\frac{d}{dx}l_x\right] = \mu_x,$$

by Equation (6.8). ❐

6.3.2 THE PROBABILITY DENSITY FUNCTION OF T_0

With the force of failure, which is the same as the hazard rate, now defined, the next function to develop from l_x is the PDF of the age-at-failure random variable T_0 (Remember that we wish to show that the life table is a representation of the distribution of this random variable.)

From Equation (5.10) we have $f_0(x) = \lambda_0(x) \cdot S_0(x)$. In the life table context, $\lambda_0(x) = \mu_x$ and $S_0(x) = \frac{l_x}{l_0}$. Thus we have, for $x \geq 0$,

$$f_0(x) = \mu_x\left(\frac{l_x}{l_0}\right) = {}_xp_0\mu_x. \tag{6.11}$$

Also, from Equation (6.8), $\frac{d}{dx}l_x = -l_x\mu_x$. Dividing both sides by l_0 gives

$$\frac{d}{dx}\,{}_xp_0 = -{}_xp_0\mu_x. \tag{6.12}$$

EXAMPLE 6.4

Show that $\int_0^\infty f_0(x)\,dx = 1$.

SOLUTION

Since $f_0(x) = {}_xp_0\mu_x$, we have $\int_0^\infty {}_xp_0\mu_x\,dx = -{}_xp_0\,\big|_0^\infty$, from Equation (6.12). Thus we have ${}_0p_0 - {}_\infty p_0 = 1$, since ${}_0p_0 = 1$ and ${}_\infty p_0 = 0$. ❒

With the PDF in hand, we can now find $E[T_0]$, which we recall is denoted by $\overset{\circ}{e}_0$. (Throughout this and the following section, all expectations are assumed to exist.) We have

$$\overset{\circ}{e}_0 = E[T_0] = \int_0^\infty x \cdot f_0(x)\,dx = \int_0^\infty x \cdot {}_xp_0\mu_x\,dx. \tag{6.13}$$

Integration by parts produces the alternative formula

$$\overset{\circ}{e}_0 = E[T_0] = \int_0^\infty {}_xp_0\,dx = \frac{1}{l_0} \cdot \int_0^\infty l_x\,dx. \tag{6.14}$$

The second moment of T_0 is found from

$$E[T_0^{\,2}] = \int_0^\infty x^2 \cdot {}_xp_0\mu_x\,dx. \tag{6.15a}$$

Integration by parts produces

$$E[T_0^{\,2}] = 2\int_0^\infty x \cdot {}_xp_0\,dx = \frac{2}{l_0} \cdot \int_0^\infty x \cdot l_x\,dx. \tag{6.15b}$$

Then the variance of T_0 is given by

$$Var(T_0) = E[T_0^2] - \{E[T_0]\}^2 = \frac{2}{l_0} \cdot \int_0^\infty x \cdot l_x \, dx - \left(\frac{1}{l_0} \cdot \int_0^\infty l_x \, dx \right)^2. \tag{6.16}$$

6.3.3 CONDITIONAL PROBABILITIES AND DENSITIES

We have already discussed the conditional probabilities ${}_n p_x$ and ${}_n q_x$ in terms of both $S_0(x)$ and l_x.

Another conditional probability of some interest is denoted by ${}_{n|m} q_x$. It represents the probability that an entity known to be alive at age x will fail between ages $x+n$ and $x+n+m$. In terms of the probability notation of Chapter 5, ${}_{n|m} q_x = Pr[(x+n) < T_0 \leq (x+n+m) \,|\, T_0 > x]$. This can also be expressed as the probability that an entity age x will survive n years, but then fail within the next m years. This way of stating the probability suggests that we can write

$$ {}_{n|m} q_x = {}_n p_x \cdot {}_m q_{x+n}. \tag{6.17}$$

Here ${}_m q_{x+n}$ is the conditional probability of failing between ages $x+n$ and $x+n+m$, given alive at age $x+n$. In turn, ${}_n p_x$ is the conditional probability of surviving to age $x+n$, given alive at age x. Their product gives the probability of failing between ages $x+n$ and $x+n+m$, given alive at age x. In terms of l_x, we have, from Equations (6.6) and (6.5),

$$ {}_{n|m} q_x = \frac{l_{x+n}}{l_x} \cdot \frac{{}_m d_{x+n}}{l_{x+n}} = \frac{{}_m d_{x+n}}{l_x}. \tag{6.18a}$$

When $m = 1$ we use the notation

$$ {}_{n|} q_x = \frac{d_{x+n}}{l_x}. \tag{6.18b}$$

Recall that ${}_{n|} q_x$ was defined in Section 5.3.6 as $Pr(K_x = n)$ or $Pr(K_x^* = n+1)$, the probability that an entity alive at age x would fail in the $(n+1)^{st}$ year.

EXAMPLE 6.5

Show that ${}_{n|m} q_x = {}_n p_x - {}_{n+m} p_x$, and give an interpretation of this result.

SOLUTION

Since, from Equation (6.3), ${}_m d_{x+n} = l_{x+n} - l_{x+n+m}$, then Equation (6.18a) becomes ${}_{n|m} q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x} = {}_n p_x - {}_{n+m} p_x$. Since ${}_n p_x$ is the probability of surviving to age $x+n$, we can think of it as containing the probability of surviving to any age beyond $x+n$. If we remove from ${}_np_x$ the probability of surviving to $x+n+m$, which is ${}_{n+m}p_x$, we have the probability of surviving to $x+n$, but not to $x+n+m$, which is ${}_{n|m}q_x$. ❒

Next we wish to explore the conditional PDF for death at age y, given alive at age x, where $y > x$. From Equation (5.44) we know this conditional PDF is $f_0(y \mid T_0 > x) = \frac{f_0(y)}{S_0(x)}$. Now from Equation (6.11) we have $f_0(y) = \frac{1}{l_0} \cdot l_y \mu_y$, and from Equation (6.1) we have $S_0(x) = \frac{l_x}{l_0}$. Thus

$$f_0(y \mid T_0 > x) = \frac{l_y \mu_y}{l_x} = {}_{y-x}p_x \mu_y. \tag{6.19a}$$

Letting $t = y-x$, so $y = x+t$, we have

$$f_0(x+t \mid T_0 > x) = {}_tp_x \mu_{x+t}, \tag{6.19b}$$

the conditional PDF of the random variable for the length of future lifetime of an entity alive at age x. This conditional PDF is a very useful function for developing other results.

