THEORY OF INTEREST
AND LIFE CONTINGENCIES
WITH PENSION APPLICATIONS

A Problem-Solving Approach

Third Edition

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To my Mother and the memory of my Father

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PREFACE

It is impossible to escape the practical implications of compound interest in our modern society. The consumer is faced with a bewildering choice of bank accounts offering various rates of interest, and wishes to choose the one which will give the best return on her savings. A home-buyer is offered various mortgage plans by different companies, and wishes to choose the one most advantageous to him. An investor seeks to purchase a bond which pays coupons on a regular basis and is redeemable at some future date; again, there are a wide variety of choices available.

Comparing possibilities becomes even more difficult when the payments involved are dependent on the individual's survival. For example, an employee is offered a variety of different pension plans and must decide which one to choose. Also, most people purchase life insurance at some point in their lives, and a bewildering number of different plans are offered.

The informed consumer must be able to make an intelligent choice in situations like those described above. In addition, it is important that, whenever possible, she be able to make the appropriate calculations herself in such cases. For example, she should understand why a given series of mortgage payments will, in fact, pay off a certain loan over a certain period of time. She should also be able to decide which portion of a given payment is paying off the balance of the loan, and which portion is simply paying interest on the outstanding loan balance.

The first goal of this text is to give the reader enough information so that he can make an intelligent choice between options in a financial situation, and can verify that bank balances, loan payments, bond coupons, etc. are correct. Too few people in today's society understand how these calculations are carried out.

In addition, however, we are concerned that the student, besides being able to carry out these calculations, understands why they work. It is not enough to memorize a formula and learn how to apply it; you should understand why the formula is correct. We also wish to present the material in a proper mathematical setting, so the student will see how the theory of interest is interrelated with other branches of mathematics.

Let me explain why the phrase “Problem-Solving Approach” appears in the title of this text. We will prove a very small number of formulae and then concentrate our attention on showing how these formulae can be applied to a wide variety of problems. Skill will be needed to take the data presented in a particular problem and see how to rearrange it so the formulae can be used. This approach differs from many texts, where a large number of formulae are presented, and the student tries to memorize which problems can be solved by direct application of a particular formula. We wish to emphasize understanding, not rote memorization.

A working knowledge of elementary calculus is essential for a thorough understanding of all the material. However, a large portion of this book can be read by those without such a background by omitting the sections dependent on calculus. Other required background material such as geometric sequences, probability and expectation, is reviewed when it is required.

Each chapter in this text includes a large number of examples and exercises. It should be obvious that the most efficient way for a student to learn the material is for her to work all the exercises.

Finally, let us stress that it is assumed that every student has a calculator (with a y^x button) and knows how to use it. It is because of our ability to use a calculator that many formulae mentioned in older texts on the subject are now unnecessary.

This book is naturally divided into two parts. Chapters 1-5 are concerned solely with the Theory of Interest, and Life Contingencies is introduced in Chapters 6-11.

In Chapter 1 we present the basic theory concerning the study of interest. Our goal here is to give a mathematical background for this area, and to develop the basic formulae which will be needed in the rest of the book. Students with a weak calculus background may wish to omit Section 1.6 on the force of interest, as it is of more theoretical than practical importance. In Chapter 2 we show how the theory in Chapter 1 can be applied to practical problems. The important concept of equation of value is introduced, and many worked examples of numerical problems are presented. Chapter 3 discusses the extremely important concept of annuities. After developing a few basic formulae, our main emphasis in this chapter is on practical problems, seeing how data for such problems can be substituted in the basic formulae. It is in this section especially that we have left out many of the formulae presented in other texts, preferring to concentrate on problem-solving techniques rather than rote memorization. Chapters 4 and 5 deal with further applications of the material in Chapters 1 through 3, namely amortization, sinking funds and bonds.

Chapter 6 begins with a review of the important concepts of probability and expectation, and then illustrates how probability can be combined with the theory of interest. In Chapter 7 we introduce life tables, discussing how they are constructed and how they can be applied. Chapter 8 is concerned with life annuities, that is annuities whose payment are conditional on survival, and Chapter 9 discusses life insurance. These ideas are generalized to multi-life situations in Chapter 10.

Finally, Chapter 11 demonstrates how many of these concepts are applied in the extremely important area of pension plans.

Chapters 1 through 6 have been used for several years as the text material for a one semester undergraduate course in the Theory of Interest, and I would like to thank those students who pointed out errors in earlier drafts. In addition, I am deeply indebted to Brenda Crewe and Wanda Heath for an excellent job of typing the manuscript, and to my colleague, Dr. P. P. Narayanaswami, for his invaluable technical assistance.

Chuck Vinsonhaler, University of Connecticut, was strongly supportive of this project, and introduced me to the people at ACTEX Publications, for which I owe him a great deal. Dick London did the technical content editing, Marilyn Baleshiski provided the electronic typesetting, and Marlene Lundbeck designed the text's cover. I would like to thank them for taking such care in turning a very rough manuscript into what I hope is a reasonably comprehensive yet friendly and readable text book for actuarial students.

St. John’s, Newfoundland Michael M. Parmenter
December, 1988

PREFACE TO THE REVISED EDITION

In the fifteen months since the original edition of this text was published, a number of comments have been received from teachers and students regarding that edition.

We are pleased to note that most of the comments have been quite complimentary to the text, and we are making no substantial modifications at this time.

A significant, and thoroughly justified, criticism of the original edition is that time diagrams were not used to illustrate the examples given in the second half of the text, and that deficiency has been rectified by the inclusion of thirty-five additional figures in the Revised Edition.

Thus it is fair to say that there are no *new topics* contained in the Revised Edition, but rather that the pedagogy has been strengthened. For this reason we prefer to call the new printing a Revised Edition, rather than a Second Edition.

In addition we have corrected the errata in the original edition. We would like to thank all those who took the time to bring the various errata to our attention.

February, 1990 M.M.P.

PREFACE TO THE THIRD EDITION

It is now more than ten years since the original publication of this textbook. In that time, several very significant developments have occurred to suggest that a new edition of the text is now needed, and those developments are reflected in the modifications and additions made in this Third Edition.

First, improvements in calculator technology give us better approaches to reach numerical results. In particular, many calculators now include iteration algorithms to permit direct calculation of unknown annuity interest rates and bond yield rates. Accordingly, the older approximate methods using interpolation have been deleted from the text.

Second, with the discontinued publication of the classic textbook *Life Contingencies* by C.W. Jordan, our text has become the only one published in North America which provides the traditional presentation of contingency theory. To serve the needs of those who still prefer this traditional approach, including the use of commutation functions and a deterministic life table model, we have chosen to include various topics contained in Jordan's text but not contained in our earlier editions. These include insurances payable at the moment of death (Section 9.3), life contingent accumulation functions (Section 8.2), the table of uniform seniority concept for use with Makeham and Gompertz annuity values (Section 11.1), simple contingent insurance functions (Section 11.1), and an expansion of the material regarding multiple-decrement theory (Section 7.6).

