Ambrose Lo's Study Manual for SOA Exam FAM-L

(Fundamentals of Actuarial Mathematics – Long-Term)

Ambrose Lo, PhD, FSA, CERA

Second Edition

Contents

Preface

NOTE TO STUDENTS

Please read this preface carefully, even if it looks long. It contains **VERY** important information that will help you make the most of this study manual and ease your learning.

Thank you very much for choosing to use this study manual, which is designed to provide complete coverage of Exam FAM-L (*Fundamentals of Actuarial Mathematics – Long-Term*) and prepare you (more than!) adequately for this challenging exam.

P.1 About Exam FAM-L

Exam Administrations

Exam FAM-L is a computer-based exam that lasts for 1 hour 45 minutes and consists of 20 multiple-choice questions. Offered for the first time in October 2022 by the Society of Actuaries (SOA), it is currently delivered via computer-based testing (CBT) in a Prometric exam center in March, July, and November. You can find the specifics of each testing window (e.g., exam dates, registration deadlines) at:

`https://www.soa.org/education/exam-req/exam-day-info/exam-schedules/`.

When the registration window is open, you will register online at

`https://www.soa.org/education/exam-req/registration/edu-registration/`,

receive an email confirmation letter from the SOA containing your candidate ID, then schedule an appointment at Prometric (`https://www.prometric.com/soa`).

Exam Theme: What's Exam FAM-L Like?

Here is the SOA's official web page for Exam FAM:

`https://www.soa.org/education/exam-req/edu-exam-fam/`.

Exam FAM-L, which is one half of FAM, consists of Topics 7 to 11. These five topics, along with their weight ranges and where they are covered in this study manual, are shown in the table overleaf.

Topic		Weight range	Relevant chapter(s) of this manual
7.	Insurance Coverages and Retirement Financial Security Programs	2.5-7.5%	8
8.	Mortality Models	7.5-12.5%	1, 2
9.	Parametric and Non-Parametric Estimation	2.5-7.5%	3
10.	Present Value Random Variables for Long-Term Insurance Coverages	10-15%	4, 5
11.	Premium and Policy Value Calculation for Long-Term Insurance Coverages	15-20%	6, 7
	Total	50%[1]	

From the syllabus of Exam FAM:

> "The syllabus for the long-term section of the examination develops the candidate's knowledge of the theoretical basis of *contingent payment* models and the application of those models to insurance and other financial risks."

The word "contingent" means "dependent on something random" and therefore carries uncertainty. Broadly speaking, Exam FAM-L is about the study of *contingent cash flows* **$**, especially those associated with life insurance and annuities. Traditionally called *life contingencies*, this fundamental subject in actuarial science has been on the SOA exam syllabus for long (as far as I know, there were already life contingencies exam questions in the 1940s). It tests skills for dealing with random events (because the cash flows of life insurance and annuity products are contingent on the random lifetimes of their policyholders) and the time value of money (because the cash flows arise in the future), which are the topics of Exams P and FM. Using tools from these two preliminary exams, we will set up a probabilistic framework for doing *pricing* (setting the premium charged on policyholders) and *reserving* (determining how much to set aside to fund future obligations) for common life insurance and annuity policies. For the most part, the exam topics are mathematically interesting and practically useful.

Mathematical Prerequisites

The exam syllabus says that:

> "A thorough knowledge of calculus, *probability*, mathematical statistics and *interest theory* is assumed."

I assume that you have passed Exams P and FM, and are no stranger to concepts like the mean, variance, and distribution/density/mass function of a random variable, and how to do discounting using the effective annual interest rate i vs. the force of interest δ, all of which will be intensively used in Exam FAM-L. There will be instances where you will do some simple differentiation and integration, so you are also assumed to be reasonably good at calculus. (Don't worry, integration by parts, which most students hate, is very rarely used in FAM-L).

[1]The total is 50% because FAM-L is exactly 50% of the entire FAM exam.

Historical Pass Rates and Pass Marks %

The table below shows the number of sitting candidates, number of passing candidates, pass rates, and pass marks (= the actual percentage score you have to get to pass the exam) for Exam FAM-L since it was offered in October 2022:

Sitting	# Candidates	# Passing Candidates	Pass Rate	Pass Mark
November 2023	540	359	66.5%	69.4% (I took this FAM-L! 😍)
July 2023	734	493	67.2%	73.5% (I took FAM-S in this sitting! 😅)
March 2023	1063	784	73.8%	74.3%
October 2022	1833	1373	74.9%	75.0% (I took FAM in this sitting! 😉)

Perhaps to your astonishment, the pass rates of Exam FAM-L have been anomalously high, typically close to 70%, compared to only 40-50% for other ASA-level exams. The probable reason is that FAM-L can only be taken by students who have passed Exam STAM (which is known to be a beast!) and are rather well-prepared students (they are not a novice!).

Syllabus Text

Exam FAM-L has a single textbook:

> *Actuarial Mathematics for Life Contingent Risks* (3rd Edition), 2020, by Dickson, Hardy, and Waters, Cambridge University Press, ISBN: 978-1-108-47808-3.

This book has been on the SOA exam syllabus since Spring 2012. Referred to as AMLCR in this manual, it covers the theory and applications of life contingencies with a rather practical flavor. Although I find the book a pleasant read, I streamline the presentation and sequence of some topics to make our learning more coherent and effective for exam preparation.

This manual is self-contained in the sense that reading the manual carefully is more than sufficient to pass the exam, and you need not refer to AMLCR for the purposes of learning. However, at the beginning of each chapter, I reference the relevant chapters and sections of AMLCR for the benefit of students who have access to the book and want to read more. For in-text examples and end-of-section problems that are motivated from exercises in AMLCR, the relevant exercises are properly cited to give them due credit. As the exam syllabus says,

"Exercises [in AMLCR] are considered part of the required readings."

In fact, some of the exercises in AMLCR were the theme of some recent exam questions and may continue to be so in future exams. ⚠

Exam Tables and the Standard Normal Calculator

During the exam, you will have access to a few tables (collectively called the "FAM-L tables" in the rest of this manual), which are available on the Prometric computers. Useful, if not necessary for solving some of the exam questions, these tables save your memory burden and include:

- The Standard Ultimate Life Table (to be introduced in Section 2.1)
- A table of various quantities corresponding to the effective interest rate $i = 0.05$

- Selected formulas which are harder to remember

Here is the link to download these tables, which can be found on the last page of the exam syllabus:

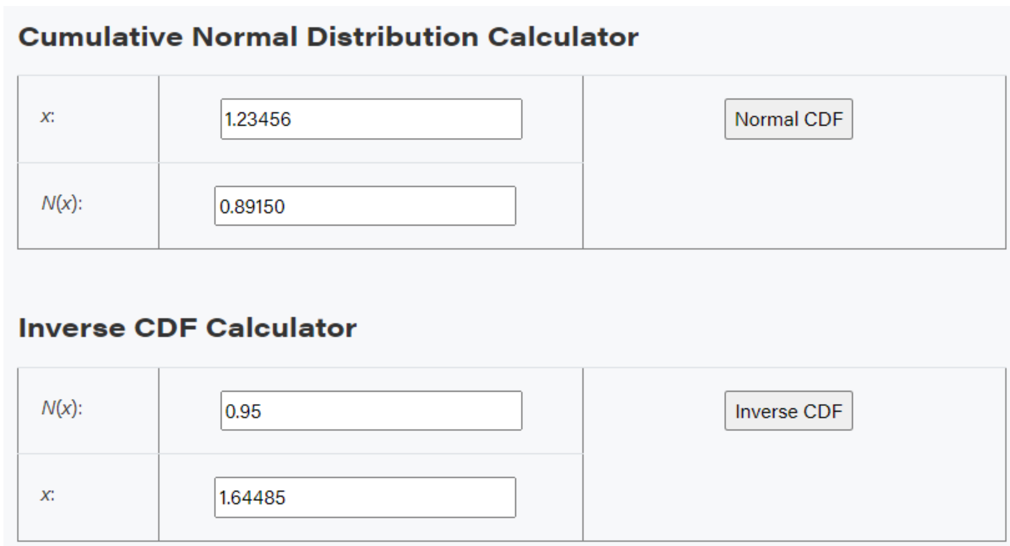
`www.soa.org/4a1b80/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-tables.pdf`

I strongly suggest that you print a copy of these tables and formulas as they will be used intensively as you work through this study manual and when you take the real exam.

Although page 5 of the FAM-L tables is a table of values for the standard normal distribution, the CBT environment also includes *Prometric's standard normal calculator*, accessible from

`https://prometric.com/soa`.

Here is a screenshot of the calculator:



The calculator has two parts:

- In the upper part of the table, you can get the value of the standard normal distribution function $\Phi(x)$ to 5 decimal places, e.g., $\Phi(1.23456) = 0.89150$ above. The argument x can be any real value (positive or negative) and expressed in whatever precision you like. In the rest of this manual, I will input the argument to 5 decimal places, consistent with the output.
- The lower part is for the *percentiles* of the standard normal distribution, which are the values that lead to different prescribed values of the distribution function, e.g., the 95th percentile is 1.64485, as shown above. (You may have used the less accurate value of 1.645 in your earlier studies.)

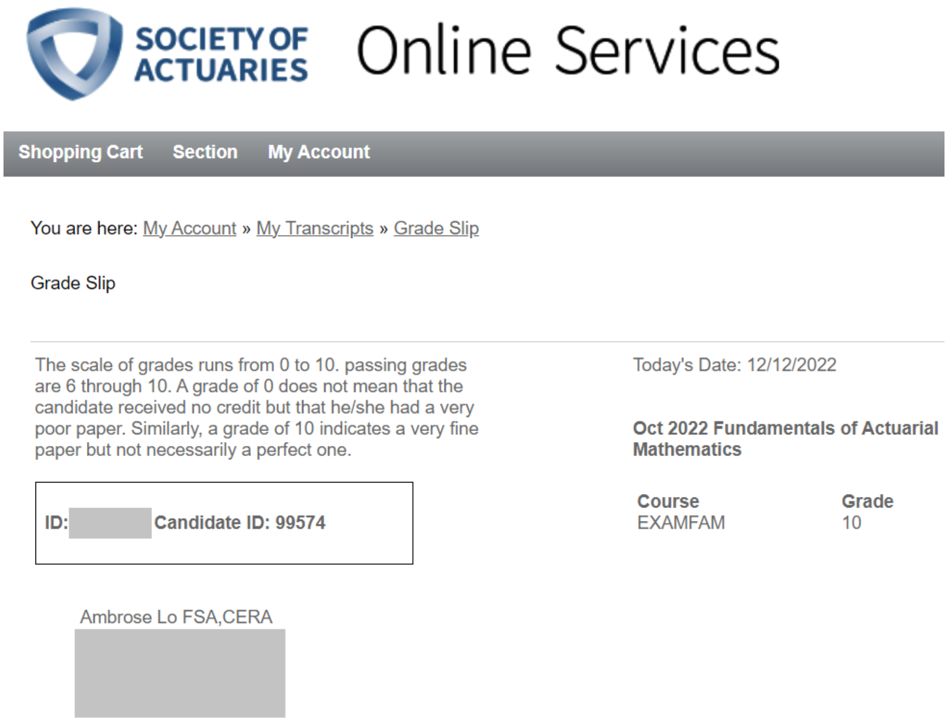
P.2 About this Study Manual

What is Special about This Study Manual?