If both numerator and denominator on the right side of Equation (6.8a) are divided by l_x, we obtain

$$\mu_{x+t} = \frac{-\frac{d}{dt}\,{}_tp_x}{{}_tp_x}, \tag{6.20}$$

which is equivalent to

$$\frac{d}{dt}\,{}_tp_x = -{}_tp_x \mu_{x+t}. \tag{6.21}$$

The expected future lifetime of an entity alive at age x is given by

$$\overset{\circ}{e}_x = E[T_x] = \int_0^\infty t \cdot {}_tp_x \mu_{x+t}\, dt = \int_0^\infty {}_tp_x\, dt, \tag{6.22}$$

by evaluating the first integral using integration by parts. The second moment of T_x is

$$E[T_x^2] = \int_0^\infty t^2 \cdot {}_tp_x \mu_{x+t}\, dt = 2\int_0^\infty t \cdot {}_tp_x \; dt, \tag{6.23}$$

again by using integration by parts on the first integral, so its variance is

$$Var(T_x) = E[T_x^2] - \{E[T_x]\}^2 = 2\int_0^\infty t \cdot {}_tp_x \; dt - \left(\int_0^\infty {}_tp_x \; dt\right)^2. \tag{6.24}$$

CHAPTER FIFTEEN

MODELS WITH VARIABLE INTEREST RATES

Thus far in the text, when calculating the actuarial present value (APV) for contingent payment models, including insurance products, we have treated time until failure and mode of failure as random variables. But we have always assumed that a single interest rate was valid throughout the life of the model, however long that might be. It can be risky to assume that interest rates will remain constant at today's rates. Indeed some insurance companies around the world have experienced severe losses as a result of pricing products at interest rates that proved to be too optimistic.

In this chapter we address contingent payment models using interest rates that vary with time. Sections 15.1 and 15.2 address models with deterministic contingent payment amounts evaluated using non-deterministic interest rates. The term structure of interest rates and implied forward rates of interest are introduced in Sections 15.3 and 15.4.

The treatment of topics in Chapter 15 follows a heuristic approach. To simplify the discussion in Sections 15.1 and 15.2, we make the assumption that the market consists only of *one-period securities*. For our discussion of interest rates, the only securities available for investment are one-period bonds that pay a single coupon plus principal at the end of the period. This assumption enables us to introduce features of interest rate variability without having to deal with issues such as a term structure or partial-period payments. In addition, there is no distinction (in the absence of default) between the interest rate of a bond and the rate of return on that bond. In Sections 15.3 and 15.4 we broaden the discussion to include multi-period bonds, including those with partial-period payments (coupons). This will enable us to develop the term structure of spot interest rates along with implied forward rates of interest.

15.1 ACTUARIAL PRESENT VALUES USING VARIABLE INTEREST RATES

Interest rates in the United States have varied substantially over time. Table 15.1 shows sample one-year U.S. Treasury interest rates between 1962 and 2009.[1] This table gives a good indication of just how variable interest rates can be over time. In this section, we discuss one method for incorporating this variability into calculating the actuarial present value for contingent payment models. This method involves the construction of *interest rate scenarios* for the future. An interest rate scenario is a possible future path for interest rates. For example, Table 15.2 shows three illustrative interest rate scenarios for one-year interest rates in the first five years of a contingent contract. Each row represents a different scenario for the one-year interest rate in each year over the next five years. The pre-subscript j on the interest rate symbol

[1] Source: www.ustreas.gov.

indicates the scenario from which that rate was taken. For example, ${}_3i_3 = .04$ means that the interest rate in the third interest rate scenario in the third year is 4%.

TABLE 15.1

Year	Rate	Year	Rate	Year	Rate
1962	3.10%	1978	8.34%	1994	5.32%
1963	3.36	1979	10.65	1995	5.94
1964	3.85	1980	12.00	1996	5.52
1965	4.15	1981	14.80	1997	5.63
1966	5.20	1982	12.27	1998	5.05
1967	4.88	1983	9.58	1999	5.08
1968	5.69	1984	10.91	2000	6.11
1969	7.12	1985	8.42	2001	3.49
1970	6.90	1986	6.45	2002	2.00
1971	4.89	1987	6.77	2003	1.24
1972	4.95	1988	7.65	2004	1.31
1973	7.32	1989	8.53	2005	2.79
1974	8.20	1990	7.89	2006	4.38
1975	6.78	1991	5.86	2007	5.00
1976	5.88	1992	3.89	2008	3.17
1977	6.08	1993	3.43	2009	0.40

TABLE 15.2

Scenario j	${}_ji_1$	${}_ji_2$	${}_ji_3$	${}_ji_4$	${}_ji_5$
1	6%	7%	8%	9%	10%
2	6	6	6	6	6
3	6	5	4	3	2

EXAMPLE 15.1

For each of the three interest rate scenarios in Table 15.2, find the actuarial present value of a five-year pure endowment issued at age $x = 65$ for amount \$1000. The mortality rates for each year of age are $q_{65} = .03$, $q_{66} = .04$, $q_{67} = .05$, $q_{68} = .06$, and $q_{69} = .07$.