Third, actuaries today are interested in various concepts of finance beyond those included in traditional interest theory. To that end we have introduced the ideas of real rates of return, investment duration, modified duration, and so on, in this Third Edition.

Fourth, the new edition provides a gentle introduction to the more modern stochastic view of contingency theory, in the completely new Chapter 10, to supplement the traditional presentation.

In connection with the expansion of topics, the new edition contains over forty additional exercises and examples. As well, the numerical answers to the exercises have been made more precise and the errata in the previous edition have been corrected. We would like to thank everyone who brought such errata to our attention.

With the considerable modifications made in the new edition, we believe this text is now appropriate for two major audiences: pension actuaries, who wish to understand the use of commutation functions and deterministic contingency theory in pension mathematics, and university students, who seek to understand basic contingency theory at an introductory level before undertaking a study of the more mathematically sophisticated stochastic contingency theory.

As with the original edition of this text, the staff at ACTEX Publications has been invaluable in the development of this new edition. Specifically I would like to thank Denise Rosengrant for her text composition and typesetting work, and Dick London, FSA, for his technical content editing.

February, 1999 M.M.P

CHAPTER ONE
INTEREST: THE BASIC THEORY

1.1 ACCUMULATION FUNCTION

The simplest of all financial transactions is one in which an amount of money is invested for a period of time. The amount of money initially invested is called the *principal* and the amount it has grown to after the time period is called the *accumulated value* at that time.

This is a situation which can easily be described by functional notation. If t is the length of time for which the principal has been invested, then the amount of money at that time will be denoted by $A(t)$. This is called the *amount function.* For the moment we will only consider values $t \geq 0$, and we will assume that t is measured in years. We remark that the initial value $A(0)$ is just the principal itself.

In order to compare various possible amount functions, it is convenient mathematically to define the *accumulation function* from the amount function as $a(t) = \frac{A(t)}{A(0)}$. We note that $a(0) = 1$ and that $A(t)$ is just a constant multiple of $a(t)$, namely $A(t) = k \cdot a(t)$ where $k = A(0)$ is the principal.

What functions are possible accumulation functions? In theory, any function $a(t)$ with $a(0) = 1$ could represent the way in which money accumulates with the passage of time. Certainly, however, we would hope that $a(t)$ is increasing. Should $a(t)$ be continuous? That depends on the situation; if $a(t)$ represents the amount owing on a loan t years after it has been taken out, then $a(t)$ may be continuous if interest continues to accumulate for non-integer values of t. However, if $a(t)$ represents the amount of money in your bank account t years after the initial deposit (assuming no deposits or withdrawals in the meantime), then $a(t)$ will stay constant for periods of time, but will take a jump whenever interest is paid into the account. The graph of such an $a(t)$ will be a step function. We will normally assume in this text that $a(t)$ is continuous; it is easy to make allowances for other situations when they turn up.

In Figure 1.1 we have drawn graphs of three different types of accumulation functions which occur in practice:

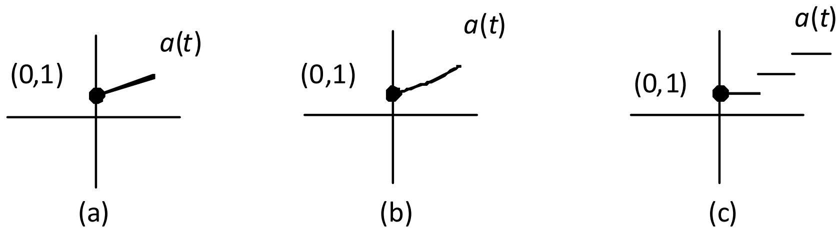

FIGURE 1.1

Graph (a) represents the case where the *amount* of interest earned is constant over each year. On the other hand, in cases like (b), the amount of interest earned is increasing as the years go on. This makes more sense in most situations, since we would hope that as the principal gets larger, the interest earned also increases; in other words, we would like to be in a situation where “interest earns interest”. There are many different accumulation functions which look roughly like the graph in (b), but the exponential curve is the one which will be of greatest interest to us.

We remarked earlier that a situation like (c) can arise whenever interest is paid out at fixed periods of time, but no interest is paid if money is withdrawn between these time periods. If the amount of interest paid is constant per time period, then the “steps” will all be of the same height. However, if the amount of interest increases as the accumulated value increases, then we would expect the steps to get larger and larger as time goes on.

We have used the term interest several times now, so perhaps it is time to define it!

$$\boxed{\textit{Interest} = \textit{Accumulated Value} - \textit{Principal}}$$

This definition is not very helpful in practical situations, since we are generally interested in comparing different financial situations to determine which is most profitable. What we require is a standardized measure for interest, and we do this by defining the *effective rate of interest* i (per year) to be the interest earned on a principal of amount 1 over a period of one year. That is,

$$i = a(1) - 1. \tag{1.1}$$

We can easily calculate i using the amount function $A(t)$ instead of $a(t)$, if we recall that $A(t) = k \cdot a(t)$. Thus

$$i = a(1) - 1 = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)}. \tag{1.2}$$

Verbally, the effective rate of interest per year is the amount of interest earned in one year divided by the principal at the beginning of the year. There is nothing sacred about the term "year" in this definition. We can calculate an effective rate of interest over any time period by simply taking the numerator of the above fraction as being the interest earned over that period.

More generally, we define the effective rate of interest in the n^{th} year by

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}. \tag{1.3}$$

Note that i_1, calculated by (1.3), is the same as i defined by either (1.1) or (1.2).

Example 1.1

Consider the function $a(t) = t^2 + t + 1$.

(a) Verify that $a(0) = 1$.
(b) Show that $a(t)$ is increasing for all $t \geq 0$.
(c) Is $a(t)$ continuous?
(d) Find the effective rate of interest i for $a(t)$.
(e) Find i_n.

Solution

(a) $a(0) = (0)^2 + (0) + 1 = 1$.
(b) Note that $a'(t) = 2t + 1 > 0$ for all $t \geq 0$, so $a(t)$ is increasing.
(c) The easiest way to solve this is to observe that the graph of $a(t)$ is a parabola, and hence $a(t)$ is continuous (or recall from calculus that all polynomial functions are continuous).
(d) $i = a(1) - 1 = 3 - 1 = 2$.
(e)
$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{n^2 + n + 1 - [(n-1)^2 + (n-1) + 1]}{(n-1)^2 + (n-1) + 1}$$
$$= \frac{2n}{n^2 - n + 1}. \qquad \square$$

1.2 SIMPLE INTEREST

There are two special cases of the accumulation function $a(t)$ that we will examine closely. The first of these, *simple interest*, is used occasionally, primarily between integer interest periods, but will be discussed mainly for historical purposes and because it is easy to describe. The second of these, *compound interest*, is by far the most important accumulation function and will be discussed in the next section. Keep in mind that in both of these cases $a(t)$ is continuous, and also that there are some practical settings where modifications must be made.

