S. BROVERMAN STUDY GUIDE

FOR

SOA EXAM FM

2018 EDITION

Samuel Broverman, ASA, PHD

2brove@rogers.com
www.sambroverman.com

2brove@rogers.com
www.sambroverman.com

S. BROVERMAN EXAM FM/2 STUDY GUIDE
2018

TABLE OF CONTENTS

NOTES, EXAMPLES

AND PROBLEM SETS

SOA EXAM FM NOTES

©S. Broverman, 2018

Calculus Review

Natural log and exponential functions: $ln(x)$ (sometimes written $log(x)$) is the log to the base e ($e \approx 2.7183$) ; $ln(e) = 1$, $ln(1) = 0$, $e^0 = 1$, $ln(e^y) = y$, $e^{ln(x)} = x$, $ln(a^y) = y \cdot ln(a)$, $ln(y \cdot z) = ln(y) + ln(z)$, $ln(\frac{y}{z}) = ln(y) - ln(z)$, $e^x e^w = e^{x+w}$, $(e^x)^w = e^{x \cdot w}$

Differentiation: product rule $\frac{d}{dx}[g(x) \cdot h(x)] = g'(x)h(x) + g(x)h'(x)$;
quotient rule $\frac{d}{dx}[\frac{h(x)}{g(x)}] = \frac{h'(x)g(x) - g'(x)h(x)}{[g(x)]^2}$;
$\frac{d}{dx} ln[g(x)] = \frac{g'(x)}{g(x)}$; $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$; $\frac{d}{dx} a^x = a^x \cdot ln(a)$

Integration: $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$; $\int a^x \, dx = \frac{a^x}{ln(a)} + c$; $\int \frac{1}{a+bx} \, dx = \frac{1}{b} ln(a+bx) + c$;
$\frac{d}{dx} \int_a^x g(t) \, dt = g(x)$; $\frac{d}{dx} \int_x^b g(t) \, dt = -g(x)$;
if $k > 0$ then $\int_0^\infty e^{-kx} \, dx = \frac{1}{k}$;
if $k > 0$ and n is an integer ≥ 0 then $\int_0^\infty x^n e^{-kx} \, dx = \frac{n!}{k^{n+1}}$

Some useful finite and infinite series: The following series summations may arise on an exam question:

(i) **sum of the first n positive integers**: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(ii) **finite geometric series:** $1 + r + r^2 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$, $r + r^2 + \cdots + r^k = \frac{r-r^{k+1}}{1-r}$

(iii) **infinite geometric series:** if $|r| < 1$ then $1 + r + r^2 + \cdots = \frac{1}{1-r}$, $r + r^2 + \cdots = \frac{r}{1-r}$

(iv) **increasing geometric series:** infinite series, $1 + 2r + 3r^2 + \cdots = \frac{1}{(1-r)^2}$,

(v) **some series that are less likely to arise:**

exponential series, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

natural log series, if $|x| < 1$ then $ln(1+x) = x - + \frac{x^2}{2} + \frac{x^3}{3} -$

SECTION 1 - EFFECTIVE RATES OF INTEREST AND DISCOUNT

SECTION 1 - EFFECTIVE RATES OF INTEREST AND DISCOUNT

Sections 1.1-1.3 of "Mathematics of Investment and Credit"

Of the various dictionary definitions of **interest** (in the context of a financial transaction) that can be found, a typical one is that interest is the charge for or cost of borrowing money over a period of time. The **sum or amount of interest charged** is related to the **rate of interest** that is applied to the amount borrowed over the time period. Interest rates are generally described in one of two equivalent ways:

(i) as a percentage, such as 10%, or (ii) as a decimal, such as .10 (equivalent to 10%).

A financial transaction can take place over any period of time, and interest rates can be quoted for any time period, but the conventional way in which an interest rate is quoted is as an annual rate.

In the case of a simple loan or investment transaction in which an amount is borrowed (or invested) at a specified interest rate for a specified period of time and then repaid at the end of that period, the amount repaid is

Initial amount $borrowed +$ *Amount of Interest for the time period*, where

Amount of Interest for the time period $=$ *Amount borrowed* $\times$ *Interest rate for the time period* .

Alternatively, the interest rate for the time period is equal to

$$\frac{\textit{Amount of Interest for the time period}}{\textit{Amount borrowed at the start of the period}} .$$

The "interest rate for the time period" mentioned above is generally found from the quoted annual interest rate and applied over a period of time that may not be one year. It is important to note that the phrase "quoted annual interest rate" must be more precisely defined in order to know how to apply it over a specified period of time.