I fully understand that you have an acutely limited amount of study time and that the syllabus of Exam FAM-L, even only as one half of the full FAM exam, is still enormous. With this in mind, the overriding objective of this study manual is to help you grasp the material in Exam FAM-L as effectively and efficiently as possible, so that you will pass the exam on your first try easily and go on to more advanced exams like (AT)PA confidently. Here are some unique features of this manual to make this possible.

Feature 1: The Coach DID Play!

Usually coaches don't play 😆, but as a study manual author, I took the initiative to write the **October 2022 Exam FAM**, the **July 2023 Exam FAM-S**, and the **November 2023 Exam FAM-L** to experience first-hand what the real exams were like, despite having been an FSA since 2013 (and technically free from SOA exams thereafter!). I made this decision in the belief that *teaching* an exam and *taking* an exam are rather different activities, and braving the exam myself is the best way to ensure that my manual is indeed useful for exam preparation. (If the manual is useful, then at the minimum the author himself can do well, right?)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The scale of grades runs from 0 to 10. passing grades are 6 through 10. A grade of 0 does not mean that the candidate received no credit but that he/she had a very poor paper. Similarly, a grade of 10 indicates a very fine paper but not necessarily a perfect one.

ID: **Candidate ID: 99574**

Ambrose Lo FSA,CERA

Today's Date: 12/12/2022

Oct 2022 Fundamentals of Actuarial Mathematics

Course	Grade
EXAMFAM	10


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Grade Slip

The scale of grades runs from 0 to 10. passing grades are 6 through 10. A grade of 0 does not mean that the candidate received no credit but that he/she had a very poor paper. Similarly, a grade of 10 indicates a very fine paper but not necessarily a perfect one.

ID: **Candidate ID: 67888**

Ambrose Lo FSA,CERA

Today's Date: 9/11/2023

Jul 2023 Fundamentals of Actuarial Mathematics Short Term

Course	Grade
EXAMFAMS	10


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Grade Slip

The scale of grades runs from 0 to 10. passing grades are 6 through 10. A grade of 0 does not mean that the candidate received no credit but that he/she had a very poor paper. Similarly, a grade of 10 indicates a very fine paper but not necessarily a perfect one.

Today's Date: 1/29/2024

Nov 2023 Fundamentals of Actuarial Mathematics Long Term

ID: **Candidate ID: 17433**

Course	Grade
EXAMFAML	10

Ambrose Lo FSA,CERA

If you use this study manual, you can rest assured that it is written from an exam taker's perspective by a professional instructor who has experienced the "pain" of FAM(-L and -S) candidates and truly understands their needs. Drawing upon his "real battle experience" and firm grasp of the exam topics, the author will go to great lengths to help you prepare for this challenging exam in the best possible way. You are in capable hands.

Feature 2: Exam-focused Content

To maximize your learning effectiveness and efficiency, I have divided this study manual into two parts:

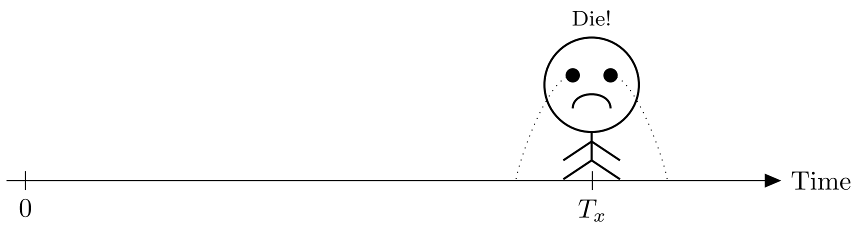
- **Part I: Core (Chapters 1 to 8)**
  - ▷ *(Let's start with the syllabus)* Each chapter/section starts by explicitly stating which learning objectives and outcomes of the FAM-L exam syllabus we are going to study, to assure you that we are on track and hitting the right target.
  - ▷ *(In-text explanations)* The explanations in each chapter are thorough, but **exam-focused** and **learning-oriented**. Besides having a coherent narrative flow that shows the connections ⇄ between different exam topics, this manual strives to keep you motivated by showing how the concepts are typically tested, how the formulas are used, and where the exam focus lies in each section. Instead of showing unnecessary mathematical proofs that add little value to exam preparation, I grasp the chance to explain the intuitive meaning and mathematical structure of various formulas in the syllabus, to help you better remember and apply these formulas on the exam.
  - ▷ *(In-text illustrative examples)* The main text of the manual is interspersed with carefully chosen past **SOA/CAS exam questions**, with full bibliographic details given (name of exam, year of examination, question number). Complementing the in-text explanations, these illustrative examples are meant to show you how the concepts you have just learned are usually tested and are an essential part of your reading.

- *(Boxed formulas and exam notes)* Formulas that you will use all the time are $\boxed{\text{boxed}}$ for easy identification and retention, and numbered (in the (X.X.X) format) for later references. Important exam items and common mistakes committed by students are highlighted by boxes that look like:

  EXAM NOTE

  Be sure to pay special attention to boxes like this!

- *(Timeline diagrams)* Timeline diagrams are an indispensable tool for illustrating cash flows and formulas in life contingencies. In the main text of this manual (especially Chapters 4 and 5), you will see the timeline diagrams for various life insurance and annuity products. The cute guy below will sacrifice himself and die multiple times to help us understand life contingencies!



- *(End-of-section problems)* Besides the **234 in-text examples**, this study manual features **752 end-of-section/chapter practice problems**. In particular, all FAM-L sample questions have been included in this manual. Here are the links to these sample questions and solutions in case you want to look at the original source:

  `https://www.soa.org/4a1b79/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-quest.pdf`

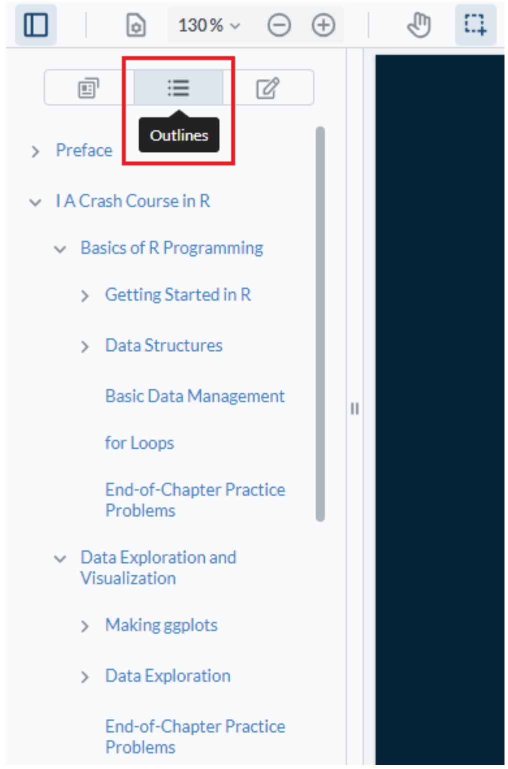
  `https://www.soa.org/4a1b83/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-sol.pdf`

  Designed to reinforce what you have learned in the main text of the manual and provide additional opportunities, these practice problems are either real exam problems taken/adapted from relevant SOA/CAS past exams, or are original problems intended to illustrate less commonly tested items, all with step-by-step solutions (and many with problem-solving remarks). The harder ones are labeled as [**HARDER!**].

  While almost all exam prep resources boast a large number of exercises, the arrangement of this manual is unique in the following aspects:

  - The practice problems, including the original ones, are in the same multiple-choice format as a real FAM-L exam question, with (A), (B), (C), (D), and (E). As you can see in my solutions, many of the wrong answers correspond to some common mistakes that students make (I tried to put myself in students' shoes), which is also the case in the real exam. I hope you won't pick those distractors! ☺

- ☐ To ease exam preparation, the practice problems are categorized by **theme** rather than by year, and you can navigate different themes using the "Outlines" tab on *Actuarial University*. Here is a screenshot:



  Within each theme, problems testing similar items are grouped, so even a cursory glance at these problems will show you how those topics are typically tested and exam items that repeatedly emerge. You will find that many SOA questions are simple variations of some older exam questions, which shows that doing past exam questions is really important.

- **Part II: Final preparation**

  The second part concludes this comprehensive manual with the following resources:

  ▷ Four (4) original full-length **practice exams** designed to mimic the real FAM-L exam in terms of coverage and style immediately follow the core of this study manual. These practice exams give you a holistic review of the syllabus material and assess your readiness to take and pass the real exam. Detailed illustrative solutions are provided for each exam.

  ▷ Appendix A of the manual is a 27-page **cheat sheet** (the exam syllabus is enormous!) that provides a "helicopter" view of the entire FAM-L exam, and is useful for both regular review and last-minute exam preparation. A downloadable and printable version of the cheat sheet is available on my personal web page.

A Study Guide: How to Use This Manual?

Having taught in a CAE university for about 10 years and got in touch with thousands of actuarial students, I have come up with the following study tips (which are useful even beyond FAM-L) and suggestions on how to make the most of this manual:

Step 1. Read the main text of each section of the manual carefully, including *all* of the in-text examples.

When you work out the in-text examples, be sure to get a paper, do the math, and try to replicate my solutions (I mean it!). For any actuarial exams,

> keep in mind that it is not enough to be able to do problems; you need to do problems *accurately* and *efficiently*,

and the "learning by doing" approach above is the best way to help you absorb the material effectively and solve problems quickly. You need to get your hands dirty!

Step 2. Use the cheat sheet in Appendix A (and available on my personal web page) to look back on the important formulas and results in the section you have just read. This is an important way to reinforce your learning.

Step 3. Proceed to the end-of-section practice problems.

To succeed in any actuarial exam, I can't overemphasize the importance of practicing a wide variety of exam-type problems to consolidate your understanding and develop proficiency. This is the only path to true mastery of the material. However, life contingencies is a classical subject in SOA/CAS exams, which explains the abundance of old exam problems included in this manual (some of them date back to early 1980s...before I was born!). There are so many that it would be virtually impossible (neither is it desirable) for a student to work out all of them—you would be overwhelmed!

To maximize the effectiveness and efficiency of your learning,

> I have marked the most representative and instructive practice problems in each section with an **asterisk (*)** and recommend that you do **ALL** of these **asterisked** problems in your first round of reading.

