# Ambrose Lo's
# Study Manual for SOA Exam FAM
## (Fundamentals of Actuarial Mathematics)

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3rd Edition

# Contents

# Preface

> **NOTE TO STUDENTS**
>
> Please read this preface carefully, even if it looks long. It contains **VERY** important information that will help you navigate this study manual smoothly and study for Exam FAM effectively.

Thank you very much for choosing to use this study manual, which is designed to provide complete coverage of Exam FAM (*Fundamentals of Actuarial Mathematics*) and prepare you more than adequately for this challenging exam.

## P.1 About Exam FAM

### Exam Administrations

Exam FAM is a 3.5-hour computer-based exam that consists of **34 multiple-choice questions**.[1] Offered for the first time in October 2022 by the Society of Actuaries (SOA), it is currently delivered via computer-based testing (CBT) in a Prometric exam center in March, July, and October effective from 2025. You can find the specifics of each testing window (e.g., exam dates, registration deadlines) at:

`https://www.soa.org/education/exam-schedule/`.

When the registration window is open, you will register for the exam online at

`https://www.soa.org/education/exam-req/registration/edu-registration/`,

receive an email confirmation letter from the SOA containing your candidate ID, then schedule an appointment at Prometric (`https://www.prometric.com/soa`).

### Exam Theme: What's Exam FAM Like?

Here is the SOA's official web page for Exam FAM:

`https://www.soa.org/education/exam-req/edu-exam-fam/`

[1] Effective from the November 2024 sitting, the exam syllabus has be trimmed and the number of questions reduced from 40 to 34. Please see the SOA's official announcement here: `https://www.soa.org/education/exam-req/syllabus-study-materials/edu-updates-exam-fam/`.

In short, FAM is an **ENORMOUS** exam (much more so than P and FM combined! ☹) that is made up of two largely independent and equally important parts: The short-term part (FAM-S) and the long-term part (FAM-L), and a total of ten (10) topics. On the exam, there will be about $34/2 = 17$ questions from each part, but they will be presented at random with no distinction regarding which part they relate to. (In fact, the real exam, even with 34 questions, will likely only test a very small subset of the whole exam syllabus.) The 10 topics, along with their weight ranges and where they are covered in this study manual, are shown in the table below.

| | Topic | Weight range | Relevant chapter(s) of this manual |
|---|---|---|---|
| | FAM-S Part | | |
| 1. | Short-Term Insurance and Reinsurance Coverages | 5-10% | 1 |
| 2. | Severity, Frequency, and Aggregate Models | 12.5-17.5% | 2 |
| 3. | Parametric Estimation | 2.5-7.5% | 3 |
| 4. | Introduction to Credibility | 2.5-5% | 4 |
| 5. | Pricing and Reserving for Short-Term Insurance Coverages | 10-15% | 5 |
| 6. | Option Pricing Fundamentals | 2.5-7.5% | 6 |
| | Total | 48.75% | |
| | FAM-L Part | | |
| 7. | Long-Term Insurance Coverages and Retirement Financial Security Programs | 2.5-5% | 13 |
| 8. | Mortality Models | 10-15% | 7, 8 |
| 9. | Present Value Random Variables for Long-Term Insurance Coverages | 12.5-20% | 9, 10 |
| 10. | Premium and Policy Value Calculation for Long-Term Insurance Coverages | 15-22.5% | 11, 12 |
| | Total | 51.25% | |

Let me give you a quick taste 🍸 of what the two parts of Exam FAM are like.

**FAM-S.** From the syllabus of Exam FAM:

> "The syllabus for the short-term section of the examination provides an introduction to *modeling* and covers important actuarial methods that are useful in modeling. It will also introduce students to the foundational principles of ratemaking and reserving for short-term coverages."

We will delve into the details of the six FAM-S exam topics when we get to specific chapters of this manual. Simply put,

> FAM-S is largely an extension and applications of the fundamental concepts and tools you learned in Exam P to actuarial modeling.

The main theme is to analyze probabilistic quantities of actuarial models constructed for the *size of loss*, *number of losses*, and *total losses* faced by an insurance company, which are the three types of random variable that figure most prominently in short-term insurance, a.k.a. *property and casualty* (P&C) *insurance*. You will be busy calculating such things as probabilities, percentiles, means, and variances of various probability distributions used to model these loss random variables, such as exponential, uniform, normal, lognormal, Pareto, and many more. In one topic (Topic 3), you will also learn how to estimate the parameters of these distributions. Blending probability, statistics, and insurance, the exam materials have a strong probabilistic and actuarial flavor. If you love probability and statistics, then I'm sure you will love ❤ FAM-S (at least, you won't hate it!).

**FAM-L.** The FAM-L part is largely separate from the FAM-S part. According to the exam syllabus:

> "The syllabus for the long-term section of the examination develops the candidate's knowledge of the theoretical basis of *contingent payment* models and the application of those models to insurance and other financial risks."

The word "contingent" means "dependent on something random" and therefore carries uncertainty. Broadly speaking, Exam FAM-L is about the study of *contingent cash flows* $, especially those associated with life insurance and annuities. Traditionally called *life contingencies*, this fundamental subject in actuarial science has been on the SOA exam syllabus for long (as far as I know, there were already life contingencies exam questions in the 1940s). It tests skills for dealing with random events (because the cash flows of life insurance and annuity products are contingent on the random lifetimes of their policyholders) and the time value of money (because the cash flows arise in the future), which are the topics of Exams P and FM. Using tools from these two preliminary exams, we will set up a probabilistic framework for doing *pricing* (setting the premium charged on policyholders) and *reserving* (determining how much to set aside to fund future obligations) for common life insurance and annuity policies. For the most part, the exam topics are mathematically interesting and practically useful.

## Mathematical Prerequisites

The exam syllabus says that:

> "A thorough knowledge of calculus, *probability*, mathematical statistics and *interest theory* is assumed."

I assume that you have passed Exams P and FM, and are no stranger to concepts like the mean, variance, and distribution/density/mass function of a random variable, and how to do discounting using the effective annual interest rate $i$ vs. the force of interest $\delta$ (also denoted by $r$), all of which will be intensively used in Exam FAM. There will be instances where you will do some simple differentiation and integration, so you are also assumed to be reasonably good at calculus.

(Don't worry, integration by parts, which most students hate, or tabular integration, is very rarely used in FAM. ☺)

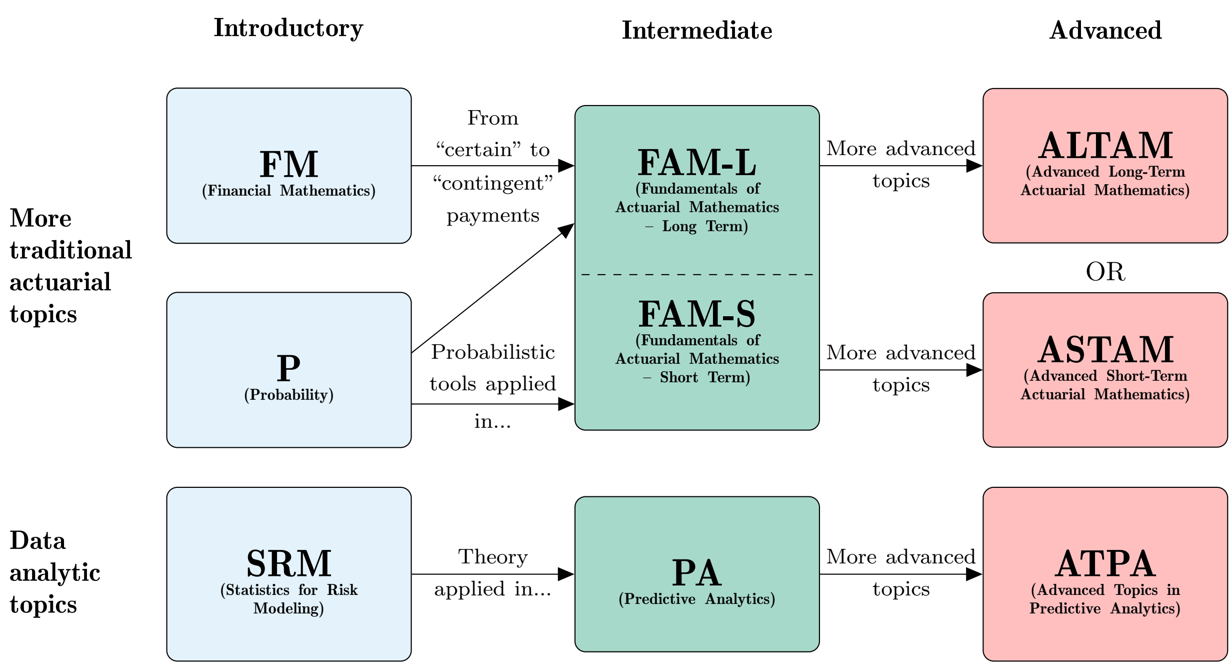
## Relationship with other ASA Exams

In the current ASA curriculum, Exam FAM is typically taken by students who have passed Exams P and FM, and followed by Exam ALTAM (*Advanced Long-Term Actuarial Mathematics*) or Exam ASTAM (*Advanced Short-Term Actuarial Mathematics*). These two exams are 3-hour written-answer exams that develop the fundamental knowledge you learn in FAM to a more "advanced" level. Fortunately,

> it is enough to take either ALTAM or ASTAM, depending on whether you like the short-term part or the long-term part more (more precisely, which part you hate less!). You have a choice!

The flowchart below shows how these exams (and other ASA exams for your information) are related. While there is no set order in which the exams should be taken, students typically attempt exams from left to right, or from introductory, intermediate, to advanced.

### Flowchart of ASA Exams Effective from 2022



## Historical Pass Rates and Pass Marks %

The table below shows the number of sitting candidates, number of passing candidates, pass rates, and pass marks (= the actual percentage score you have to get to pass the exam) for Exam FAM since it was offered in October 2022:

| Sitting | # Candidates | # Passing Candidates | Pass Rate | Pass Mark |
|---|---|---|---|---|
| November 2024 | | *(To be posted on the SOA web page 🔗)* | | |
| July 2024 | 1479 | 857 | 57.9% | 56.9% |
| March 2024 | 1169 | 640 | 54.7% | 56.9% |
| November 2023 | 1762 | 1055 | 59.9% | 55.6% (I took FAM-L in this sitting! 😅) |
| July 2023 | 1353 | 794 | 58.7% | 63.2% (I took FAM-S in this sitting! 😅) |
| March 2023 | 1200 | 753 | 62.8% | 58.9% |
| October 2022 | 853 | 423 | 49.6% | 62.2% (I took this FAM! 😉) |

The pass rates and pass marks have fluctuated by quite a bit over the past sittings, perhaps because FAM is still in its infancy. On average, they are both close to 60%. In the current format, Exam FAM consists of 34 multiple-choice questions, so to play safe:

You should aim to get **at least 22 questions** correct.

## Syllabus Readings 📕

Exam FAM has a few required readings:

- *Loss Models: From Data to Decisions* (5th Edition), 2019, by Klugman, Panjer, and Willmot, Wiley, ISBN: 978-1-119-52378-9

  Having been on the SOA exam syllabus since 2000, this college-style, mathematically oriented textbook accounts for the majority of the material of Exam FAM-S (and ASTAM). Topics 2, 3, 4, and a large part of Topic 1 of the exam are based on this textbook.

- *Introduction to Ratemaking and Loss Reserving for Property and Casualty Insurance* (5th Edition),[2] 2022, by Brown and Lennox, ACTEX Learning, ISBN: 978-1-64756-787-3

  Referred to simply as IRLR in this study manual, this textbook is the main source of the qualitative (or descriptive) material of Exam FAM-S. It is the backbone of Topic 5 and part of Topic 1 of the exam.

- *Actuarial Mathematics for Life Contingent Risks* (3rd Edition), 2020, by Dickson, Hardy, and Waters, Cambridge University Press, ISBN: 978-1-108-47808-3

  Abbreviated to AMLCR in this study manual, this book covers Topic 6 of the FAM-S part and the entire FAM-L part (Topics 7-10) with a rather practical flavor. Although I find the book a pleasant read, I have streamlined the presentation and sequence of some topics to make our learning more coherent and our exam preparation more effective.

[2]To be honest, there are better resources for learning loss reserving and ratemaking, e.g., those published by the Casualty Actuarial Society (CAS), which, of course, specializes in P&C insurance. For obvious reasons, the SOA avoids using the CAS's resources as syllabus material. 😉

As far as possible, I have followed the notation in the syllabus readings because exam questions may use the symbols and shorthand in the readings without further definitions, and expect you to know what the symbols mean.

This manual is self-contained in the sense that studying the manual carefully is more than sufficient to pass the exam, and you need not refer to any of the syllabus readings for the purposes of learning. However, at the beginning of each chapter/section, I reference the relevant readings for the benefit of students who have access to these resources and want to read more. For in-text examples and end-of-section problems that are motivated from exercises in the syllabus readings, the relevant readings are properly cited to give them due credit. As the exam syllabus says,

"Exercises [in AMLCR] are considered part of the required readings."

⚠ In fact, many exercises in the syllabus readings were the theme of some recent exam questions and may continue to be so in future exams.

## Exam Tables and the Standard Normal Calculator

During the exam, you will have access to a set of tables, which are available on the Prometric computers. Here is the link to download these tables, which can be found on the last page of the exam syllabus:

`https://www.soa.org/globalassets/assets/files/edu/2024/fall/2024-10-exam-fam-tables.pdf`

Useful, if not necessary for solving some of the exam questions, these tables save your memory burden and include:

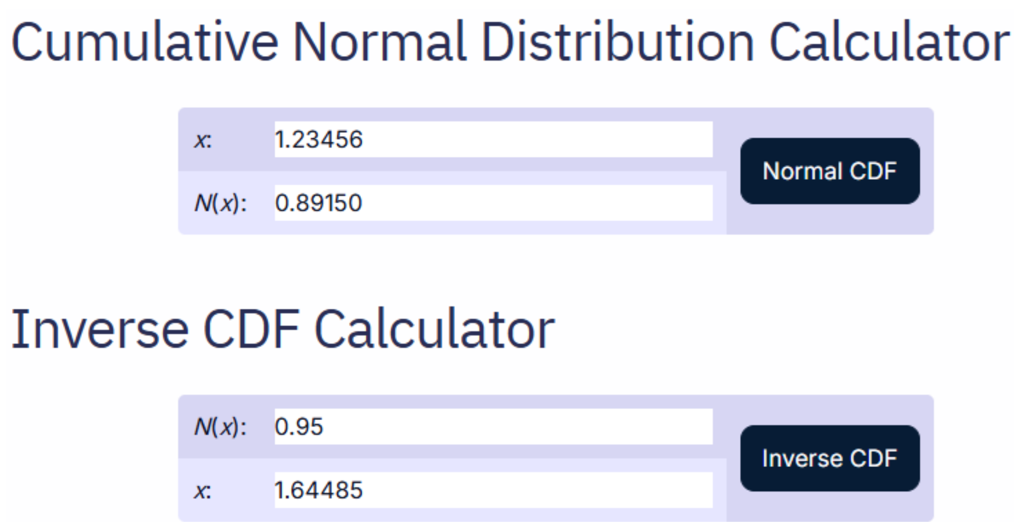
- FAM-S part (the tables are collectively called the "FAM-S tables" in the rest of this manual and heavily used in Chapters 1-4)
  - ▷ *Appendix A:* An inventory of continuous probability distributions
  - ▷ *Appendix B:* An inventory of discrete probability distributions
- FAM-L part (the tables are collectively called the "FAM-L tables")
  - ▷ The Standard Ultimate Life Table (to be introduced in Section 8.1)
  - ▷ A table of various quantities corresponding to the effective interest rate $i = 0.05$
  - ▷ Selected formulas which are harder to remember

I strongly suggest that you print a copy of these tables and formulas as they will be used intensively as you work through this study manual and when you take the real exam.

The CBT environment also includes *Prometric's standard normal calculator*, accessible from

`https://prometric.com/soa`.

Here is a screenshot of the calculator:




The calculator has two parts:

- In the upper part of the table, you can get the value of the standard normal distribution function $\Phi(x)$ to 5 decimal places, e.g., $\Phi(1.23456) = 0.89150$ above. The argument $x$ can be any real value (positive or negative) and expressed in whatever precision you like. In the rest of this manual, I will input the argument to 5 decimal places, consistent with the output.

- The lower part is for the *percentiles* of the standard normal distribution, which are the values that lead to different prescribed values of the distribution function, e.g., the 95th percentile is 1.64485, as shown above. (You may have used the less accurate value of 1.645 in your earlier studies.)