Simple Interest

At **annual simple interest rate** $\boldsymbol{i}$, an initial investment of amount 1 accumulates to $\boldsymbol{1 + i \times t}$ at time t. In this expression, t represents time and is usually measured in years. When simple interest at annual rate i is specified, the interest rate for a period of time of length t years is $i \times t$. If the amount of an investment or loan is C (at time 0), then at time t years, the amount of interest is $C \times i \times t$, and the total value of the investment or amount owing on the loan will be $C \times [1 + i \times t]$.

When simple interest is specified, there are variations in the way in which t can be measured. The two most common forms of simple interest are the following.
(i) **Ordinary simple interest**, in which $t = \frac{m}{12}$, where m is the number of months in the time period of the loan or investment. If a time period specifies a number of months in the context of simple interest, then this would be the way that t is measured.
(ii) **Exact simple interest**, in which $t = \frac{d}{365}$, where d is the exact number of days of the loan.
A variation of exact simple interest is the **Bankers Rule**, in which $t = \frac{d}{360}$.
Simple interest accumulation is usually restricted to periods of less than one year.

Compound Interest
At **annual effective compound interest rate $\boldsymbol{i}$**, an initial investment of 1 accumulates to $\boldsymbol{(1+i)^t}$ at time t (years). Often, t is a positive integer, but t can also involve a fractional value with accumulation sometimes referred to as **true or exact compound interest**. Although the phrase "effective interest rate" usually refers to a rate which is compounded annually, an effective interest rate can also be specified for any period of time (such as monthly effective rate, quarterly effective rate, etc.) and compounding can be done accordingly, in which case, t will be measured in appropriate units of time (months, quarters, etc.).

Note the following relationship between compound and simple interest for $i > 0$:
$(1+i)^t < 1 + it$ for $0 < t < 1$ and $(1+i)^t > 1 + it$ for $t > 1$.
For instance, with $i = .10$ (10%) and $t = 0.4$, we have
$(1.1)^{.4} = 1.03886 < 1.04 = 1 + (.4)(.1)$.

Another point to note is that the phrase "annual effective interest rate" may also be seen in some references as "effective annual interest rate".

Accumulation Function
For an investment of 1 made at time 0, the value of the investment at time t can be described in terms of an **accumulation function $\boldsymbol{A(t)}$**. For the case of simple interest at annual rate i the accumulation function is $A(t) = 1 + it$, and for compound interest it is $A(t) = (1+i)^t$. The accumulated value (AV) is also called the **future value**, and can be applied to the combined value of several amounts at a particular time point in the future.

Example 1 (SOA): Money accumulates in a fund at an annual effective interest rate of i during the first 5 years, and at an annual effective interest rate of $2i$ thereafter. A deposit of 1 is made into the fund at time 0. It accumulates to 3.09 at the end of 10 years and to 13.62 at the end of 20 years. What is the value of the deposit at the end of 7 years?

Solution: The accumulated value at time 10 is $(1+i)^5 \times (1+2i)^5 = 3.09$,
accumulated value at time 20 is $(1+i)^5 \times (1+2i)^{15} = 13.62$.
Then $\frac{(1+i)^5 \times (1+2i)^{15}}{(1+i)^5 \times (1+2i)^5} = (1+2i)^{10} = \frac{13.62}{3.09} = 4.4078 \rightarrow 2i = .16$, $i = .08$.
AV at the end of 7 years is $(1+i)^5 \times (1+2i)^2 = (1.08)^5 \times (1.16)^2 = 1.98$. □

Example 2 (SOA): Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal is made. Carl withdraws K at the end of each of years 4, 5, 6 and 7. The balance in the account at the end of year 10 is 10,000. Calculate K.

Solution: There are two (equivalent) ways to approach this problem. As a first approach, we can update the balance in the account at the time of each transaction until we reach the end of 10 years, and set the balance equal to 10,000 to solve for K from the resulting algebraic expression:
balance at $t=4$ (after interest and withdrawal) is $B_4 = 10,000 \times (1.04)^4 - (1.05)K$;
balance at $t=5$ is $B_5 = B_4 \times 1.04 = [10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K$;
at $t=6$, $B_6 = [[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K$;
at $t=7$, $B_7 = [[[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K](1.04) - K$;
at $t=10$, there is 3 years of compounding from time 7, so that $B_{10} = B_7 \times (1.04)^3$
$= [[[[10,000(1.04)^4 - (1.05)K](1.04) - (1.05)K](1.04) - K](1.04) - K](1.04)^3 = 10,000.$
Solving for K from this equation results in $K = 979.93$.