These selected problems, which are generally not more than 50% of the whole set of problems, span different themes and will add most value to your learning. Focusing on these problems should be a good learning strategy for developing a level of proficiency and confidence necessary for exam taking, while avoiding burn-out. As with the in-text examples, try to get a paper, do the math, write your own solutions, and compare yours with mine.

Step 4. Repeat Steps 1 to 3 until you finish all of the eight chapters in Part I (core) of the manual. For FAM-L, the whole process can take **THREE months** or more!

Step 5. (This step is optional.) If the real exam is more than two weeks away from today, then consider doing a small random sample of the non-asterisked end-of-section problems, especially those related to your weak spots; otherwise, go on to Step 6.

Step 6. Move on to Part II of the manual and attempt the practice exams; see the prelude to these exams on page 933 for more information.

If you have the perseverance and discipline to follow this study strategy closely, I have every confidence that you will pass the exam easily (the only question being whether you will get Grade 9 or 10! ☺).

Announcements about this Manual

As time goes by, I may post news and announcements about this study manual and Exam FAM-L on my personal web page:

`https://sites.google.com/site/ambroseloyp/publications/FAM`.

An errata list will also be maintained. I would greatly appreciate it if you could bring any potential errors, typographical or otherwise, to my attention via email (see below) so that they can be fixed in a future edition of the manual.

Contact Us

If you encounter problems with your learning, we stand ready to help.

- For **technical issues** (e.g., not able to access the manual on Actuarial University, extending your digital license, upgrading your product to include printing), please email ACTEX Learning's Customer Service at `support@actexlearning.com`. The list of FAQs available on `https://www.actuarialuniversity.com/help/faq` may also be useful.
- Questions related to **specific contents** of this manual, including potential errors, can be directed to me (Ambrose) by emailing `amblo201011@gmail.com`. Please note:
  - ▷ Remember to check out the errata list on my personal web page. It may happen that the errors you discover have already been addressed.
  - ▷ Please identify the specific page(s) of the manual your questions are about. This will provide a concrete context and make our discussion much more fruitful.

⚠ NOTE ⚠

- To expedite the resolution process, it would be greatly appreciated if you could reach out to the appropriate email address. ☺
- I will strive to get back to you ASAP. ↩ Please check your spam folder if you don't hear back from me within 2-3 days.

Acknowledgments

I would like to thank the SOA for kindly allowing me to reproduce its past and sample exam problems, for which it owns the sole copyright. These problems have been invaluable in illustrating a number of concepts in the FAM-L exam syllabus. I am also grateful to students in my ACTS:4130 (*Quantitative Methods for Actuaries*) class at The University of Iowa in Fall 2019-2022 for class testing earlier versions of this study manual.

About the Author

Ambrose Lo, PhD, FSA, CERA, was formerly Associate Professor of Actuarial Science with tenure at the Department of Statistics and Actuarial Science, The University of Iowa. He earned his B.S. in Actuarial Science (first class honors) and PhD in Actuarial Science from The University of Hong Kong in 2010 and 2014, respectively, and attained his Fellowship of the Society of Actuaries (FSA) in 2013. He joined The University of Iowa as Assistant Professor of Actuarial Science in August 2014, and was tenured and promoted to Associate Professor in July 2019. His research interests lie in dependence structures, quantitative risk management as well as optimal (re)insurance. His research papers have been published in top-tier actuarial journals, such as *ASTIN Bulletin: The Journal of the International Actuarial Association*, *Insurance: Mathematics and Economics*, and *Scandinavian Actuarial Journal*.

Besides dedicating himself to actuarial research, Ambrose attaches equal importance to teaching and education, through which he nurtures the next generation of actuaries and serves the actuarial profession. He has taught courses on financial derivatives, mathematical finance, life contingencies, and statistics for risk modeling. He is the (co)author of the *ACTEX Study Manuals for Exams ATPA, MAS-I, MAS-II, PA, and SRM*, a Study Manual for Exam FAM, and the textbook *Derivative Pricing: A Problem-Based Primer* (2018) published by Chapman & Hall/CRC Press. Although helping students pass actuarial exams is an important goal of his teaching, inculcating students with a thorough understanding of the subject and concrete problem-solving skills is always his top priority. In recognition of his outstanding teaching, Ambrose has received a number of awards and honors ever since he was a graduate student, including the 2012 Excellent Teaching Assistant Award from the Faculty of Science, The University of Hong Kong, public recognition in the Daily Iowan as a faculty member “making a positive difference in students’ lives during their time at The University of Iowa” for eight years in a row (2016 to 2023), and the 2019-2020 Collegiate Teaching Award from the College of Liberal Arts and Sciences, The University of Iowa.

Part I

Core

Prelude to FAM-L

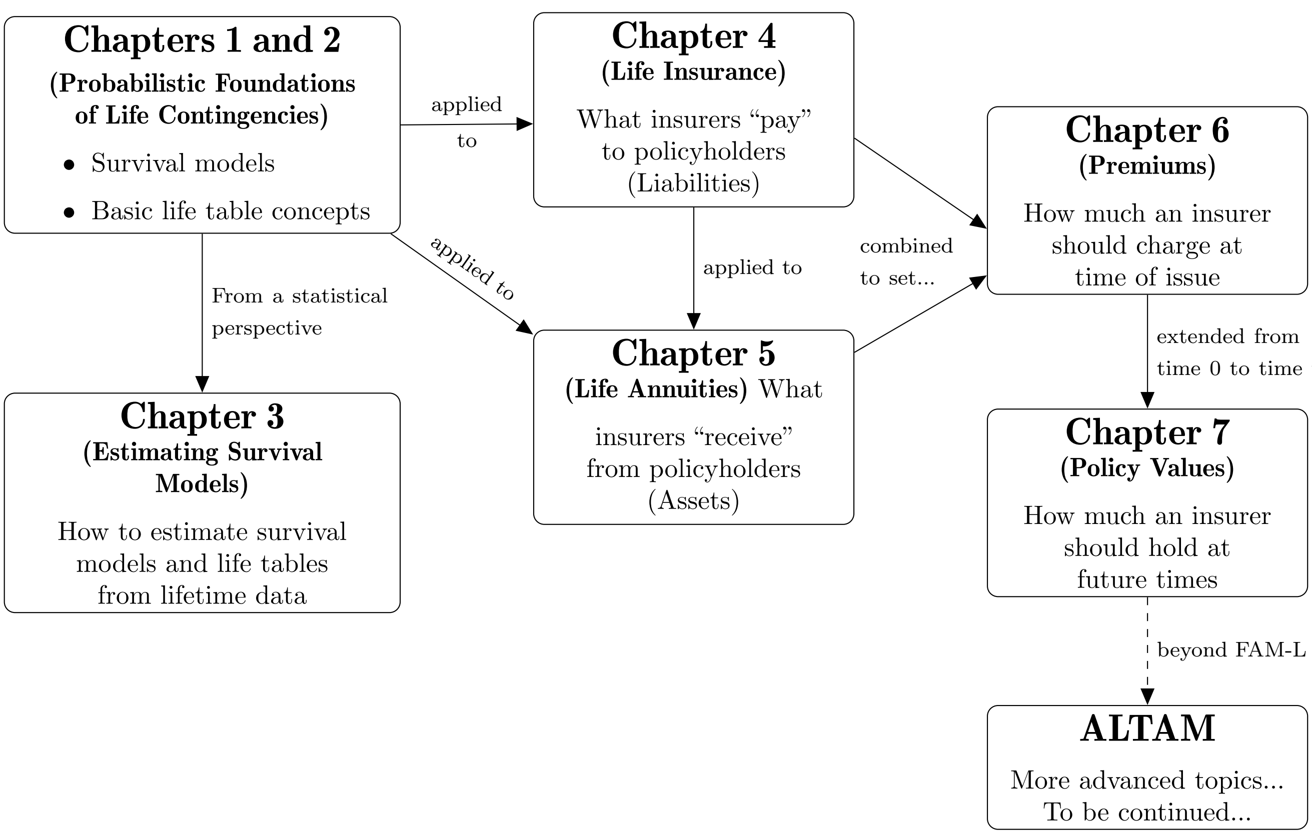
As I said in the preface, Exam FAM-L is mainly about *life contingencies*, which is largely a generalization of Exam FM from the study of certain (definite) cash flows to the study of *contingent* cash flows, with the contingency stemming from the random lifetime of a person—we can't tell how long a person is going to survive with absolute certainty (an actuary is not a fortune teller). The randomness matters a lot when it comes to the pricing and management of virtually all insurance and annuity products in practice.

As a simple example, imagine that you are going to receive \$1,000,000 in 20 years. What is the fair value of this amount today?

- A student who has passed Exam FM with grade 10 would take the time value of money into account and compute the present value $1,000,000v^{20}$, where v^{20} is the 20-year discount factor, implicitly assuming that the receipt of the \$1,000,000 payment in 20 years is certain.

- In Exam FAM-L, you will refine this present value and consider both the time value of money and the *contingency* or *randomness* of receiving the payment—you have to be alive after 20 years (hopefully so!) to receive \$1,000,000. Quantifying the randomness by a probability distribution of your future lifetime, you will calculate the "actuarial" present value $1,000,000v^{20}{}_{20}p_x$, where ${}_{20}p_x$ is a symbol denoting the probability that you survive 20 years from now. You will learn the details in Chapters 1 and 4.

We will get to the details of the five exam topics in FAM-L when we get to specific chapters of this manual. For now, the following flowchart shows you the "big picture" and how different chapters relate to one another:



Having taught life contingencies over the years, I have put together the following learning tips that will help you study for FAM-L more effectively (and enjoyably!):

- *Things are inter-connected* ⇄

  As you can see from the flowchart, a salient feature of FAM-L is that the material closely builds on itself and makes up a coherent "story." The concepts and formulas in each chapter (with the exception of Chapter 3, which is largely a statistics chapter) will be used again and again in later chapters, so make sure that you spend time developing a firm grasp of the material of each chapter and appreciating the logical development, or problems will just snowball and you will get lost.

- *Try to understand and interpret, not memorize*

  Another distinguishing feature of FAM-L is that you will see "numerous" symbols and formulas, so many that you are bound to feel overwhelmed at some point. Instead of memorizing the formulas by heart, you would be much better off trying to understand what the formulas mean by general reasoning. As this study manual shows, there are ideas and concepts behind most formulas in life contingencies. As soon as you understand the verbal interpretations, which this manual goes to great lengths to foster, many formulas become much more intuitive. In brief:

  Interpretation is the key to mastering and applying formulas in life contingencies!

- *Drilling problems*

  Although most concepts in life contingencies (from the right perspective) are quite simple, exam questions are not. As you will see, many life contingencies problems assess your understanding in multiple parts of the exam syllabus and come with tricks. To succeed in the exam, it is essential that you work out a wide range of exam-type problems to develop the level of understanding and proficiency necessary for exam taking.