# P.2 About this Study Manual

## What is Special about This Study Manual?

I fully understand that you have an acutely limited amount of study time and that the syllabus of Exam FAM is enormous, to say the least. With this in mind, the overriding objective of this study manual is to help you grasp the material in Exam FAM as effectively and efficiently as possible, so that you will pass the exam on your first try easily and go on to more advanced exams like ALTAM, ASTAM, and (AT)PA confidently. (A secondary but still important objective is to let you have some fun along the way. 😜) Here are some unique features of this manual to make this possible.

### Feature 1: The Coach DID Play!

Usually coaches don't play 😆, but as a study manual author, I took the initiative to write the **October 2022 Exam FAM**, the **July 2023 Exam FAM-S**, and the **November 2023 Exam FAM-L** to experience first-hand what the real exams were like, despite having been an FSA since 2013 (and technically free from SOA exams thereafter). I made this decision in the belief that *teaching* an exam and *taking* an exam are rather different activities, and braving the exam myself is the best way to ensure that my manual is indeed useful for exam preparation. If the manual is useful, then at the minimum the author himself can do well, right? I am thrilled that...

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The scale of grades runs from 0 to 10. passing grades are 6 through 10. A grade of 0 does not mean that the candidate received no credit but that he/she had a very poor paper. Similarly, a grade of 10 indicates a very fine paper but not necessarily a perfect one.

**ID: Candidate ID: 99574**

Today's Date: 12/12/2022

**Oct 2022 Fundamentals of Actuarial Mathematics**

| Course | Grade |
|---|---|
| EXAMFAM | 10 |

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Grade Slip

The scale of grades runs from 0 to 10. passing grades are 6 through 10. A grade of 0 does not mean that the candidate received no credit but that he/she had a very poor paper. Similarly, a grade of 10 indicates a very fine paper but not necessarily a perfect one.

**ID: Candidate ID: 67888**

Today's Date: 9/11/2023

**Jul 2023 Fundamentals of Actuarial Mathematics Short Term**

| Course | Grade |
|---|---|
| EXAMFAMS | 10 |

Ambrose Lo FSA,CERA


If you use this FAM study manual, you can rest assured that it is written from an exam taker's perspective by a professional instructor who has experienced the "pain" of FAM(-L and -S) candidates and truly understands their needs. Drawing upon his "real battle experience" and firm grasp of the exam topics, the author will go to great lengths to help you prepare for this challenging exam in the best possible way. You are in capable hands. 👍

**Feature 2: Exam-focused Content**

To maximize your learning effectiveness and efficiency, I have divided this study manual into three parts, with different characteristics:

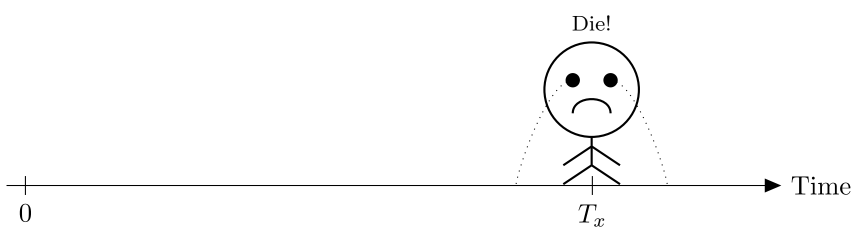
- **Parts I and II: Core of FAM-S (Chapters 1 to 6) and Core of FAM-L (Chapters 7 to 13)**
  - ▷ *(Let's start with the syllabus!)* Each chapter/section starts by explicitly stating which learning objectives and outcomes of the FAM exam syllabus we are going to study, to assure you that we are on track and hitting the right target.
  - ▷ *(In-text explanations)* The explanations in each chapter are thorough, but exam-focused and learning-oriented. Besides having a coherent narrative flow that shows the connections ⇄ between different exam topics, this manual strives to keep you motivated by showing how the concepts are typically tested, how the formulas are used, and where the exam focus lies in each section. Instead of showing unnecessary mathematical proofs that add little value to exam preparation, I grasp the chance to explain the intuitive meaning and mathematical structure of various formulas in the syllabus, to help you better remember and apply these formulas on the exam.

▷ *(In-text illustrative examples)* The main text of the manual is interspersed with carefully chosen past SOA/CAS exam questions, with full bibliographic details given (name of exam, year of examination, question number). Complementing the in-text explanations, these illustrative examples are meant to show you how the concepts you have just learned are usually tested and are an essential part of your reading.

▷ *(Boxed formulas and exam notes)* Formulas that you will use all the time are boxed for easy identification and retention, and numbered (in the (X.X.X) format) for later references. Important exam items and common mistakes committed by students are highlighted by boxes that look like:

> **⚠ EXAM NOTE ⚠**
>
> Be sure to pay special attention to boxes like this!

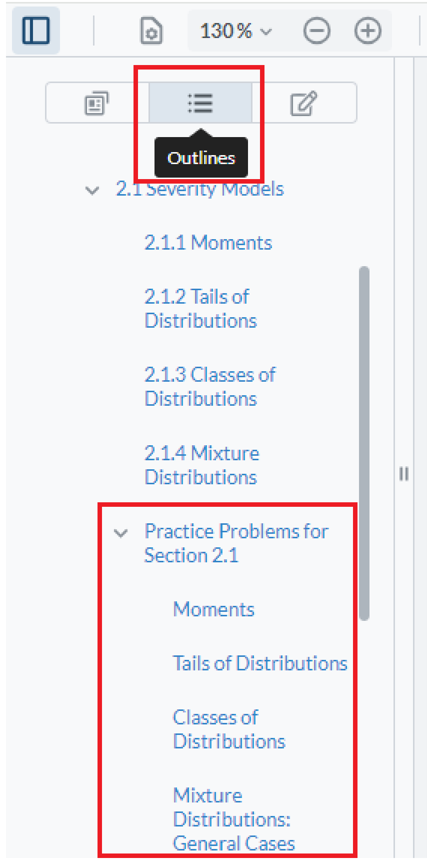
▷ *(Timeline diagrams)* Timeline diagrams are an indispensable tool for illustrating cash flows and formulas in life contingencies. In the main text of this manual (especially Chapters 9 and 10), you will see the timeline diagrams for various life insurance and annuity products. The cute guy below will sacrifice himself and die multiple times to help us understand life contingencies!




▷ *(End-of-section problems)* Besides the **332 in-text examples**, this study manual features about **1300 end-of-section/chapter practice problems**. In particular, all FAM sample questions have been included in this manual. Designed to reinforce what you have learned in the main text of the manual and provide additional opportunities, these practice problems are either real exam problems taken/adapted from relevant SOA/CAS past exams, or are original problems intended to illustrate less commonly tested items, all with step-by-step solutions and many with problem-solving remarks. The harder ones are labeled as [**HARDER!**].

While almost all exam prep resources boast a large number of exercises, the arrangement of this manual is unique in the following aspects:

- ☐ The practice problems, including the original ones, are in the same multiple-choice format as a real FAM exam question, with (A), (B), (C), (D), and (E). As you can see in my solutions, many of the wrong answers correspond to some common mistakes that students make (I tried to put myself in students' shoes), which is also the case in the real exam. I hope you won't pick those distractors! ☺
- ☐ To ease exam preparation, the practice problems are categorized by **theme** rather than by year, and you can navigate different themes using the "Outlines" tab on *Actuarial University*. 🏛 Here is a screenshot for Section 2.1:




Within each theme, problems testing similar items are grouped, so even a cursory glance at these problems will show you how those topics are typically tested and exam items that repeatedly emerge. You will find that many SOA questions are simple variations of some older exam questions, which shows that doing past exam questions is really important.

- **Part III: Final preparation**

  The last part concludes this comprehensive manual with the following resources:

  ▷ Five (5) original full-length **practice exams** designed to mimic the real FAM exam in terms of coverage, style, and difficulty immediately follow the core of this study manual. These practice exams give you a holistic review of the syllabus material and assess your readiness to take and pass the real exam. Detailed illustrative solutions are provided for each exam.

  ▷ Appendix A of the manual is a 42-page **cheat sheet** (the exam syllabus is enormous!) that provides a "helicopter" view of the entire FAM exam, and is useful for both regular review and last-minute exam preparation. A downloadable and printable version of the cheat sheet is available on my personal web page.

## A Study Guide: How to Use This Manual?

Having taught in a CAE university for about 10 years and got in touch with thousands of actuarial students, I have come up with the following study tips (which are useful even beyond FAM) and suggestions on how to make the most of this manual:

Step 1. Read the main text in each section of the manual carefully, including *all* of the in-text examples.

(The FAM-S and FAM-L parts are largely independent of each other, so you may start with either part. There are a handful of topics that appear in both parts, and these common topics are noted in the main text of the manual.)

When you work out the in-text examples, be sure to get a paper, do the math, and try to "replicate" my solutions (I mean it!). For any actuarial exams,

> keep in mind that it is not enough to be able to do problems; you need to do problems *accurately* and *efficiently*,

and the "learning by doing" approach above is the best way to help you absorb the material effectively and solve problems quickly. You need to get your hands dirty!

Step 2. Use the cheat sheet in Appendix A (and available on my personal web page) to look back on the important formulas and results in the section you have just read. This is an important way to reinforce your learning.

Step 3. Proceed to the end-of-section practice problems.

To succeed in any actuarial exam, I can't overemphasize the importance of practicing a wide variety of exam-type problems to consolidate your understanding and develop proficiency. This is the only path to true mastery of the material. However, loss modeling and life contingencies are classical subjects in SOA/CAS exams, which explains the abundance of old exam problems included in this manual (some of them date back to early 1980s...before I was born! ). There are so many that it would be virtually impossible (neither is it desirable) for a student to work out all of them—you would be overwhelmed!

To maximize the effectiveness and efficiency of your learning,

> I have marked the most representative and instructive practice problems in each section with an **asterisk (*)** and recommend that you do **ALL** of these **asterisked** problems in your first round of reading.

These selected problems, which are generally not more than 50% of the whole set of problems, span different themes and will add most value to your learning. Focusing on these problems should be a good learning strategy for developing a level of proficiency and confidence necessary for exam taking, while avoiding burn-out. As with the in-text examples, try to get a paper, do the math, write your own solutions, and compare yours with mine. (If you can come up with solutions that are shorter and neater than mine, let me know!)

Step 4. Repeat Steps 1 to 3 until you finish all of the 13 chapters in Parts I and II of the manual. For FAM, the whole process can take **FIVE months**, if not more!

Step 5. (This step is optional.) If the real exam is more than two weeks away from today, then consider doing a small random sample of the non-asterisked end-of-section problems, especially those related to your weak spots; otherwise, go on to Step 6.

Step 6. Move on to Part III of the manual and attempt the practice exams; see the prelude to these exams on page 1577 for more information.

If you have the perseverance and discipline to follow this study strategy closely, I have every confidence that you will pass the exam easily (the only question being whether you will get Grade 9 or 10! ☺).

## Announcements

As time goes by, I may post news and announcements about this study manual and Exam FAM on my personal web page:

`https://sites.google.com/site/ambroseloyp/publications/FAM`.

An errata list will also be maintained. I would greatly appreciate it if you could bring any potential errors, typographical or otherwise, to my attention via email (see below) so that they can be fixed in a future edition of the manual.

## Contact Us

If you encounter problems with your learning, we always stand ready to help.

- For **technical issues** (e.g., not able to access the manual on Actuarial University, extending your digital license, upgrading your product to include printing), please email ACTEX Learning's Customer Service at

  `support@actexlearning.com`

  The list of FAQs available on `https://www.actuarialuniversity.com/help/faq` may also be useful.

- Questions related to **specific contents** of this manual, including potential errors, can be directed to me (Ambrose) by emailing `amblo201011@gmail.com`. Please note:

  ▷ Remember to check out the errata list on my personal web page. It may happen that the errors you discover have already been addressed.

  ▷ Please identify the specific page(s), example(s), or problem(s) your questions are about. This will provide a concrete context and make our discussion much more fruitful.

**NOTE**

- To expedite the resolution process, it would be greatly appreciated if you could reach out to the appropriate email address. ☺
- I will strive to get back to you ASAP. Please check your spam folder if you don't hear back from me within 2-3 days.

## Acknowledgments

I would like to thank the SOA for kindly allowing me to reproduce its past and sample exam problems, for which it owns the sole copyright. These problems have been invaluable in illustrating a number of concepts in the FAM exam syllabus. I am also grateful to students in my ACTS:4130 (*Quantitative Methods for Actuaries*) class at The University of Iowa in Fall 2019-2022 for class testing earlier versions of this study manual.

## About the Author

**Ambrose Lo**, PhD, FSA, CERA, is the author of several study manuals for professional actuarial examinations and an Adjunct Associate Professor at the Department of Statistics and Actuarial Science, the University of Hong Kong (HKU). He earned his BSc in Actuarial Science (first class honors) and PhD in Actuarial Science from HKU in 2010 and 2014, respectively, and attained his Fellowship of the Society of Actuaries (FSA) in 2013. He joined the Department of Statistics and Actuarial Science, the University of Iowa (UI) as Assistant Professor of Actuarial Science in August 2014, and was promoted to Associate Professor with tenure in July 2019. His research interests lie in dependence structures, quantitative risk management as well as optimal (re)insurance. His research papers have been published in top-tier actuarial journals, such as *ASTIN Bulletin: The Journal of the International Actuarial Association*, *Insurance: Mathematics and Economics*, and *Scandinavian Actuarial Journal*. He left the UI and returned to Hong Kong in July 2023.

Besides dedicating himself to actuarial research, Ambrose attaches equal (if not more!) importance to teaching and education, through which he nurtures the next generation of actuaries and serves the actuarial profession. He has taught courses on a wide range of actuarial science topics, such as financial derivatives, mathematics of finance, life contingencies, and statistics for risk modeling. He is also the (co)author of the *ACTEX Study Manuals for Exams ATPA, MAS-I, MAS-II, PA, and SRM*, a *Study Manual for Exam FAM*, and the textbook *Derivative Pricing: A Problem-Based Primer* (2018) published by Chapman & Hall/CRC Press. Although helping students pass actuarial exams is an important goal of his teaching, inculcating students with a thorough understanding of the subject and logical reasoning is always his top priority. In recognition of his outstanding teaching, Ambrose has received a number of awards and honors ever since he was a graduate student, including the 2012 Excellent Teaching Assistant Award from the Faculty of Science, HKU, public recognition in the Daily Iowan as a faculty member "making a positive difference in students' lives during their time at UI" for nine years in a row (2016 to 2024), and the 2019-2020 Collegiate Teaching Award from the UI College of Liberal Arts and Sciences.

## 2.4 Risk Measures

*****FROM THE FAM-S EXAM SYLLABUS*****

**2. Topic: Severity, Frequency, and Aggregate Models (12.5-17.5%)**

**Learning Objectives**

The Candidate will understand the characteristics of and uses for commonly used severity, frequency, and aggregate models.

**Learning Outcomes**

The candidate will be able to:

m) Calculate Value at Risk and Tail Value at Risk.

n) Determine whether a given risk measure has certain desirable properties.

**OPTIONAL SYLLABUS READINGS**

*Loss Models*, Chapter 3 (Sections 2 and 5)

### 2.4.1 General Properties of Risk Measures

**What are risk measures?** Whether we are considering the ground-up loss $X$ (Sections 1.1 and 2.1), the claim frequency $N$ (Section 2.2), or the aggregate loss $S$ (Section 2.3), the primary risk an insurer faces with respect to these random variables is that they take large positive values. To "measure" your exposure to these "risk" variables (in particular, to their extreme values), *risk measures* can be very useful quantitative[36] tools and are the subject of this section. While risk measures are general concepts that apply to all kinds of random variables, in Exam FAM-S we will concern ourselves mostly with non-negative, possibly heavy-tailed random variables, which abound in actuarial contexts. (You may see risk measures again when you study for FSA exams...maybe in one or two years? ☺)

Technically speaking, a risk measure is a mapping, usually denoted by $\rho$, from a set of random variables[37] to the real line (the set of all real numbers). For a given random variable $X$, $\rho(X)$ is a real number capturing certain distributional characteristics of $X$. The mean and variance, $\mathrm{E}[X]$ and $\mathrm{Var}(X)$, are perhaps the simplest examples of risk measures used to summarize the distribution of a random variable. In actuarial science, the distributional characteristics of greatest interest are the magnitude and uncertainty (or variability) of loss variables: How large is the loss typically, and how risky is the loss? If $\rho(X)$ quantifies such characteristics of a loss variable $X$, then it is useful to think of $\rho(X)$ as the amount of capital needed to protect against adverse outcomes of $X$.

[36] In contrast, the discussion in Subsection 2.1.2 on how to measure tail weight is mostly qualitative.