Simple interest is the case where the graph of $a(t)$ is a straight line. Since $a(0) = 1$, the equation must therefore be of the general form $a(t) = 1 + bt$ for some b. However, the effective rate of interest i is given by $i = a(1) - 1 = b$, so the formula is

$$a(t) = 1 + it, \quad t \geq 0. \tag{1.4}$$

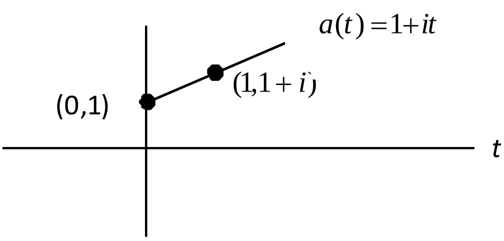

FIGURE 1.2

Remarks

1. This is case (a) graphed in Figure 1.1. In this situation, the amount of interest earned each year is constant. In other words, only the original principal earns interest from year to year, and interest accumulated in any given year does not earn interest in future years.
2. The formula $a(t) = 1 + it$ applies to the case where the principal is $A(0) = a(0) = 1$. More generally, if the principal at time 0 is equal to k, the amount at time t will be $A(t) = k(1 + it)$.

3. We noted above that the "i" in $a(t) = 1 + it$ is also the effective rate of interest for this function. Note however that

$$i_n = \frac{1 + in - [1 + i(n-1)]}{1 + i(n-1)}$$

$$= \frac{i}{1 + i(n-1)}. \tag{1.5}$$

Observe that i_n is not constant. In fact, i_n decreases as n gets larger, a fact which should not surprise us. If the amount of interest stays constant as the accumulated value increases, then clearly the effective rate of interest is going down.

4. Clearly $a(t) = 1 + it$ is a formula which works equally well for all values of t, integral or otherwise. However, problems can develop in practice, as illustrated by the following example.

Example 1.2

Assume Jack borrows 1000 from the bank on January 1, 1996 at a rate of 15% simple interest per year. How much does he owe on January 17, 1996?

Solution

The general formula for the amount owing at time t in general is $A(t) = 1000(1 + .15t)$, but the problem is to decide what value of t should be substituted into this formula. An obvious approach is to take the number of days which have passed since the loan was taken out and divide by the number of days in the year, but should we count the number of days as 16 or 17? Getting really picky, should we worry about the time of day when the loan was taken out, or the time of day when we wish to find the value of the loan? Obviously, any value of t is only a convenient approximation; the important thing is to have a consistent rule to be used in practice. Two techniques are common:

(a) The first method is called *exact simple interest*, and with it we use

$$t = \frac{\textit{number of days}}{365}. \tag{1.6}$$

When counting the number of days it is usual to count the last day, but not the first. In our case this would lead to $t = \frac{16}{365}$ so Jack owes $1000\left[1 + (.15)\left(\frac{16}{365}\right)\right] = 1006.58$.

(b) The second method is called *ordinary simple interest* (or the *Banker's Rule*), and with it we use

$$t = \frac{\textit{number of days}}{360}. \qquad (1.7)$$

The same procedure as above is used for calculating the number of days. In our case, we would have $t = \frac{16}{360}$ so the debt is

$$1000\left[1 + (.15)\left(\frac{16}{360}\right)\right] = 1006.67.$$

The common practice in Canada is to use exact simple interest, whereas ordinary simple interest is used in the United States and in international markets. □

1.3 COMPOUND INTEREST

The most important special case of the accumulation function $a(t)$ is the case of compound interest. Intuitively speaking, this is the situation where money earns interest at a fixed effective rate; in this setting, the interest earned in one year earns interest itself in future years.

If i is the effective rate of interest, we know that $a(1) = 1 + i$, so 1 becomes $1 + i$ after the first year. What happens in the second year? Consider the $1 + i$ as consisting of two parts, the initial principal 1 and the interest i earned in the first year. The principal 1 will earn interest in the second year and will accumulate to $1 + i$. The interest i will also earn interest in the second year and will grow to $i(1 + i)$. Hence the total amount after two years is $1 + i + i(1+ i) = (1 + i)^2$. By continuing this reasoning, we see that the formula for $a(t)$ is

$$a(t) = (1 + i)^t, \quad t \geq 0. \qquad (1.8)$$

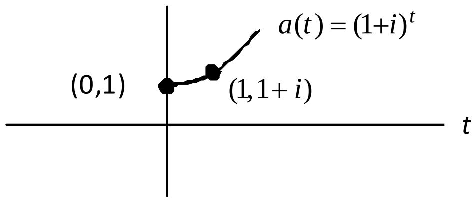

FIGURE 1.3

Remarks

1. This is an example of the type of function shown in part (b) of the graph in Figure 1.1.
2. The formula $a(t) = (1 + i)^t$ applies to the case where the principal is $A(0) = a(0) = 1$. More generally, if the principal at time 0 is equal to k, the amount at time t will be $A(t) = k(1 + i)^t$.
3. Observe that the "i" in $(1 + i)^t$ is the effective rate of interest. More generally,

$$i_n = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = 1 + i - 1 = i. \tag{1.9}$$

   Hence, in this case i_n is the same for all positive integers n. We shouldn't be surprised, since this fits with our idea that, in compound interest, the effective rate of interest is constant.
4. Mathematically, any value of t, whether integral or not, can be substituted into $a(t) = (1 + i)^t$. This is an easier task for us today than it was fifty years ago; we just have to press the appropriate buttons on our calculators! Again, there are problems determining what value of t should be used, but we can deal with them as we did in the last section.

   In practical situations, however, a very different solution is sometimes used in the case of compound interest. To find the amount of a loan (for example) when t is a fraction, first find the amounts for the integral values of t immediately before and immediately after the fractional value in question. Then use linear interpolation between the two computed amounts to calculate the required answer.

   This is equivalent to saying that compound interest is used for integral values of t, and simple interest is used between integral values. In Figure 1.4, the solid line represents $a(t) = (1 + i)^t$, whereas the dotted lines indicate the graph of $a(t)$ if linear interpolation is used.

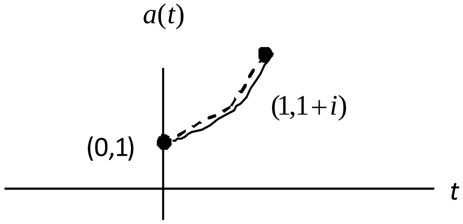

FIGURE 1.4

As we will see later, this common procedure benefits the lender in a financial transaction, and (consequently) is detrimental to the borrower if she has to repay the loan at a duration between integral values.

Example 1.3

Jack borrows 1000 at 15% compound interest.

(a) How much does he owe after 2 years?

(b) How much does he owe after 57 days, assuming compound interest between integral durations?