Let's get started.

Chapter 1

Survival Models

FROM THE FAM-L EXAM SYLLABUS

8. Topic: Mortality Models (7.5-12.5%)

Learning Objectives

The Candidate will understand key concepts concerning parametric and non-parametric mortality models for individual lives.

Learning Outcomes

The Candidate will be able to:

a) Understand parametric survival models, life tables, and the relationships between them.

b) Given a parametric survival model, calculate survival and mortality probabilities, the force of mortality function, and moments of the curtate and complete future lifetime random variable.

c) Identify and apply standard actuarial notation for future lifetime distributions and moments, including select and ultimate functions.

OPTIONAL SYLLABUS READINGS

AMLCR, Chapter 2

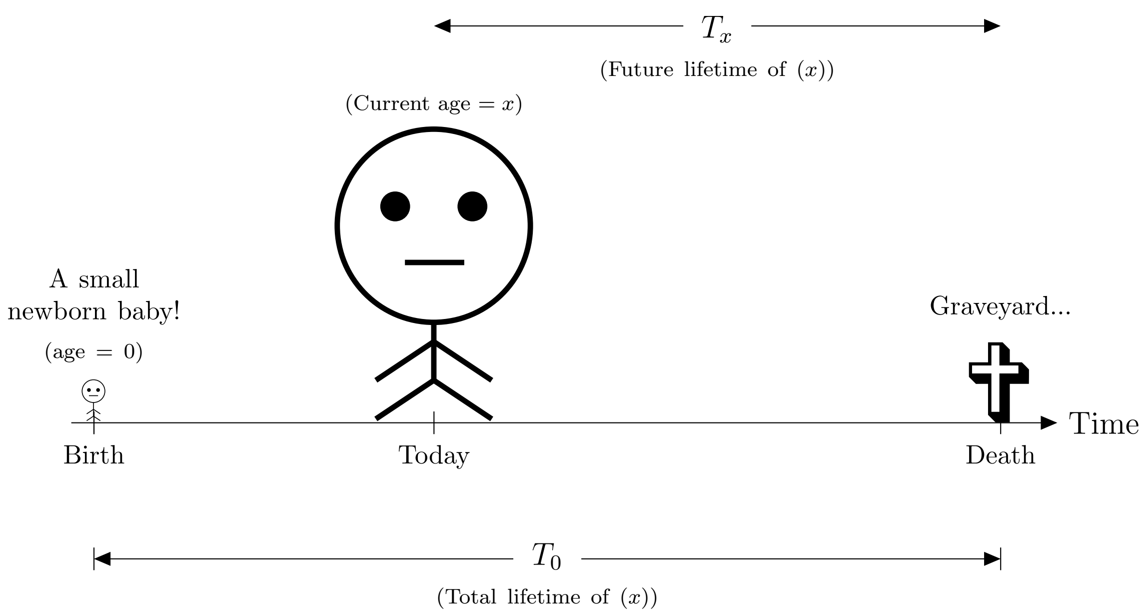
Chapter overview: This introductory chapter marks the beginning of our study of life contingencies. Treating the future lifetime of an individual as a random variable, we study its distributional quantities, including probabilities and moments. In Section 1.1, we formally define the future lifetime random variable and learn how to manipulate its probabilities. One of the highlights in this section is how to use actuarial notation to represent different kinds of probabilities compactly. Section 1.2 introduces the force of mortality, a distributional quantity that conveniently describes and characterizes a survival distribution. Moments associated with the future lifetime random variable are covered in Section 1.3, where we will see how to calculate different types of moments in terms of survival probabilities. In Section 1.4, we devote our attention to several mortality laws that either are mathematically tractable or provide a reasonably good fit to human mortality. Finally, Section 1.5 develops the all-important tool of recursive formulas, which are useful for both practical and conceptual reasons, and will accompany you throughout your study of life contingencies.

EXAM NOTE

- Even though this is just the first chapter in this study manual, the chances are that you will find the material here not entirely straightforward and you may even feel overwhelmed at some point (if not, you are really good! ☺). Life contingencies does have a rather steep learning curve and is a subject that "haunts" generations of actuaries. Nevertheless, it is absolutely essential that you master this chapter. As I mentioned in the preface, things in life contingencies are **inter-dependent** and everything in the rest of this manual closely builds on previous material.
- As Topic 8 carries a weight of 7.5-12.5% in the entire FAM exam, you can expect to see $40(7.5\%)$ to $40(12.5\%)$, or **3 to 5 questions** set on this and the next chapter combined.

1.1 The Future Lifetime Random Variable

The basic building block in life contingencies. Throughout our study of life contingencies, we use the symbol (x) to represent a life currently aged x. In other words, (1) refers to a 1-year-old baby, (20) represents a 20-year-old youngster, and Ambrose's name in the life contingencies arena is (3$\underline{x}$) (not young anymore!). Corresponding to (x), we model their future lifetime by a positive continuous random variable[1] denoted by T_x, where "T" stands for "$\underline{t}$ime" and the subscript indicates the current age of the life. When we sum the current age and the future lifetime of (x), we see that $x + T_x$ is the age at death, or total lifetime T_0, of the individual. The figure below provides a pictorial illustration.



[1]It makes sense to think of T_x as a random variable because the exact time of death of (x) is unknown or "random." A survival model for T_x allows us to quantify the uncertainty in its behavior using tools in probability theory.

Note that every time we speak of (x), we implicitly assume[2] that this individual is alive at age x. With this implicit assumption, we can define the future lifetime T_x in terms of the total lifetime of the same person via[3]

$$\boxed{T_x := T_0 - x \mid T_0 > x,} \tag{1.1.1}$$

where the vertical bar is the same as the one in the conditional probability notation $\mathrm{P}(A \mid B)$, meaning "conditional on" or "given that."

Much of this chapter is to study the probability distribution of the future lifetime random variable. Our focus in this section is on different kinds of probabilities associated with the future lifetime. As we will see in Chapters 4 and 5, these probabilities are the building blocks of the valuation formulas for virtually all life insurance and annuity policies.

Connections between T_x's with different x's. To study the distribution of T_x, we can investigate its *distribution function* and *survival function*, denoted respectively by

$$F_x(t) := \mathrm{P}(T_x \le t) \quad \text{and} \quad S_x(t) := \mathrm{P}(T_x > t),$$

where the subscript "x" indicates that the two functions are defined for a life currently aged x and t is the argument of the function. By definition, $F_x(t)$ is the probability that (x) dies in the coming t years, before reaching age $x+t$, and $S_x(t)$ is the probability that (x) survives at least t years and reaches age $x+t$.

As we learned in Exam P, $S_x(t)$, as a survival function for a non-negative random variable, satisfies the following three necessary and sufficient conditions:

(1) $S_x(0) = 1$

(*Meaning:* Future lifetime must be positive.)

(2) $\lim_{t\to\infty} S_x(t) = 0$

(*Meaning:* everyone eventually dies...no human being is immortal!)

(3) $S_x(t)$ is a decreasing (more precisely, non-increasing) function of t, i.e., $S_x(u) \ge S_x(v)$ if $u \le v$.

(*Meaning:* The probability of surviving a shorter duration must be greater than or equal to the probability of surviving a longer duration.)

We will see a number of concrete survival distributions fulfilling these conditions in the rest of this study manual. For now, our task is to turn F_x and S_x into some useful expressions for the purposes of computations. In doing so, we also explore the distributional relationships among future lifetimes of different current ages.

- *Relating T_x to T_0 for any $x \ge 0$:* According to (1.1.1),

$$S_x(t) = \mathrm{P}(T_x > t) = \mathrm{P}(T_0 - x > t \mid \underbrace{T_0 > x}_{\substack{T_x \text{ defined only} \\ \text{in this event}}}) = \frac{\mathrm{P}(T_0 > x+t)}{\mathrm{P}(T_0 > x)} = \boxed{\frac{S_0(x+t)}{S_0(x)}}, \tag{1.1.2}$$

[2] If the individual dies before reaching age x, then there is no point in studying their future lifetime beyond age x...(x) is a ghost! 👻

[3] Throughout this study manual, the symbol ":=" means "is defined as."

where the third equality uses the definition of the conditional probability

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \text{ and } B)}{\mathrm{P}(B)} \quad \text{for } \mathrm{P}(B) > 0.$$

The usefulness of (1.1.2) is that it allows us to calculate the survival probabilities of T_x for any age x using those of the age-at-death random variable T_0, i.e., we can go from $S_0(\cdot)$ to $S_x(\cdot)$.

Example 1.1.1. (SOA Exam FAM-L Sample Question 2.7 / Exam MLC Fall 2016 Multiple-Choice Question 2: Going from $S_0(\cdot)$ to $S_x(\cdot)$) You are given the following survival function of a newborn:

$$S_0(x) = \begin{cases} 1 - x/250, & 0 \le x < 40 \\ 1 - (x/100)^2, & 40 \le x \le 100 \end{cases}$$

Calculate the probability that (30) dies within the next 20 years.

(A) 0.13	(B) 0.15	(C) 0.17
(D) 0.19	(E) 0.21	

Solution. By (1.1.2), the probability that (30) survives the next 20 years is

$$S_{30}(20) = \frac{S_0(50)}{S_0(30)} = \frac{1 - (50/100)^2}{1 - 30/250} = \frac{75}{88},$$

where the denominator $S_0(30)$ is from the first expression of $S_0(x)$ given above and the numerator $S_0(50)$ is from the second expression. The complement of this probability is the probability that (30) dies within the next 20 years:

$$F_{30}(20) = 1 - S_{30}(20) = 1 - \frac{75}{88} = \frac{13}{88} = \boxed{0.1477}. \quad \textbf{(Answer: (B))}$$

□

- *Relating T_{x+t} to T_x for any $x \ge 0$ and $t \ge 0$:* For any non-negative t and u, applying (1.1.2) three times gives

$$S_{x+t}(u) \overset{(1.1.2)}{=} \frac{S_0(x+t+u)}{S_0(x+t)} = \frac{S_0(x+t+u)/S_0(x)}{S_0(x+t)/S_0(x)} \overset{(1.1.2)\times 2}{=} \frac{S_x(t+u)}{S_x(t)},$$

which generalizes (1.1.2) in the sense that we can calculate the survival probabilities of a future lifetime (T_{x+t}) from those of a future lifetime at *any earlier age* (T_x), not necessarily age 0.