[37] Following *Loss Models*, we assume that the set of random variables is such that if $X$ and $Y$ belong to the set, then so do $cX$ and $X + Y$ for any positive constant $c$.

**Coherent risk measures.** There are a myriad of risk measures actuaries can use to quantify risk exposure, but some are more commonly used in practice because they satisfy certain desirable properties. Here are four such properties:

- *Monotonicity:* If $X \le Y$, then $\rho(X) \le \rho(Y)$. (The values of the risk measures are ordered the same way as the random variables.)

  Meaning: If one risk is always bigger than the other, then the risk measure for the larger risk should also be larger, which makes perfect sense from an economic point of view—we need a larger capital to support a larger risk.

- *Positive homogeneity:* For any positive constant $c$, $\rho(cX) = c\rho(X)$.

  Meaning: A positively homogeneous risk measure is independent of the currency in which risks are measured. In other words, changing the unit of loss changes the risk measure in proportion, e.g., doubling the exposure to a risk leads to double the capital, which is sensible.

  An intuitive consequence of positive homogeneity is that $\rho(0) = 0$,[38] i.e., we need nothing (no capital) to support nothing (no loss)!

- *Translation invariance:* For any constant $c$ (not necessarily positive), $\rho(X + c) = \rho(X) + c$.

  Meaning: Adding a constant amount (positive or negative) to a risk adds the same amount to the required capital. With $X$ set equal to 0, this property, along with $\rho(0) = 0$, means that $\rho(0 + c) = \rho(0) + c$, or $\rho(c) = c$, i.e., the capital required to support a certain outcome is exactly the value of that outcome.

- *Subadditivity:* $\rho(X + Y) \le \rho(X) + \rho(Y)$.

  Meaning: The risk measure for two risks combined, $\rho(X + Y)$, will not be greater than the sum of the risk measures for the two individual risks taken separately, $\rho(X)+\rho(Y)$, due to the *diversification benefit* from combining risks (one risk may hedge against the other risk). This property is desirable for measuring risks at the corporate level. Without this property, companies may find it advantageous to disintegrate into smaller companies in an attempt to lower the total risk.

  Subadditivity is perhaps the most debatable property among the four properties above. As noted in *Loss Models*, some believe that merging several small companies exposes each to the reputational risk of others and should lead to a larger total risk than when they are treated in isolation.

If a risk measure satisfies all of these four properties, then we say that it is a *coherent* risk measure.

**⚠ EXAM NOTE ⚠**

Exam questions may test whether you understand these properties by giving you a certain risk measure and asking you **which of the four properties is/are satisfied/violated**. If all of the four properties are satisfied, then the risk measure is coherent.

[38] To see this, we set $X = 0$ to get $\rho(0) = \rho(c \cdot 0) = c\rho(0)$ for any positive constant $c$. This forces $\rho(0) = 0$.

**Example 2.4.1. (SOA Exam FAM-S Sample Question 87 / October 2022 Exam FAM-S: Is the expected value a coherent risk measure?)** You are asked to consider whether the risk measure $\rho(X) = \mathrm{E}(X)$ is coherent.

Determine which of the following statements is correct.

(A) $\rho(X)$ does NOT possess subadditivity.

(B) $\rho(X)$ does NOT possess monotonicity.

(C) $\rho(X)$ does NOT possess positive homogeneity.

(D) $\rho(X)$ does NOT possess translation invariance.

(E) $\rho(X)$ is a coherent risk measure.

*Solution.* Let's check each of the four properties for the expected value:

- *Monotonicity:* If $X \le Y$, then $\mathrm{E}[X] \le \mathrm{E}[Y]$, so the expected value is monotonic.
- *Positive homogeneity:* For any positive constant $c$, $\mathrm{E}[cX] = c\mathrm{E}[X]$, so the expected value is positively homogeneous.

  *Remark.* In fact, $\mathrm{E}[cX] = c\mathrm{E}[X]$ is true for any constant $c$, not necessarily positive.
- *Translation invariance:* For any constant $c$, $\mathrm{E}[X + c] = \mathrm{E}[X] + c$, so the expected value is translation invariant.
- *Subadditivity:* For any $X$ and $Y$, $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] \overset{\text{(still true!)}}{\le} \mathrm{E}[X] + \mathrm{E}[Y]$, so the expected value is subadditive.

Since the expected value satisfies all of the four properties, it is a coherent risk measure. **(Answer: (E))** □

*Remark.* In a similar spirit, Problems 2.4.1 and 2.4.3 explore the coherence of the standard deviation $\rho(X) = \sqrt{\mathrm{Var}(X)}$ and the standard deviation principle $\rho(X) = \mathrm{E}[X] + k\sqrt{\mathrm{Var}(X)}$. Be sure to work out these problems!

In the next two subsections, we will look at two specific risk measures listed in Learning Outcome 2 m) of the FAM-S exam syllabus.

### 2.4.2 Value at Risk

**Definition and motivation.** The *Value at Risk* (VaR) is a widely used risk measure, especially in the banking industry. In essence, it is just a glorified way to refer to a *percentile*, a familiar concept from Exam P. For a given number $p \in (0, 1)$, the VaR of a random variable $X$ at the $100p\%$ security level is the $100p$th percentile of $X$, denoted by $\mathrm{VaR}_p(X)$, $\pi_p(X)$, or simply $\pi_p$ if the random variable is understood. It is formally defined as

$$\boxed{\mathrm{VaR}_p(X) = \min\{x : F_X(x) \ge p\}.}$$

If $X$ is a loss random variable, then by definition, it is the size of loss for which there is a small probability of exceedence (not more than $p$). We can therefore interpret the VaR as the amount of capital required so that the insurer can withstand $100p\%$ of possible losses and stay solvent with a high probability.

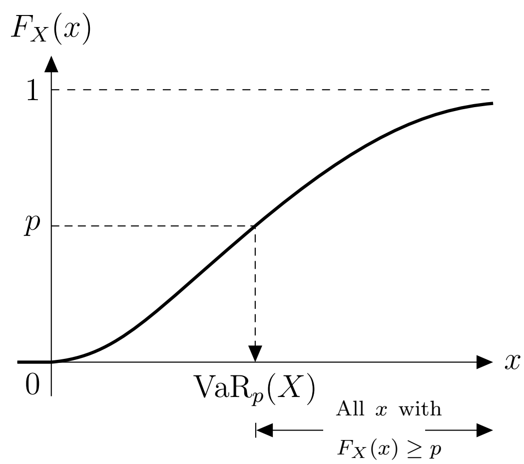
You may wonder: Why define the VaR in such a complex way, involving the minimum of all $x$ such that $F_X(x) \geq p$? Why not simply take $\text{VaR}_p(X)$ as the value of $x$ such that $F_X(x) = p$? The problem is that there may be no $x$, or more than one $x$ such that $F_X(x) = p$, and the "complex" definition above is necessary to deal with all kinds of distributions, especially discrete distribution functions, which jump at each possible value. Let's distinguish three cases.

*Case* 1. *(Most common case)* $F_X$ *is continuous and strictly increasing.*

This regular case is the case you will most likely see on the exam. For a continuous and strictly increasing distribution function $F_X$ and $p \in (0, 1)$, there is one and only one value of $x$ such that

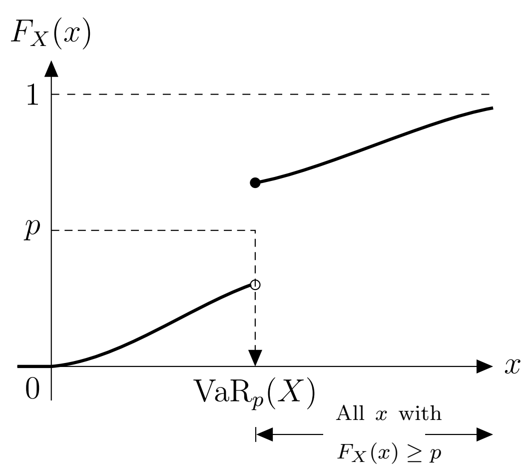
$$\boxed{F_X(x) = p;} \tag{2.4.1}$$

see the figure below. In this case, $\text{VaR}_p(X)$ is the usual inverse of the distribution function $F_X$ corresponding to $p$—it is the unique value of $x$ that makes $F_X(x)$ equal to $p$—and solving (2.4.1) for $x$ gives the desired VaR.



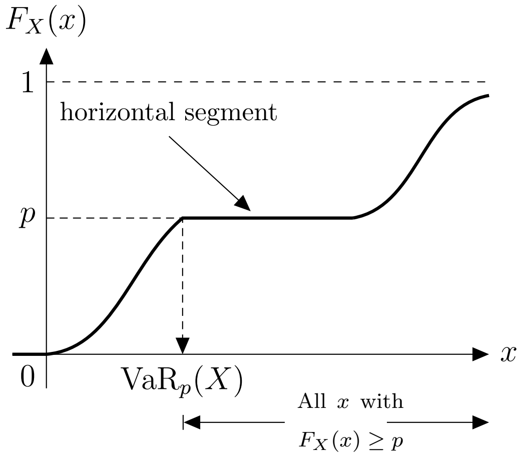

*Case* 2. $p$ *corresponds to a jump point of* $F_X$*.*

If $F_X$ jumps immediately from a value below $p$ to a value above $p$ at a certain point, then there is no $x$ such that $F_X(x) = p$. However, the VaR, by definition, is the smallest $x$ such that $F_X(x) \geq p$, and this smallest value is exactly the location of the jump, which will serve as the VaR, as the figure below shows.



*Case* 3. *$p$ corresponds to a horizontal part of $F_X$.*

If $p$ touches a horizontal segment of $F_X$, then there is an interval of values of $x$ such that $F_X(x) = p$. The smallest $x$ such that $F_X(x) \geq p$ is the *leftmost* point in the interval, which will then serve as the VaR; see the figure below.




Combining the three cases, we can see that $F_X[\text{VaR}_p(X)] \geq p$ (not $= p$ in general ⚠), and equality holds if $F_X$ is continuous at $\text{VaR}_p(X)$.

**Example 2.4.2. (Calculating the VaR of a mixed distribution)** A random variable $X$ has distribution function

$$F_X(x) = 1 - 0.2e^{-0.01x}, \quad x \geq 0.$$

(a) Calculate $\text{VaR}_{0.9}(X)$.

(A) 59 (B) 69 (C) 79
(D) 89 (E) 99

*Solution.* This is Case 1 above. Solving the equation $F_X(x) = 0.9$ yields a unique solution

$$x = -\frac{1}{0.01}\ln\left(\frac{1-0.9}{0.2}\right) = \boxed{69.3147},$$

which serves as $\text{VaR}_{0.9}(X)$. **(Answer: (B))** □

(b) Calculate the median of $X$.

(A) 0 (B) 10 (C) 20
(D) 30 (E) 40

*Solution.* The median of $X$ is its 50th percentile. At $x = 0$, $F_X(x)$ jumps instantaneously from $0\,(< 0.5)$ to $1 - 0.2e^{-0.01(0)} = 0.8\,(> 0.5)$, so Case 2 above applies and the median is $\text{VaR}_{0.5}(X) = \boxed{0}$. **(Answer: (A))** □

*Remark.* ⚠ Notice that $X$ is *not* an exponential random variable. In fact, $F_X(x) = 0.2(1 - e^{-x/100}) + 0.8(1)$ for $x \geq 0$, so the distribution of $X$ is a 20/80 mixture of an exponential distribution with $\theta = 100$ and a probability mass at 0.

**Example 2.4.3. (CAS Part 4B Fall 1996 Question 3 / *Loss Models* Exercise 3.20: Given two VaR's, deduce the parameters of a distribution)** You are given the following:

- Losses follow a Weibull distribution with parameters $\tau$ and $\theta$.
- The 25th percentile of the distribution is 1,000.
- The 75th percentile of the distribution is 100,000.

Determine $\tau$.

(A) Less than 0.4
(B) At least 0.4, but less than 0.6
(C) At least 0.6, but less than 0.8
(D) At least 0.8, but less than 1.0
(E) At least 1.0

**Ambrose's comments:** Section A.2 of the FAM-S tables includes the formula for the VaR of a large number of common continuous distributions, so you probably won't see an exam question that asks you to directly compute the VaR of one of those distributions given the parameter(s)—such a question is too easy! Rather, an exam question may turn things around and ask you to **deduce the distribution parameter(s) from one or several VaR's**.

*Solution.* From Section A.3.2.3 of the FAM-S tables, the VaR of a Weibull distribution is $\text{VaR}_p(X) = \theta[-\ln(1-p)]^{1/\tau}$. Therefore, we solve

$$\begin{cases} \text{VaR}_{0.25}(X) = \theta[-\ln(1-0.25)]^{1/\tau} = 1,000 \\ \text{VaR}_{0.75}(X) = \theta[-\ln(1-0.75)]^{1/\tau} = 100,000 \end{cases}.$$

Dividing the second equation by the first one, we eliminate $\theta$ and get

$$\left[\frac{\ln 0.25}{\ln 0.75}\right]^{1/\tau} = \frac{100,000}{1,000} = 100.$$

Taking logarithm of both sides further gives

$$\frac{1}{\tau}\ln\left(\frac{\ln 0.25}{\ln 0.75}\right) = \ln 100 \quad \Rightarrow \quad \tau = \boxed{0.3415}. \quad \textbf{(Answer: (A))}$$

□

*Remark.* From either equation, you can also get $\theta = 38,421.63$.

**VaR of a function of a random variable.** There are two notable continuous distributions for which the FAM-S tables do not provide a VaR formula: The normal and lognormal distributions. To derive and better understand the VaR of these two distributions, the following general properties concerning the VaR of *a function of* a random variable are useful: (These properties will also be used in Chapter 9 of the FAM-L part.)

*Case* 1. If $g$ is an *increasing* continuous[39] function applied to a random variable $Y$, then

$$\mathrm{VaR}_p\left(g(Y)\right) = g\left(\mathrm{VaR}_p(Y)\right). \tag{2.4.2}$$

In other words, the $100p\%$ VaR of $g(Y)$ equals the $g$ function evaluated at the $100p\%$ (same probability level) VaR of $Y$.[40]

*Case* 2. If $h$ is a *decreasing* continuous function, then

$$\mathrm{VaR}_p\left(h(Y)\right) = h\left(\mathrm{VaR}_{1-p}(Y)\right). \tag{2.4.3}$$

In other words, the $100p\%$ VaR of $h(Y)$ equals the $h$ function evaluated at the $100\boxed{(1-p)}\%$ VaR of $Y$.

To derive the VaR of a normal random variable $X$ with mean $\mu$ and variance $\sigma^2$, note that $X = \mu + \sigma Z$, which is an increasing function of the standard normal random variable $Z$ (keep in mind that $\sigma > 0$). Using (2.4.2), we can relate the VaR of $X$ to the VaR of $Z$ at the same level $p$ via

$$\boxed{\mathrm{VaR}_p(X) = \mu + \sigma \mathrm{VaR}_p(Z) = \mu + \sigma z_p,} \tag{2.4.4}$$

where $z_p$ is the $100p$th percentile of the standard normal distribution.

**Example 2.4.4. (SOA Exam P Sample Question 164: VaR of normal random variables)** The working lifetime, in years, of a particular model of bread maker is normally distributed with mean 10 and variance 4.

Calculate the 12th percentile of the working lifetime, in years.

(A) 5.30 (B) 7.65 (C) 8.41

(D) 12.35 (E) 14.70

*Solution.* By (2.4.4) with $\mu = 10$, $\sigma = \sqrt{4}$, $p = 0.12$, and $z_{0.12} = -1.17499$ (from the Prometric standard normal calculator), the 12th percentile is

$$\pi_{0.12} = \mu + \sigma z_{0.12} = 10 + \sqrt{4}(-1.17499) = \boxed{7.6500}. \quad \textbf{(Answer: (B))}$$

□

Deriving the VaR of a lognormal random variable $X'$ with parameters $\mu$ and $\sigma$ is just as easy. By definition, $X' = \mathrm{e}^X$, which is an increasing continuous function of $X \sim \mathrm{N}(\mu, \sigma^2)$, so we can use (2.4.2) again to get

$$\boxed{\mathrm{VaR}_p(X') = \mathrm{e}^{\mathrm{VaR}_p(X)} \overset{(2.4.4)}{=} \mathrm{e}^{\mu + \sigma z_p}.} \tag{2.4.5}$$

[39]Virtually all, if not all, of the functions you encounter in Exam FAM-S are continuous, so in the rest of this manual, I will drop "continuous."