SOLUTION

In each scenario, the APV is

$$1000\,{}_5E_{65} \;=\; 1000({}_jv^5 \cdot {}_5p_{65}),$$

where ${}_j v^5$ represents five years of discounting at interest rates given by Scenario j. Regardless of the chosen scenario,

$${}_5p_{65} = (.97)(.96)(.95)(.94)(.93) = .7734.$$

We can find ${}_1v^5$, for example, as

$${}_1v^5 = \left(\frac{1}{1.06}\right)\left(\frac{1}{1.07}\right)\left(\frac{1}{1.08}\right)\left(\frac{1}{1.09}\right)\left(\frac{1}{1.10}\right) = .6809,$$

so the APV under Scenario 1 is $(1000)(.7734)(.6809) = 526.61$. Under Scenarios 2 and 3 the APV's are 577.93 and 635.97, respectively. (The reader is asked to verify these results in Exercise 15-1.) ❐

We can imagine that an insurer who has priced a pure endowment contract assuming level interest rates of 6% (Scenario 2) will be unhappy if it chooses to invest the net single premium in one-year bonds, and the interest rates then emerge similarly to Scenario 3.[2]

EXAMPLE 15.2

Using the same mortality and interest assumptions as in Example 15.1, find the actuarial present value for a five-year term insurance of unit amount issued at age $x = 65$, with benefit paid at the end of the year of failure. Find a separate APV for each of the three scenarios.

SOLUTION

We adapt Equation (7.8) to find the actuarial present value for the five-year term insurance under Scenario j, and we denote this APV by ${}_j A^{1}_{65:\overline{5}|}$.

TABLE 15.3

t	Year t Rate	${}_1v^t$	${}_{t-1\|}q_{65}$	${}_1v^t \cdot {}_{t-1\|}q_{65}$
1	.06	.9434	.0300	.0283
2	.07	.8817	.0388	.0342
3	.08	.8164	.0466	.0380
4	.09	.7490	.0531	.0398
5	.10	.6809	.0582	.0396
${}_1A^{1}_{65:\overline{5}\|}$				**.1799**

[2] In practice, the situation is more complicated than that presented here, because the insurer will generally try to invest in securities with a maturity similar to that of the product from which the net single premium arose. In this case the insurer will only have to worry about current interest rates for bond cash flows requiring reinvestment. However, for some very long-term contracts such as whole life contingent annuities, whole life insurance, or long term care insurance, this problem can be serious.

The results of the calculation for Scenario 1 are shown in Table 15.3 in spreadsheet form. Note how the term v^k in Equation (7.8), which assumes a constant interest rate, is generalized to ${}_j v^k = \prod_{t=1}^{k}\left(1+{}_j i_t\right)^{-1}$ in the case of the j^{th} variable interest rate scenario. The APV under Scenario 1 is ${}_1 A^{1}_{65:\overline{5}|} = .1799$. (The reader should repeat the steps depicted in Table 15.3 under Scenarios 2 and 3 to verify that ${}_2 A^{1}_{65:\overline{5}|} = .1875$ and ${}_3 A^{1}_{65:\overline{5}|} = .1958$.) Note that the APV is higher for the lower interest rate scenarios. ❐

15.2 DETERMINISTIC INTEREST RATE SCENARIOS

Interest rate scenarios used in actuarial analysis are of two distinct types. *Deterministic scenarios*, described in this section, are determined *a priori* and are often used to "stress" a product's profitability in the event future interest rates are unfavorable. Scenarios of this type are sometimes prescribed by regulatory agencies to provide a test of sensitivity to interest rates that is common across products and companies. *Stochastic scenarios* are scenarios that are created using a stochastic interest rate simulator based on an assumed probability distribution for future interest rates.

We address the deterministic scenarios in this section by studying a sample regulatory policy designed to test the interest sensitivity of insurance products. If a product "fails" the interest sensitivity test, the company selling the product must hold additional capital as contingent funds for adverse changes in interest rates. Although the example here is fictional, similar deterministic scenarios are performed in some jurisdictions as part of cash flow testing of products for interest sensitivity.

EXAMPLE 15.3

An annuity company sells the following two products:

(a) A five-year annual payment temporary immediate annuity

(b) A five-year pure endowment

The national regulatory authority requires the following two-step interest rate test in order to determine if the annuity company must hold additional capital:

(1) The net single premium (NSP) for each product is calculated under three deterministic interest rate scenarios:

   (i) Rates remain level at the current rate.
   (ii) Rates rise 1% per year until they reach twice the current rate, and then remain level in succeeding years.
   (iii) Rates fall 1% per year until they reach one-half the current rate, and then remain level in succeeding years.

(2) If the NSP under the falling interest rate scenario is 5% or more above the NSP in the level rate case, the company must hold additional capital.

If the probability of death in any given year remains constant at $q_x = .02$ and the current interest rate is 6%, determine whether this annuity company must hold additional capital for either product.

SOLUTION

(a) For the five-year temporary immediate annuity, we first calculate the NSP (or APV) in the level rate case, using Equation (8.21). We obtain

$$ {}_l a_{x:\overline{5}|} = \sum_{t=1}^{5} {}_l v^t \cdot {}_t p_x = \sum_{t=1}^{5} \left(\frac{1}{1.06}\right)^t \cdot (.98)^t = 3.9756, $$

where the pre-subscript l denotes the level interest rate case. For the falling interest rate case, denoted by the pre-subscript f, the NSP is given by

$$ {}_f a_{x:\overline{5}|} = \sum_{t=1}^{5} {}_f v^t \cdot {}_t p_x = \sum_{t=1}^{5} {}_f v^t \cdot (.98)^t. $$

The calculation is summarized in Table 15.4 below. The reader should repeat the steps depicted in Table 15.4 to calculate the APV under the rising interest rate scenario, obtaining the value ${}_r a_{x:\overline{5}|} = 3.8461$ (see Exercise 15-3(a)). Since the falling interest rate scenario does not produce an APV more than 5% greater than under the level rate case, the annuity company does not need to hold additional reserves for its five-year temporary immediate annuity product.