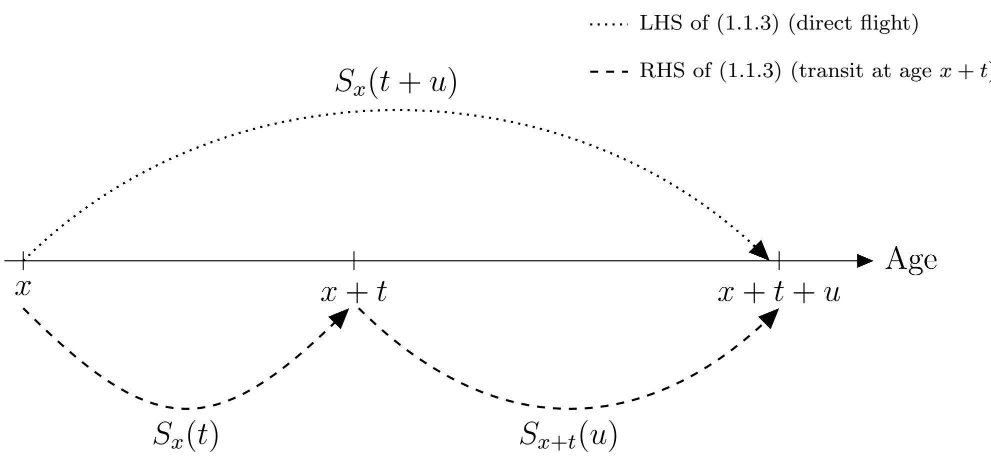
The formula above is often represented in the following "factorization" form:

$$\boxed{S_x(t+u) = S_x(t)S_{x+t}(u),} \tag{1.1.3}$$

We can interpret this formula conveniently this way:

The LHS[4] is the probability that (x) survives to age $x+t+u$, while the RHS captures the probability of the same event in a two-step sequence. In the first step, (x) survives for t years to age $x+t$, with a probability of $S_x(t)$. In the second step, the life, now aged $x+t$, further survives u more years to age $x+t+u$, with a probability of $S_{x+t}(u)$. What (1.1.3) says is that the probability of taking a "direct flight" from age x to age $x+t+u$ is the same as the probability of taking a "transit" at age $x+t$, then further "traveling" to age $x+t+u$.

The figure below visualizes the equivalence of the two ways to move from age x to age $x+t+u$. Exam FAM, as a multiple-choice exam, will not directly test interpretations, but keeping the diagram in mind can help you remember and apply the factorization formula much more easily on the exam!



[IMPORTANT!] Actuarial notation. The notation, $F_x(t)$ and $S_x(t)$, is commonly used in probability and statistics. In actuarial science, there is a separate set of notation, known as *International Actuarial Notation* (IAN), dedicated to representing many actuarial quantities of interest. Symbols under IAN usually take the form of a base symbol decorated by subscripts, pre-subscripts, superscripts, and/or pre-superscripts indicating various sorts of information: (Actuaries make good use of spaces!)

$${}^{\text{pre-superscript}}_{\text{pre-subscript}}\text{base symbol}^{\text{superscript}}_{\text{subscript}}.$$

In Exam FAM-L, you will come across numerous actuarial symbols (more will come in Exam ALTAM, if you choose to take it!). This section introduces the three most fundamental symbols, all of which denote certain probabilities of the future lifetime random variable:

(1) *Survival probability:* ${}_tp_x := S_x(t) = \mathrm{P}(T_x > t)$, as we discussed above, is the probability that (x) survives t years to age $x+t$. The subscript x denotes the current age of the life of interest, the pre-subscript t identifies the duration of survival, and the letter "p" suggests survival. When $t=1$, we often suppress the pre-subscript t and simply write p_x (instead of ${}_1p_x$).

In actuarial notation, (1.1.3) can be rewritten in the factorized form

$$\boxed{{}_{t+u}p_x = {}_tp_x \times {}_up_{x+t}.} \tag{1.1.4}$$

[4]Throughout this study manual, LHS (resp. RHS) stands for "left-hand side" (resp. "right-hand side").

This decomposition formula, which connects three p's with different ages and lengths of survival, will be used intensively in your entire study in Exam FAM-L. Applied inductively, (1.1.4) takes the following special but important form in terms of one-year survival probabilities:

$$\boxed{{}_np_x = \underbrace{p_x\, p_{x+1} \cdots p_{x+n-1}}_{n \text{ 1-year survival probabilities}} \quad \text{for any positive integer } n.} \tag{1.1.5}$$

This special form is most commonly used when mortality tables (to be introduced in Chapter 2) are provided in an exam question.

Example 1.1.2. (SOA Exam LTAM Fall 2019 Multiple-Choice Question 1: Going from ${}_?p_x$ to ${}_?p_{x+t}$) You are given that ${}_tp_{40} = \exp(-0.06(1.12^t - 1))$ for all $t \geq 0$.

Calculate the probability that a life age 45 survives 10 years.

(A) 0.80 (B) 0.83 (C) 0.86

(D) 0.90 (E) 0.93

Solution. Note that the given survival function is for age 40, but we are interested in a life of a different age, which is 45. This is where factorization formula (1.1.4) comes in useful. Using (1.1.4) with $x = 40$, $t = 5$, $u = 10$, we relate ${}_?p_{40}$ and ${}_?p_{45}$ via

$${}_{10}p_{45} = \frac{{}_{15}p_{40}}{{}_5p_{40}} = \frac{\exp[-0.06(1.12^{15} - 1)]}{\exp[-0.06(1.12^5 - 1)]} = \boxed{0.8004}. \quad \textbf{(Answer: (A))}$$

□

Remark. The given survival distribution is a Gompertz distribution, which will be covered in Subsection 1.4.4.

(2) *Mortality probability:* ${}_tq_x := F_x(t) = 1 - {}_tp_x$ is the probability that (x) dies within the coming t years. The letter "q" generally denotes mortality in life contingencies. Like ${}_tp_x$, we write q_x to mean ${}_tq_x$ in the special case when $t = 1$. We often call the 1-year mortality probability q_x the *mortality rate* at age x. We will see in Chapter 2 that mortality rates play an important role in life tables.

⚠ WARNING ⚠

While survival probabilities satisfy factorization formula (1.1.4), mortality probabilities do not. In other words,

$${}_{t+u}q_x \neq {}_tq_x \times {}_uq_{x+t} \quad !$$

Example 1.1.3. (Going from mortality rates to $_nq_x$) You are given the following mortality rates:

x	q_x
50	0.20
51	0.22
52	0.25
53	0.30

Calculate $_3q_{50}$.

(A) 0.01 (B) 0.15 (C) 0.36
(D) 0.53 (E) 0.67

Solution. Although $_nq_x$ cannot be factorized into one-year pieces, $_np_x$ can be, so let's apply factorization formula (1.1.5) with $x = 50$ and $n = 3$ to get

$$_3p_{50} = p_{50}p_{51}p_{52} = (1 - 0.20)(1 - 0.22)(1 - 0.25) = 0.468.$$

The complement of $_3p_{50}$ is $_3q_{50}$, so $_3q_{50} = 1 - 0.468 = \boxed{0.532}$. **(Answer: (D))** □

Remark. ⚠ Here are possible reasons for getting some of the wrong answers: (Avoid making these mistakes on the real exam!)

(A) It corresponds to $q_{50}q_{51}q_{52} = 0.011$.

(E) It is for $_4q_{50} = 1 - {_4p_{50}} = 1 - (1 - q_{50})(1 - q_{51})(1 - q_{52})(1 - q_{53}) = 0.6724$. (In fact, the value of q_{53} is not needed for solving this example.)

(3) *Deferred mortality probability:* As a generalization of $_tq_x$, the *deferred mortality probability* $_{u|t}q_x := \mathrm{P}(u < T_x \le u + t)$[5] is the probability that (x) survives u years, then dies in the following t years. In other words, it is the probability that (x) dies between age $x + u$ and age $x + u + t$. The fact that death takes place following a deferred period of u years justifies why we call it a "deferred" mortality probability. Of course, if $u = 0$ (i.e., the deferred period vanishes), then $_{u|t}q_x = {_tq_x}$, which is simply a mortality probability. If $t = 1$, we again omit it and write $_{u|}q_x$. However, even if $u = 1$, it is still written, so you may see a symbol like $_{1|}q_x$, which is the probability that (x) dies between ages $x + 1$ and $x + 2$.

Deferred mortality probabilities are the most complex among the three types of probability in this section. There are three different ways to express $_{u|t}q_x$. The three formulas are algebraically equivalent, but the interpretations are different.

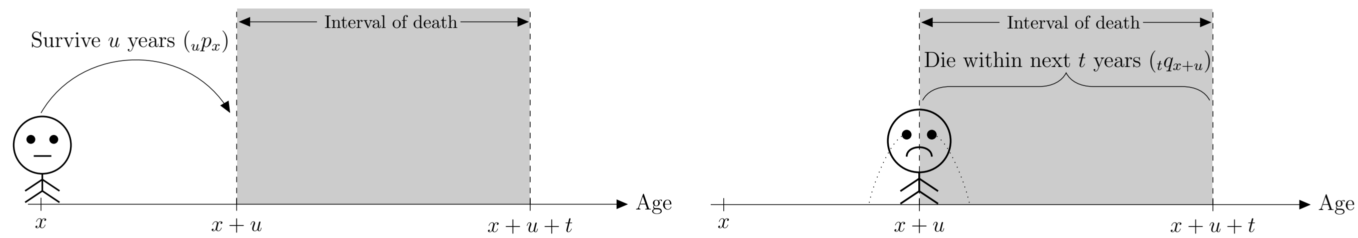
- *$p\times q$ form:* The first way is to factorize $_{u|t}q_x$ as a product of the u-year survival probability $_up_x$ and the t-year mortality probability $_tq_{x+u}$ based on the updated age $x + u$:

$$\boxed{_{u|t}q_x = {_up_x} \times {_tq_{x+u}}.} \tag{1.1.6}$$

[5]For some unknown reason, AMLCR places a long vertical line between u and t and writes $_{u}|_{t}q_x$, which is somewhat awkward. Recent LTAM/MLC exams also use $_{u|t}q_x$ instead of following AMLCR's typographical style.

To make sense of this $p \times q$ formula, remember that ${}_{u|t}q_x$ is the probability of dying between ages $x+u$ and $x+u+t$. To die in this age interval, it is necessary for (x) to first survive u years to reach age $x+u$. The probability of this movement is ${}_up_x$. Following the u-year survival, the life is now aged $x+u$ and free to die anywhere in the coming t years. The probability of doing so is ${}_tq_{x+u}$ (note that the subscript has been updated from x to $x+u$ after the first step). Multiplying the two probabilities gives (1.1.6).

Here is a graphical illustration of this two-step movement.



(a) Step 1: Survive u years to age $x+u$, with a probability of ${}_up_x$.

(b) Step 2: Standing at age $x+u$, die in the next t years, with a probability of ${}_tq_{x+u}$. (The life is crying because death is impending!)