[40]Intuitively, (2.4.2) makes sense because if $g$ is an increasing function, then large values of $g(Y)$ correspond to large values of $Y$ and small values of $g(Y)$ correspond to small values of $Y$. The order is reversed (i.e., large values of $h(Y)$ go with small values of $Y$) if $h$ is a decreasing function.

**Example 2.4.5. (VaR of a lognormal distribution)** You are given that a loss random variable $X$ follows a lognormal distribution with parameters $\mu = 5$ and $\sigma = 2$.

Calculate $\text{VaR}_{0.9}(X)$.

(A) 1000 (B) 2000 (C) 3000

(D) 4000 (E) 5000

*Solution.* By (2.4.5) with $z_{0.9} = 1.28155$,

$$\text{VaR}_{0.9}(X) = \text{e}^{5+2(1.28155)} = \text{e}^{7.5631} = \boxed{1925.81}. \quad \textbf{(Answer: (B))}$$

□

*Remark.* ⚠ Option (D) is for the 95% VaR, $\text{VaR}_{0.95}(X) = \text{e}^{5+2(1.64485)} = 3982.64$, which is more than double of $\text{VaR}_{0.9}(X)$.

**Is VaR coherent?** Although VaR is an economically intuitive and mathematically tractable risk measure, it is not a coherent risk measure. While it satisfies monotonicity, positive homogeneity, and translation invariance, it violates *subadditivity* (this can be the theme of a "Determine which of the following statements is true/correct" exam question, like Example 2.4.1). The following example provides a simple illustration of the non-subadditivity.

**Example 2.4.6. (Non-subadditivity of VaR)** Suppose that $X$ and $Y$ are independent and identically distributed loss random variables with the following probability mass function:

$$f(x) = \begin{cases} 0.96, & \text{if } x = 0, \\ 0.04, & \text{if } x = 50. \end{cases}$$

(a) Calculate $\text{VaR}_{0.95}(X)$.

(A) 0 (B) 10 (C) 25

(D) 50 (E) $\text{VaR}_{0.95}(X)$ is not well-defined.

*Solution.* The common distribution function of $X$ and $Y$ is

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.96, & \text{if } 0 \le x < 50, \\ 0.96 + 0.04 = 1, & \text{if } x \ge 50. \end{cases}$$

As 0.95 lies between 0 and 0.96, and $F(x)$ jumps immediately from 0 to 0.96 at $x = 0$, Case 2 on page 353 applies and $\text{VaR}_{0.95}(X) = \boxed{0}$. **(Answer: (A))** □

(b) Calculate $\text{VaR}_{0.95}(X + Y)$.

(A) 0 (B) 10 (C) 50

(D) 100 (E) $\text{VaR}_{0.95}(X + Y)$ is undefined.

*Solution.* Here are the possible values of $S := X + Y$ along with the corresponding probabilities: (Keep in mind that $X$ and $Y$ are independent.)

| Value | Probability |
|---|---|
| 0 | $0.96^2 = 0.9216$ |
| 50 | $2(0.96)(0.04) = 0.0768$ |
| 100 | $0.04^2 = 0.0016$ |

Then the distribution function of $S$ is

$$F_S(s) = \begin{cases} 0, & \text{if } s < 0, \\ 0.9216, & \text{if } 0 \leq s < 50, \\ 0.9984, & \text{if } 50 \leq s < 100, \\ 1, & \text{if } 100 \leq s. \end{cases}$$

As 0.95 lies between 0.9216 and 0.9984, and $F_S(s)$ jumps immediately from 0.9216 to 0.9984 at $s = 50$, we get $\text{VaR}_{0.95}(S) = \boxed{50}$. **(Answer: (C))** □

*Remark.* This example shows that $\text{VaR}_{0.95}(X+Y) = 50 > \text{VaR}_{0.95}(X) + \text{VaR}_{0.95}(Y) = 0$, so VaR is not subadditive, hence not coherent.

*Loss Models* briefly mentions that when restricted to *normal* random variables, VaR becomes subadditive and therefore coherent. This is because for normal random variables, VaR is just a form of the standard deviation principle $\rho(X) = \text{E}[X] + k\sqrt{\text{Var}(X)}$ with $k = z_p$, which, as Problem 2.4.3 will show, is subadditive. Nevertheless, the normal distribution is not often used for modeling insurance losses, which are typically non-negative and right-skewed.

The incoherence of VaR in general leads many practitioners to consider the risk measure in the next subsection.

### 2.4.3 Tail Value at Risk

**Definition and motivation.** *Tail Value at Risk* (TVaR)[41] is a modification of VaR that is coherent (in particular, subadditive).[42] For a given number $p \in (0,1)$, the TVaR of a random variable $X$ at the $100p\%$ security level, denoted by $\text{TVaR}_p(X)$, is the average of all VaR values above level $p$:

$$\boxed{\text{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 \text{VaR}_u(X)\,\mathrm{d}u.} \tag{2.4.6}$$

[41] Alternative names for TVaR briefly mentioned on page 41 of *Loss Models* include Conditional Value at Risk (CVaR), Average Value at Risk (AVaR), Conditional Tail Expectation (CTE), and Expected Shortfall (ES), although there are subtle differences among these concepts.

[42] To prove that TVaR is subadditive is far from trivial. There is a recent research paper providing a total of SEVEN (7) proofs of the subadditivity. If you are interested, take a look at *Seven Proofs for the Subadditivity of Expected Shortfall* 🔗 (2015), by Embrechts and Wang.

By definition, $\text{TVaR}_p(X)$ uses $\text{VaR}_p(X)$ as the definition of an adverse loss (losses exceeding this threshold are deemed adverse), and quantifies the average size of the adverse loss. This interpretation will be reinforced on page 362 when we discuss a more convenient computing formula of the TVaR.

> ⚠ **Note:** At odds with *Loss Models*, quite a few exam prep resources "define" $\text{TVaR}_p(X)$ as $\mathrm{E}[X \mid X > \text{VaR}_p(X)]$; see (2.4.9) below. While commonly used for continuous random variables, this formula is **NOT** the definition of the TVaR according to *Loss Models*, and it may not work for **discrete** random variables (see Problems 2.4.39 and 2.4.40), which also play an important role in loss modeling.

**Computing formulas.** The FAM-S tables provide the formula for the TVaR of only three distributions: Pareto (Section A.2.3.1), exponential (Section A.3.3.1), and single-parameter Pareto (Section A.5.1.4). Can we calculate the TVaR of other distributions, using (2.4.6) or otherwise?

While (2.4.6) is the definition of the TVaR, it is often not the best formula to use on the exam because it is not easy to integrate $\text{VaR}_p(X)$ with respect to $p$ for many distributions. As an example, consider the exponential distribution with parameter $\theta$, which is one of the simplest continuous distributions. From Section A.3.3.1 of the FAM-S tables, $\text{VaR}_p(X) = -\theta \ln(1-p)$. By (2.4.6), we would need to deal with

$$\text{TVaR}_p(X) = \frac{1}{1-p}\int_p^1 [-\theta \ln(1-u)]\,\mathrm{d}u = \cdots \text{(nasty calculus 💩)} \cdots$$

using integration by parts, which most students hate! We need more convenient computing formulas.

- *(In terms of the limited expected value)* To begin with, the following general formula for TVaR that expresses it as the VaR added by an adjustment term is useful:[43]

$$\boxed{\text{TVaR}_p(X) = \pi_p + \frac{\mathrm{E}[(X-\pi_p)_+]}{1-p} \overset{(1.1.7)}{=} \pi_p + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_p]}{1-p}.} \tag{2.4.7}$$

  For a distribution whose $\mathrm{E}[(X-d)_+]$ or $\mathrm{E}[X \wedge x]$ is easy to deal with, (2.4.7) provides a good way to calculate $\text{TVaR}_p(X)$. Here are two such distributions:

  Normal. For a normal random variable with mean $\mu$ and variance $\sigma^2$, Example 3.15 of *Loss Models* says that "with a bit of calculus,[44] it can be shown that"

$$\boxed{\text{TVaR}_p(X) = \mu + \sigma\left[\frac{\phi(z_p)}{1-p}\right],} \tag{2.4.8}$$

[43] (2.4.7) can be found at the bottom of page 492 of *Loss Models*.

[44] *(If you are interested)* By (2.4.4) and (2.4.7), and noting that $X = \mu + \sigma Z$, where $Z$ is standard normal,

$$\text{TVaR}_p(X) = (\mu + \sigma z_p) + \frac{\mathrm{E}\left[((\mu+\sigma Z) - (\mu + \sigma z_p))_+\right]}{1-p} = \mu + \sigma z_p + \frac{\sigma \mathrm{E}[(Z-z_p)_+]}{1-p}.$$

By definition,

$$\mathrm{E}[(Z-z_p)_+] = \int_{z_p}^{\infty} (z-z_p)\,\phi(z)\,\mathrm{d}z \overset{(\phi'(z) = -z\phi(z))}{=} -\int_{z_p}^{\infty} \mathrm{d}\phi(z) - z_p \int_{z_p}^{\infty} \phi(z)\,\mathrm{d}z = \phi(z_p) - z_p(1-p).$$

Thus $\text{TVaR}_p(X) = \mu + \sigma z_p + \frac{\sigma[\phi(z_p) - z_p(1-p)]}{1-p} = \mu + \sigma\left[\frac{\phi(z_p)}{1-p}\right]$, which is (2.4.8).

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$$

is the standard normal density function, and is "hiding" on page 2 of the FAM tables. You can see that like the VaR, the TVaR of normal random variables is a form of the standard deviation principle $\rho(X) = \mathrm{E}[X] + k\sqrt{\mathrm{Var}(X)}$, with $k = \phi(z_p)/(1-p)$.

Lognormal. The lognormal distribution is another perfect example for which the use of (2.4.7) is most suitable as you can find the formula for $\mathrm{E}[X]$ and $\mathrm{E}[X \wedge \pi_p]$ in Section A.5.1.1 of the FAM-S tables, and $\pi_p$ is easy to calculate by (2.4.5). The following example provides an illustration.

**Example 2.4.7. (Example 2.4.5 continued: TVaR of a lognormal distribution)** You are given that a loss random variable $X$ follows a lognormal distribution with parameters $\mu = 5$ and $\sigma = 2$.

Calculate $\mathrm{TVaR}_{0.9}(X)$.

(A) 5000 (B) 6000 (C) 7000

(D) 8000 (E) 9000

*Solution.* In Example 2.4.5, we found that $\pi_{0.9} = \mathrm{e}^{7.5631}$. Using the formulas in Section A.5.1.1 of the FAM-S tables, $\mathrm{E}[X] = \mathrm{e}^{\mu+\sigma^2/2} = \mathrm{e}^{5+2^2/2} = \mathrm{e}^7$ and

$$\begin{aligned}
\mathrm{E}[X \wedge \pi_{0.9}] &= \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}\Phi\left(\frac{\ln \pi_{0.9} - \mu - \sigma^2}{\sigma}\right) + \pi_{0.9}\left[1 - \Phi\left(\frac{\ln \pi_{0.9} - \mu}{\sigma}\right)\right] \\
&= \mathrm{e}^7\Phi\left(\frac{\ln \mathrm{e}^{7.5631} - 5 - 2^2}{2}\right) + \mathrm{e}^{7.5631}\left[1 - \Phi\left(\frac{\ln \mathrm{e}^{7.5631} - 5}{2}\right)\right] \\
&= \mathrm{e}^7\Phi(-0.71845) + \mathrm{e}^{7.5631}[1 - \Phi(1.28155)] \\
&= \mathrm{e}^7(0.23624) + \mathrm{e}^{7.5631}(1 - 0.9) \\
&= 451.6492.
\end{aligned}$$

Then by (2.4.7),

$$\begin{aligned}
\mathrm{TVaR}_{0.9}(X) &= \pi_{0.9} + \frac{\mathrm{E}[X] - \mathrm{E}[X \wedge \pi_{0.9}]}{1 - 0.9} \\
&= \mathrm{e}^{7.5631} + \frac{\mathrm{e}^7 - 451.6492}{0.1} \\
&= \boxed{8376}. \quad \textbf{(Answer: (D))}
\end{aligned}$$

□

*Remark.* (i) In general, for $X \sim \mathrm{LN}(\mu, \sigma)$, we can show that

$$\mathrm{TVaR}_p(X) = \mathrm{E}[X]\left[\frac{\Phi(\sigma - z_p)}{1-p}\right] = \mathrm{e}^{\mu+\sigma^2/2}\left[\frac{\Phi(\sigma - z_p)}{1-p}\right].$$

This formula is not shown in *Loss Models* or tested in released past exams, so there is probably no need to memorize it.

(ii) ⚠ Unlike the VaR, note that $\text{TVaR}_{0.9}(X) \neq e^{\text{TVaR}_{0.9}(X')}$, even though $X = e^{X'}$, where $X' \sim \text{N}(5, 2^2)$. To check this,

$$\begin{aligned}\text{TVaR}_{0.9}(X') &\overset{(2.4.8)}{=} 5 + 2\left[\frac{\phi(1.28155)}{1-0.9}\right] = 5 + 2\left[\frac{0.175499}{0.1}\right] = 8.509974,\\ e^{\text{TVaR}_{0.9}(X')} &= e^{8.509974} = 4964.0324,\end{aligned}$$

which is not the same as $\text{TVaR}_{0.9}(X) = 8376$. Yes, the TVaR is (far!) more difficult to deal with than the VaR.

- *(For continuous distributions: In terms of the mean excess loss)* If the distribution function of $X$ is continuous at $\pi_p$ (likely the case in exam questions), then $S_X(\pi_p) = 1 - p$, so the second term in (2.4.7) can be further written as

$$\frac{\text{E}[(X-\pi_p)_+]}{1-p} = \frac{\text{E}[(X-\pi_p)_+]}{S_X(\pi_p)} \overset{(1.1.19)}{=} e_X(\pi_p),$$

which is the mean excess loss introduced in Section 1.1 with the $100p\%$ VaR serving as the "deductible,"[45] and (2.4.7) becomes

$$\boxed{\begin{aligned}\text{TVaR}_p(X) &= \pi_p + e_X(\pi_p)\\ &= \pi_p + \text{E}[X - \pi_p \mid X > \pi_p]\\ &= \text{E}[X \mid X > \pi_p].\end{aligned}} \tag{2.4.9}$$

Formula (2.4.9), which exhibits $\text{TVaR}_p(X)$ as the conditional expectation of $X$ given that $X > \pi_p$, is useful in two aspects:

▷ *(Computations)* As a computing formula, (2.4.9) is most useful when the distribution of the excess loss variable $X - \pi_p \mid X > \pi_p$, or its shifted counterpart $X \mid X > \pi_p$, is simple. Consider again the exponential distribution with parameter $\theta$. Recall from (1.1.16) that $e_X(d) = \theta$, which is the same value for any $d > 0$, so

$$\text{TVaR}_p(X) \overset{(2.4.9)}{=} \pi_p + e_X(\pi_p) = -\theta\ln(1-p) + \theta,$$

which is the TVaR formula in Section A.3.1.1 of the FAM-S tables.

As another example, consider $X \sim \text{U}[a, b]$, whose TVaR is *not* given in the FAM-S tables. Because $X \mid X > \pi_p \sim \text{U}[\pi_p, b]$, i.e., the left end-point is shifted upward from $a$ to $\pi_p$, we can use (2.4.9) to immediately get

$$\boxed{\text{TVaR}_p(X) = \text{E}[X \mid X > \pi_p] = \frac{\pi_p + b}{2} = \pi_{(p+1)/2},} \tag{2.4.10}$$

which is the VaR evaluated at the mid-point of $p$ and 1 as the security level.

[45] We can think of $\text{E}[X \mid X > \pi_p]$ as the expected cost per payment when there is a *franchise* deductible of $\pi_p$.

▷ *(Insights)* From a conceptual point of view, (2.4.9) also sheds further light on the meaning of TVaR. Unlike $\text{VaR}_p(X)$, which only captures a certain point of the distribution of $X$, $\text{TVaR}_p(X) = \text{E}[X \mid X > \text{VaR}_p(X)]$ reflects the entire right "tail" of the distribution beyond $\text{VaR}_p(X)$ (remember that TVaR stands for "Tail Value at Risk"). Using $\text{VaR}_p(X)$ as the starting point, $\text{TVaR}_p(X)$ brings in the mean excess loss $e_X(\pi_p)$, which measures the average excess loss given that the VaR threshold has been exceeded.

**Example 2.4.8. (TVaR of a uniform distribution)** You are given:

(i) $X$ is uniformly distributed on the interval $[0, \theta]$.

(ii) $\text{Var}(X) = 120,000$

Calculate $\text{TVaR}_{0.95}(X)$.