(c) How much does he owe after 1 year and 57 days, under the same assumption as in part (b)?

(d) How much does he owe after 1 year and 57 days, assuming linear interpolation between integral durations?

(e) In how many years will his principal have accumulated to 2000?

Solution

(a) $1000(1.15)^2 = 1322.50$.

(b) The most suitable value for t is $\frac{57}{365}$, and the accumulated value is $1000(1.15)^{\frac{57}{365}} = 1022.07$.

(c) $1000(1.15)^{1\frac{57}{365}} = 1175.38$

(d) We must interpolate between $A(1) = (1000)(1.15) = 1150.00$ and $A(2) = 1000(1.15)^2 = 1322.50$. The difference between these values is $A(2) - A(1) = 172.50$. The portion of this difference which will accumulate in 57 days, assuming simple interest, is $\left(\frac{57}{365}\right)(172.50) = 26.94$. Thus the accumulated value after 1 year and 57 days is $1150.00 + 26.94 = 1176.94$. Observe that the borrower owes more money in this case than he does in part (c).

(e) We seek t such that $1000(1.15)^t = 2000$, or that $(1.15)^t = 2$. Using logs we obtain

$$t = \frac{\log 2}{\log 1.15} = 4.9595 \text{ years.} \qquad \square$$

To close this section, we will compare simple interest and compound interest to see which gives the better return. In Figure 1.5, graphs for both simple interest and compound interest are drawn on the same set of axes.

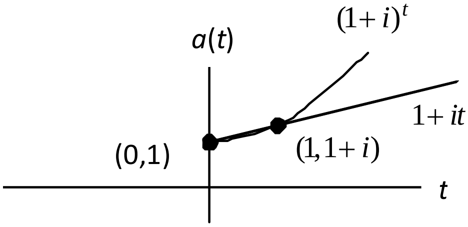

FIGURE 1.5

We know that the exponential function $(1+i)^t$ is always concave up (because the second derivative is $(1+i)^t[\ln(1+i)]^2$, which is greater than zero), whereas $1+it$ is a straight line. These facts tell us that the only points of intersection of these graphs are the obvious ones, namely (0,1) and $(1, 1+i)$. They also give us the two important relationships

$$(1+i)^t < 1+it, \quad \text{for} \quad 0<t<1 \tag{1.10}$$

and

$$(1+i)^t > 1+it, \quad \text{for} \quad t>1. \tag{1.11}$$

Hence we conclude that compound interest yields a higher return than simple interest if $t>1$, whereas simple interest yields more if $0<t<1$. The first of these statements does not surprise us, since for $t>1$, we have interest as well as principal earning interest in the $(1+i)^t$ case. The second statement reminds us that, for periods of less than a year, simple interest is more beneficial to the lender than compound interest, a fact which was illustrated in Example 1.3.

1.4 PRESENT VALUE AND DISCOUNT

In Section 1.1 we defined accumulated value at time t as the amount that the principal accumulates to over t years. We now define the *present value t* years in the past as the amount of money that will accumulate to the principal over t years. In other words, this is the reverse procedure of that which we have been discussing up to now.

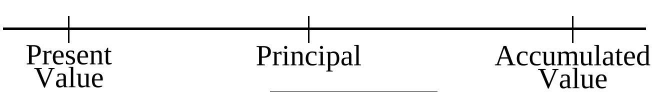

FIGURE 1.6

For example, 1 accumulates to $1 + i$ over a single year. How much money is needed, at the present time, to accumulate to 1 over one year? We will denote this amount by v, and, recalling that v accumulates to $v(1 + i)$, we have $v(1 + i) = 1$. Therefore

$$v = \frac{1}{1+i}. \tag{1.12}$$

These two accumulations are shown in Figure 1.7.

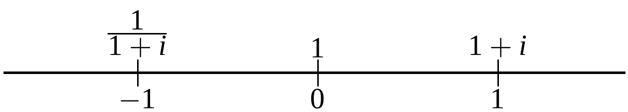

FIGURE 1.7

From now on, unless explicitly stated otherwise, we will assume that we are in a compound interest situation, where $a(t) = (1 + i)^t$. In this case, the present value of 1, t years in the past, will be $v^t = \frac{1}{(l+i)^t}$. We summarize this on the time diagram shown in Figure 1.8.

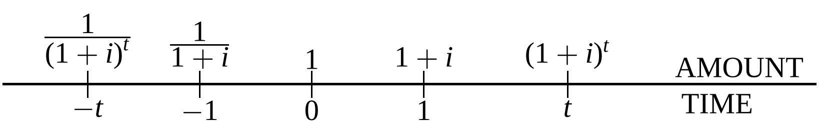

FIGURE 1.8

Observe that, since $v^t = (1 + i)^{-t}$, the function $a(t) = (1 + i)^t$ expresses all these values, for both positive and negative values of t. Hence $(1 + i)^t$ gives the value of one unit (at time 0) at any time t, past or future. The graph is shown in Figure 1.9.

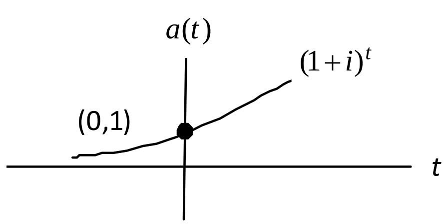

FIGURE 1.9

Example 1.4

The Kelly family buys a new house for 93,500 on May 1, 1996. How much was this house worth on May 1, 1992 if real estate prices have risen at a compound rate of 8% per year during that period?

Solution

We seek the present value, at time $t = -4$, of 93,500 at time 0. This is $93{,}500\left(\frac{1}{1.08}\right)^4 = 68{,}725.29$. □

What happens to the calculation of present values if simple interest is assumed instead of compound interest? The accumulation function is now $a(t) = 1 + it$. Hence, the present value of one unit t years in the past is given by x, where $x(1 + it) = 1$. Thus the present value is

$$x = \frac{1}{1 + it}. \tag{1.13}$$

The time diagram for this case is shown in Figure 1.10.

$\frac{1}{1+it}$	$\frac{1}{1+i}$	1	$1+i$	$1+it$	AMOUNT
$-t$	-1	0	1	t	TIME

FIGURE 1.10

In Exercise 15, you are asked to sketch the graph of this situation. Unlike the compound interest case, this graph changes dramatically as it passes through the point $(0, 1)$.

We now turn our attention to the concept of discount. For the moment we will not assume compound interest, since any accumulation function will be satisfactory.

Imagine that 100 is invested, and that one year later it has accumulated to 112. We have been viewing the 100 as the "starting figure," and have imagined that interest of 12 is added to it at the end of the year. However, we could also view 112 as the basic figure, and imagine that 12 is deducted from that value at the start of the year. From the latter point of view, the 12 is considered an amount of *discount.*

Students sometimes get confused about the difference between interest and discount, but the important thing to remember is that the only difference is in the point of view, not in the underlying financial transaction. In both situations we have 100 accumulating to 112, and nothing can change that.