- *$p-p$ form (in terms of survival probabilities only):* It is possible to express ${}_{u|t}q_x$ solely in terms of survival probabilities of the same current age x, but different survival durations:

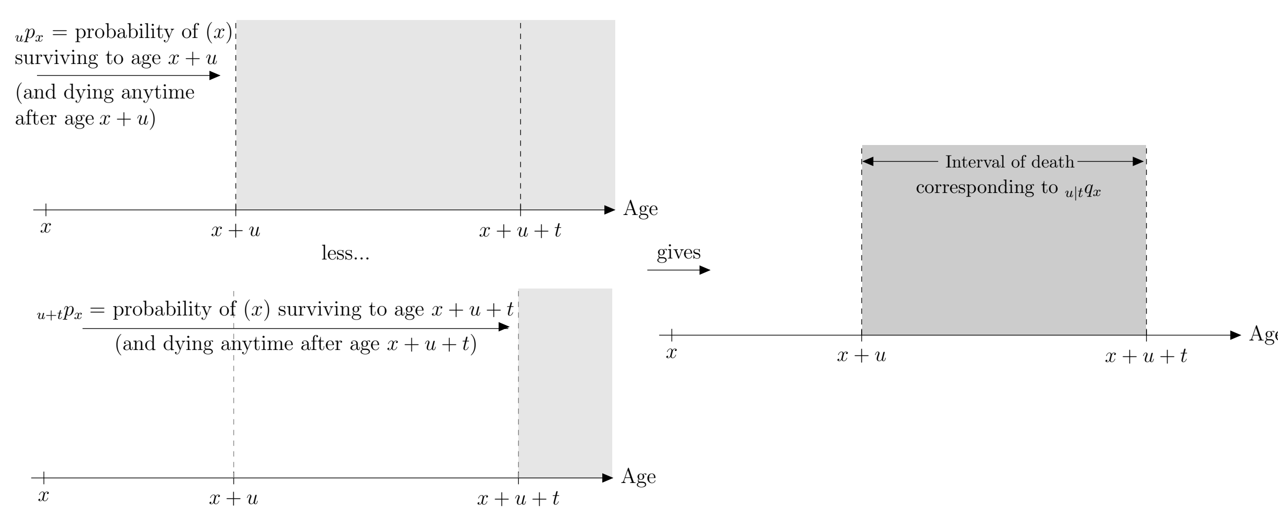
$$\boxed{{}_{u|t}q_x = {}_up_x - {}_{u+t}p_x.} \tag{1.1.7}$$

Algebraically, we can easily understand this $p-p$ form by writing

$$ {}_{u|t}q_x = \mathrm{P}(u < T_x \le u+t) = \mathrm{P}(T_x > u) - \mathrm{P}(T_x > u+t)$$

and noting that $\mathrm{P}(T_x > u) = {}_up_x$ and $\mathrm{P}(T_x > u+t) = {}_{u+t}p_x$ by definition.

We can also understand the $p-p$ formula graphically this way. While ${}_up_x$ corresponds to (x) surviving u years (and dying anytime after age $x+u$), ${}_{u+t}p_x$ corresponds to (x) surviving $u+t$ years (and dying anytime after age $x+u+t$). By subtraction, the interval of death corresponding to ${}_up_x - {}_{u+t}p_x$ is the age interval $(x+u, x+u+t]$, which is what ${}_{u|t}q_x$ covers.



- *$q - q$ form (in terms of mortality probabilities only):* The companion formula for (1.1.7) is a formula for ${}_{u|t}q_x$ solely in terms of mortality probabilities of the same current age x, but different death durations:

$$\boxed{{}_{u|t}q_x = {}_{u+t}q_x - {}_uq_x.} \tag{1.1.8}$$

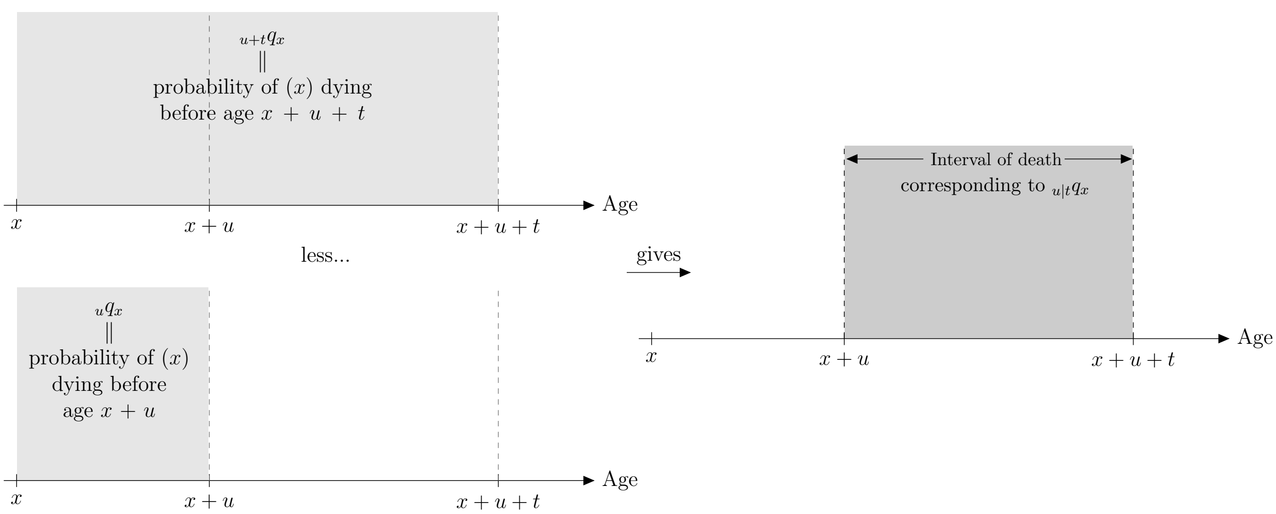
Again, we can understand this $q - q$ form easily by writing

$${}_{u|t}q_x = \mathrm{P}(u < T_x \leq u + t) = \mathrm{P}(T_x \leq u + t) - \mathrm{P}(T_x \leq u) = {}_{u+t}q_x - {}_uq_x,$$

or switching the p's in (1.1.7) to the q's by taking complements:

$${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x = (1 - {}_uq_x) - (1 - {}_{u+t}q_x) = {}_{u+t}q_x - {}_uq_x.$$

Here is a graphical illustration of (1.1.8). Subtracting the interval of death corresponding to ${}_{u+t}q_x$ and that corresponding to ${}_uq_x$ gives the age interval $(x + u, x + u + t]$, which is what ${}_{u|t}q_x$ covers.



EXAM NOTE

- All of the three formulas, (1.1.6), (1.1.7), and (1.1.8), may be used on the exam, depending on which formula is most convenient in the context of an exam question, so be sure to make an effort to master all of them. Instead of memorizing the formulas blindly, try to understand their intuitive meaning with the aid of the timeline diagrams above.
- At this stage, you may feel overwhelmed to see so many formulas (three formulas for ${}_{u|t}q_x$ and the factorization formula for ${}_tp_x$) thrown at you at one time, but fear not! These formulas will be used repeatedly in the rest of this study manual, so often that they will (hopefully!) become second nature to you as time goes by.

Example 1.1.4. (SOA Part 4B May 1984 Question 7 (Modified): Manipulating several q's) You are given:

(i) ${}_{1|}q_{x+1} = 0.095$

(ii) ${}_{2|}q_{x+1} = 0.171$

(iii) $q_{x+3} = 0.200$

(a) Calculate q_{x+1}.

Solution. In (ii), the current age plus the deferred period equals $(x+1)+2 = x+3$, which is the current age in (iii), so let's use the $p \times q$ formula to shift the age from $x+1$ to $x+3$: (Remember that ${}_{2|}q_{x+1}$ is the same as ${}_{2|1}q_{x+1}$.)

$$\underset{\text{(ii)}}{0.171} = {}_{2|}q_{x+1} = {}_2p_{x+1}q_{x+3} = {}_2p_{x+1}\underset{\text{(iii)}}{(0.2)} \quad \Rightarrow \quad {}_2p_{x+1} = 0.855.$$

This survival probability has the same current age $(x+1)$ as the deferred mortality probability ${}_{1|}q_{x+1}$ in (i), and $u + t = 1 + 1 = 2$, matching the survival duration, so let's try to relate ${}_{1|}q_{x+1}$ and ${}_2p_{x+1}$ using the $p - p$ formula:

$$\underset{\text{(i)}}{0.095} = {}_{1|}q_{x+1} = p_{x+1} - {}_2p_{x+1} = p_{x+1} - \underset{\text{(just found)}}{0.855} \quad \Rightarrow \quad p_{x+1} = 0.95.$$

Finally, q_{x+1} is the complement of p_{x+1}, so $q_{x+1} = 1 - 0.95 = \boxed{0.05}$. □

(b) Calculate q_{x+2}.

Solution 1. One way to find q_{x+2} is to use factorization formula (1.1.5):

$$\underset{\text{from (a)}}{0.855} = {}_2p_{x+1} = p_{x+1}p_{x+2} = \underset{\text{from (a)}}{0.95}\,(p_{x+2}) \quad \Rightarrow \quad p_{x+2} = 0.9.$$

Then $q_{x+2} = 1 - 0.9 = \boxed{0.1}$. □

Solution 2. Another way is to apply the $p \times q$ formula to ${}_{1|}q_{x+1}$ in (i):

$$\underset{\text{(i)}}{0.095} = {}_{1|}q_{x+1} \overset{(1.1.6)}{=} p_{x+1}q_{x+2} = \underset{\text{from (a)}}{0.95}\,(q_{x+2}) \quad \Rightarrow \quad q_{x+2} = \boxed{0.1}.$$

□

Remark. In the original version of this exam question, candidates were asked to calculate $q_{x+1} + q_{x+2}$ and given five answer choices:

(A) 0.15 *(correct)* (B) 0.20 (C) 0.25

(D) 0.27 (E) 0.30

Practice Problems for Section 1.1

⚠ IMPORTANT NOTE ⚠

Don't be scared by the abundance of practice problems in this study manual. As I said in the preface, it is really important to work out a wide variety of non-trivial exam-style problems to master the material in Exam FAM(-L). As a general guide, it would be a good idea to:

- Do **ALL** of the problems marked with an **asterisk (*)** in your first round of reading. These representative problems are of appropriate level of difficulty and highly recommended!
- Do a **random sample** of the rest, when you come back to this section after finishing the whole manual. (Of course, the more you do, the better...hard work pays!)

Survival and Mortality Probabilities

Problem 1.1.1. (SOA Course 150 November 1990 Multiple-Choice Question 25: Possible survival functions) Which of the following can serve as survival functions for $x \geq 0$?

I. $S_0(x) = \exp(x - 0.7(2^x - 1))$

II. $S_0(x) = \dfrac{1}{(1+x)^2}$

III. $S_0(x) = \exp(-x^2)$

(A)	I and II only	(B)	I and III only	(C)	II and III only
(D)	I, II and III	(E)	The correct answer is not given by (A), (B), (C) or (D).		