(A) 1160 (B) 1170 (C) 1180

(D) 1190 (E) 1200

*Solution.* From (ii), $\text{Var}(X) = \theta^2/12 = 120,000$, so $\theta = 1200$. Solving $F_X(x) = x/1200 = 0.95$, we get $\pi_{0.95} = 0.95(1200) = 1140$. By (2.4.10),

$$\text{TVaR}_{0.95}(X) = \frac{\pi_{0.95} + \theta}{2} = \frac{1140 + 1200}{2} = \boxed{1170}. \quad \textbf{(Answer: (B))}$$

(Alternatively, $\text{TVaR}_{0.95}(X) = \pi_{0.975}(X) = 0.975(1200) = 1170$.) □

*Remark.*

(i) For a general uniform distribution on the interval $[a, b]$, we have $\pi_p = (1-p)a + pb$, which is a weighted average of the two end-points.

(ii) Here is an alternative solution based on the TVaR's definition, (2.4.6):

By solving $F_X(x) = x/\theta = p$, we get $\text{VaR}_p(X) = p\theta = 1200p$ for any $0 < p < 1$. Then by (2.4.6), we integrate the VaR from $p = 0.95$ to $p = 1$ and get

$$\text{TVaR}_{0.95}(X) = \frac{1}{1 - 0.95}\int_{0.95}^{1} 1200p\,\text{d}p = \frac{1200(1^2 - 0.95^2)}{2(0.05)} = \boxed{1170}.$$

The uniform distribution (and its power transformations) is one of the rare distributions for which $\text{VaR}_p(X)$ is easy to integrate and (2.4.6) works well. Problem 2.4.32 is another illustration.

## Practice Problems for Section 2.4

*—Highly recommended problems are marked with an asterisk (*).—*

### General Properties of Risk Measures

**Problem 2.4.1. * [HARDER!] (Is the standard deviation coherent?)** You are asked to consider whether the standard deviation $\rho(X) = \sqrt{\mathrm{Var}(X)}$ is a coherent risk measure.

Determine the number of properties of a coherent risk measure satisfied by $\rho(X)$.

(A) 0 (B) 1 (C) 2

(D) 3 (E) 4

*Solution.*

I. ✘ *Monotonicity:* Even if $X \le Y$, there is no guarantee that $\sqrt{\mathrm{Var}(X)} \le \sqrt{\mathrm{Var}(Y)}$, so $\rho$ is NOT monotonic in general.

II. ✓ *Positive homogeneity:* For any positive constant $c$,

$$\rho(cX) = \sqrt{\mathrm{Var}(cX)} = \sqrt{c^2\mathrm{Var}(X)} = c\sqrt{\mathrm{Var}(X)} = c\rho(X),$$

so $\rho$ is positively homogeneous.

*Remark.* The variance, however, is not positively homogeneous.

III. ✘ *Translation invariance:* Let $c$ be a constant. Because

$$\rho(X+c) = \sqrt{\mathrm{Var}(X+c)} = \sqrt{\mathrm{Var}(X)} = \rho(X) \ne \rho(X) + c$$

in general, $\rho$ is NOT translation invariant.

IV. ✓ *Subadditivity:* Let $r$ be the correlation coefficient between $X$ and $Y$. From Exam P, we know that $r \le 1$, so

$$\begin{aligned}
\rho(X+Y)^2 &= \mathrm{Var}(X+Y) \\
&= \rho(X)^2 + \rho(Y)^2 + 2r\rho(X)\rho(Y) \\
&\le \rho(X)^2 + \rho(Y)^2 + 2\rho(X)\rho(Y) \\
&= [\rho(X) + \rho(Y)]^2.
\end{aligned}$$

Upon taking the square root, $\rho(X+Y) \le \rho(X) + \rho(Y)$, so $\rho$ is subadditive.

To conclude, $\rho$ satisfies two of the four properties (positive homogeneity and subadditivity) of a coherent risk measure. **(Answer: (C))** □

**Problem 2.4.2. (Implementing the standard deviation principle)** You are given:

(i) $\rho(\cdot)$ is the standard deviation principle with $k = 2$.

(ii) $X$ is a Pareto random variable with parameters $\alpha = 3$ and $\theta = 1000$.

Calculate $\rho(X)$.

(A) 1600 (B) 1800 (C) 2000
(D) 2200 (E) 2400

**Ambrose's comments:** A demanding exam question may expect you to remember the definition of the standard deviation principle, which is not unreasonable because this risk measure is discussed in an in-text example (Example 3.13) of *Loss Models*.

*Solution.* Using the formulas in Section A.2.3.1 of the FAM-S tables,

$$\mathrm{E}[X] = \frac{\theta}{\alpha-1} = \frac{1000}{3-1} = 500 \quad \text{and} \quad \mathrm{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = \frac{2(1000)^2}{(3-1)(3-2)} = 1000^2,$$

so $\mathrm{Var}(X) = 1000^2 - 500^2 = 750,000$. Then

$$\rho(X) = \mathrm{E}[X] + k\sqrt{\mathrm{Var}(X)} = 500 + 2\sqrt{750,000} = \boxed{2232.05}. \quad \textbf{(Answer: (D))}$$

□

**Problem 2.4.3. * (Based on Example 3.13 and Exercise 3.31 of *Loss Models*: Is the standard deviation principle coherent?)** You are asked to consider whether the *standard deviation principle*

$$\rho(X) = \mathrm{E}[X] + k\sqrt{\mathrm{Var}(X)},$$

where $k$ is a constant, is a coherent risk measure.

Determine which of the following statements is correct.

(A) $\rho(X)$ fails monotonicity. (B) $\rho(X)$ fails positive homogeneity.
(C) $\rho(X)$ fails translation invariance. (D) $\rho(X)$ fails subadditivity.
(E) It is a coherent risk measure.

*Solution.* To simplify presentation, let's write $\rho(X) = \mu_X + k\sigma_X$ for a random variable $X$ with mean $\mu_X$ and standard deviation $\sigma_X$, and check each of the four properties.

- ✘ *Monotonicity:* If $X \le Y$, then $\mu_X \le \mu_Y$ always, but it is not always true that $\sigma_X \le \sigma_Y$. As a simple example, consider $(X, Y)$ jointly distributed as follows:

| Scenario | Probability | $X$ | $Y$ |
|---|---|---|---|
| 1 | 0.5 | 0 | 1 |
| 2 | 0.5 | 1 | 1 |

Then:

- ▷ $X$ is a Bernoulli random variable with mean $\mu_X = 0.5$ and variance $\sigma_X^2 = 0.5(0.5) = 0.25$, and $Y$ is a random variable with mean $\mu_Y = 1$ and variance $\sigma_Y^2 = 0$ (because $Y = 1$ always).
- ▷ $X \le Y$ always.

With $k = 10$,

$$\rho(X) = 0.5 + 10(0.25) = 3 > \rho(Y) = 1 + 10(0) = 1,$$

so $\rho$ violates monotonicity in general.

- ✓ *Positive homogeneity:* For any positive constant $c$, $\sigma_{cX} = \sqrt{\text{Var}(cX)} = \sqrt{c^2\text{Var}(X)} = c\sigma_X$, so

$$\rho(cX) = \mu_{cX} + k\sigma_{cX} = c\mu_X + k(c\sigma_X) = c(\mu_X + k\sigma_X) = c\rho(X),$$

so $\rho$ is positively homogeneous.

- ✓ *Translation invariance:* For any constant $c$,

$$\rho(X + c) = \mu_{X+c} + k\sigma_{X+c} = (\mu_X + c) + k\sigma_X = \rho(X) + c,$$

so $\rho$ is translation invariant.

- ✓ *Subadditivity:* Because $\mu_{X+Y} = \mu_X + \mu_Y$ and $\sigma_{X+Y} \le \sigma_X + \sigma_Y$ (see the solution to Problem 2.4.1), we have

$$\rho(X + Y) = \mu_{X+Y} + k\sigma_{X+Y} \le (\mu_X + \mu_Y) + k(\sigma_X + \sigma_Y) = \rho(X) + \rho(Y),$$

so $\rho$ is subadditive.

To conclude, $\rho$ violates monotonicity (and only monotonicity). **(Answer: (A))** □

*Remark.* As remarked in Example 3.13 of *Loss Models*, the multiple $k$ is usually a large number chosen to ensure that a loss will exceed the risk measure with some specified small probability.

**Problem 2.4.4. * (SOA Exam FAM-S Sample Question 102: More on the monotonicity property of the standard deviation principle)** You are given:

(i) For any random variable $X$, the risk measure is $\rho(X) = \mu_X + k\sigma_X$.

(ii) The variables $Y$ and $Z$ are defined as follows:

| $y$ | $\text{P}(Y = y)$ |
|---|---|
| 0 | 0.2 |
| 1 | 0.8 |

and $Z$ is constant with $Z = 1$.

Calculate the largest value of $k$ such that $\rho$ has the monotonicity property with respect to $Y$ and $Z$.

(A) 0.50 (B) 0.75 (C) 1.00
(D) 1.25 (E) 1.50

*Solution.* Observe that $Y$ takes values of either 0 or 1, while $Z$ is constant at 1, so $Y \le Z$ is always true. Then $\rho$ satisfying the monotonicity property is equivalent to $\rho(Y) \le \rho(Z)$.

Let's calculate $\rho(Y)$ and $\rho(Z)$ from the mean and variance of each of $Y$ and $Z$.

- $Y$ is Bernoulli with parameter $q = 0.8$, so $\mu_Y = 0.8$, $\sigma_Y = \sqrt{0.8(1-0.8)} = 0.4$, and
$$\rho(Y) = \mu_Y + k\sigma_Y = 0.8 + 0.4k.$$
- $Z = 1$ always, so $\mu_Z = 1$, $\sigma_Z = 0$, and
$$\rho(Z) = \mu_Z + k\sigma_Z = 1 + k(0) = 1.$$

Then $\rho(Y) \le \rho(Z)$ if and only if

$$0.8 + 0.4k \le 1 \quad \Leftrightarrow \quad k \le \boxed{0.5}. \quad \textbf{(Answer: (A))}$$

□

**Problem 2.4.5. (Based on SOA Exam QFIIRM Spring 2022 Question 6: About the use of the standard deviation principle)** To quantify potential losses on its life insurance business, your company is considering the risk measure $\rho$ defined by $\rho(X) = \mathrm{E}[X] + 3\sqrt{\mathrm{Var}(X)}$, for loss random variable $X$.

Determine which of the following statements is/are true.

I. If $X$ is a uniformly distributed random variable on the interval $(-8, 2)$, then $\rho(X) = 5.66$, correct to 2 decimal places.

II. $\rho(X)$ is desirable because the risk metric for the sum of two risks will not exceed the sum of the risk metrics of each individual risk.

III. $\rho(X)$ is desirable for economic capital purposes because it captures the tail risk well.

(A) None (B) I and II only (C) I and III only
(D) II and III only (E) The correct answer is not given by (A), (B), (C), or (D).

*Solution.*

I. True. If $X \sim \mathrm{U}(-8, 2)$, then

$$\rho(X) = \mathrm{E}[X] + 3\sqrt{\mathrm{Var}(X)} = \frac{-8+2}{2} + 3\sqrt{\frac{[2-(-8)]^2}{12}} = 5.6603.$$

II. True, because $\rho$ is subadditive; recall Problem 2.4.3.

III. False, because $\rho$ only captures the expected value and variance of the underlying risk, which reflect limited information about tail risk. **(Answer: (B))** □

**VaR**

**Problem 2.4.6. * (Is VaR a coherent risk measure?)** You are asked to consider whether Value at Risk is a coherent risk measure.

Determine which of the following statements is correct.

(A) It does NOT possess subadditivity.

(B) It does NOT possess monotonicity.

(C) It does NOT possess positive homogeneity.

(D) It does NOT possess translation invariance.

(E) It is a coherent risk measure.

*Solution.* As discussed in the main text, VaR satisfies all of the four properties of a coherent risk measure, with the exception of subadditivity. **(Answer: (A))** □

*Remark.* If the problem statement is changed to "You are asked to consider whether Value at Risk is a coherent risk measure within the class of normal random variables," then the answer will become (E).

**Problem 2.4.7. (SOA Exam P Sample Question 421: VaR of a distribution in the FAM-S tables – I; exponential)** The lifetimes of televisions of a certain model are exponentially distributed with a median of 2.7 years.

Calculate the 87.5th percentile of the lifetimes for these televisions.

(A) 3.08

(B) 4.73

(C) 8.10

(D) 10.80

(E) 19.68

*Solution.* From Section A.3.3.1 of the FAM-S tables,

$$\pi_{0.5} = -\theta \ln(1-0.5) \overset{\text{(given)}}{=} 2.7 \quad \Rightarrow \quad \theta = 3.895277.$$

Then the 87.5th percentile is $\pi_{0.875} = -3.895277 \ln(1-0.875) = \boxed{8.1}$. **(Answer: (C))** □

**Problem 2.4.8. (CAS Exam 3 Fall 2003 Question 17: VaR of a distribution in the FAM-S tables – II; inverse exponential)** Losses have an Inverse Exponential distribution. The mode is 10,000.

Calculate the median.

(A) Less than 10,000
(B) At least 10,000, but less than 15,000
(C) At least 15,000, but less than 20,000
(D) At least 20,000, but less than 25,000
(E) At least 25,000

*Solution.* From Section A.3.3.2 of the FAM-S tables, the mode of an inverse exponential distribution is $\theta/2$, so $\theta = 2(10,000) = 20,000$. The median of the distribution is the VaR of $X$ at the 50% level, so using the formula for $\text{VaR}_p(X)$ there, $\text{VaR}_{0.5}(X) = \theta(-\ln p)^{-1} = 20,000(-\ln 0.5)^{-1} = \boxed{28,853.90}$. **(Answer: (E))** □

**Problem 2.4.9. (SOA Exam P Sample Question 54: VaR of a distribution in the FAM-S tables – III; single-parameter Pareto)** An insurer's annual weather-related loss, $X$, is a random variable with density function

$$f(x) = \begin{cases} \frac{2.5(200)^{2.5}}{x^{3.5}}, & x > 200 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the difference between the 30th and 70th percentiles of $X$.

(A) 35
(B) 93
(C) 124
(D) 231
(E) 298

*Solution.* This is a single-parameter Pareto distribution with parameters $\alpha = 2.5$ and $\theta = 200$. From Section A.5.1.4 of the FAM-S tables, $\text{VaR}_p(X) = \theta(1-p)^{-1/\alpha}$, so

$$\text{VaR}_{0.7}(X) - \text{VaR}_{0.3}(X) = 200(1-0.7)^{-1/2.5} - 200(1-0.3)^{-1/2.5} = \boxed{93.06}. \quad \textbf{(Answer: (B))}$$

□

**Problem 2.4.10. * (VaR of a distribution in the FAM-S tables – IV; Weibull)** You are given that a loss random variable $X$ has the probability density function

$$f(x) = 0.02x\mathrm{e}^{-0.01x^2}, \quad x \geq 0.$$

Calculate $\mathrm{VaR}_{0.8}(X)$.

(A) 11 (B) 12 (C) 13
(D) 14 (E) 15

*Solution.* Inside the exponential function, $x$ is raised to a certain power, suggesting that this is a Weibull distribution. Matching the given density with

$$f(x) = \frac{\tau(x/\theta)^{\tau}\mathrm{e}^{-(x/\theta)^{\tau}}}{x}$$

in Section A.3.2.3 of the FAM-S tables, we deduce that $\theta = 10$ and $\tau = 2$. Using the formula for $\mathrm{VaR}_p(X)$ there, we get

$$\mathrm{VaR}_{0.8}(X) = \theta[-\ln(1-p)]^{1/\tau} = 10[-\ln(1-0.8)]^{1/2} = \boxed{12.69}. \quad \textbf{(Answer: (C))}$$

□

*Remark.* Even if you can't recognize that this is a Weibull distribution, you can still work out this problem from first principles. Solving

$$F_X(x) = \int_0^x 0.02t\mathrm{e}^{-0.01t^2}\,\mathrm{d}t \overset{\text{(substitution)}}{=} -\int_{t=0}^{t=x} \mathrm{d}\mathrm{e}^{-0.01t^2} = 1 - \mathrm{e}^{-0.01x^2} = 0.8$$

gives $x = \sqrt{-\frac{1}{0.01}\ln(1-0.8)} = 12.69$.

**Note:** The next few problems deal with the VaR of a distribution **not** found in the FAM-S tables. You will have to work out the VaR using specialized formulas or from first principles.

**Problem 2.4.11. (SOA Exam P Sample Question 401: VaR of a normal distribution – I)** Let $X$ be a normally distributed random variable representing the amount of an individual claim of a policyholder covered by a group health policy.