TABLE 15.4

t	Year t Rate	${}_f v^t$	${}_t p_x$	${}_f v^t \cdot {}_t p_x$
1	.06	.9434	.9800	.9245
2	.05	.8985	.9604	.8629
3	.04	.8639	.9412	.8131
4	.03	.8388	.9224	.7737
5	.03	.8143	.9039	.7360
${}_f a_{x:\overline{5}|}$				**4.1102**

(b) For the five-year pure endowment, we again calculate first the APV in the level interest case, obtaining

$$ {}_l A_{x:\overline{5}|}^{\;\;1} = {}_l v^5 \cdot {}_5 p_x = \left(\frac{1}{1.06}\right)^5 \cdot (.98)^5 = .6755. $$

Under the falling interest rate scenario, we have ${}_f v^5 = .8143$ (from Table 15.4) along with the value ${}_5 p_x = (.98)^5 = .9039$, so the APV for the five-year pure endowment is $(.8143)(.9039) = .7361$. The ratio of the falling rate APV to the level rate APV is $\frac{.7361}{.6755} = 1.0897$.

Since the falling rate APV is more that 5% above the level rate APV, the annuity company is required to hold additional capital for each five-year pure endowment product that it sells. ❒

15.3 SPOT INTEREST RATES AND THE TERM STRUCTURE OF INTEREST RATES

We now drop the assumption made in Sections 15.1 and 15.2 that the market consists only of one-period securities, and move to a more realistic set of investment products. We assume that it is possible to buy interest-bearing securities of varying maturities. Also, we assume that some of these interest-bearing securities make periodic interest payments every six months and make an interest and principal payment at maturity. For the sake of simplicity, we assume that all of these securities are risk-free (i.e., they are certain to pay interest and principal with no chance of default), and we refer to all of them as *bonds*. Bonds with periodic interest payments are called *coupon bonds* whereas bonds with no periodic payments and a single payment at maturity are called *zero-coupon bonds*. There is a large market in United States Treasury securities fitting these descriptions.

Table 15.5 shows available interest rates for coupon-bearing treasury securities of varying maturities on a particular date.[3]

TABLE 15.5

Maturity (in years)	Nominal Annual Yield for Coupon-bearing Bonds ($i^{(2)}$)
0.5	2.44%
1.0	2.60
1.5	2.76
2.0	2.93

This table suggests that on the day in question, we could expect to purchase a treasury security with a maturity of six months at a yield of 2.44%.[4] In other words, for an investment of $1000, we would expect to receive $1012.20 in six months. Note that the coupon payment, made in addition to the principal, is half the stated yield. A one-year bond purchased the same day would pay $13 in six months and $1013 at the end of one year.

[3] Source: Daily Treasury Yield Curve Rates at www.ustreas.gov; 1.5 year yield is interpolated.

[4] In reality, such a security with exact yield and maturity dates may not be available on that day.

Similarly, a two-year bond would entitle the purchaser to three semi-annual payments of \$14.65 and a final payment of \$1014.65.

The first important feature of this table is that bonds with differing maturities offer differing rates of interest. The extra yield for longer-term bonds reflects the loss of liquidity that investors suffer by committing their money for a longer period of time, and can be thought of as a type of *liquidity premium*. Differences in yield also reflect market expectations for what the future short-term rates of interest will be. On occasion, expectations for lower short-term rates in the future will offset the liquidity premium and longer maturities will have lower yields than shorter maturities.

A second important feature of the table is that there is an implied set of zero-coupon bond interest rates for each maturity listed in the table, which can be derived from the coupon-bearing bond yields using a method called *bootstrapping*. First, the zero-coupon bond yield for a maturity of six-months must equal that of the coupon-bearing bond, since both consists of only a single payment at that maturity. Therefore the nominal annual yield, convertible semiannually, for a six-month zero-coupon bond, denoted $z_{0.5}$, is $z_{0.5} = 2.440\%$, or 1.220% as an effective semiannual rate.

To calculate the yield for a zero-coupon bond which matures in one year, we use the six-month zero-coupon rate to value the six-month coupon payment and the original price of the bond to determine the implied one-year zero-coupon rate. For example, the one-year bond described above pays \$13 in six months and \$1013 in one year for the price of \$1000. Therefore the implied one-year zero-coupon yield, denoted $z_{1.0}$, must satisfy

$$1000 \ = \ \frac{13}{1.01220} + \frac{1013}{\left(1+\frac{z_{1.0}}{2}\right)^2},$$

where .01220 is effective semiannual and $z_{1.0}$ is nominal annual, convertible semiannually. From this we obtain $z_{1.0} = 2.601\%$. Similarly, $z_{1.5}$ must satisfy

$$1000 \ = \ \frac{13.80}{1.01220} + \frac{13.80}{\left(1+\frac{.02601}{2}\right)^2} + \frac{1013.80}{\left(1+\frac{z_{1.5}}{2}\right)^3}.$$

(The reader should note that the one-year zero-coupon rate was used for that maturity, rather than the one-year coupon-bearing rate.) From this we obtain $z_{1.5} = 2.763\%$. Finally, using

$$1000 \ = \ \frac{14.65}{1.01220} + \frac{14.65}{\left(1+\frac{.02601}{2}\right)^2} + \frac{14.65}{\left(1+\frac{.02763}{2}\right)^3} + \frac{1014.65}{\left(1+\frac{z_{2.0}}{2}\right)^4},$$

we determine that $z_{2.0} = 2.936\%$. In summary, the bootstrap method produces the results shown in Table 15.6.