Alternatively, we can accumulate to time 10 the initial deposit and the withdrawals separately, subtracting the accumulated withdrawals from the accumulated deposits. The balance at time 10 is
$10,000(1.04)^{10} - K(1.05)(1.04)^6 - K(1.05)(1.04)^5 - K(1.04)^4 - K(1.04)^3 = 10,000$.
This is the same equation as in the first approach (and must result in the same value of K).

In general, when using compound interest, for a series of deposits and withdrawals that occur at various points in time, the balance in an account at any given time point is the accumulated values of all deposits minus the accumulated values of all withdrawals to that time point. This is also the idea behind the "dollar-weighted rate of return", which will be discussed in a later section of this study guide. □

Present Value

The **present value (PV) of 1 due in one year** is the amount required now to accumulate to amount 1 as of the end of one year from now. For instance, at an annual effective interest rate of 10%, X invested now accumulates to $X \times 1.1$ one year from now. In order for this accumulated value to be 1, we must have $X \times 1.1 = 1$, or equivalently, $X = \frac{1}{1.1} = .9091$; this is the present value of 1 due in 1 year. In general, at annual effective rate of interest i, the present value of 1 due in one year is $\frac{1}{1+i}$, which is denoted $\boldsymbol{v_i}$ in actuarial notation. Often the subscript i is dropped from the v-factor, if the interest rate being used is obvious from the context of the situation being considered. $v = \frac{1}{1+i}$ is sometimes referred to as the **present value factor** or **discount factor**. Note that if $i > 0$, then $v < 1$.

The **present value of 1 due in *t* years** is the amount required now to accumulate to amount 1 as of the end of t years from now. Present value is usually formulated in the context of compound interest, with the present value of 1 due in t years equal to $\boldsymbol{\frac{1}{(1+i)^t} = (1+i)^{-t} = v_i^t}$ (or, more simply, v^t). If the compound period is a month or a quarter, we can still define a present value factor for the appropriate compounding period. For instance, with an monthly effective interest rate of 2% (compounding at 2% every month), the one-month present value factor (or discount factor) would be $\frac{1}{1.02} = v_{.02} = .9804$ (this is the amount needed now to accumulate to 1 in one month), and the present value of 1 due in t months is $\frac{1}{(1.02)^t} = v_{.02}^t$ (this is the amount needed now to accumulate to 1 in t months).

Present value can also be formulated on the basis of simple interest, where the present value of 1 due in t years is equal to $\frac{1}{1+it}$. More generally, if an amount of 1 invested now accumulates to $A(t)$ in t years, then the PV of 1 due in t years based on that accumulation function is $\frac{1}{A(t)}$.

An **equation of value** for a financial transaction equates, at a particular point in time, the present and accumulated value of all amounts received with the present and accumulated values of all amounts to be disbursed (or paid out).

An investment will consist of cash outflows and inflows over time. There may a single large investment (outflow) at time 0 followed by income (inflows) in the future, or the investment may involve several outflows at different times. Suppose we denote by C_k the net outflow/inflow at time point k, including time 0, where a negative value of C_k indicates an outflow. If an investor has a target rate of return i per time period for the investment, then the **net present value** of the investment based on that return is $\sum_{k=0}^{n} C_k \times v_i^k$. The net present value can be a basis for comparing investment options.

Example 3 (SOA): At an annual effective interest rate of i, $i > 0$, each of the following two sets of payments has present value K:

(i) A payment of 121 immediately and another payment of 121 at the end of one year.

(ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

Calculate K.

Solution: At time 0 (now) the equation of value which equates the present values of the two sets of payments is $K = 121 + 121v = 144v^2 + 144v^3$. After factoring, this becomes
$121(1+v) = 144v^2(1+v) \rightarrow v^2 = \frac{121}{144} \rightarrow v = \frac{11}{12} \rightarrow K = 121 + 121(\frac{11}{12}) = 231.92$.
Note that the value of i implied is the solution of $\frac{11}{12} = v = \frac{1}{1+i}$, so that $i = \frac{1}{11} = .0909$. $\square$

Calculator Note 1, Accumulated and Present Value

Accumulated values and present values of single payments using annual effective interest rates can be made in the following way. Clear calculator registers before starting the keystroke sequence with $\boxed{\textbf{2nd}}$ $\boxed{\textbf{CLR WORK}}$ **.**

Accumulated Value: A deposit of 100 made at time 0 grows at annual effective interest rate 5%. The accumulated value at the end of 10 years is $100(1.05)^{10} = 162.89$ **. This can be found using the calculator in two ways.**

