SOA Exam FM Study Manual

With *Study*Plus+

***Study*Plus+ gives you digital access* to:**

- Flashcards & Formula Sheet
- Actuarial Exam & Career Strategy Guides
- Technical Skill eLearning Tools
- Samples of Supplemental Texts & Study Tools
- And more!

**See inside for keycode access and login instructions*

Spring 2017 Edition, Second Printing | **Volume I**

John B. Dinius, FSA | Matthew J. Hassett, Ph.D.
Michael I. Ratliff, Ph.D., ASA | Toni Coombs Garcia
Amy C. Steeby, MBA, MEd

ACTEX Learning | Learn Today. Lead Tomorrow.

ACTEX

SOA Exam FM Study Manual

Spring 2017 Edition, Second Printing

John B. Dinius, FSA | Matthew J. Hassett, Ph.D.
Michael I. Ratliff, Ph.D., ASA | Toni Coombs Garcia
Amy C. Steeby, MBA, MEd

ACTEX Learning
New Hartford, Connecticut

ACTEX Learning
Learn Today. Lead Tomorrow.

Actuarial & Financial Risk Resource Materials
Since 1972

ISBN: 978-1-63588-061-8

Printed in the United States of America.

Your Opinion is Important to Us

ACTEX is eager to provide you with helpful study material to assist you in gaining the necessary knowledge to become a successful actuary. In turn we would like your help in evaluating our manuals so we can help you meet that end. We invite you to provide us with a critique of this manual by sending this form to us at your convenience. We appreciate your time and value your input.

Publication:

ACTEX FM Study Manual, Spring 2017 Edition, Second Printing

I found Actex by: (Check one)

☐ A Professor ☐ School/Internship Program ☐ Employer ☐ Friend ☐ Facebook/Twitter

In preparing for my exam I found this manual: (Check one)

☐ Very Good ☐ Good ☐ Satisfactory ☐ Unsatisfactory

I found the following helpful:

____

____

____

____

____

I found the following problems: (Please be specific as to area, i.e., section, specific item, and/or page number.)

____

____

____

____

To improve this manual I would:

____

____

____

____

Name: ________________________________

Address: ________________________________

____

Phone: ________________ E-mail: ________________

(Please provide this information in case clarification is needed.)

Send to: Stephen Camilli
ACTEX Learning
P.O. Box 715
New Hartford, CT 06057

Or visit our website at www.ActexMadRiver.com to complete the survey on-line. Click on the "Send Us Feedback" link to access the online version. You can also e-mail your comments to Support@ActexMadRiver.com.

Preface

ACTEX first published a study manual for the Society of Actuaries' Exam FM ("Financial Mathematics") in 2004. That manual was prepared by lead author Matthew Hassett, assisted by Michael Ratliff, Toni Coombs Garcia, and Amy Steeby. The manual has been regularly updated and expanded to keep pace with changes in the syllabus for Exam FM and to increase the number of sample problems and practice exams. Now this new edition of the ACTEX Study Manual for Exam FM, edited by lead author John Dinius, has been extensively revised and edited to reflect changes in the SOA's Exam FM syllabus effective with the June 2017 administration of the exam.

The revised syllabus for Exam FM has eliminated most of the material on Financial Derivatives, and has added sections on "The Determination of Interest Rates" and "Interest Rate Swaps." The SOA has issued new study notes on these two topics, as well as a new study note on "Using Duration and Convexity to Approximate Present Value." This material is covered in two new modules in this manual (Modules 8 and 9) plus an expansion of the Asset-Liability Management module (Module 7). In addition, many sections of the other modules in the manual have been revised to provide additional explanations, examples, and solutions.

This manual has 9 modules that are arranged in 3 groups of 3 modules each. The first 3 modules present basic concepts (the time value of money, annuities, and loan repayment). The next 3 modules apply these concepts to a range of topics (bonds, yield rates, and the term structure of interest rates). The final 3 modules examine more complex real-world concepts (asset-liability management, the factors that influence market interest rates, and interest rate swaps). After each group of three modules there is a "midterm exam," providing the student an opportunity to check his/her progress. Also included at the end of the manual are 11 practice exams of 35 problems each. These are intended to provide realistic exam-taking experience to complete the student's preparation for Exam FM.