TABLE 15.6

Maturity (in years)	Nominal Annual Yield for Coupon-bearing Bonds	Nominal Annual Yield for Zero-coupon Bonds
0.5	2.44%	2.440%
1.0	2.60	2.601
1.5	2.76	2.763
2.0	2.93	2.936

Regarding Table 15.6, the dependence of available yields on years to maturity is referred to as the *term structure of interest rates*. The associated zero-coupon bond rates are often referred to as *spot rates*. Once a set of spot rates has been obtained, it is easy to value any set of cash flows, whether or not those cash flows are uniform.

EXAMPLE 15.4

To finance the construction of an auditorium, a college has agreed to make the following payments at the following maturity times:

Payment	\$200,000	\$50,000	\$50,000	\$100,000
Maturity	Today	6 months	12 months	24 months

Using the term structure of interest rates in Table 15.6, calculate the net present value of these payments.

SOLUTION

We directly find

$$NPV = 200,000 + \frac{50,000}{1.01220} + \frac{50,000}{\left(1+\frac{.02601}{2}\right)^2} + \frac{100,000}{\left(1+\frac{.02936}{2}\right)^4} = 392,459.12. \quad ❒$$

EXAMPLE 15.5

A client age 60 purchases a five-year term life insurance policy that will pay \$1,000,000 at the end of the year of death. The client will fund the policy with level annual premiums, and the insurance company has the ability to lock in appropriate forward rates of interest on those premiums. Using the information in Table 15.7, calculate the net level annual premium for the policy.

TABLE 15.7

Maturity (in years)	Annual Yield for Zero-coupon Bonds	x	q_x
1.0	3.0%	60	.02
2.0	4.0	61	.03
3.0	5.0	62	.04
4.0	6.0	63	.05
5.0	7.0	64	.06

SOLUTION

The most straightforward solution to this problem is to calculate the APV of the premium payments and set it equal to the APV of the insurance death benefit, using the equivalence principle. For a level annual premium, P, the APV of premium is

$$P\cdot\ddot{a}_{60:\overline{5}|} = P(1+vp_{60}+v^2\ {}_2p_{60}+v^3\ {}_3p_{60}+v^4\ {}_4p_{60}),$$

where each v^t value is calculated using the t-year spot rate. Using this and the mortality rates shown above, we have

$$P\cdot\ddot{a}_{60:\overline{5}|} = P\left[1+\frac{.98}{1.03}+\frac{(.98)(.97)}{(1.04)^2}+\frac{(.98)(.97)(.96)}{(1.05)^3}+\frac{(.98)(.97)(.96)(.95)}{(1.06)^4}\right].$$

From this we find the APV of the premiums to be $4.3054P.$ The APV of the death benefit is

$$A^{1}_{60:\overline{5}|} = vq_{60}+v^2p_{60}\cdot q_{61}+v^3\ {}_2p_{60}\cdot q_{62}+v^4\ {}_3p_{60}\cdot q_{63}+v^5\ {}_4p_{60}\cdot q_{64},$$

where, again, spot interest rates are used. (As an exercise, the reader should verify that $A^{1}_{60:\overline{5}|}=.1527.$) Then the net level premium is

$$P = \frac{(1{,}000{,}000)(.1527)}{4.3054} = 35{,}467.09.$$ ❐

15.4 FORWARD INTEREST RATES

For this section, we assume a financial environment in which investors can buy and sell zero-coupon bonds that pay interest at current spot rates in any dollar amount and with no transaction costs. In such an environment, current spot rates imply another set of interest rates that can be locked in today for future deposits. For example, suppose an investor simultaneously undertakes the following pair of transactions:

Transaction A:	Buy a \$1000 par value two-year zero-coupon bond paying 2.96% interest.
Transaction B:	Sell a \$1000 par value one-year zero-coupon bond paying 2.62% interest.

With this pair of transactions, the investor has a net cash flow of zero today. In one year he must pay principal and interest on the one-year bond, and in two years he will receive principal and interest on the two-year bond. The resulting net cash flows experienced by the investor are shown in Table 15.8.

TABLE 15.8

Time (in years)	Net Cash Flow
0	$ 0.00
1	− 1026.20
2	1060.08

These are the same cash flows that would be experienced by an investor who agrees one year in advance to invest $1026.20 in a zero-coupon bond at 3.30% interest (except for some small round-off error). Therefore by purchasing and selling securities of differing maturities today, an investor can "lock in" a return on an investment one or more periods from now. In the current example, we say that the 3.30% interest rate obtained for an investment one year from now is the *one-year forward one-year rate*, since the interest rate obtained is for an investment one year from now (i.e., one year forward) and is obtained for a one-year security. When a similar set of transactions is implemented to lock in a rate n years from now on a k-year zero coupon bond, the resulting rate is called the *n-year forward k-year rate.* We denote this rate by $f_{n,k}$.

EXAMPLE 15.6

Using the yields in Table 15.9, find all possible forward rates for forward securities with maturities of one, two, three, and four years.