1. Calculator in standard-calculator mode.
Key in 1.05 $\boxed{y^x}$ **, key in 10 ,** $\boxed{=}$ $\boxed{\times}$ **, key in 100 ,** $\boxed{=}$
The screen should display 162.8894627. In this function, $y = 1.05$ **and** $x = 10$**.**
2. Calculator in prompted-worksheet mode.
Key in $\boxed{\textbf{2nd}}$ $\boxed{\textbf{P/Y}}$ $\boxed{\downarrow}$ **key in 1 (this sets 1 compounding period per year).**
Key in $\boxed{\textbf{2nd}}$ $\boxed{\textbf{QUIT}}$ **(this returns calculator to standard-calculator mode)**
Key in 100 $\boxed{\textbf{PV}}$ $\boxed{\textbf{ENTER}}$ **, key in 5** $\boxed{\textbf{I/Y}}$ $\boxed{\textbf{ENTER}}$ **,**
key in 10 $\boxed{\textbf{N}}$ $\boxed{\textbf{ENTER}}$ **, key in** $\boxed{\textbf{CPT}}$ $\boxed{\textbf{FV}}$
The screen should display $-$ **162.8894627.**

The calculator interprets the PV of 100 as an amount received (an in-cashflow) and the FV as the amount that must be paid back (an out-cashflow), so the FV is a "negative" cashflow. If the PV had been entered as -100, then FV would have been positive.

Present Value: The present value of 500 due in 8 years at annual effective rate of interest 8% is $500v_{.08}^{8}=270.13$. This can be found using the calculator in two ways.
1. Calculator in standard-calculator mode.
Key in 1.08 $\boxed{y^x}$, key in 8 , $\boxed{+/-}$ $\boxed{=}$ $\boxed{\times}$, key in 500 $\boxed{=}$
The screen should display 270.1344423. This keystroke sequence can be replaced by
Key in 1.08 $\boxed{1/x}$ $\boxed{y^x}$, key in 8 , $\boxed{=}$ $\boxed{\times}$, key in 500 , $\boxed{=}$.
2. Calculator in prompted-worksheet mode.
Key in $\boxed{\text{2nd}}$ $\boxed{\text{P/Y}}$ $\boxed{\downarrow}$ key in 1 (this sets 1 compounding period per year).
Key in $\boxed{\text{2nd}}$ $\boxed{\text{QUIT}}$ (this returns calculator to standard-calculator mode)
Key in 500 $\boxed{\text{FV}}$ $\boxed{\text{ENTER}}$, key in 8 $\boxed{\text{I/Y}}$ $\boxed{\text{ENTER}}$,
key in 8 , $\boxed{\text{N}}$ $\boxed{\text{ENTER}}$, key in $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$
The screen should display -270.1344423. (same comment applies about the negative value).

As a more general procedure, in the equation $(PV)(1+i)^N=FV$, if any 3 of the 4 variables PV , i , N , FV are entered, then the 4th can be found using the $\boxed{\text{CPT}}$ function. As an example, suppose that an initial investment of 100 at annual effective rate of interest i grows to 300 in 10 years. Then $100(1+i)^{10}=300$, from which we get $i=(3)^{1/10}-1=.1161$ (11.61%). The keystroke sequence solving for i is
key in $\boxed{\text{2nd}}$ $\boxed{\text{P/Y}}$ $\boxed{\downarrow}$ key in 1 ,
key in 100 $\boxed{\text{PV}}$ $\boxed{\text{ENTER}}$, key in 300 $\boxed{+/-}$ $\boxed{\text{FV}}$ $\boxed{\text{ENTER}}$,
key in 10 , $\boxed{\text{N}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{CPT}}$ $\boxed{\text{I/Y}}$
The screen should display 11.61 (this is the % measure).

The function described in Calculator Note 1 is also valid for fractional exponents. For instance, $100\times(1.05)^{10.5}$ can be found by keying in 10.5 instead of 10 in the accumulated value calculation above.

Rate of Discount d

Just as miles and kilometers are alternative measures for distance, there are alternative ways for measuring investment behavior. Actuarial terminology has a phrase to describe alternative ways of measuring investment growth. To say that two measures are **equivalent rates** of growth means that the measure describe the same pattern of investment growth.

The **annual effective rate of discount d** is a way of measuring investment growth but it is usually applied in the context of formulating present value: $1-d$ is the present value of 1 due in 1 year. We have already seen that given an annual effective interest rate of i. the present value of 1 due in one year is $v=\frac{1}{1+i}$. Now using an annual effective discount rate d, we can also formulate the present value of 1 due in one year as $1-d$. If we equate the two formulations we have for present value, we get $v=\frac{1}{1+i}=1-d$. If these relationships are satisfied, we say that i and d are **equivalent rates**, as they both describe the same pattern of investment behavior. For equivalent rates of interest i and discount d, their numerical will generally not be the same. For instance, if $i=.10$, then $v=\frac{1}{1.1}=.9091=1-.0909$, so in order for the discount rate d to be equivalent to $i=.10$ the relationship $.9091=1-d$ must be satisfied, and therefore, $d=.0909$. Thus, the annual effective interest rate of 10% is equivalent to the annual effective discount rate of 9.09%. Just as a mile and a kilometer are not equal in length, neither are the annual effective interest rate and the annual effective discount rate equal numerically. Miles or kilometers can be used to describe equal lengths (one mile = 1.625 km) and interest and discount can be used to describe the same investment behavior: $i=.10$ and $d=.0909$ both refer to a PV factor of $.9091$, and they are referred to as equivalent rates.