You are given that $\mathrm{Var}(X) = 250,000$ and $\mathrm{P}[X < 1000] = 0.3446$.

Calculate the difference between the 90th percentile of $X$ and the median of $X$.

(A) 24 (B) 441 (C) 641
(D) 822 (E) 980

*Solution.* The difference between the 90th percentile of $X$ and the median of $X$ is

$$\pi_{0.9} - \pi_{0.5} \overset{(2.4.4)}{=} \sigma(z_{0.9} - z_{0.5}) = \sqrt{250,000}(1.28155 - 0) = \boxed{640.775}. \quad \textbf{(Answer: (C))}$$

□

*Remark.* The fact that $\mathrm{P}(X < 1000) = 0.3446$, or equivalently, 1000 is the 34.46th percentile of $X$, is not required for solving this problem. From this fact, we can deduce the mean of $X$:

$$\pi_{0.3446} \overset{(2.4.4)}{=} \mu + \sqrt{250,000}(-0.39994) = 1000 \quad \Rightarrow \quad \mu = 1199.97,$$

which can be used to calculate the individual value of $\pi_{0.9}$.

**Problem 2.4.12. (SOA Exam P Sample Question 378 (Reworded): VaR of a normal distribution – II)** In a certain year, an insurance company's profit is modeled by a normal distribution with mean 6.72. The 80% VaR of the profit is 8.40.

Calculate the 90% VaR of the insurance company's profit in the year.

(A) 8.61 (B) 8.96 (C) 9.28
(D) 9.45 (E) 12.80

*Solution.* From the 80% VaR, we can deduce the standard deviation via

$$\pi_{0.8} \overset{(2.4.4)}{=} 6.72 + 0.84162\sigma = 8.40 \quad \Rightarrow \quad \sigma = 1.996150.$$

Now based on $p = 0.9$,

$$\pi_{0.9} \overset{(2.4.4)}{=} 6.72 + 1.28155(1.996150) = \boxed{9.2782}. \quad \textbf{(Answer: (C))}$$

□

**Problem 2.4.13. (SOA Exam P Sample Question 391: VaR of a normal distribution – III)** The annual profits of each of two car insurance companies, A and B, are normally distributed with the same standard deviation.

The mean annual profit of company A is 30.

A profit of 214 is both the 96th percentile of company A's annual profit and the 90th percentile of company B's annual profit.

Calculate the mean annual profit of company B.

(A) 33 (B) 42 (C) 54
(D) 79 (E) 105

*Solution.* Let $\sigma$ be the common standard deviation. We are given that

$$\pi^A_{0.96} \overset{(2.4.4)}{=} 30 + 1.75069\sigma = 214 \quad \Rightarrow \quad \sigma = 105.1014.$$

Turning to company B,

$$\pi^B_{0.90} \overset{(2.4.4)}{=} \mu^B + 1.28155(105.1014) = 214 \quad \Rightarrow \quad \mu^B = \boxed{79.31}. \quad \textbf{(Answer: (D))}$$ □

**Problem 2.4.14. (VaR of an unfamiliar continuous distribution)** You are given that a loss random variable $X$ has the distribution function

$$F_X(x) = \exp(-\mathrm{e}^{-x}), \quad x \in \mathbb{R}.$$

Calculate $\mathrm{VaR}_{0.99}(X)$.

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

*Solution.* Solving the equation $F_X(x) = 0.99$ for $x$, we get

$$\begin{aligned} & \exp(-\mathrm{e}^{-x}) = 0.99 \\ \Rightarrow \quad & \mathrm{e}^{-x} = -\ln 0.99 \\ \Rightarrow \quad & x = -\ln(-\ln 0.99) = \boxed{4.60}. \quad \textbf{(Answer: (E))} \end{aligned}$$

□

*Remark.* *(If you are interested)* This strange-looking distribution is called a *Gumbel* distribution, which can be found in Section A.4.1.1 of *Loss Models* (not part of the FAM-S tables).

**Problem 2.4.15. * (Based on SOA Course 3 Fall 2003 Question 18: VaR of a mixture of two exponentials)** The unlimited severity $X$ for claim amounts under an auto liability insurance policy is given by the cumulative distribution function:

$$F(x) = 1 - 0.3\mathrm{e}^{-0.2x} - 0.7\mathrm{e}^{-0.1x}, \quad x \geq 0.$$

Calculate $\mathrm{VaR}_{0.75}(X)$.

(A) 10.7 (B) 11.0 (C) 11.2
(D) 11.6 (E) 11.8

*Solution.* This is a mixture of exponential distributions, and we do not have a formula for its VaR from the FAM-S tables, so let's work out this problem from first principles. By definition, $\mathrm{VaR}_{0.75}(X)$ is the value of $x$ such that

$$F(x) = 1 - 0.3\mathrm{e}^{-0.2x} - 0.7\mathrm{e}^{-0.1x} = 0.75 \quad \Rightarrow \quad 0.3\mathrm{e}^{-0.2x} + 0.7\mathrm{e}^{-0.1x} - 0.25 = 0,$$

which is a quadratic equation in $\mathrm{e}^{-0.1x}$. Using the quadratic formula,

$$\mathrm{e}^{-0.1x} = \frac{-0.7 \pm \sqrt{0.7^2 - 4(0.3)(-0.25)}}{2(0.3)} = 0.314699 \text{ or } \underset{\text{(rejected } \because \mathrm{e}^{-0.1x} > 0)}{-2.648032}$$

so $\mathrm{VaR}_{0.75}(X) = -\dfrac{\ln 0.314699}{0.1} = \boxed{11.56}$. **(Answer: (D))** □

*Remark.* ⚠ Unlike the raw moments, the VaR of a mixture distribution is *not* the mixture of the component VaR's.

**Problem 2.4.16. [HARDER!] (VaR of a discrete distribution)** You are given the following discrete loss distribution:

$$f_X(x) = \begin{cases} 0.80, & \text{if } x = 0, \\ 0.10, & \text{if } x = 20, \\ 0.06, & \text{if } x = 50, \\ 0.02, & \text{if } x = 80, \\ 0.02, & \text{if } x = 100. \end{cases}$$

(a) Calculate $\text{VaR}_{0.95}(X)$.

(A) 0 (B) 20 (C) 50
(D) 80 (E) 100

*Solution.* From the probability mass function of $X$, we can get its distribution function, which is a step function:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.8, & \text{if } 0 \le x < 20, \\ 0.9, & \text{if } 20 \le x < 50, \\ 0.96, & \text{if } 50 \le x < 80, \\ 0.98, & \text{if } 80 \le x < 100, \\ 1, & \text{if } x \ge 100. \end{cases}$$

As $F_X(x)$ jumps from $0.9\,(< 0.95)$ to $0.96\,(> 0.95)$ at $x = 50$, we have $\text{VaR}_{0.95}(X) = \boxed{50}$. **(Answer: (C))** □

(b) Calculate $\text{VaR}_{0.96}(X)$.

(A) 0 (B) 20 (C) 50
(D) 80 (E) 100

*Solution.* Since $F_X(x) = 0.96$ for all $x \in [50, 80)$, $\text{VaR}_{0.96}(X)$, which is the smallest such $x$, equals $\boxed{50}$. **(Answer: (C))** □

(c) Calculate $\text{VaR}_{0.99}(X)$.

(A) 0 (B) 20 (C) 50
(D) 80 (E) 100

*Solution.* As $F_X(x)$ jumps from $0.98\,(< 0.99)$ to $1\,(> 0.99)$ at $x = 100$, we have $\text{VaR}_{0.99}(X) = \boxed{100}$. **(Answer: (E))** □

**Note:** Another possible type of exam question is to provide you with the values of the **VaR at two different levels** and ask you to deduce the parameter(s) of the underlying distribution or calculate other probabilistic quantities.

**Problem 2.4.17. (CAS Part 4B Spring 1996 Question 17 / *Loss Models* Exercise 3.19: Given two VaR's – I; Pareto)** You are given the following:

- Losses follow a Pareto distribution with parameters $\alpha$ and $\theta$.
- The 10th percentile of the distribution is $\theta - k$, where $k$ is a constant.
- The 90th percentile of the distribution is $5\theta - 3k$.

Determine $\alpha$.

(A) Less than 1.25
(B) At least 1.25, but less than 1.75
(C) At least 1.75, but less than 2.25
(D) At least 2.25, but less than 2.75
(E) At least 2.75

*Solution.* From Section A.2.3.1 of the FAM-S tables, the VaR of a Pareto distribution is $\text{VaR}_p(X) = \theta[(1-p)^{-1/\alpha} - 1]$. Therefore, we solve

$$\begin{cases} \text{VaR}_{0.1}(X) = \theta[(1-0.1)^{-1/\alpha} - 1] = \theta - k \\ \text{VaR}_{0.9}(X) = \theta[(1-0.9)^{-1/\alpha} - 1] = 5\theta - 3k \end{cases} \Rightarrow \begin{cases} 0.9^{-1/\alpha} = \dfrac{2\theta - k}{\theta} \\ 0.1^{-1/\alpha} = \dfrac{6\theta - 3k}{\theta} = \dfrac{3(2\theta - k)}{\theta} \end{cases}.$$

Dividing the second equation from the first, we eliminate $\theta$ and get

$$9^{-1/\alpha} = \left(\frac{0.9}{0.1}\right)^{-1/\alpha} = \left(\frac{2\theta - k}{\theta}\right)\left(\frac{\theta}{3(2\theta - k)}\right) = \frac{1}{3} \quad \Rightarrow \quad \alpha = \boxed{2}. \quad \textbf{(Answer: (C))}$$

□

**Problem 2.4.18. * (SOA Exam C Spring 2007 Question 24 (Simplified): Given two VaR's – II; Weibull)** For a portfolio of policies, you are given:

(i) Losses follow a Weibull distribution with parameters $\theta$ and $\tau$.

(ii) The Value at Risk at the 20% level is 77.4.

(iii) The Value at Risk at the 70% level is 124.7.

Calculate the estimate of $\theta$.

(A) Less than 100
(B) At least 100, but less than 105
(C) At least 105, but less than 110
(D) At least 110, but less than 115
(E) At least 115

*Solution.* From Section A.3.2.3 of the FAM-S tables, $\mathrm{VaR}_p(X) = \theta[-\ln(1-p)]^{1/\tau}$. Matching the VaR's at the 20% and 70% levels, we solve

$$\begin{cases} \theta[-\ln(1-0.2)]^{1/\tau} = 77.4 \\ \theta[-\ln(1-0.7)]^{1/\tau} = 124.7 \end{cases}.$$

Dividing the second equation by the first equation, we eliminate $\theta$ and get

$$\left(\frac{\ln 0.3}{\ln 0.8}\right)^{1/\tau} = \frac{124.7}{77.4}.$$

Taking natural logarithms on both sides yields

$$\frac{1}{\tau}\ln\left(\frac{\ln 0.3}{\ln 0.8}\right) = \ln\left(\frac{124.7}{77.4}\right),$$

resulting in $\tau = 3.534245$. Putting this back to either equation, we find $\theta = \boxed{118.3195}$. **(Answer: (E))** □

**Problem 2.4.19. (SOA Exam C Fall 2006 Question 1 (Simplified): Given two VaR's – III; Burr)** You are given:

(i) Losses follow a Burr distribution with $\alpha = 2$.

(ii) The Value at Risk at the 30% level is 336.

(iii) The Value at Risk at the 65% level is 466.

Calculate $\gamma$.

(A) Less than 2.9

(B) At least 2.9, but less than 3.2

(C) At least 3.2, but less than 3.5

(D) At least 3.5, but less than 3.8

(E) At least 3.8

*Solution.* From Section A.2.2.2 of the FAM-S tables, $\mathrm{VaR}_p(X) = \theta[(1-p)^{-1/2} - 1]^{1/\gamma}$ for a Burr distribution with $\alpha = 2$. Matching the VaR's at the 30% and 65% levels, we solve

$$\begin{cases} \theta[(1-0.3)^{-1/2} - 1]^{1/\gamma} = 336 \\ \theta[(1-0.65)^{-1/2} - 1]^{1/\gamma} = 466 \end{cases} \Rightarrow \left(\frac{0.690309}{0.195229}\right)^{1/\gamma} = \frac{466}{336}$$

$$\Rightarrow \gamma = \boxed{3.8614}.$$ **(Answer: (E))**

□

**Problem 2.4.20. (SOA Course 4 Fall 2003 Question 2 (Simplified): Given two VaR's – IV; loglogistic)** You are given:

(i) Losses follow a loglogistic distribution.

(ii) The 40% Value at Risk is 89.2.

(iii) The 80% Value at Risk is 206.

Calculate $\theta$.

(A) Less than 77

(B) At least 77, but less than 87

(C) At least 87, but less than 97

(D) At least 97, but less than 107

(E) At least 107

*Solution.* From Section A.2.3.3 of the FAM-S tables, $\text{VaR}_p(X) = \theta(p^{-1} - 1)^{-1/\gamma}$. Matching the VaR's at the 40% and 80% levels, we solve

$$\begin{cases} \theta(0.4^{-1} - 1)^{-1/\gamma} = 89.2 \\ \theta(0.8^{-1} - 1)^{-1/\gamma} = 206 \end{cases} \Rightarrow \left(\frac{0.25}{1.5}\right)^{-1/\gamma} = \frac{206}{89.2}$$
$$\Rightarrow \gamma = 2.140705.$$

Then from either equation, we can get $\theta = \boxed{107.80}$. **(Answer: (E))** □

**Problem 2.4.21. * (SOA Course 4 Fall 2000 Question 39 (Reworded): Given two VaR's – V; lognormal)** You are given the following information about a study of individual claims:

(i) The Value at Risk at the 20% level is 18.25.

(ii) The Value at Risk at the 80% level is 35.80.

Determine the probability that a claim is greater than 30 using a lognormal distribution.

(A) 0.34

(B) 0.36

(C) 0.38

(D) 0.40

(E) 0.42

*Solution.* Using (2.4.5) for the VaR of lognormal random variables with $z_{0.2} = -0.84162$ and $z_{0.8} = 0.84162$, we solve

$$\begin{cases} e^{\mu - 0.84162\sigma} = 18.25 \\ e^{\mu + 0.84162\sigma} = 35.80 \end{cases} \Rightarrow \begin{cases} \mu - 0.84162\sigma = \ln 18.25 \\ \mu + 0.84162\sigma = \ln 35.80 \end{cases}.$$

Solving these two equations yields $\sigma = 0.4003$ and $\mu = 3.2411$. The probability that a claim is greater than 30 is

$$\text{P}(X > 30) = 1 - \Phi\underbrace{\left(\frac{\ln 30 - 3.2411}{0.4003}\right)}_{0.39994} = 1 - 0.65540 = \boxed{0.3446}. \quad \textbf{(Answer: (A))}$$

□

**Note:** The next few problems nicely combine VaR with other parts of the FAM-S exam syllabus, e.g., coverage modifications (Section 1.1), stop-loss premiums (Section 2.3), option payoffs (after you have finished Chapter 6).

**Problem 2.4.22. * (SOA Exam P Sample Question 61: VaR of a function of a random variable – I; policy limit)** An insurance policy reimburses dental expense, $X$, up to a maximum benefit of 250. The probability density function for $X$ is:

$$f(x) = \begin{cases} ce^{-0.004x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $c$ is a constant.

Calculate the median benefit for this policy.

(A) 161 (B) 165 (C) 173

(D) 182 (E) 250

*Solution.* Observe from the form of $f(x)$ that $X$ is exponential with parameter $\theta = 0.004^{-1} = 250$. Because $Y^L = X \wedge 250$ is an increasing function of $X$, the median of $Y^L$ is

$$\pi_{0.5}(Y^L) \overset{(2.4.2)}{=} \pi_{0.5}(X) \wedge 250 \overset{\substack{\text{(Section A.3.3.1} \\ \text{of FAM-S tables)}}}{=} [-250\ln(1-0.5)] \wedge 250 = \boxed{173.29}. \quad \textbf{(Answer: (C))}$$

□

*Remark. (Not required for solving this problem)* To make sure that $\int_0^\infty f(x)\,\mathrm{d}x = 1$, we need $c = 0.004$.

**Problem 2.4.23. (SOA Exam P Sample Question 215: VaR of a function of a random variable – II; deductible)** Losses under an insurance policy are exponentially distributed with mean 4. The deductible is 1 for each loss.

Calculate the median amount that the insurer pays a policyholder for a loss under the policy.