This manual is written so that it can be read without reference to any other text. However, we strongly recommend that the student obtain and read one of the official textbooks for SOA Exam FM, and use that text *in combination with* this manual. The following pages provide recommendations on how to prepare for actuarial exams and suggestions on how to use this manual most effectively.

A note about Errors:
If you find a possible error in this manual, please let us know. Use the "Feedback" link on the ACTEX homepage (www.actexmadriver.com) and describe the issue. We will review all comments and respond to you with an answer. Any confirmed errata will be posted on the ACTEX website under the "Errata" link.

On passing exams

How to Learn Actuarial Mathematics and Pass Exams

On the next page you will find a list of study tips for learning the material in the Exam FM syllabus and passing Exam FM. But first it is important to state the basic learning philosophy that we are using in this guide:

You must master the basics before you proceed to the more difficult problems

Think about your basic calculus course. There were some very challenging applications in which you used derivatives to solve hard max-min problems.

It is important to learn how to solve these hard problems, but if you did not have the basic skills of taking derivatives and doing algebraic simplification, you could not do the more advanced problems. Thus every calculus book has you practice derivative skills before presenting the tougher sections on applied problems.

You should approach interest theory the same way. The first 2 or 3 modules give you the basic tools you will need to solve the problems in the later modules. Learn these concepts and methods (and the related formulas) very well, as you will need them in each of the remaining modules.

This guide is designed to progress from simpler problems to harder ones.

In each module we start with the basic concepts and simple examples, and then progress to more difficult material so that you will be prepared to attack actual exam problems by the end of the module.

The same philosophy is used in our practice exams at the end of this manual. The first few practice exams have simpler problems, and the problems become more difficult as you progress through the practice exams.

A good strategy when taking an exam is to answer all of the easier problems before you tackle the harder ones.

An exam is scored in percentage terms, and a multiple choice exam like Exam FM will have a mix of problems at different difficulty levels.

If an exam has ten problems and three are very hard, getting the right answers to only the three hard problems and missing the others gets you a score of 30%. This is actually a possibility if the very hard problems are the first ones on the exam and you try to solve them first.

A useful exam strategy is to go through the exam and quickly solve all the more basic problems before spending extra time on the hard ones. Strive to answer <u>all</u> of the easy problems correctly.

Study Tips

1) Develop a schedule so that you will complete your studying in time for the exam. Divide your schedule into time for each module, plus time at the end to review and to solve practice problems. Your schedule will depend on how much time you have before the exam, but a reasonable approach might be to complete one module per week.

2) If possible, join a study group of your peers who are studying for Exam FM.

3) For each module:
   a) Read the module in the FM manual.
   b) As you read through the examples in the text, make sure that you can correctly compute the answers.
   c) Summarize each concept you learn in the manual's margins or in a notebook.
   d) Understand the main idea of each concept and be able to summarize it in your own words. Imagine that you are trying to teach someone else this concept.
   e) While reading, create flash cards for the formulas, to facilitate memorization.
   f) Learn the calculator skills thoroughly and know *all* of your calculator functions.
   g) Do a review of the corresponding chapter in the recommended text.
   h) Do the Basic Review Problems and review your solutions.
   i) Do the Sample Exam Problems and review your solutions.
      i) If you have been stuck on a problem for more than 20 minutes, it is OK to refer to the solutions. Just make sure that when you are finished with the problem, you can recite the concept that you missed and summarize it in your own words. If you get stuck on a problem, think about what principles were used in this question and see if you could write a different problem with similar structure (as if you were the exam writer).
      ii) Mark each sample exam problem as an Easy, Medium, or Hard problem.

4) After learning the material in each module, it is a good idea to go back to previous modules and do a quick half-hour or 1-hour review, so that information isn't forgotten.

5) Go back and redo the sample exam problems that you have marked as Medium or Hard when you looked through them the first time.