Although they are different numbers, i and d can refer to the same growth pattern for invested money. To say that .9091 grows to 1 in 1 year means that the money has grown by 10% (that is the interest rate i). It also means that the present value at the start of the year is 9.09% less than the accumulated value of 1 at the end of the year, which is described by saying that .9091 is the discounted value of 1 at discount rate 9.09%.

Note the following relationships that come from equivalent i and d:
i and d are equivalent rates (although different numerically) if $\boldsymbol{1-d=v=\frac{1}{1+i}}$, or alternatively, if $\boldsymbol{d=\frac{i}{1+i}}$ or $\boldsymbol{i=\frac{d}{1-d}}$.
Thus, equivalent rates i and d satisfy the inequality $d<i$ (for positive rates).

Another way of looking at the discount rate and its connection the equivalent interest rate is as follows. It was mentioned earlier that the interest rate for the time period is equal to $\frac{\textit{Amount of Interest for the time period}}{\textit{Amount borrowed at the start of the period}}$.

The discount rate for that same time period is $\frac{\textit{Amount of Interest for the time period}}{\textit{Amount owing at the end of the period}}$.

We can see this in very basic case in which the loan is of amount 1 with interest for the period at rate i. The interest rate is, or course $\frac{i}{1}$, or just i for the period. The amount owing at the end of the period is $1+i$ and the discount rate is $\frac{i}{1+i}$, which we see is d in the equivalence relationship mentioned in the previous paragraph. Again, i and d are different numerically. They are two different, but equivalent ways of describing the same transaction.

Note also that for equivalent i and d, we have the relationship $id = i - d$, and $\frac{1}{d} = \frac{1}{i} + 1$. The algebraic link between interest and discount rates applies to any periodic rates. If j is the compound rate of interest per month and d_j is the equivalent compound rate of discount per month, then $\boldsymbol{v_j = \frac{1}{1+j} = 1 - d_j}$, $\boldsymbol{d_j = \frac{j}{1+j}}$ and $\boldsymbol{j = \frac{d_j}{1-d_j}}$.

Compound discount can also be formulated, with the present value of 1 due in n years being $\boldsymbol{(1-d)^n = v^n}$. It is possible (although somewhat unnatural) to use compound discount to formulate accumulation. At annual effective discount rate d, the accumulated value of 1 in n years is $(1-d)^{-n}$, since $\boldsymbol{(1-d)^{-1} = 1+i}$ for the equivalent annual effective rate of interest i. For instance, if the annual effective rate of discount is 6%, then the equivalent annual effective rate of interest is $\frac{.06}{1-.06} = .063830$ (6.3830%), and $1 - .06 = .94 = \frac{1}{1.06383} = (1.06383)^{-1}$. The accumulation factor for 5 years would be $(1.063830)^5 = 1.3626 = (1-.06)^{-5} = (.94)^{-5}$.

An annual **simple discount rate** $\boldsymbol{d}$, refers to the situation in which the present value of 1 due in t years is $1 - dt$, with similar considerations given to measuring t as in the case of simple interest.

When an amount of 1 is invested at time 0 and the investment growth is represented by the accumulation function $A(t)$, the investment growth from time t_1 to time t_2 is $A(t_2) - A(t_1)$. This may also be described as the **amount of interest** earned from time t_1 to time t_2.

Calculator Note 2 , Accumulated and Present Values Using a Discount Rate

Accumulated values and present values of single payments using an annual effective rate of discount can be made in the following way. Clear calculator registers before starting the keystroke sequence.

Accumulated Value: A deposit of 25 made at time 0 grows at annual effective discount rate 6%. The accumulated value at the end of 5 years is
$25(1-.06)^{-5} = 25(.94)^{-5} = 34.06.$
This can be found using the calculator in two ways.
Calculator is in standard calculator mode.

1. Key in .94 $\boxed{y^x}$ **, key in 5** $\boxed{+/-}$ $\boxed{=}$ $\boxed{\times}$ **, key in 25 ,** $\boxed{=}$
The screen should display 34.06 (rounded to nearest .01).