(A) 1.77 (B) 2.08 (C) 2.12

(D) 2.77 (E) 3.12

*Solution.* The payment for each loss is $Y^L = (X-1)_+$, which is an increasing function of $X$, so the median of $Y^L$ is

$$\pi_{0.5}(Y^L) \overset{(2.4.2)}{=} (\pi_{0.5}(X) - 1)_+ \overset{\substack{\text{(Section A.3.3.1} \\ \text{of FAM-S tables)}}}{=} (-4\ln(1-0.5) - 1)_+ = \boxed{1.7726}. \quad \textbf{(Answer: (A))}$$

□

**Problem 2.4.24. * (SOA Exam P Sample Question 181: VaR of a function of a random variable – III; deductible)** Losses covered by an insurance policy are modeled by a uniform distribution on the interval $[0, 1000]$. An insurance company reimburses losses in excess of a deductible of 250.

Calculate the difference between the median and the 20th percentile of the insurance company reimbursement, over all losses.

| | | |
|---|---|---|
| (A) 225 | (B) 250 | (C) 300 |
| (D) 375 | (E) 500 | |

*Solution.* Let $Y^L = (X-250)_+$ be the insurance company reimbursement, which is an increasing function of $X$. By (2.4.2)

$$\begin{aligned}\pi_{0.2}(Y^L) &= (\pi_{0.2}(X) - 250)_+ = (0.2(1000) - 250)_+ = 0,\\ \pi_{0.5}(Y^L) &= (\pi_{0.5}(X) - 250)_+ = (0.5(1000) - 250)_+ = 250,\end{aligned}$$

so $\pi_{0.5}(Y^L) - \pi_{0.2}(Y^L) = 250 - 0 = \boxed{250}$. **(Answer: (B))** □

**Problem 2.4.25. (SOA Exam QFIIRM Spring 2023 Question 2 (a)(ii) (Reworded): VaR of a function of a random variable – IV; option payoff)** Company ABC would like to model the economic capital for a product using the following information:

- Losses mimic a put option: $L = 10,000\max(1 - S_{10}, 0)$, where $S_{10}$ is the price at time 10 of an underlying equity investment.
- By using historical data, ABC estimates $S_{10}$ has a lognormal distribution where $\mu = 1$, $\sigma = 0.7$.

Calculate the 95% VaR of loss L.

*Solution.* Because $L$ is a *decreasing* function of $S_{10}$, we have $\pi_{0.95}(L) \overset{(2.4.3)}{=} 10,000\,(1 - \pi_{0.05}(S_{10}))_+$, where the VaR of $S_{10}$ is based on the $1 - 95\% = 5\%$ security level. By (2.4.5),

$$\pi_{0.05}(S_{10}) = \mathrm{e}^{1+0.7z_{0.05}} = \mathrm{e}^{1+0.7(-1.64485)} = 0.859508,$$

so $\pi_{0.95}(L) = 10,000(1 - 0.859508)_+ = \boxed{1404.92}$. □

**Problem 2.4.26. * (Based on Problem 2.3.73: Deducing VaR from stop-loss premiums)** For aggregate claims $S$ of an insurance policy, you are given:

(i) All claim amounts are non-negative integers.

(ii) The following net stop-loss premiums:

$$\begin{aligned} \mathrm{E}[(S-16)_+] &= 3.89 & \mathrm{E}[(S-25)_+] &= 2.75 \\ \mathrm{E}[(S-20)_+] &= 3.33 & \mathrm{E}[(S-26)_+] &= 2.69 \\ \mathrm{E}[(S-24)_+] &= 2.84 & \mathrm{E}[(S-27)_+] &= 2.65 \end{aligned}$$

Calculate $\mathrm{VaR}_{0.95}(S)$.

(A) 20 (B) 24 (C) 25
(D) 26 (E) 27

*Solution.* Applying $\mathrm{E}[(S-(d+1))_+] \overset{(2.3.14)}{=} \mathrm{E}[(S-d)_+] - S_S(d)$ inductively, we get

$$\begin{cases} S_S(24) = 2.84 - 2.75 = 0.09 \\ S_S(25) = 2.75 - 2.69 = 0.06 \\ S_S(26) = 2.69 - 2.65 = 0.04 \end{cases} \Rightarrow \begin{cases} F_S(24) = 0.91 \\ F_S(25) = 0.94 \\ F_S(26) = 0.96 \end{cases}.$$

Since $S$ can only take non-negative integers and $F_S(x)$ jumps from 0.94 ($<0.95$) to 0.96 ($>0.95$) at $x = 26$, we have $\mathrm{VaR}_{0.95}(S) = \boxed{26}$. **(Answer: (D))** □

**TVaR**

**Problem 2.4.27. (Is TVaR a coherent risk measure?)** You are asked to consider whether Tail Value at Risk is a coherent risk measure.

Determine which of the following statements is correct.

(A) It does NOT possess subadditivity.
(B) It does NOT possess monotonicity.
(C) It does NOT possess positive homogeneity.
(D) It does NOT possess translation invariance.
(E) It is a coherent risk measure.

*Solution.* As discussed in the main text, TVaR is a coherent risk measure. In particular, it is subadditive, unlike VaR. **(Answer: (E))** □

**Problem 2.4.28. (TVaR of a uniform distribution)** You are given:

(i) $X$ is uniformly distributed on the interval $[0, \theta]$.

(ii) $\text{TVaR}_{0.90}(X) = 19,000$

Calculate $\text{TVaR}_{0.95}(X)$.

(A) 19,500 (B) 19,600 (C) 19,700
(D) 19,800 (E) 19,900

*Solution.* From (ii),

$$\text{TVaR}_{0.90}(X) \overset{(2.4.10)}{=} \pi_{0.95} = 0.95\theta = 19,000 \quad \Rightarrow \quad \theta = 20,000.$$

Then

$$\text{TVaR}_{0.95}(X) \overset{(2.4.10)}{=} \pi_{0.975} = 0.975\theta = \boxed{19,500}. \quad \textbf{(Answer: (A))}$$

□

**Problem 2.4.29. * (Based on Exercise 3.36 of *Loss Models*: Comparing the tail weight of exponential and Pareto through VaR and TVaR)** You are given:

(i) $X$ is an exponential random variable with parameter $\theta = 500$.

(ii) $Y$ is a Pareto random variable with parameters $\alpha = 3$ and $\theta = 1000$.

Determine which of the following statements is/are true.

I. $X$ and $Y$ have the same mean.

II. $Y$ has a higher VaR and TVaR at the 95% security level than $X$.

III. $Y$ has a heavier right tail than $X$.

(A) I only (B) II only (C) III only
(D) I, II, and III (E) The correct answer is not given by (A), (B), (C), or (D).

*Solution.* I. True. $\text{E}[X] = 500 = \frac{1000}{3-1} = \text{E}[Y]$.

II. True. Using the formulas in Sections A.2.3.1 and A.3.3.1 of the FAM-S tables,

$$\begin{aligned}
\text{VaR}_{0.95}(X) &= -500\ln(1-0.95) = 1497.8661, \\
\text{TVaR}_{0.95}(X) &= 1497.8661 + 500 = 1997.8661, \\
\text{VaR}_{0.95}(Y) &= 1000[(1-0.95)^{-1/3} - 1] = 1714.4176, \\
\text{TVaR}_{0.95}(Y) &= 1714.4176 + \frac{1000(1-0.95)^{-1/3}}{3-1} = 3071.6264.
\end{aligned}$$

Therefore, $Y$ has a higher VaR and TVaR at the 95% security level than $X$.

III. True. The fact that $Y$ has a higher VaR and TVaR shows that $Y$ has a heavier right tail than $X$. **(Answer: (D))** □

**Problem 2.4.30. * (Given two VaR's/TVaR's, find another – I; normal)** For a normal random variable $X$, you are given:

(i) $\text{TVaR}_{0.7}(X) = 84.7693$

(ii) $\text{TVaR}_{0.8}(X) = 91.9943$

Calculate $\text{TVaR}_{0.9}(X)$.

(A) 101 (B) 102 (C) 103
(D) 104 (E) 105

*Solution.* Using (2.4.8) for the TVaR of normal random variables, we solve (recall that $\phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}$)

$$\begin{cases} \mu + \sigma\left[\dfrac{\phi(0.52440)}{1-0.7}\right] = \mu + \sigma\left(\dfrac{0.347693}{0.3}\right) = \text{TVaR}_{0.7}(X) = 84.7693 \\ \mu + \sigma\left[\dfrac{\phi(0.84162)}{1-0.8}\right] = \mu + \sigma\left(\dfrac{0.279962}{0.2}\right) = \text{TVaR}_{0.8}(X) = 91.9943 \end{cases}.$$

Solving these two equations yields $\sigma = 30$ and $\mu = 50$. By (2.4.8) again, this time with $p = 0.9$,

$$\text{TVaR}_{0.9}(X) = \mu + \sigma\left[\frac{\phi(1.28155)}{1-0.9}\right] = 50 + 30\left(\frac{0.175499}{0.1}\right) = \boxed{102.65}. \quad \textbf{(Answer: (C))}$$

□

**Problem 2.4.31. [HARDER!] (Given two VaR's/TVaR's, find another – II; Pareto)** For a Pareto random variable $X$, you are given:

(i) $\text{VaR}_{0.8}(X) = 70.9976$

(ii) $\text{VaR}_{0.96}(X) = 192.4018$

Calculate $\text{TVaR}_{0.96}(X)$.

(A) 260 (B) 280 (C) 300
(D) 320 (E) 340

*Solution.* From Section A.2.3.1 of the FAM-S tables, $\text{VaR}_p(X) = \theta[(1-p)^{-1/\alpha} - 1]$ for a Pareto distribution. Therefore, we solve

$$\begin{cases} \theta(0.2^{-1/\alpha} - 1) = \theta(5^{1/\alpha} - 1) = 70.9976 \\ \theta(0.04^{-1/\alpha} - 1) = \theta(5^{2/\alpha} - 1) = 192.4018 \end{cases}.$$

Using the identity $x^2 - 1 \equiv (x+1)(x-1)$, we further get

$$5^{1/\alpha} + 1 = \frac{(5^{1/\alpha}+1)(5^{1/\alpha}-1)}{5^{1/\alpha}-1} = \frac{192.4018}{70.9976} \quad \Rightarrow \quad \alpha = 3.$$

Then from either equation, we get $\theta = 100$.

Now using the TVaR formula, we have

$$\begin{aligned} \text{TVaR}_{0.96}(X) &= \text{VaR}_{0.96}(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha - 1} \\ &= 192.4018 + \frac{100(1-0.96)^{-1/3}}{3-1} \\ &= \boxed{338.60}. \quad \textbf{(Answer: (E))} \end{aligned}$$

□

*Remark.* It would be very difficult, if not impossible, to solve for $\alpha$ and $\theta$ analytically from $\text{TVaR}_{0.8}(X)$ and $\text{TVaR}_{0.96}(X)$, which are complex functions of $\alpha$ and $\theta$, so this problem provides $\text{VaR}_{0.8}(X)$ and $\text{VaR}_{0.96}(X)$ instead.

**Note:** The next few problems test the calculation of the TVaR using (2.4.6), (2.4.7), or (2.4.9). They involve either distributions other than exponential, Pareto, uniform, normal (so you don't have a readily available TVaR formula) or familiar distributions in non-standard settings (e.g., coverage modifications are imposed).

**Problem 2.4.32. * (Based on Example 1.1.14: TVaR of a beta distribution)** Loss amounts have the distribution function

$$F_X(x) = \begin{cases} (x/100)^2, & 0 \le x \le 100, \\ 1, & 100 < x. \end{cases}$$

Calculate $\text{TVaR}_{0.9}(X)$.

(A) 95 (B) 96 (C) 97

(D) 98 (E) 99

*Solution 1 (By definition of TVaR, (2.4.6)).* Solving $F_X(x) = (x/100)^2 = p$, we get $\text{VaR}_p(X) = 100\sqrt{p}$ for $0 < p < 1$. Then by (2.4.6),

$$\text{TVaR}_{0.9}(X) = \frac{1}{1-0.9}\int_{0.9}^{1} 100\sqrt{p}\,\mathrm{d}p = \frac{2(100)}{3(0.1)}(1^{3/2} - 0.9^{3/2}) = \boxed{97.46}. \quad \textbf{(Answer: (C))}$$

□

*Solution 2 (By (2.4.7)).* Let's begin by finding $\pi_{0.9}$ by solving

$$F_X(x) = \left(\frac{x}{100}\right)^2 = 0.9 \quad \Rightarrow \quad \pi_{0.9} = 94.8683.$$

Then by (1.1.5) and direct integration,

$$\begin{aligned} \text{E}[(X - \pi_{0.9})_+] &= \int_{94.868330}^{100} \left[1 - \left(\frac{x}{100}\right)^2\right] \mathrm{d}x \\ &= (100 - 94.8683) - \frac{1}{3(100)^2}(100^3 - 94.8683^3) \\ &= 0.2588. \end{aligned}$$

Finally, (2.4.7) yields

$$\mathrm{TVaR}_{0.9}(X) = \pi_{0.9} + \frac{\mathrm{E}[(X-\pi_{0.9})_+]}{1-0.9} = 94.8683 + \frac{0.2588}{0.1} = \boxed{97.46}. \quad \textbf{(Answer: (C))}$$

□

*Remark.* In fact, $X$ has the same distribution as $\sqrt{Y}$, where $Y$ is uniformly distributed on the interval $(0, 100^2)$.

**Problem 2.4.33. * (Problem 2.4.15 continued: TVaR of a mixture distribution – I)** You are given:

(i) The unlimited severity $X$ for claim amounts under an auto liability insurance policy is given by the cumulative distribution function:

$$F(x) = 1 - 0.3\mathrm{e}^{-0.2x} - 0.7\mathrm{e}^{-0.1x}, \quad x \geq 0.$$

(ii) $\mathrm{VaR}_{0.75}(X) = 11.5614$ (from Problem 2.4.15).

Calculate $\mathrm{TVaR}_{0.75}(X)$.

(A) 18 (B) 19 (C) 20
(D) 21 (E) 22

*Solution.* By (1.1.5) and direct integration,

$$\begin{aligned}
\mathrm{E}[(X-\pi_{0.75})_+] &= \int_{11.5614}^{\infty} S_X(x)\,\mathrm{d}x \\
&= \int_{11.5614}^{\infty} (0.3\mathrm{e}^{-0.2x} + 0.7\mathrm{e}^{-0.1x})\,\mathrm{d}x \\
&= 0.3\left[\frac{\mathrm{e}^{-0.2(11.5614)}}{0.2}\right] + 0.7\left[\frac{\mathrm{e}^{-0.1(11.5614)}}{0.1}\right] \\
&= 2.351446.
\end{aligned}$$

Then by (2.4.7),

$$\mathrm{TVaR}_{0.75}(X) = \pi_{0.75} + \frac{\mathrm{E}[(X-\pi_{0.75})_+]}{1-0.75} = 11.5614 + \frac{2.351446}{0.25} = \boxed{20.97}. \ \textbf{(Answer: (D))}$$

□

**Problem 2.4.34. (SOA Exam QFIIRM Fall 2022 Question 3 (e): TVaR of a mixture distribution – II)** Simple Life Insurance Company uses the following loss distribution in setting up economic capital for its operational risk:

- $L = 0$ million, with probability of 0.4.
- $L = 50$ million, with probability of 0.56.
- $L = U$ with probability of 0.04, where $U$ is a random variable uniformly distributed on the interval $(50, 200)$.

Calculate the 95% Tail Value at Risk of $L$.

*Solution.* Note that $L$ is a mixture of three distributions, and its distribution function is

$$F_L(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.4, & \text{if } 0 \leq x < 50, \\ 0.96 + 0.04\left(\frac{x-50}{200-50}\right), & \text{if } 50 \leq x < 200. \end{cases}$$

Because $F_L$ jumps momentarily from $0.4\,(< 0.95)$ to $0.96\,(> 0.95)$ at $x = 50$, we have $\pi_{0.95} = 50$. Then we can find $\text{TVaR}_{0.95}(L)$ in two ways:

- *(Solution 1: By definition of TVaR, (2.4.6))* For $p \geq 0.95$, the $100p$% VaR of $L$ is

$$\pi_p(L) = \begin{cases} 50, & \text{if } 0.95 \leq p \leq 0.96, \\ 50 + \frac{150}{0.04}(p - 0.96), & 0.96 < p < 1. \end{cases}$$

  By (2.4.6),

$$\begin{aligned} \text{TVaR}_{0.95}(L) &= \frac{\int_{0.95}^{0.96} 50\,\mathrm{d}p + \int_{0.96}^{1} \left[50 + \frac{150}{0.04}(p - 0.96)\right] \mathrm{d}p}{1 - 0.95} \\ &= \frac{50(0.96 - 0.95) + 50(1 - 0.96) + \frac{150}{0.04}\left(\frac{0.04^2}{2}\right)}{0.05} \\ &= \boxed{110}. \end{aligned}$$

- *(Solution 2: By (2.4.7))* Integrating the survival function of $L$,

$$\begin{aligned} \text{TVaR}_{0.95}(L) &= \pi_{0.95} + \frac{\mathrm{E}[(L - \pi_{0.95})_+]}{1 - 0.95} \\ &= 50 + \frac{0.04 \int_{50}^{200} \left(1 - \frac{x-50}{150}\right) \mathrm{d}x}{0.05} \\ &= 50 + \frac{0.04\left[(200 - 50) - \frac{150^2}{2(150)}\right]}{0.05} \\ &= \boxed{110}. \end{aligned}$$

□

**Problem 2.4.35. (Problem 2.4.26 continued: Calculating TVaR from stop-loss premiums)** For aggregate claims $S$ of an insurance policy, you are given:

(i) All claim amounts are non-negative integers.