TABLE 15.9

Maturity (in years)	Annual Yield for Zero-coupon Bonds
1.0	3.0%
2.0	4.0
3.0	5.0
4.0	6.0
5.0	7.0

SOLUTION

We show here the calculations for $f_{1,4}$ and $f_{2,2}$. (Calculations for other forward rates are similar, and are left to the reader as Exercise 15-12.) $f_{1,4}$ is the only forward four-year rate that can be calculated from the rates in the table. This rate is most easily calculated using the logic that an investor obtains the same total return *either* by buying a five-year zero-coupon bond, *or* by investing in a one-year bond and then investing the proceeds for four years at the one-year forward four-year rate. That is,

$$(1+z_5)^5 = (1+z_1)^1 \cdot (1+f_{1,4})^4. \tag{15.1}$$

In this case we have

$$(1.07)^5 = (1.03)^1 \cdot (1+f_{1,4})^4,$$

from which we find $f_{1,4} = 8.024\%$. Similarly, $f_{2,2}$ must satisfy

$$(1+z_4)^4 = (1+z_2)^2 \cdot (1+f_{2,2})^2, \tag{15.2}$$

from which we find $f_{2,2} = 8.038\%$. All of the forward rates, rounded to four decimal places, are shown in Table 15.10. ❒

TABLE 15.10

n	$f_{n,1}$	$f_{n,2}$	$f_{n,3}$	$f_{n,4}$
1.0	5.01%	6.01%	7.02%	8.02%
2.0	7.03	8.04	9.05	–
3.0	9.06	10.07	–	–
4.0	11.10	–	–	–

EXAMPLE 15.7

A five-year pure endowment contract issued to a person age 60 is funded with level annual premiums and has a maturity benefit of \$10,000. Premiums are payable at the beginning of each year, and the benefit is payable at the end of the fifth year. Table 15.11 shows mortality rates for a 60-year-old and forward rates that are currently available. Use this information to calculate the net level annual premium for the pure endowment. Note that $f_{0,5} = z_5$.

TABLE 15.11

y	$f_{y,5-y}$	x	q_x
0	4.0%	60	.02
1	5.0	61	.03
2	6.0	62	.04
3	7.0	63	.05
4	8.0	64	.06

SOLUTION

Since we are given forward rates, it will be easiest to determine the level annual premium retrospectively. The premiums must accumulate with interest and survivorship to total \$10,000 at the end of the fifth year. That is,

$$10{,}000 = P \cdot \ddot{s}_{60:\overline{5}|} = P\left(\frac{1}{{}_5E_{60}} + \frac{1}{{}_4E_{61}} + \frac{1}{{}_3E_{62}} + \frac{1}{{}_2E_{63}} + \frac{1}{{}_1E_{64}}\right),$$

where, for example,

$$_3E_{62} \;=\; \frac{_3p_{62}}{(1+f_{2,3})^3} \;=\; \frac{(.96)(.95)(.94)}{(1.06)^3} \;=\; .7198.$$

Similar calculations produce

$$10{,}000 \;=\; P\cdot \ddot{s}_{60:\overline{5}|} \;=\; P\left(\frac{1}{.6698}+\frac{1}{.6841}+\frac{1}{.7198}+\frac{1}{.7800}+\frac{1}{.8704}\right),$$

from which we find $P=1476.02$.

Note that we could also have found the net level annual premium prospectively by first converting the forward rates to current spot rates. We first note that $z_5 = f_{0,5} = 4.0\%$. Then to calculate z_n for $n<5$, we use the relationship

$$(1+z_n)^n\cdot(1+f_{n,5-n})^{5-n} \;=\; (1+z_5)^5. \tag{15.3}$$

The resulting spot rates, rounded to four decimal places, are shown in Table 15.12. (The reader should verify that they are correct.)

TABLE 15.12

n	z_n
1	0.094%
2	1.071
3	2.049
4	3.023
5	4.000

Then prospectively we have

$$10{,}000\,{}_5E_{60} \;=\; P\cdot\ddot{a}_{60:\overline{5}|} \;=\; P(1+v\cdot{}_1p_{60}+v^2\cdot{}_2p_{60}+v^3\cdot{}_3p_{60}+v^4\cdot{}_4p_{60}).$$

The left side of this equation evaluates to

$$10{,}000v^5\cdot{}_5p_{60} \;=\; \frac{(10{,}000)(.98)(.97)(.96)(.95)(.94)}{(1.04)^5} \;=\; 6698.13,$$

where the five-year spot rate has been used. For the right side of the equation, each v^n is calculated using the corresponding spot rate z_n. From this we find $\ddot{a}_{60:\overline{5}|}=4.53795$, from which we again find

$$P \;=\; \frac{6698.13}{4.53795} \;=\; 1476.02. \qquad ❐$$

15.5 TRANSFERRING THE INTEREST RATE RISK

The overriding theme of this text is that persons facing financial risks can be relieved of those risks by paying an insurer to assume them. From the insurer's perspective, there are three primary risks associated with a contract of life insurance, namely those of expenses, mortality, and interest.

The insurer charges for the expenses of doing business by increasing the net premiums to reach the contract premiums (or gross premiums), actually paid. If operational expenses turn out to be less than assumed in setting the contract premiums, the insurer makes a profit on the expense element. If the opposite turns out to be the case, then the insurer loses money on the expense element. Generally insurers are fairly good at charging for their expenses, so the *expense risk* is not very great.

For many years the view was held that the major risk to the insurer was the *mortality risk*. If failures occurred earlier than, or at greater rates than, as predicted by the underlying survival model, the insurer suffered losses on the mortality element under life insurance contracts. Under annuities, the opposite would be true; the insurer would suffer a loss if mortality was lighter (i.e., if annuitants lived longer) than as predicted by the survival model.