Since discount focuses on the total at the end of the year, it is natural to define the effective rate of discount, d, as

$$d = \frac{a(1) - 1}{a(1)}. \tag{1.14}$$

In other words, standardization is achieved by dividing by $a(1)$ instead of $a(0)$, as was done in (1.2) to define the effective rate of interest i.

More generally, the effective rate of discount in the n^{th} year is given by

$$d_n = \frac{a(n) - a(n-1)}{a(n)}. \tag{1.15}$$

(Compare this with the definition of i_n, given by (1.3).)

Now we will derive some basic identities relating d to i. One identity follows immediately from the definition of d, namely,

$$d = \frac{a(1) - 1}{a(1)} = \frac{(1+i) - 1}{1+i} = \frac{i}{1+i}. \tag{1.16}$$

Since $1 + i > 1$, this tells us that $d < i$.

Immediately from the above we obtain

$$1 - d = 1 - \frac{i}{1+i} = \frac{1}{1+i} = v. \tag{1.17}$$

Actually, this identity is exactly what we would expect from the definition of d. The fact that $1 - d$ accumulates to 1 over one year is the exact analogy of 1 accumulating to $1 + i$ over the same period.

Solving either of the above identities for i, we obtain

$$i = \frac{d}{1-d}. \tag{1.18}$$

The reader will be asked to derive other identities in the exercises and to give verbal arguments in support of them. We note that all identities derived so far hold for any accumulation function. For the rest of this section, it will be assumed that $a(t) = (1 + i)^t$.

In Section 1.3 we learned that to find the accumulated value t years in the future we multiply by $(1+i)^t$, whereas to find the present value t years in the past we multiply by $\frac{1}{(1+i)^t}$. However, identity (1.17) tells us that $1-d=\frac{1}{1+i}$. Hence, if d is involved, the rules for present and accumulated value are reversed: present value is obtained by multiplying by $(1-d)^t$, and accumulated value by multiplying by $\frac{1}{(1-d)^t}$.

Example 1.5

1000 is to be accumulated by January 1, 1995, at a compound rate of discount of 9% per year.

(a) Find the present value on January 1, 1992.

(b) Find the value of i corresponding to d.

Solution

(a) $1000(1-.09)^3 = 753.57.$

(b) $i=\frac{d}{1-d}=\frac{.09}{.91}=.0989.$ □

Example 1.6

Jane deposits 1000 in a bank account on August 1, 1996. If the rate of compound interest is 7% per year, find the value of this deposit on August 1, 1994.

Solution

Some students think that the answer to this question should be 0, because the money hasn't been deposited yet! However, in a mathematical sense, we know that money has value at all times, past or future, so the correct answer is $1000\left(\frac{1}{1.07}\right)^2 = 873.44.$ □

1.5 NOMINAL RATE OF INTEREST

We will assume $a(t)=(1+i)^t$ throughout this section and, unless stated otherwise, in all remaining sections of the book.

Example 1.7

A man borrows 1000 at an effective rate of interest of 2% per month. How much does he owe after 3 years?

$\boxed{\text{Solution}}$

What we want is the amount of the debt after three years. Since the effective interest rate is given per month, three years is 36 interest periods. Thus the answer is $1000(1.02)^{36} = 2039.89$. □

The point of the above example is to illustrate that effective rates of interest need not be given per year, but can be defined with respect to any period of time. To apply the formulae developed to this point, we must be sure that t is the number of *effective interest periods* in any particular problem.

In many real-life situations, the effective interest period is not a year, but rather some shorter period. Perhaps the lender tries to keep this fact hidden, as it might be to his benefit to do so! For example, suppose you want to take out a mortgage on a house and you discover a rate of 12% per year. When you dig a little, however, what you find out is that this rate is "convertible semiannually", which means that it is really 6% effective per half-year. Is that the same thing? Not at all. Consider what happens to an investment of 1. After half a year it has accumulated to 1.06. After one year (two interest periods) it has become $(1.06)^2 = 1.1236$. So, over a one-year period, the amount of interest gained is .1236, which means the effective rate of interest per year is actually 12.36%. Although it may not be clear from the advertising, many mortgage loans are convertible semiannually, so the effective rate of interest is higher than the rate quoted.

As another example, consider a well-known credit card which charges 18% per year convertible monthly. This means that the actual rate of interest is $\frac{.18}{12} = .015$ effective per month. Over the course of a year, 1 will accumulate to $(1.015)^{12} = 1.1956$, so the effective rate of interest per year is actually 19.56%.

The 18% in the last example is called a *nominal* rate of interest, which means that it is convertible over a period other than one year. In general, we use the notation $i^{(m)}$ to denote a nominal rate of interest convertible m times per year, which implies an effective rate of interest of $\frac{i^{(m)}}{m}$ per m^{th} of a year. If i is the effective rate of interest per year, it follows that

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m. \tag{1.19}$$

Example 1.8

Find the accumulated value of 1000 after three years at a rate of interest of 24% per year convertible monthly.

Solution

This is really 2% effective per month, so the answer is the same as Example 1.7, namely $1000(1.02)^{36} = 2039.89$. □

Remark

An alternative method of solving Example 1.8 is to find i, the effective rate of interest per year, and then proceed as in Section 1.3. We would have $i = \left(1+\frac{i^{(m)}}{m}\right)^m - 1 = (1+.02)^{12} - 1 = .26824$, and the answer would be $1000\,(1.26824)^3 = 2039.88$.

Notice the difference of .01 in the two answers. This is because not enough decimal places were kept in the value of i, and some error crept in. Of course, if you use the memory in your calculator it is unlikely that this type of error will occur. Nevertheless, the first solution is still preferable; time spent on unnecessary calculations can be significant in examination situations.

It will be extremely important in later sections of the text to be able to convert from one nominal rate of interest to another whose convertible frequency is different. Here is an example of this.

Example 1.9

If $i^{(6)} = .15$, find the equivalent nominal rate of interest convertible semiannually.

Solution

We have $\left(1 + \frac{i^{(2)}}{2}\right)^2 = \left(1 + \frac{.15}{6}\right)^6$, so $i^{(2)} = 2[(1.025)^3 - 1] = .15378$.

□

In the same way that we defined a nominal rate of interest, we could also define a nominal rate of discount, $d^{(m)}$, as meaning an effective rate of discount of $\frac{d^{(m)}}{m}$ per m^{th} of a year. Analogous to identity (1.19), it is easy to see that

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m. \tag{1.20}$$

Since $1 - d = \frac{1}{1+i}$, we conclude that

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = 1 + i = (1-d)^{-1} = \left[1 - \frac{d^{(n)}}{n}\right]^{-n}, \qquad (1.21)$$

for all positive integers m and n.

Example 1.10

Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible monthly.