(ii) The following net stop-loss premiums:

$$\begin{aligned} \mathrm{E}[(S-16)_+] &= 3.89 & \quad \mathrm{E}[(S-25)_+] &= 2.75 \\ \mathrm{E}[(S-20)_+] &= 3.33 & \quad \mathrm{E}[(S-26)_+] &= 2.69 \\ \mathrm{E}[(S-24)_+] &= 2.84 & \quad \mathrm{E}[(S-27)_+] &= 2.65 \end{aligned}$$

Calculate $\mathrm{TVaR}_{0.95}(S)$.

(A) 72 (B) 74 (C) 76

(D) 78 (E) 80

*Solution.* In Problem 2.4.26, we found that $\mathrm{VaR}_{0.95}(S) = 26$. By (2.4.7),

$$\mathrm{TVaR}_{0.95}(S) = 26 + \frac{\mathrm{E}[(S-26)_+]}{1-0.95} = 26 + \frac{2.69}{0.05} = \boxed{79.8}. \quad \textbf{(Answer: (E))}$$

□

**Problem 2.4.36. * [HARDER!] (TVaR of a function of a random variable – I; policy limit)** The ground-up loss for an insurance policy follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 1000$.

A policy limit of 5000 is imposed on each loss.

Calculate the Tail Value at Risk of the cost per loss at the 95% security level.

(A) 2800 (B) 2900 (C) 3000

(D) 3100 (E) 3200

*Solution.* In this challenging problem, we are asked to find the TVaR of $Y^L = X \wedge 5000$, which is an increasing function of $X$. While $\pi_{0.95}(Y^L) \overset{(2.4.2)}{=} \pi_{0.95}(X) \wedge 5000$, do note that it is *not* true that $\mathrm{TVaR}_{0.95}(Y^L) = \mathrm{TVaR}_{0.95}(X) \wedge 5000$; recall remark (ii) in Example 2.4.7. ⚠

Here are two ways to solve this problem, both of which are instructive.

- *(Solution 1: By definition of TVaR, (2.4.6))* Let's start with the VaR of $Y^L$ and express it in terms of the VaR of $X$:

$$\begin{aligned} \pi_p(Y^L) &\overset{(2.4.2)}{=} \pi_p(X) \wedge 5000 \\ &= 1000[(1-p)^{-1/3} - 1] \wedge 5000 \\ &= \begin{cases} 1000[(1-p)^{-1/3} - 1], & \text{if } 0 < p < 0.995370, \\ 5000, & \text{if } 0.995370 \leq p < 1. \end{cases} \end{aligned}$$

Note that if $p$ is large enough, then $\pi_p(Y^L)$ equals the 5000 policy limit and stops increasing with $p$.

Using (2.4.6) and splitting the integral of the VaR according to whether $p < 0.995370$ or $p \geq 0.995370$, we get

$$\begin{aligned}
\text{TVaR}_{0.95}(Y^L) &= \frac{1}{1-0.95}\int_{0.95}^{1} \pi_p(Y^L)\,\mathrm{d}p \\
&= \frac{1}{0.05}\left(\int_{0.95}^{0.995370} 1000[(1-p)^{-1/3}-1]\,\mathrm{d}p + \int_{0.995370}^{1} 5000\,\mathrm{d}p\right) \\
&= \frac{1}{0.05}\left(1000\left[-\frac{3}{2}(1-p)^{2/3}-p\right]_{0.95}^{0.995370} + 5000(1-0.995370)\right) \\
&= \frac{1}{0.05}[116.542432 + 5000(1-0.995370)] \\
&= \boxed{2793.85}. \quad \textbf{(Answer: (A))}
\end{aligned}$$

- *(Solution 2: In terms of* $\mathrm{E}[(Y^L - \pi_{0.95}(Y^L))_+]$*)* By (2.4.7),

$$\text{TVaR}_{0.95}(Y^L) = \pi_{0.95}(Y^L) + \frac{\mathrm{E}[(Y^L - \pi_{0.95}(Y^L))_+]}{1-0.95},$$

where $\pi_{0.95}(Y^L) = 1000[(1-0.95)^{-1/3} - 1] = 1714.4176$. The survival function of $Y^L$ is

$$S_{Y^L}(y) \overset{(1.1.1)}{=} \begin{cases} S_X(y) = \left(\frac{1000}{y+1000}\right)^3, & \text{if } y < 5000, \\ 0, & \text{if } y \geq 5000. \end{cases}$$

By (1.1.5), we integrate the survival function of $Y^L$ and get

$$\begin{aligned}
\mathrm{E}[(Y^L - 1714.4176)_+] &= \int_{1714.4176}^{\infty} S_{Y^L}(y)\,\mathrm{d}y \\
&= \int_{1714.4176}^{5000} \left(\frac{1000}{y+1000}\right)^3 \mathrm{d}y \\
&= \left[-\frac{1000^3}{2(y+1000)^2}\right]_{1714.4176}^{5000} \\
&= 53.971552.
\end{aligned}$$

Therefore, $\text{TVaR}_{0.95}(Y^L) = 1714.4176 + \dfrac{53.971552}{0.05} = \boxed{2793.85}$. **(Answer: (A))** □

*Remark.* ⚠ Option (D) is for $\text{TVaR}_{0.95}(X) \wedge 5000 = 3071.63 \wedge 5000 = 3071.63$.

**Problem 2.4.37. [HARDER!] (TVaR of a function of a random variable – II; deductible)** The ground-up loss for an insurance policy follows an exponential distribution with parameter $\theta = 1000$.

An ordinary deductible of 500 is imposed on each loss.

Calculate the Tail Value at Risk of the cost per loss at the 90% security level.

(A) 1800 (B) 2300 (C) 2800
(D) 3300 (E) 3800

*Solution.* As in the preceding problem, there are different ways to find the TVaR of $Y^L = (X-500)_+$, but given that there is only a deductible (not a policy limit) and $X$ is exponential, which is memoryless, the easiest solution is to use $\text{TVaR}(Y^L) \overset{(2.4.9)}{=} \text{E}[Y^L \mid Y^L > \pi_{0.9}(Y^L)]$. It turns out that the solution has a special mathematical structure that will be better revealed if we use general notation.

To begin with, observe that (assume that $\pi_p(X) > d$, which is the case for $p$ close to 1)

$$Y^L = (X-d)_+ > \pi_p(Y^L) \overset{(2.4.2)}{=} (\pi_p(X)-d)_+ = \pi_p(X) - d \quad \Leftrightarrow \quad X > \pi_p(X).$$

Using (2.4.9) and switching from $Y^L$ to $X$, we have

$$\begin{aligned} \text{TVaR}_p(Y^L) &= \text{E}[Y^L \mid Y^L > \pi_p(Y^L)] \\ &= \text{E}[X-d \mid X > \pi_p(X)] \\ &= \text{E}[X \mid X > \pi_p(X)] - d \\ &= \text{TVaR}_p(X) - d. \end{aligned}$$

For $p = 0.9$, $d = 500$, and $X \sim \text{Exp}(\theta = 1000)$, this equals $\text{TVaR}_{0.9}(Y^L) = -1000\ln(1-0.9) + 1000 - 500 = \boxed{2802.59}$. **(Answer: (C))** □

*Remark.* ⚠ The solution above shows that there is a simple relationship between $\text{TVaR}_p((X-d)_+)$ and $\text{TVaR}_p(X)$: Provided that $\text{TVaR}_p(X) > d$,

$$\text{TVaR}_p((X-d)_+) = \text{TVaR}_p(X) - d = (\text{TVaR}_p(X) - d)_+,$$

i.e., we can bring "TVaR" inside the $(? - d)_+$ function. However, $\text{TVaR}_p(g(X)) \neq g(\text{TVaR}_p(X))$ for a general increasing function $g$, e.g., $\text{TVaR}_p(X \wedge u) \neq \text{TVaR}_p(X) \wedge u$; see Problem 2.4.36.

**Problem 2.4.38. (TVaR of a discrete random variable – I)** Determine which of the following formulas for Tail Value at Risk must be true for a discrete random variable $X$.

I. $\text{TVaR}_p(X) = \dfrac{1}{1-p}\displaystyle\int_p^1 \text{VaR}_u(X)\,\text{d}u$

II. $\text{TVaR}_p(X) = \pi_p + \dfrac{\text{E}[(X-\pi_p)_+]}{1-p}$

III. $\text{TVaR}_p(X) = \text{E}[X \mid X > \text{VaR}_p(X)]$

(A) None (B) I and II only (C) I and III only
(D) II and III only (E) The correct answer is not given by (A), (B), (C), or (D).

*Solution.* I and II are general formulas for TVaR, while III works when the distribution function is continuous at $\pi_p$ and may fail for discrete distributions. **(Answer: (B))** □

**Problem 2.4.39. [HARDER!] (Problem 2.4.16 continued: TVaR of a discrete random variable – II)** You are given the following discrete loss distribution:

$$f_X(x) = \begin{cases} 0.80, & \text{if } x = 0, \\ 0.10, & \text{if } x = 20, \\ 0.06, & \text{if } x = 50, \\ 0.02, & \text{if } x = 80, \\ 0.02, & \text{if } x = 100. \end{cases}$$

(a) Calculate $\text{TVaR}_{0.95}(X)$.

(A) 82 (B) 84 (C) 86
(D) 88 (E) 90

*Solution.* As in Problem 2.4.16, $X$ is discrete, so (2.4.9) may not work, and we have to proceed very carefully. In Problem 2.4.16, we found that

$$\text{VaR}_p(X) = \begin{cases} 50, & \text{if } 0.95 \le p \le 0.96, \\ 80, & \text{if } 0.96 < p \le 0.98, \\ 100, & \text{if } 0.98 < p \le 1. \end{cases}$$

In other words, $\text{VaR}_p(X)$ is a piecewise constant function of $p$. Then using (2.4.6) and splitting the integral $\int_{0.95}^1 \text{VaR}_p(X)\,\mathrm{d}p$ into three pieces, we get

$$\begin{aligned} \text{TVaR}_{0.95}(X) &= \frac{1}{1-0.95}\int_{0.95}^{1} \text{VaR}_p(X)\,\mathrm{d}p \\ &= \frac{(0.96-0.95)(50)+(0.98-0.96)(80)+(1-0.98)(100)}{0.05} \\ &= \boxed{82}. \quad \textbf{(Answer: (A))} \end{aligned}$$

Alternatively, we can use (2.4.7), which works for general distributions, to get

$$\begin{aligned} \text{TVaR}_{0.95}(X) &= \pi_{0.95} + \frac{\text{E}[(X-\pi_{0.95})_+]}{1-0.95} \\ &= 50 + \frac{0.02(80-50)+0.02(100-50)}{0.05} \\ &= \boxed{82}. \quad \textbf{(Answer: (A))} \end{aligned}$$

□

*Remark.* ⚠ Option (E) is for

$$\mathrm{E}[X \mid X > \pi_{0.95}] = \mathrm{E}[X \mid X > 50] = \frac{0.02(80) + 0.02(100)}{0.04} = 90,$$

which is different from $\mathrm{TVaR}_{0.95}(X)$.

(b) Calculate $\mathrm{TVaR}_{0.96}(X)$.

(A) 82 (B) 84 (C) 86
(D) 88 (E) 90

*Solution.* Similar to part (a),

$$\begin{aligned}\mathrm{TVaR}_{0.96}(X) &\overset{(2.4.6)}{=} \frac{1}{1-0.96}\int_{0.96}^{1} \mathrm{VaR}_p(X)\,\mathrm{d}p \\ &= \frac{(0.98-0.96)(80) + (1-0.98)(100)}{0.04} \\ &= \boxed{90}. \quad \textbf{(Answer: (E))}\end{aligned}$$

Alternatively,

$$\begin{aligned}\mathrm{TVaR}_{0.96}(X) &\overset{(2.4.7)}{=} \pi_{0.96} + \frac{\mathrm{E}[(X-\pi_{0.96})_+]}{1-0.96} \\ &= 50 + \frac{0.02(80-50) + 0.02(100-50)}{0.04} \\ &= \boxed{90}. \quad \textbf{(Answer: (E))}\end{aligned}$$

□

(c) Calculate $\mathrm{TVaR}_{0.99}(X)$.

(A) 90 (B) 92 (C) 95
(D) 98 (E) 100

*Solution.* Similar to part (a),

$$\mathrm{TVaR}_{0.99}(X) \overset{(2.4.6)}{=} \frac{1}{1-0.99}\int_{0.99}^{1} \mathrm{VaR}_p(X)\,\mathrm{d}p = \frac{(1-0.99)(100)}{0.01} = \boxed{100}. \quad \textbf{(Answer: (E))}$$

Alternatively,

$$\begin{aligned}\mathrm{TVaR}_{0.99}(X) &\overset{(2.4.7)}{=} \pi_{0.99} + \frac{\mathrm{E}[(X-\pi_{0.99})_+]}{1-0.99} \\ &\overset{(X\le\pi_{0.99}=100 \text{ always})}{=} 100 + \frac{0}{0.01} \\ &= \boxed{100}. \quad \textbf{(Answer: (E))}\end{aligned}$$

□

**Problem 2.4.40. * [HARDER!] (TVaR of a discrete random variable – III)** You are given that $N$ is a zero-modified Poisson random variable with $\lambda = 2$ and $p_0^M = 0.4$.

(a) Calculate the 75% Value at Risk of $N$.

(A) 0 (B) 1 (C) 2
(D) 3 (E) 4

*Solution.* By (2.2.5), $c = \dfrac{1-p_0^M}{1-p_0} = \dfrac{1-0.4}{1-\mathrm{e}^{-2}} = 0.693911$. Then by (2.2.4),

$$\begin{aligned} p_1^M &= cp_1 = 0.693911\left(\frac{\mathrm{e}^{-2}2^1}{1!}\right) = 0.187821, \\ p_2^M &= cp_2 = 0.693911\left(\frac{\mathrm{e}^{-2}2^2}{2!}\right) = 0.187821. \end{aligned}$$

Since $F_N(1) = p_0^M + p_1^M = 0.587821 < 0.75$ and $F_N(2) = F_N(1) + p_2^M = 0.775642 > 0.75$, we have $\mathrm{VaR}_{0.75}(N) = \boxed{2}$. **(Answer: (C))** □

(b) Calculate the 75% Tail Value at Risk of $N$.

(A) 3.5 (B) 3.6 (C) 3.7
(D) 3.8 (E) 3.9

*Solution.* By (2.4.7),

$$\mathrm{TVaR}_{0.75}(N) = \mathrm{VaR}_{0.75}(N) + \frac{\mathrm{E}[(N-\mathrm{VaR}_{0.75}(N))_+]}{1-0.75} = 2 + \frac{\mathrm{E}[N]-\mathrm{E}[N\wedge 2]}{0.25}.$$

The two expected values can be determined as

$$\begin{aligned} \mathrm{E}[N] &\overset{(2.2.9)}{=} c(2) = 1.387822, \\ \mathrm{E}[N\wedge 2] &\overset{(2.3.11)}{=} p_1^M + 2(1-p_0^M-p_1^M) = 1.012179, \end{aligned}$$

so $\mathrm{TVaR}_{0.75}(N) = 2 + \dfrac{1.387822 - 1.012179}{0.25} = \boxed{3.5026}$. **(Answer: (A))** □

*Remark.* ⚠ Option (C) is for

$$\mathrm{TVaR}_{0.75}(N) = \mathrm{VaR}_{0.75}(N) + e_N(\mathrm{VaR}_{0.75}(N)) = 2 + \frac{\mathrm{E}[N]-\mathrm{E}[N\wedge 2]}{1-p_0^M-p_1^M-p_2^M} = 3.6743,$$

which does not work because $N$ is not a continuous random variable ($\mathrm{P}(N > \pi_{0.75}) \neq 0.25$).