2. Key in 25 , $\boxed{\text{PV}}$ $\boxed{\text{ENTER}}$**, key in 6 ,** $\boxed{+/-}$ $\boxed{\text{I/Y}}$ $\boxed{\text{ENTER}}$**,**
key in 5 , $\boxed{+/-}$ $\boxed{\text{N}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$
The screen should display $-$ **34.06 (negative sign indicating outflow).**

Present Value: The present value of 500 due in 8 years at annual effective rate of discount 8% is $500(1-.08)^8 = 500(.92)^8 = 256.61.$ **This can be found using the calculator in two ways.**

Calculator is in standard calculator mode.

1. Key in .92 $\boxed{y^x}$ **, key in 8 ,** $\boxed{=}$ $\boxed{\times}$ **, key in 500 ,** $\boxed{=}$
The screen should display 256.61.

2. Key in 500 , $\boxed{\text{FV}}$ $\boxed{\text{ENTER}}$**, key in 8 ,** $\boxed{+/-}$ $\boxed{\text{I/Y}}$ $\boxed{\text{ENTER}}$**,**
key in 8 , $\boxed{+/-}$ $\boxed{\text{N}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{CPT}}$ $\boxed{\text{PV}}$
The screen should display $-$ **256.61.**

Example 4 (SOA): A deposit of X is made into a fund which pays an annual effective interest rate of 6% for 10 years. At the same time, $X/2$ is deposited into another fund which pays an annual effective rate of discount of d for 10 years. The amounts of interest earned over the 10 years are equal for both funds. Calculate d.

Solution: Amount of interest earned during ten years is equal to
accumulated value at time 10 $-$ initial amount invested :
the amount of interest earned by first fund is $X(1.06)^{10} - X = .790848X$
the amount of interest earned by the second fund is $\frac{X}{2}(1-d)^{-10} - \frac{X}{2} = \frac{X}{2}[(1-d)^{-10} - 1]$.
Since the amount of interest earned over the 10 years is the same for both funds, it follows that $.790848X = \frac{X}{2}[(1-d)^{-10} - 1] \rightarrow d = .0905.$ □

Average Rate of Interest

The conventional notion of the average of a set of numbers is the "simple average". This is found by adding the numbers and dividing the total by how many number are the set. When looking at compound interest over several compounding periods with differing effective rates for the periods, the notion used for average compound rate per period is not the simple average. For instance, suppose an investment of 1 grows to a value of $A(2)$ at time 2 years. If i is the average annual effective rate of interest for each of the two years, then $A(2) = (1+i)^2$ would be used to find i. If the annual effective rates of interest in the first and second years were 5% and 15%, then $A(2) = 1.05 \times 1.15 = 1.2075 = (1+i)^2$, so that $i = .098863$. This is less than the simple average of 5% and 15%, which is 10%. The average rate i is $\sqrt{1.05 \times 1.15}$. This is called the "geometric average" of 1.05 and 1.15. In general, the average annual effective growth factor for a period of years is the geometric average of the growth factors for the various years: $1 + i = [(1+i_1) \times (1+i_2) \times \cdots (1+i_n)]^{1/n}$. In general, the geometric average of positive numbers is less than the simple average unless all the numbers are the same, in which case the geometric average is equal to the simple average.

PROBLEM SET 1
Effective Rates of Interest and Discount

1. (SOA) Gertrude deposits 10,000 in a bank. During the first year, the bank credits an annual effective interest rate of i. During the second year, the bank credits an annual effective rate of interest $(i-5\%)$. At the end of two years, she has 12,093.75 in the bank. What would Gertrude have in the bank at the end of three years, if the annual effective rate of interest were $(i+9\%)$ for each of the three years?

A) 16,851 B) 17,196 C) 17,499 D) 17,936 E) 18,113

2. (SOA) At an annual effective interest rate of i, $i>0$, the following are all equal:

(i) the present value of 10000 a the end of 6 years:

(ii) the sum of the present values of 6000 at the end of year t and 56000 at the of year $2t$; and

(iii) 5000 immediately.

Calculate the present value of a payment of 8000 at the end of year $t+3$ using the same annual effective interest rate.

A) 1334 B) 1414 C) 1604 D) 1774 E) 2004

3. Smith invests $10,000 in a 120-day short-term guaranteed investment certificate at a bank, based on exact simple interest at annual rate of 9.5%. After 60 days, the interest rate has risen to 12% and Smith would like to redeem the certificate early and reinvest in a 60-day certificate at the higher interest rate. In order for Smith to have no advantage in redeeming early and reinvesting at the higher rate, what early redemption penalty (deducted from the accumulated value of the investment certificate to that point) should the bank charge Smith at the time of early redemption (answer to nearest $1)?