Solution. Recall from page 7 that a survival function $S_0(\cdot)$ for lifetime must satisfy the following three conditions:

- $S_0(0) = 1$
- $\lim_{t\to\infty} S_0(t) = 0$
- $S_0(t)$ is a non-increasing function of t.

II and III both satisfy these three conditions. For I, note that $S_0(1) = \exp(1 - 0.7(2^1 - 1)) = e^{0.3} > 1 = S_0(0)$, so I cannot be a survival function. **(Answer: (C))** □

Problem 1.1.2. * (CAS Exam 3L Spring 2011 Question 2 (Modified): Going from $S_0(\cdot)$ to $S_x(\cdot)$) Given that $S_0(x) = (1 - x/10)^{2/3}$ for $0 \le x \le 10$.

Calculate the probability that a life aged 1.5 survives to age 7.5.

(A) Less than 0.441
(B) At least 0.441 but less than 0.442
(C) At least 0.442 but less than 0.443
(D) At least 0.443 but less than 0.444
(E) At least 0.444

Solution. By (1.1.2), the required probability is

$$S_{1.5}(6) = \frac{S_0(7.5)}{S_0(1.5)} = \left(\frac{0.25}{0.85}\right)^{2/3} = \boxed{0.4423}. \quad \textbf{(Answer: (C))}$$

□

Problem 1.1.3. (SOA Part 4 November 1986 Morning Multiple-Choice Question 14: Percentile of a future lifetime – I) You are given $S_0(x) = 1/(1 + x)$.

Determine the median future lifetime of (y).

(A) $y + 1$
(B) y
(C) 1
(D) $1/y$
(E) $1/(1 + y)$

Solution. By (1.1.2), the survival function of T_y is

$$S_y(t) = \frac{S_0(y+t)}{S_0(y)} = \frac{1/(1+y+t)}{1/(1+y)} = \frac{1+y}{1+y+t}.$$

Setting $S_y(t) = 0.5$ leads to $t = \boxed{1 + y}$, which is the median future lifetime of (y). **(Answer: (A))** □

Remark. This survival distribution, called the Pareto distribution, is not a plausible distribution for human life—the median future lifetime of (y) increases as (y) becomes older. The older a person, the healthier they become!

Problem 1.1.4. * (Percentile of a future lifetime – II) You are given the following probability density function of the lifetime of a *new* machine:

$$f_0(x) = x\mathrm{e}^{-x^2/2} \quad \text{for } x \ge 0.$$

Calculate the 30th percentile of the future lifetime of a *two-year old* machine.

(A) 0.17
(B) 0.35
(C) 0.53
(D) 0.71
(E) 0.89

Ambrose's comments: Recall from Exam P that the $100p$th percentile of a random variable X is the number $\pi_p(X)$ such that $\mathrm{P}(X \le \pi_p(X)) = p$, or equivalently, $\mathrm{P}(X > \pi_p(X)) = 1 - p$.

Solution. By integration (or by inspection), the survival function of the lifetime of a new machine is $S_0(x) = \mathrm{e}^{-x^2/2}$ for $x \geq 0$. By (1.1.2), the survival function of the future lifetime of a two-year old machine can be related to $S_0(\cdot)$ via

$$S_2(x) = \frac{S_0(x+2)}{S_0(2)} = \frac{\mathrm{e}^{-(x+2)^2/2}}{\mathrm{e}^{-2^2/2}}, \quad x \geq 0.$$

The 30th percentile of T_2 is the value of x such that $F_2(x) = 0.3$, or equivalently, $S_2(x) = 1 - 0.3 = 0.7$. Solving $S_2(x) = 0.7$ gives

$$\mathrm{e}^{-(x+2)^2/2} = 0.7\mathrm{e}^{-2^2/2} \quad \Rightarrow \quad (x+2)^2 = 4.7133 \quad \overset{(x>0)}{\Rightarrow} \quad x = \sqrt{4.7133} - 2 = 0.1710,$$

so the 30th percentile of T_2 is $\boxed{0.1710}$. **(Answer: (A))** □

Remark.

(i) It is unnecessary to use the quadratic formula (which many students hate!) to solve $\mathrm{e}^{-(x+2)^2/2} = 0.0947$.

(ii) ⚠ You will get (C) if you calculate the 70th percentile of T_2 by solving $S_2(x) = 0.3$.

Problem 1.1.5. (Manipulating $_tp_x$ and $_tq_x$) You are given:

(i) The probability that (50) will die before age 80 is 0.3.

(ii) The probability that (70) will live to age 80 is 0.8.

Calculate the probability that (50) will die before age 70.

(A) 0.13 (B) 0.32 (C) 0.51

(D) 0.69 (E) 0.88

Solution. Given that $_{30}q_{50} = 0.3$ and $_{10}p_{70} = 0.8$, we have

$$0.7 = {}_{30}p_{50} \overset{(1.1.4)}{=} {}_{20}p_{50}({}_{10}p_{70}) = {}_{20}p_{50}(0.8) \quad \Rightarrow \quad {}_{20}p_{50} = 0.875.$$

Then the required probability is $_{20}q_{50} = 1 - {}_{20}p_{50} = 1 - 0.875 = \boxed{0.125}$. **(Answer: (A))** □

Remark. ⚠ Answer (E) is for $_{20}p_{50} = 0.875$.

Problem 1.1.6. [HARDER!] (SOA Exam FAM-L Sample Question 2.2 / Exam MLC Spring 2013 Question 20: Moments of the number of survivors) Scientists are searching for a vaccine for a disease. You are given:

(i) 100,000 lives age x are exposed to the disease.

(ii) Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1.

(iii) The probability that the vaccine will be available is 0.2.

(iv) For each life during year 1, $q_x = 0.02$.

(v) For each life during year 2, $q_{x+1} = 0.01$ if the vaccine has been given, and $q_{x+1} = 0.02$ if it has not been given.

Calculate the standard deviation of the number of survivors at the end of year 2.

(A) 100 (B) 200 (C) 300
(D) 400 (E) 500

Solution. Let N be the number of survivors at the end of year 2.

Case 1. *The vaccine is available.*

The probability of surviving to the end of year 2 is ${}_2p_x = p_x p_{x+1} = 0.98(0.99) = 0.9702$, so $N \sim \mathrm{B}(100,000,\ 0.9702)$, i.e., N follows a binomial distribution with parameters $m = 100,000$ and $q = 0.9702$ (using the FAM-S parameterization). Its first two moments are

$$\begin{aligned} \mathrm{E}[N] &= 100,000(0.9702) = 97,020, \\ \mathrm{E}[N^2] &\overset{(\mathrm{Var}(N)+\mathrm{E}[N]^2)}{=} 100,000(0.9702)(1-0.9702) + 97,020^2 = 9,412,883,291. \end{aligned}$$

Case 2. *The vaccine is not available.*

The probability of surviving to the end of year 2 is ${}_2p_x = p_x p_{x+1} = 0.98^2 = 0.9604$, so $N \sim \mathrm{B}(100,000,\ 0.9604)$. Its first two moments are

$$\begin{aligned} \mathrm{E}[N] &= 100,000(0.9604) = 96,040, \\ \mathrm{E}[N^2] &= 100,000(0.9604)(1-0.9604) + 96,040^2 = 9,223,685,403. \end{aligned}$$

With the two cases combined, N follows a *mixture distribution* (a concept in FAM-S/STAM), with a weight of 0.2 to a $\mathrm{B}(100,000,\ 0.9702)$ distribution and a weight of 0.8 to a $\mathrm{B}(100,000,\ 0.9702)$ distribution. The raw moments of a mixture distribution equal a mixture of the individual raw moments, so

$$\begin{aligned} \mathrm{E}[N] &= 0.2(97,020) + 0.8(96,040) = 96,236, \\ \mathrm{E}[N^2] &= 0.2(9,412,883,291) + 0.8(9,223,685,403) = 9,261,524,981, \end{aligned}$$

so the unconditional standard deviation of N is $\sqrt{\mathrm{E}[N^2] - \mathrm{E}[N]^2} = \boxed{396.59}$. **(Answer: (D))** □

Deferred Mortality Probabilities

Problem 1.1.7. (CAS Exam LC Spring 2014 Question 3: Calculating deferred mortality probabilities from $f_0(\cdot)$) You are given the following density function for T_0:

$$f_0(x) = \begin{cases} 1/100 & \text{for } 0 \le x \le 45 \\ 0.055\mathrm{e}^{-0.1(x-45)} & \text{for } x \ge 45 \end{cases}$$

Calculate ${}_{45|11}q_0$.

(A) Less than 0.250
(B) At least 0.250, but less than 0.300
(C) At least 0.300, but less than 0.350
(D) At least 0.350, but less than 0.400
(E) At least 0.400

Solution. By definition, ${}_{45|11}q_0 = \mathrm{P}(45 < T_0 < 56)$. Given the density function of T_0, this probability can be evaluated directly by integration as

$$\int_{45}^{56} f_0(x)\,\mathrm{d}x = 0.055\int_{45}^{56} \mathrm{e}^{-0.1(x-45)}\,\mathrm{d}x = 0.055\left[\frac{1-\mathrm{e}^{-0.1(11)}}{0.1}\right] = \boxed{0.3669}. \quad \textbf{(Answer: (D))}$$

□

Problem 1.1.8. * (CAS Exam 3L Spring 2010 Question 2: Calculating deferred mortality probabilities from $S_0(\cdot)$ – I) You are given the following survival function:

$$S_0(x) = \left(1 - \frac{x}{90}\right)^{1/2}, \quad 0 \le x \le 90.$$

Calculate ${}_{10|25}q_{30}$.

(A) Less than 0.15
(B) At least 0.20, but less than 0.25
(C) At least 0.25, but less than 0.30
(D) At least 0.30, but less than 0.35
(E) At least 0.35

Solution. By (1.1.7), ${}_{10|25}q_{30} = {}_{10}p_{30} - {}_{35}p_{30}$. Then by (1.1.2),

$${}_{10}p_{30} = \frac{S_0(40)}{S_0(30)} = \left(\frac{50}{60}\right)^{1/2} = 0.912871 \quad \text{and} \quad {}_{35}p_{30} = \frac{S_0(65)}{S_0(30)} = \left(\frac{25}{60}\right)^{1/2} = 0.645497.$$

Thus ${}_{10|25}q_{30} = 0.912871 - 0.645497 = \boxed{0.2674}$. **(Answer: (C))** □

Remark. In this problem, you have to use both the $p - p$ formula for ${}_{u|t}q_x$ and the factorization formula for survival probabilities. In general,

$${}_{u|t}q_x = \frac{S_0(x+u) - S_0(x+u+t)}{S_0(x)}.$$

Problem 1.1.9. (CAS Exam MAS-I Fall 2018 Question 5: Calculating deferred mortality probabilities from $S_0(\cdot)$ – II) You are given the following survival function for (new) LED light bulbs 💡 in years:

$$S_0(x) = \left(1 - \frac{x}{30}\right)^{1/2}, \quad \text{for } 0 \le x < 30$$

A homeowner replaced his kitchen light with an LED light bulb 24 months ago and it is still functioning today.