Solution

$\left[1 - \frac{d^{(2)}}{2}\right]^{-2} = \left[1 + \frac{i^{(12)}}{12}\right]^{12}$, so $1 - \frac{d^{(2)}}{2} = (1.01)^{-6} = .942045$, from which we find $d^{(2)} = 2(1 - .942045) = .11591$. □

1.6 FORCE OF INTEREST

We note before starting this section that it is somewhat theoretical, and is independent of the rest of the text. Anyone wishing to proceed directly to more practical problems can safely omit this material. In particular, more background knowledge is required for a full understanding here than is required for any other section; students with only a sketchy knowledge of calculus might omit this on first reading.

Assume that the effective annual rate of interest is $i = .12$, and that we want to find nominal rates $i^{(m)}$ equivalent to i. The formula $i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$, which comes from identity (1.19), is used to calculate these values which are shown in Table 1.1.

TABLE 1.1

m	1	2	5	10	50
$i^{(m)}$	.12	.1166	.1146	.1140	.1135

We observe that $i^{(m)}$ decreases as m gets larger, a fact which we will be able to prove later in this section. We also observe that the values of $i^{(m)}$ are decreasing very slowly as we go further and further along; in the language of calculus, $i^{(m)}$ seems to be approaching a limit. This is, in fact, what is happening, and we can use L'Hopital's rule to see what the limit is. There is no need to assume $i = .12$ in our derivation, so we proceed with arbitrary i.

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} m\left[(1+i)^{1/m} - 1\right] = \lim_{m\to\infty} \frac{(1+i)^{1/m} - 1}{\frac{1}{m}} \tag{1.22}$$

Since (1.22) is of the form $\frac{0}{0}$, we take derivatives top and bottom, cancel, and obtain

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} \left[(1+\ i)^{1/m} \cdot ln\,(1+i)\right] = ln\,(1+i), \tag{1.23}$$

since $\lim\limits_{m\to\infty} (1+i)^{1/m} = 1$.

This limit is called the *force of interest* and is denoted by δ, so we have

$$\delta = ln\,(1+\ i). \tag{1.24}$$

In our example, $\delta = ln(1.12) = .11333$. The reader should compare this with the entries in Table 1.1.

Intuitively, δ represents a nominal rate of interest which is convertible *continuously*, a notion of more theoretical than practical importance. However, δ can be a very good approximation for $i^{(m)}$ when m is large (for example, a nominal rate convertible daily), and has the advantage of being very easy to calculate.

We note that identity (1.24) can be rewritten as

$$e^{\delta} = 1 + i. \tag{1.25}$$

The usefulness of this form is shown in the next example. Again we stress the importance of being able to convert a rate of interest with a given conversion frequency to an equivalent rate with a different conversion frequency.

Example 1.11

A loan of 3000 is taken out on June 23, 1997. If the force of interest is 14%, find each of the following:

(a) The value of the loan on June 23, 2002.

(b) The value of i.

(c) The value of $i^{(12)}$.

Solution

(a) The value 5 years later is $3000(1+i)^5$. Using $e^{\delta}=1+i$, we obtain $3000(e^{.14})^5 = 3000\, e^{.7} = 6041.26$.

(b) $i = e^{.14} - 1 = .15027$.

(c) $\left(1+\frac{i^{(12)}}{12}\right)^{12} = 1+i = e^{.14}$, so we have the result

$$i^{(12)} = 12(e^{.14/12} - 1) = .14082. \qquad \square$$

Remark

Note that if we tried to solve part (a) by first obtaining $i = .15027$ (as in part (b)), and then calculating $3000(1.15027)^5$, we would get 6041.16, an answer differing from our first answer by .10. There is nothing wrong with this second method, except that not enough decimal places were carried in the value of i to guarantee an accurate answer. Let us repeat an earlier admonition: it is always wise to do as few calculations as necessary.

Observe that $D[(1+i)^t] = (1+i)^t \cdot \, ln\,(1+i)$, where D stands for the derivative with respect to t. Hence we see that

$$\delta = ln(1+i) = \frac{D[(1+i)^t]}{(1+i)^t} = \frac{D[a(t)]}{a(t)}. \qquad (1.26)$$

Let us see why this fact happens to be true. Recall from the definition of the derivative that $D[a(t)] = \lim_{h\to 0} \frac{a(t+h)-a(t)}{h}$, so

$$\frac{D[a(t)]}{a(t)} = \lim_{h\to 0} \frac{a(t+h)-a(t)}{h \cdot a(t)} = \lim_{h\to 0} \frac{\frac{a(t+h)-a(t)}{a(t)}}{h}. \qquad (1.27)$$

The term $\dfrac{a(t+h)-a(t)}{a(t)}$ in (1.27) is just the effective rate of interest over a very small time period h, so $\dfrac{\frac{a(t+h)-a(t)}{a(t)}}{h}$ is the nominal annual rate corresponding to that effective rate, which agrees with our earlier definition of δ.

The above analysis does more than that, however. It also indicates how the force of interest should be defined for arbitrary accumulation functions.

First, let us observe that $\delta = ln\,(1+i)$ is independent of t. However, this is a special property of compound interest, corresponding to a constant i_n. For arbitrary accumulation functions, we define the force of interest at time t, δ_t, by

$$\delta_t = \frac{D[a(t)]}{a(t)}, \tag{1.28}$$

since we would normally expect δ_t to depend on t.

For certain functions, it is more convenient to use the equivalent definition

$$\delta_t = D[ln\,(a(t))]. \tag{1.29}$$

We also remark that, since $A(t) = k \cdot a(t)$, it follows that

$$\delta_t = \frac{D[A(t)]}{A(t)} = D[ln\,(A(t))]. \tag{1.30}$$

Example 1.12

Find δ_t in the case of simple interest.

Solution

$\delta_t = \dfrac{D(1+it)}{1+it} = \dfrac{i}{1+it}.$ □

We now have a method for finding the force of interest, δ_t, given any accumulation function $a(t)$. What if we are given δ_t instead, and wish to derive $a(t)$ from it?

To start with, let us write our definition of δ_t from (1.29) using a different variable, namely $\delta_r = D[ln\,(a(r))]$, where D now means the derivative with respect to r. Integrating both sides of this equation from 0 to t, we obtain

$$\int_0^t \delta_r\, dr = \int_0^t D[ln\,(a(r))]\, dr = ln\,(a(r))\Big|_0^t$$
$$= ln\,(a(t)) - ln(a(0))$$
$$= ln\,(a(t)), \qquad (1.31)$$

since $a(0) = 1$ and $ln\, 1 = 0$. Then taking the antilog we have

$$a(t) = e^{\int_0^t \delta_r\, dr}. \qquad (1.32)$$

Example 1.13

Prove that if δ is a constant (i.e., independent of r), then $a(t) = (1+i)^t$ for some i.