A) $39 B) $41 C) $43 D) $ 45 E) $47

4. The managers of ABC Mutual Fund have reported an average annual effective return of 14% for the 10 years ending Dec. 31, 2004, and an average annual effective return of 16% for the 5 years ending Dec. 31, 2004. Find the average annual effective return for the 5 years ending Dec. 31, 1999.

A) 11% B) 11.5% C) 12% D) 12.5% E) 13%

5. Which of the following statements are correct for equivalent rates i and d?

I. $\lim_{i \to 0} d = 0$

II. $\lim_{i \to \infty} d = \infty$

III. $\frac{1}{d} - \frac{1}{i} = i$

IV. $(1-v)i = i - d$

V. $1 + it < (1+i)^t$ for any $i > 0$

VI. $1 - dt < (1-d)^t$ for any $d > 0$

6. Smith will invest $200 today and $100 one year from now. Suppose that this year's interest rate is i_1 and next year's rate will be i_2. Suppose that we define Smith's two-year average return will be i, where Smith's accumulated investment amount at the end of two years is $100[2(1+i)^2 + (1+i)]$.

Find i under each of the following interest rate environments (assume $i > 0$):

(a) $i_1 = i_2 = .10$ (b) $i_1 = .08$, $i_2 = .10$ (c) $i_1 = .10$, $i_2 = .08$

7. You are given the following information.

Initial deposit to a fund: $35,000

Withdrawal from the fund at the end of the fourth year: $70,000

Value of the fund at the end of the eighth year: $14,000

No other deposits or withdrawals were made during the eight-year period.

In what range is the annual rate of return for the fund during the eight-year period?

A) Less than 14% B) 14% but less than 19% C) 19% but less than 24%

D) 24% but less than 29% E) 29% or more

8. (SOA May 05) Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these cash flows is 364.46 at an annual effective rate of interest i.

Calculate i.

A) 10% B) 11% C) 12% D) 13% E) 14%

9. (SOA May 05) A store is running a promotion during which customers have two options for payment. Option one is to pay 90% of the purchase price two months after the date of sale. Option two is to deduct X% off the purchase price and pay cash on the date of sale.
A customer wishes to determine X such that he is indifferent between the two options when valuing them using an annual effective interest rate of 8%. Which of the following equations of value would the customer need to solve?

A) $\left(\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=0.90$ B) $\left(1-\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=0.90$

C) $\left(\frac{X}{100}\right)(1.08)^{1/6}=0.90$ D) $\left(\frac{X}{100}\right)\left(\frac{1.08}{1.06}\right)=0.90$ E) $\left(1-\frac{X}{100}\right)(1.08)^{1/6}=0.90$

10. At an annual effective rate of interest of i, a payment of 100 right now combined with the present value of a payment of 100 to be made 3 years from now has same total present value right now as the present value now of a single payment of 190 to be made 1 year from now. Find i.

PROBLEM SET 1 SOLUTIONS

1. $10,000(1+i)(1+i-.05)=12,093.75 \rightarrow 10,000(1+i)^2-500(1+i)-12,093.75=0$. Solving the quadratic equation results in $1+i=1.125$, -1.075 . We ignore the negative root. With $i=.125$ we get $i+.09=.215$. Then $10,000(1.215)^3=17,936$. Answer: D

2. $5,000=10,000v^6=6,000v^t+56,000v^{2t} \rightarrow v^6=.5$, and solving the quadratic in v^t we get $v^t=.25$. Then $8000v^{t+3}=8000(.25)(\sqrt{.5})=1414.$ Answer: B

3. $10,000[1+\frac{120}{365}\cdot(.095)]=10,312.33$ is the maturity amount of the original GIC. At 60 days, the accumulated value (at 9.5%) is $10,000[1+\frac{60}{365}\cdot(.095)]=10,156.16$. The amount required to invest for the next 60 days at 12% to reach the original maturity amount is $\frac{10{,}312.33}{1+\frac{60}{365}\cdot(.12)}=10,112.84$. Penalty $=10,156.16-10,112.84=43.32$. Answer: C

4. If average annual effective return (or rate of interest) is i for the first 5 years, we have $(1+i)^5(1.16)^5=(1.14)^{10} \rightarrow i=.1203$.