By assuming that the lifetimes of different policyholders are independent, the insurer can *diversify* the mortality risk over the collection of policyholders. Some will fail earlier and some later, so that the aggregate risk can be better predicted. In light of this, we refer to the mortality risk as a *diversifiable risk*. (This concept was illustrated in Section 9.3.)[5]

When the insurer selects an interest rate for the premium calculation, it is assuming that it will be able to earn that rate on its invested assets backing the insurance or annuity contracts. If it earns interest on its assets at a greater rate than that assumed, it makes a profit on the interest element. On the other hand, the insurer faces an *interest rate risk* that earned rates will fall below assumed rates and it will therefore suffer a loss on the interest element. This has been a problem for many insurers in recent years.

If an interest loss occurs, due to falling interest rates in the investment marketplace, it will occur on all contracts alike. For this reason we refer to the interest rate risk as a *non-diversifiable risk*.

Although the insurer cannot diversify the interest rate risk across the collection of policyholders, it is possible for the insurer to *transfer* part or all of that risk back to the insured. When this is done under a life insurance or annuity contract, we say that the policyholder is participating in the interest rate risk.[6]

In this text we explore how this is accomplished under variable or indexed universal life insurance contracts. (See Sections 16.2.1 and 16.2.3.) For annuity contracts, transferring all or part of the interest rate risk to the annuitant occurs under *variable annuity contracts*. Such contracts are not discussed in this text.

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[5] See Appendix C for a mathematical analysis of risk diversification.

[6] Another strategy available to an insurer to reduce interest rate risk is *hedging*.

15.6 EXERCISES

15.1 Actuarial Present Values Using Variable Interest Rates

15-1 Complete Example 15.1 for Scenarios 2 and 3.

15-2 Complete Example 15.2 for Scenarios 2 and 3.

15.2 Deterministic Interest Rate Scenarios

15-3 (a) Complete part (a) of Example 15.3 for the rising interest rate scenario.

(b) Complete part (b) of Example 15.3 for the rising interest rate scenario.

15-4 A company sells insurance in a country where only one-year bonds are available as investments to back its business. Our task is to compare the interest sensitivity of the following three products in this environment.

(i) A 5-year immediate annuity-certain, where payments are made regardless of survival status.
(ii) A 5-year immediate life annuity.
(iii) A single premium 5-year term insurance contract.

The applicable failure rates are $q_x = .10, q_{x+1} = .15, q_{x+2} = .20, q_{x+3} = .25,$ and $q_{x+4} = .30.$

(a) Assuming today's interest rate is 7%, calculate the actuarial present value for each of the three products using each of the following two interest rate scenarios:

(1) Increasing: rates rise by 1% each year, but do not exceed 11% in any year.
(2) Decreasing: rates fall by 1% each year, but do not fall below 3% in any year.

(b) Which of the products is least interest sensitive in this environment? Explain.

15-5 For the same country and interest scenarios as in Exercise 15-4, we wish to evaluate the following two similar products:

(1) Single premium 10-year term insurance of face amount \$1000, with benefit paid at the end of the year of failure.
(2) Annual premium 10-year term insurance of face amount \$1000, with benefit paid at the end of the year of failure. The level annual premiums are paid at the beginning of each year.

(a) For both products, assume q_x = .05 for all years. Calculate the benefit premium for each product, assuming rates remain level over the life of the product.

(b) Calculate the actuarial present value of the gain for each product under the increasing and decreasing scenarios. (Note that the premium was chosen so that the actuarial present value in each case is zero in the event of level rates.)

(c) In terms of interest risk, which payment scheme appears less risky for the insurance company? Explain.

15.3 Spot Interest Rates and the Term Structure of Interest Rates

15-6 Verify that $A^{\,1}_{60:\overline{5|}} = .1527$ in Example 15.5.

15-7 Use the nominal annual coupon yields in the table below to calculate the corresponding zero-coupon yields of the same maturities. (In both cases the nominal annual yield rates are convertible semiannually.)

Maturity (in years)	Nominal Annual Yield for Coupon-bearing Bonds	Nominal Annual Yield for Zero-coupon Bonds
0.5	2.0%	
1.0	4.0	
1.5	6.0	
2.0	8.0	

15-8 Use the annual coupon yields in the table below to calculate the corresponding zero-coupon yields of the same maturities. (For this exercise, we assume annual-payment coupon bonds rather than semiannual-payment coupon bonds.) How does the solution compare to that of Exercise 15-7?

Maturity (in years)	Annual Yield for Coupon-bearing Bonds	Annual Yield for Zero-coupon Bonds
1	2.0%	
2	4.0	
3	6.0	
4	8.0	

15-9 Use the annual zero-coupon yields in the table below to calculate the corresponding yields for annual-payment coupon bonds of the same maturities. (We assume here that coupon bonds pay coupons annually rather than semiannually.)

Maturity (in years)	Annual Yield for Coupon-bearing Bonds	Annual Yield for Zero-coupon Bonds
1		2.0%
2		4.0
3		6.0
4		8.0

15-10 Assume the following zero-coupon rates and calculate the implied yields for coupon bonds with equivalent maturities. (In both cases the nominal annual yield rates are convertible semiannually.)

Maturity (in years)	Nominal Annual Yield for Coupon-bearing Bonds	Nominal Annual Yield for Zero-coupon Bonds
0.5		2.0%
1.0		4.0
1.5		6.0
2.0		8.0

15-11 The regents of Fantastic University provide a four-year scholarship for one incoming freshman who plans to major in actuarial science. Current tuition at Fantastic is \$26,000 per year and tuition is expected to increase 8% per year over the next four years. The first annual tuition payment is due today. Each year we assume a 25% chance that the scholarship recipient will change majors or drop out of school; either event cancels future scholarship payments. Using the table of yields from Exercise 15-9, calculate the actuarial present value of this scholarship.

15.4 Forward Interest Rates

15-12 Complete Example 15.6 by verifying the $f_{n,k}$ values shown in Table 15.10.

15-13 Verify the spot rate values shown in Table 15.12.