Solution

If $\delta_r = c$, the right hand side of (1.32) is $e^{\int_0^t c\, dr} = e^{ct} = (e^c)^t$. Hence the result is proved with $i = e^c - 1$. □

Example 1.14

Prove that $\int_0^n A(t)\,\delta_t\, dt = A(n) - A(0)$ for any amount function $A(t)$.

Solution

The left hand side is

$$\int_0^n A(t)\,\delta_t\, dt = \int_0^n A(t)\left[\frac{D[A(t)]}{A(t)}\right] dt = \int_0^n D[A(t)]\, dt$$
$$= A(t)\Big|_0^n = A(n) - A(0) \text{ as required.}$$

□

The identity in the above example has an interesting verbal interpretation. The term $\delta_t\, dt$ represents the effective rate of interest at time t for the infinitesimal "period of time" dt. Hence $A(t)\,\delta_t\, dt$ represents the amount of interest earned in this period, and $\int_0^n A(t)\,\delta_t\, dt$ represents the total amount of interest earned over the entire period, a number which is clearly equal to $A(n) - A(0)$.

We now return to the compound interest case where we have $a(t) = (1+i)^t$. It is interesting to write some of the formulae already developed as power series expansions. For example $\delta = ln\,(1+i)$ becomes

$$\delta = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \cdots. \qquad (1.33)$$

Convergence is a concern here, but as long as $|\,i\,| < 1$, which is usually the case, the above series does converge.

Another important formula was $i = e^{\delta} - 1$, which becomes

$$i \;=\; \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots. \tag{1.34}$$

Since all terms on the right hand side are positive, this allows us to conclude immediately that $i > \delta$. We note in passing that this series converges for all δ.

Next let us expand the expression $i = \dfrac{d}{1-d} = d(1-d)^{-1}$, which becomes

$$i = d(1 + d + d^2 + d^3 + \cdots) = d + d^2 + d^3 + \cdots. \tag{1.35}$$

Again this shows us very clearly that $i > d$. We also note that we must have $|\,d\,| < 1$ for this series to converge. In fact, trying to put $d = 2$ yields an amusing result: the left hand side is $i = \dfrac{2}{1-2} = -2$, whereas the right hand side becomes $2 + 2^2 + 2^3 + \cdots$, all of which are positive terms. Thus we have "proven" that -2 is a positive number!

Next let us expand $i^{(m)}$ as a function of i. From (1.19) we have $i^{(m)} = m[(1+i\,)^{1/m} - 1]$, so

$$\begin{aligned} i^{(m)} \;&=\; m\left[1 + \frac{1}{m}\,i + \frac{\frac{1}{m}(\frac{1}{m}-1)}{2!}\,i^2 \;+\; \frac{(\frac{1}{m})(\frac{1}{m}-1)(\frac{1}{m}-2)}{3!}\,i^3 \;+\; \cdots \;-1\right] \\ &=\; i + \left[\frac{\frac{1}{m} \;-\; 1}{2!}\right] i^2 \;+\; \frac{(\frac{1}{m}-1)(\frac{1}{m}-2)}{3!}\,i^3 \;+\; \cdots. \end{aligned} \tag{1.36}$$

Again, this converges for $|\,i\,| < 1$.

Why are we interested in power series expansions? Well, we have already seen that they sometimes allow us to easily conclude facts like $i > \delta$ (although they certainly aren't needed for that). They also give us a quick means of calculating some of these functions, since often only the first few terms of the series are necessary for a high degree of accuracy. If you ask your calculator to do this work for you instead, it will oblige, but the program used for the calculation will often be a variation of one of those described above.

As a final example, let us expand $d^{(m)}$ in terms of δ. We have

$$\left[1-\frac{d^{(m)}}{m}\right]^m = (1+i)^{-1} = e^{-\delta}, \tag{1.37}$$

so

$$\begin{aligned} d^{(m)} &= m[1-e^{-\delta/m}] \\ &= m\left[1 - \left(1+\left(-\frac{\delta}{m}\right)+\frac{(-\frac{\delta}{m})^2}{2!}+\frac{(-\frac{\delta}{m})^3}{3!}+\cdots\right)\right] \\ &= m\left[\frac{\delta}{m}-\frac{\delta^2}{2!m^2}+\frac{\delta^3}{3!m^3}-\cdots\right] \\ &= \delta-\frac{\delta^2}{2!m}+\frac{\delta^3}{3!m^2}-\cdots. \end{aligned} \tag{1.38}$$

From this we easily see that $\lim_{m\to\infty} d^{(m)} = \delta$. In other words, there is no need to define a force of discount, because it will turn out to be the same as the force of interest already defined.

EXERCISES

1.1 Accumulation Function; 1.2 Simple Interest; 1.3 Compound Interest

1-1. Alphonse has 14,000 in an account on January 1, 1995.
- (a) Assuming simple interest at 8% per year, find the accumulated value on January 1, 2001.
- (b) Assuming compound interest at 8% per year, find the accumulated value on January 1, 2001.
- (c) Assuming exact simple interest at 8% per year, find the accumulated value on March 8, 1995.
- (d) Assuming compound interest at 8% per year, but linear interpolation between integral durations, find the accumulated value on February 17, 1997.

1-2. Mary has 14,000 in an account on January 1, 1995.

(a) Assuming compound interest at 11% per year, find the accumulated value on January 1, 2000.

(b) Assuming ordinary simple interest at 11% per year, find the accumulated value on April 7, 2000.

(c) Assuming compound interest at 11% per year, but linear interpolation between integral durations, find the accumulated value on April 7, 2000.

1-3. For the $a(t)$ function given in Example 1.1, prove that $i_{n+1} < i_n$ for all positive integers n.

1-4. Consider the function $a(t) = \sqrt{1+(i^2+2i)t^2},\ i > 0,\ t \geq 0.$

(a) Show that $a(0) = 1$ and $a(1) = 1+i$.

(b) Show that $a(t)$ is increasing and continuous for $t \geq 0$.

(c) Show that $a(t) < 1+it$ for $0 < t < 1$, but $a(t) > 1+it$ for $t > 1$.

(d) Show that $a(t) < (1+i)^t$ if t is sufficiently large.

1-5. Let $a(t)$ be a function such that $a(0) = 1$ and i_n is constant for all n.

(a) Prove that $a(t) = (1+i)^t$ for all integers $t \geq 0$.

(b) Can you conclude that $a(t) = (1+i)^t$ for all $t \geq 0$?

1-6. Let $A(t)$ be an amount function. For every positive integer n, define $I_n = A(n) - A(n-1)$.

(a) Explain verbally what I_n represents

(b) Prove that $A(n) - A(0) = I_1 + I_2 + \cdots + I_n$.

(c) Explain verbally the result in part (b).

(d) Is it true that $a(n) - a(0) = i_1 + i_2 + \cdots + i_n$? Explain.

1-7. (a) In how many years will 1000 accumulate to 1400 at 12% simple interest?

(b) At what rate of simple interest will 1000 accumulate to 1500 in 6 years?