Note that "average annual effective rate" for the period from time 0 to time n years would result in a growth factor of $(1+i)^n$ for the n-year period. This average rate would usually be specified in the context of some other growth pattern over the n-year period to which the average rate is equivalent. Since the average rate in this case is i for the first five years, the accumulation factor over the first five years would be $(1+i)^5$. This is then followed by 5 more years at an average annual rate of 16%, so the combined 10 year growth factor is $(1+i)^5(1.16)^5$. But since we are also told that the average rate for the 10 year period is 14%, we can formulate the 10 year growth factor as $(1.14)^{10}$. This gives us two equivalent ways of representing the 10 year growth factor. The equation of value is formulated by setting these two growth factors to be equal, after which we solve for i. Answer: C

5. Which of the following statements are correct for equivalent rates i and d?
I. $d = \frac{i}{1+i} \to 0$ as $i \to 0$. True.
II. $d = \frac{i}{1+i} \to 1$ as $i \to \infty$. False.
III. $\frac{1}{d} - \frac{1}{i} = \frac{1+i}{i} - \frac{1}{i} = 1$. False
IV. $(1-v)i = i - iv = i - \frac{i}{1+i} = i - d$. True
V. $1 + it > (1+i)^t$ for any $i > 0$ and $0 < t < 1$. False.
VI. $1 - dt < (1-d)^t$ for any $d > 0$ and $t < 1$. False.

6. (a) $\text{AV} = 100[2(1.1)^2 + 1.1] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.52 = 0 \to 1+i = 1.1\,,\ -1.6 \to i = .1$
(this could have been anticipated from the simple form of the equation).
(b) $\text{AV} = 100[2(1.08)(1.1) + 1.1] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.476 = 0 \to 1+i = 1.0918$.
(c) $\text{AV} = 100[2(1.1)(1.08) + 1.08] = 100[2(1+i)^2 + (1+i)]$
$\to 2(1+i)^2 + (1+i) - 3.456 = 0 \to 1+i = 1.0881$.
Note that in both cases (b) and (c), i is not .09, the "simple" average of .08 and .1. The rate i_1 affects growth over both a one-year and two-year period, but the rate i_2 affects growth only over the second year, so the average rate is more of "weighted average". i_1 has more "weight" than i_2 in the total accumulation over the two year period. From that point of view, it seems logical in case (b) that the average rate is greater than the simple average of .09, since more weight comes from i_1 which is larger than i_2. The same reasoning applies to case (c), since more weight comes from the i_1 which is now lower than i_2.

7. $35(1+i)^8 - 70(1+i)^4 = 14$. Let $x = (1+i)^4$, and the equation becomes
$5x^2 - 10x - 2 = 0$. Solving the quadratic equation results in
$x = \frac{10 \pm \sqrt{100+40}}{2(5)} = 2.183\,,\ -.183$. We discard the negative root, since $x = (1+i)^4 \geq 0$.
Then $(1+i)^4 = 2.183 \to i = .216$. Answer: C

8. $100+200v_i+100v_i^2=364.46$. We can solve for i four ways.
(i) Trial and error. Try each interest rate until the PV is correct. We see that $i=.10$.
Note that this trial and error approach may be an efficient problem solving technique for certain problem types. In particular, if it is possible to reduce the solution of the problem to a single equation that involves the unknown variable, then it may be possible to substitute into that variable the various possible answers provided to see which one makes the equation correct.

(ii) Solve the equation $264.46=200v+100v^2$ using the calculator financial functions with $PV=264.46$, $PMT=-200$ and $FV=100$.

(iii) Solve the equation $264.46=200v+100v^2$ using the BA-II PLUS calculator cashflow spreadsheet.

(iv) Solve the quadratic equation $100v^2+200v-264.46=0$.
$v=\frac{-200\pm\sqrt{200^2-4(100)(-264.46)}}{200}=.9091$ or -2.9091 (ignore the negative root).
$v=\frac{1}{1+i}=.9091\rightarrow i=.10$. Answer: A

9. Under Option one, the customer can pay $1-\frac{X}{100}$ now. Under Option two the customer can pay $.9$ two months from now. These two options are equivalent in the sense that the customer is indifferent to picking one or the other. Therefore, the PV of a payment of .9 that would be made in two months is equal to a payment of $1-\frac{X}{100}$ now. This can be expressed as
$1-\frac{X}{100}=(.9)v_{.08}^{1/6}$, or equivalently, $(1-\frac{X}{100})(1.08)^{1/6}=.9$. Answer: E

10. The equation of value as of now is $100+100v^3=190v$.
We use the cashflow worksheet of the BA II PLUS calculator to solve this by setting
$CFo=100$, $C01=-190$, $F01=1$, $C02=0$, $F02=1$, $C03=100$, $F03=1$.
Then calculate IRR . The resulting interest rate is 17.21% .
Since this is a cubic equation, there are three solutions for i, but the calculator returns the smallest positive solution. The other solutions are 35.67% and a negative solution for v.