15-14 Using the n-year forward one-year rates in the following table, find all determinable spot rates.

n	$f_{n,1}$
0	4.0%
1	5.0
2	6.0
3	7.0
4	8.0

15-15 Using the n-year forward one-year rates from Exercise 15-14, find all available forward rates.

15-16 In connection with taking over a client's retirement account, the client agrees to invest \$300,000 of that account with your firm for three years, starting two years from now.

(a) According to the interest rates in Exercise 15-15, what rate of interest can be locked in for the investment period?

(b) What spot-rate transactions should be entered into today in order to lock in the yield found in part (a)? Include the term and principal amount of the two transactions.

15-17 Due to the demise of a distant relative, you will receive $25,000 in one year that you would like to invest at that time for two years.

(a) According to the rates in Exercise 15-15, what rate can be locked in for the investment period?

(b) What transactions should be entered into today in order to lock in the rate from part (a)? Include the terms and principal amounts of the two transactions.

15-18 Calculate all forward rates that can be inferred from the annual coupon-bearing bond yield rates in the following table.

Maturity (in years)	**Annual Yield Rates for Coupon-bearing Bonds**
1	2.0%
2	4.0
3	6.0
4	8.0

15.5 Transferring the Interest Rate Risk

15-19 Give examples of mortality risk that is not diversifiable.

15-20 Explain the differences in interest rate risk for whole life insurance versus term life insurance.

15-21 Why is interest rate risk considered a non-diversifiable risk? Give an example of the effects of interest rate risk.

(b) $APV_{51}^{ER} = \sum_{y=61}^{64}\left[1-.03\left(65-y-\frac{1}{2}\right)\right]\cdot PAB_{y+1/2}\cdot v^{y+1/2-51}\cdot {}_{y-51}p_{51}^{(\tau)}\cdot q_y^{(r)}\cdot {}^{r}\ddot{a}_{y+1/2}^{(12)},$

where

$$PAB_{y+1/2} = .01\left(y+\tfrac{1}{2}-51\right)(100{,}000)\cdot\frac{1}{3}\left(\frac{\frac{1}{2}S_{y-3}+S_{y-2}+S_{y-1}+\frac{1}{2}S_y}{S_{51}}\right)$$

(c) $APV_{51}^{W} = \sum_{y=56}^{60} PAB_{y+1/2}\cdot v^{14}\cdot {}_{y-51}p_{51}^{(\tau)}\cdot q_y^{(w)}\ {}_{65-y-1/2}\overset{w}{p}_{y+1/2}\cdot {}^{r}\ddot{a}_{65}^{(12)}$

(d) $APV_{51}^{I} = \sum_{y=56}^{64} PAB_{y+1/2}\cdot v^{y+1/2-51}\cdot {}_{y-51}p_{51}^{(\tau)}\cdot q_y^{(i)}\cdot {}^{i}\ddot{a}_{y+1/2}^{(12)}$

(e) $APV_{51}^{D} = \sum_{y=61}^{64} .50\left[1-.03\left(65-y-\frac{1}{2}\right)\right]$

$\cdot PAB_{y+1/2}\cdot v^{y+1/2-51}\cdot {}_{y-51}p_{51}^{(\tau)}\cdot q_y^{(d)}\cdot {}^{r}\ddot{a}_{y+1/2-3}^{(12)}$

14-27 (a) 1860

(b) The APV for each benefit is calculated the same as in Exercise 14-25, except that PAB_{65} in part (a) and $PAB_{y+1/2}$ in parts (b)-(e) are all replaced by the benefit accrual 1860. The unit credit normal cost is the sum of these five APVs.

(c) The APV for each benefit is calculated the same as in part (b), except that the 1860 benefit accrual is replaced by the 7500 accrued benefit. The accrued liability is the sum of these five APVs.

14-29 (a) -4362.80 (b) 28.25

CHAPTER 15

15-1 577.93; 635.97

15-2 .1875; .1958

15-3 (a) 3.8461

(b) .6155

15-4 (a)

Interest Rate Scenario	Annuity Certain	Life Annuity	Term Insurance
Increasing	3.97	2.53	.53
Decreasing	4.25	2.65	.57

(b) The life annuity

15-5 (a) 289.84; 46.73

(b) Increasing: 23.74; 6.10
Decreasing: −29.25; −7.96

(c) The annual premium product

15-7 2.0%; 4.020%; 6.082%; 8.211%

15-8 2.0%; 4.041%; 6.169%; 8.447%

15-9 2.0%; 3.960%; 5.844%; 7.615%

15-10 2.0%; 3.980%; 5.921%; 7.804%

15-11 74,020

15-14

n	1	2	3	4	5
z_n	4.000%	4.499%	4.997%	5.494%	5.991%

15-15

n	$f_{n,1}$	$f_{n,2}$	$f_{n,3}$	$f_{n,4}$	$f_{n,5}$
0	4.00%	4.499%	4.997%	5.494%	5.991%
1	5.00	5.499	5.997	6.494	--
2	6.00	6.499	6.997	--	--
3	7.00	7.499	--	--	--
4	8.00	--	--	--	--

15-16 (a) 6.997%

(b) Sell a 2-year zero-coupon bond and buy a 5-year zero-coupon bond, each of face amount 274,724.24.

15-17 (a) 5.499%

(b) Sell a 1-year zero-coupon bond and buy a 3-year zero-coupon bond, each of face amount 24,038.46.

15-18

n	$f_{n,1}$	$f_{n,2}$	$f_{n,3}$
1	6.12%	8.32%	10.685%
2	10.56	13.04	--
3	15.58	--	--

15-19, 15-20, 15-21 (See Solutions Manual.)

CHAPTER 16

16-1 2,489.89; 2,479.75; 2,469.59