(c) Repeat parts (a) and (b) assuming compound interest instead of simple interest.

1-8. At a certain rate of simple interest, 1000 will accumulate to 1300 after a certain period of time. Find the accumulated value of 500 at a rate of simple interest $\frac{2}{3}$ as great over twice as long a period of time.

1-9. Find the accumulated value of 6000 invested for ten years, if the compound interest rate is 7% per year for the first four years and 11% per year for the last six.

1-10. Annual compound interest rates are 13% in 1994, 11% in 1995 and 15% in 1996. Find the effective rate of compound interest per year which yields an equivalent return over the three-year period.

1-11. At a certain rate of compound interest, it is found that 1 grows to 2 in x years, 2 grows to 3 in y years, and 1 grows to 5 in z years. Prove that 6 grows to 10 in $z - x - y$ years.

1-12. If 1 grows to K in x periods at compound rate i per period and 1 grows to K in y periods at compound rate $2i$ per period, which one of the following is always true? Prove your answer.
(a) $x < 2y$
(b) $x = 2y$
(c) $x > 2y$
(d) $y = \sqrt{x}$
(e) $y > 2x$

1.4 Present Value and Discount

1-13. Henry has an investment of 1000 on January 1, 1998 at a compound annual rate of discount $d = .12$.
(a) Find the value of his investment on January 1, 1995.
(b) Find the value of i corresponding to d.
(c) Using your answer to part (b), rework part (a) using i instead of d. Do you get the same answer?

1-14. Mary has 14,000 in an account on January 1, 1995.
(a) Assuming compound interest at 12% per year, find the present value on January 1, 1989.
(b) Assuming compound discount at 12% per year, find the present value on January 1, 1989.
(c) Explain the relative magnitude of your answers to parts (a) and (b).

1-15. (a) Sketch a graph of $a(t)$ with its extension to present value in the case of simple interest.

(b) Explain, both mathematically and verbally, why $1 - it$ is not the correct present value t years in the past, when simple interest is assumed.

1-16. Prove that d_n is constant in the case of compound interest.

1-17. Prove each of the following identities mathematically. For parts (a), (b) and (c), give a verbal explanation of how you can see that they are correct.

(a) $d = iv$ (d) $\frac{1}{d} - \frac{1}{i} = 1$

(b) $d = 1 - v$ (e) $d\left(1 + \frac{i}{2}\right) = i\left(1 - \frac{d}{2}\right)$

(c) $i - d = id$ (f) $i\sqrt{1-d} = d\sqrt{1+i}$

1-18. Four of the following five expressions have the same value (for $i > 0$). Which one is the exception?

(a) $\dfrac{d^3}{(1-d)^2}$ (b) $\dfrac{(i-d)^2}{1-v}$ (c) $(i-d)d$ (d) $i^3 - i^3 d$ (e) $i^2 d$

1-19. The interest on L for one year is 216. The equivalent discount on L for one year is 200. What is L?

1.5 Nominal Rate of Interest

1-20. Acme Trust offers three different savings accounts to an investor.

Account A: compound interest at 12% per year convertible quarterly.

Account B: compound interest at 11.97% per year convertible 5 times per year.

Account C: compound discount at 11.8% per year convertible 10 times per year.

Which account is most advantageous to the investor? Which account is most advantageous to Acme Trust?

1-21. Phyllis takes out a loan of 3000 at a rate of 16% per year convertible 4 times a year. How much does she owe after 21 months?

1-22. The Bank of Newfoundland offers a 12% mortgage convertible semiannually. Find each of the following:
(a) i (b) $d^{(4)}$ (c) $i^{(12)}$
(d) The equivalent effective rate of interest per month.

1-23. 100 grows to 107 in 6 months. Find each of the following:
(a) The effective rate of interest per half-year.
(b) $i^{(2)}$ (c) i (d) $d^{(3)}$

1-24. Find n such that $1 + \frac{i^{(n)}}{n} = \frac{1 + \frac{i^{(6)}}{6}}{1 + \frac{i^{(8)}}{8}}$.

1-25. Express $d^{(7)}$ as a function of $i^{(5)}$.

1-26. Show that $v\left(1 + \frac{i^{(3)}}{3}\right) = \left(1 + \frac{i^{(30)}}{30}\right)\left(1 - \frac{d^{(5)}}{5}\right)\sqrt{1-d}$.

1-27. Prove that $i^{(4)}d^{(8)} \geq i^{(8)}d^{(4)}$.

1-28. (a) Prove that $i^{(m)} - d^{(m)} = \frac{i^{(m)}d^{(m)}}{m}$.

(b) Prove that $\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$.

1.6 Force of Interest

1-29. Find the equivalent value of δ in each of the following cases.
(a) $i = .13$
(b) $d = .13$
(c) $i^{(4)} = .13$
(d) $d^{(5)} = .13$

1-30. In Section 1.3, it was shown that for $0 < t < 1$, $(1+i)^t < 1 + it$. Show that $1 + it - (1+i)^t$ is maximized at $t = \frac{1}{\delta}[ln\, i - ln\, \delta]$.

1-31. Assume that the force of interest is doubled.
(a) Show that the effective annual interest rate is more than doubled.
(b) Show that the effective annual discount rate is less than doubled.

1-32. Show that $\lim\limits_{i \to 0} \dfrac{i - \delta}{\delta^2} = .50$.

1-33. Find $a(t)$ if $\delta_t = .04(1+t)^{-1}$.

1-34. Obtain an expression for δ_t if $A(t) = ka^{t+1}\, b^{t^3}\, c^{d^t}$.

1-35. Using mathematical induction, prove that for all positive integers n, $\dfrac{d^n}{dv^n}(v^{n-1}\delta) = -(1+i)(n-1)!$, where $\dfrac{d}{dv}$ denotes derivative with respect to v.

1-36. Express v as a power series expansion in terms of δ.

1-37. Express d as a power series expansion in terms of i.

1-38. Prove that $i^{(n)} < i^{(m)}$ if $n > m$.

1-39. Prove that $d < d^{(n)} < \delta < i^{(n)} < i$ for all $n > 1$.

1-40. Show that $D(\delta_t) = \dfrac{D^2 A(t)}{A(t)} - (\delta_t)^2$, where D is the derivative with respect to t.

1-41. Show that $\delta = \dfrac{d+i}{2} + \dfrac{d^2-i^2}{4} + \dfrac{d^3+i^3}{6} + \cdots$.

1-42. Which is larger, $i - \delta$ or $\delta - d$? Prove your answer.

1-43. (a) Write a computer program which will take a given value of i and output values of $i^{(m)}$ for a succession of values of m.

(b) Extend the program in part (a) to also give you a value for δ, using $\delta = \lim\limits_{m \to \infty} i^{(m)}$.

1-44. Write a computer program which will take a given value of δ and output the equivalent value of i. (*Use the power series expansion.*)