Calculate the probability that the LED light bulb will stop working between 24 months and 81 months from today.

(A) Less than 0.09
(B) At least 0.09, but less than 0.10
(C) At least 0.10, but less than 0.11
(D) At least 0.11, but less than 0.12
(E) At least 0.12

Solution. The given LED light bulb is 24 months, or two years old, and with time measured *in years* (because x in $S_0(x)$ is in years), we are asked to find ${}_{\frac{24}{12}|\frac{81-24}{12}}q_2 = {}_{2|4.75}q_2$, which equals

$$\begin{aligned}
{}_{2|4.75}q_2 &\overset{(1.1.7)}{=} {}_2p_2 - {}_{6.75}p_2 \\
&\overset{(1.1.2)}{=} \frac{S_0(4) - S_0(8.75)}{S_0(2)} \\
&= \frac{(1-4/30)^{1/2} - (1-8.75/30)^{1/2}}{(1-2/30)^{1/2}} \\
&= \boxed{0.0925}. \quad \textbf{(Answer: (B))}
\end{aligned}$$

□

Problem 1.1.10. * (A harder version of Example 1.1.2: Calculating deferred mortality probabilities from $S_x(\cdot)$ – I) You are given that ${}_tp_{40} = \exp[-0.06(1.12^t - 1)]$ for all $t \ge 0$.

Calculate the probability that a life aged $\underline{45}$ dies between age 55 and age 70.

(A) 0.26
(B) 0.38
(C) 0.50
(D) 0.62
(E) 0.74

Solution. The required probability is ${}_{10|15}q_{45} \overset{(1.1.7)}{=} {}_{10}p_{45} - {}_{25}p_{45} \overset{(1.1.4)}{=} \dfrac{{}_{15}p_{40} - {}_{30}p_{40}}{{}_5p_{40}}$, which, from the given formula for ${}_tp_{40}$, equals

$$\frac{\exp[-0.06(1.12^{15}-1)] - \exp[-0.06(1.12^{30}-1)]}{\exp[-0.06(1.12^5-1)]} = \boxed{0.6162}. \quad \textbf{(Answer: (D))}$$

□

Remark. ⚠ Possible reason(s) for getting some of the incorrect answers:

(C) Confusing the given formula for ${}_tp_{40}$ with ${}_tp_{45}$ and calculating the final answer as $\exp[-0.06(1.12^{10}-1)] - \exp[-0.06(1.12^{25}-1)] = 0.4984$.

Problem 1.1.11. * (Calculating deferred mortality probabilities from $S_x(\cdot)$ – II; connecting life contingencies and binomial probabilities) You are given:

(i) ${}_{10}p_{30} = 0.8$

(ii) ${}_{20}p_{30} = 0.6$

Calculate the probability that at least two of three persons now aged 30 with independent future lifetimes will die between ages 40 and 50.

(A) 0.04 (B) 0.08 (C) 0.10
(D) 0.12 (E) 0.16

Solution. From (i) and (ii), the probability that a person now aged 30 will die between ages 40 and 50 is the deferred mortality probability given by

$$ {}_{10|10}q_{30} \overset{(1.1.7)}{=} {}_{10}p_{30} - {}_{20}p_{30} = 0.8 - 0.6 = 0.2. $$

Let N be the number of persons now aged 30 among the three who will die between ages 40 and 50. Because the three persons have independent and identically distributed future lifetimes, N follows a $\mathrm{B}(3, 0.2)$ distribution. Therefore,

$$ \mathrm{P}(N=2) = \binom{3}{2}(0.2)^2(0.8) = 0.096 \quad \text{and} \quad \mathrm{P}(N=3) = \binom{3}{3}(0.2)^3 = 0.008 $$

and the required probability is $\mathrm{P}(N \geq 2) = \mathrm{P}(N=2) + \mathrm{P}(N=3) = 0.096 + 0.008 = \boxed{0.104}$.
(Answer: (C)) □

Remark. ⚠ You would get (A) if you miss the binomial coefficient $\binom{3}{2}$ for $\mathrm{P}(N = 2)$.

Problem 1.1.12. * (Similar to Exercise 2.1 of AMLCR: Calculating deferred mortality probabilities from other p's and q's – I) You are given:

(i) $p_x = 0.98$

(ii) $p_{x+1} = 0.97$

(iii) ${}_2p_{x+2} = 0.92$

(iv) $q_{x+3} = 0.05$

Calculate ${}_{1|2}q_x$.

(A) 0.03 (B) 0.04 (C) 0.05
(D) 0.06 (E) 0.07

Solution. From (iii) and (iv),

$$ 0.92 = {}_2p_{x+2} \overset{(1.1.5)}{=} p_{x+2}p_{x+3} = p_{x+2}(1-0.05) \quad \Rightarrow \quad p_{x+2} = \frac{92}{95}. $$

Then ${}_{1|2}q_x \overset{(1.1.6)}{=} p_x({}_2q_{x+1}) \overset{(1.1.5)}{=} p_x(1 - p_{x+1}p_{x+2}) = 0.98\left[1 - 0.97\left(\dfrac{92}{95}\right)\right] = \boxed{0.0594}$.
(Answer: (D)) □

Problem 1.1.13. * (Relating deferred mortality probabilities to other p's and q's – II) You are given:

(i) ${}_5q_x = 0.22$

(ii) ${}_5q_{x+15} = 0.25$

(iii) ${}_{10|5}q_{x+5} = 0.17$

Calculate the probability that (x) will die in the next 20 years.

(A) 0.40 (B) 0.45 (C) 0.50
(D) 0.55 (E) 0.60

Solution. From (ii) and (iii), we have

$$0.17 = {}_{10|5}q_{x+5} \overset{(1.1.6)}{=} {}_{10}p_{x+5}({}_5q_{x+15}) = {}_{10}p_{x+5}(0.25),$$

which implies that ${}_{10}p_{x+5} = 0.68$. Therefore,

$${}_{20}p_x \overset{(1.1.4)}{=} {}_5p_x({}_{10}p_{x+5})({}_5p_{x+15}) = 0.78(0.68)(0.75) = 0.3978$$

and ${}_{20}q_x = 1 - {}_{20}p_x = \boxed{0.6022}$. **(Answer: (E))** □

Problem 1.1.14. * (Relating deferred mortality probabilities to other p's and q's – III) You are given:

(i) ${}_{1|}q_x = 0.0180$

(ii) ${}_{2|}q_x = 0.0190$

(iii) $q_{x+1} = 0.0186$

Calculate q_{x+2}.

(A) 0.016 (B) 0.018 (C) 0.020
(D) 0.022 (E) 0.024

Solution. From (i) and (iii),

$$0.0180 = {}_{1|}q_x \overset{(1.1.6)}{=} p_xq_{x+1} = p_x(0.0186) \quad \Rightarrow \quad p_x = 0.967742.$$

Then from (ii),

$$0.019 = {}_{2|}q_x \overset{(1.1.6)}{=} {}_2p_x q_{x+2} \overset{(1.1.5)}{=} p_x p_{x+1} q_{x+2} = 0.967742(1 - 0.0186)q_{x+2},$$

so $q_{x+2} = \boxed{0.0200}$. **(Answer: (C))** □

Problem 1.1.15. (Calculating deferred mortality probabilities from other *p*'s and *q*'s – IV) You are given:

(i) ${}_3q_x = 0.25$

(ii) ${}_4q_x = 0.35$

(iii) $q_{x+4} = 0.15$

Consider a group of 1,000 lives aged x with independent future lifetimes.

(a) Calculate the expected number of lives in the group who will die between ages $x + 3$ and $x + 5$.

(A)	158	(B)	168	(C)	178
(D)	188	(E)	198		

Solution. Since ${}_5p_x \overset{(1.1.4)}{=} {}_4p_x(p_{x+4}) = (1 - 0.35)(1 - 0.15) = 0.5525$, we have

$${}_{3|2}q_x \overset{(1.1.7)}{=} {}_3p_x - {}_5p_x = (1 - 0.25) - 0.5525 = 0.1975.$$

Let N be the number of lives who will die between ages $x + 3$ and $x + 5$. Because the 1,000 lives have independent and identically distributed (i.i.d.) future lifetimes, N is a binomial random variable with parameters $m = 1,000$ and $q = {}_{3|2}q_x = 0.1975$. Its mean is $\mathrm{E}[N] = 1,000({}_{3|2}q_x) = 1,000(0.1975) = \boxed{197.5}$. **(Answer: (E))** □

(b) Calculate the variance of the number of lives in the group who will die between ages $x+3$ and $x + 5$.

(A)	158	(B)	168	(C)	178
(D)	188	(E)	198		

Solution. Following the solution to part (a), the variance of N is

$$\begin{aligned} \mathrm{Var}(N) &= 1,000({}_{3|2}q_x)(1 - {}_{3|2}q_x) \\ &= 1,000(0.1975)(1 - 0.1975) \\ &= \boxed{158.49}. \quad \textbf{(Answer: (A))} \end{aligned}$$

□

Problem 1.1.16. [HARDER!] (CAS Exam 3L Spring 2011 Question 5: Deferred mortality probabilities involving two lives) You are given:

- ${}_5p_x = 0.9$
- ${}_5p_y = 0.8$
- $q_{x+5} = 0.2$
- $q_{y+5} = 0.3$

Calculate the probability that exactly one of two independent lives (x) and (y) will die in year six.

(A)	Less than 0.25	(B)	At least 0.25, but less than 0.30
(C)	At least 0.30, but less than 0.35	(D)	At least 0.35, but less than 0.40
(E)	At least 0.40		

Solution. By (1.1.6), the probability that (x) will die in year six (i.e., between time 5 and time 6) is ${}_{5|}q_x = {}_5p_x q_{x+5} = 0.9(0.2) = 0.18$ and that for (y) is ${}_{5|}q_y = {}_5p_y q_{y+5} = 0.8(0.3) = 0.24$. The probability that exactly one of the two lives will die in year six is

$$\begin{aligned}\underbrace{{}_{5|}q_x(1 - {}_{5|}q_y)}_{(x)\text{ will die in year 6, but not }(y)} + \underbrace{(1 - {}_{5|}q_x)\,{}_{5|}q_y}_{(y)\text{ will die in year 6, but not }(x)} &= 0.18(1-0.24) + (1-0.18)(0.24) \\ &= \boxed{0.3336}. \quad \textbf{(Answer: (C))}\end{aligned}$$

□