6) At the end of modules 3, 6, and 9, we have included practice exams that are like midterms. Taking these tests will help you consolidate your knowledge.

7) After learning the material in all of the modules and taking the midterms, go to the practice exams.
   a) The first 6 practice exams are relatively straightforward to enable you to review the basics of each topic. You may want to attempt them in a *non-timed* environment to evaluate your skills and understanding.
   b) The final 5 practice exams introduce more difficult questions in order to replicate the exam experience. You should take each of these in a *timed* environment to give yourself experience with exam conditions.

   Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like the ones in this manual.

Contents

Introduction

As you begin your preparation for the Society of Actuaries' Exam FM, you should be aware that studying Financial Mathematics (or "interest theory," as I like to call it) is not a matter of learning mathematics. Instead, financial mathematics involves *applying* mathematics to situations that involve financial transactions. This will require you to learn a new language, the language of the financial world, and then to apply your *existing* math skills to solve problems that are presented in this new language. It is important that you spend adequate time to fully understand the meanings of all the terms that will be introduced in this manual. Nearly all of the problems on Exam FM will be word problems (rather than just formulas), and it is very difficult to solve these problems unless you understand the language that is being used.

In this manual, we assume that you have a solid working knowledge of differential and integral calculus and some familiarity with probability. We also assume that you have an excellent knowledge of algebraic methods. Depending on what mathematics courses you have taken (and how recently), you may need to review these topics in order to understand some of the material and work the problems in this manual.

Throughout the manual, a large number of the examples and practice problems are solved using the Texas Instruments BA II Plus calculator, which is the financial calculator approved for use on Exam FM. It is essential for you to have a BA II Plus calculator in order to understand the solutions presented here, and also to solve the problems on the actual exam. This calculator is available in a standard model, and also as the "BA II Plus *Professional*." The Professional model, which is somewhat more expensive, is a bit easier to work with, which could be important when taking a time-limited exam.

Over the years, most actuarial students have found that the best way to prepare for Exam FM is to work a very large number of problems (hundreds and hundreds of problems). There are many examples, exercises, problems, and practice exams included in this manual. Many more problems can be found on the Society of Actuaries website (www.soa.org) or by searching the Web. You should plan to spend a significant proportion of your study time working problems and reviewing the solutions that are provided in the manual and on the websites.

Financial mathematics is an integral part of an actuary's skill set, and you can expect to apply interest theory regularly throughout your career. A strong understanding of the topics covered in this manual will provide you a valuable tool for understanding financial and economic matters both on and off the job.

Best of luck to you in learning Financial Mathematics and passing Exam FM!

John Dinius
January 2017

Module 1

Interest Rates and the Time Value of Money

Section 1.1

Time Value of Money

Interest theory deals with the **time value of money**. For example, a dollar invested today at 6% interest per year is worth \$1.06 one year from today. Because a dollar invested today can provide *more* than one dollar a year from now, it follows that receiving a dollar today has a *greater value* than receiving one dollar a year from now. In other words, money has a "time value," and in order to assess the value of a payment, we need to know not only the *amount* of the payment, but also *when the payment occurs*. That is the underlying principle of interest theory.

In the example of the investment at 6% interest, the dollar that is invested today is called the **principal**, and the \$0.06 increase in value is called **interest**. What happens to the investment *after* the first year depends on whether it is earning **compound interest** or **simple interest**. We illustrate this with an example based on an investment of 100 that earns 6% interest for two years.

a) Compound interest: Interest is earned during each year on the total amount in the account at the beginning of that year. The amounts in the account at the end of Year 1 and Year 2 are:

   Year 1: $100+0.06(100)=100(1.06)=106$

   Year 2: $106+0.06(106)=106(1.06)=100(1.06)^2=112.36$

   Interest is "compounded" at the end of Year 1. That is, the interest earned during Year 1 is "converted" to principal at the end of Year 1 and it becomes part of the principal that earns interest during Year 2.

b) Simple interest: In each year interest is earned on only the original principal of 100. The amounts in the account at the end of Year 1 and Year 2 are:

Year 1: $100 + 0.06(100) = 100(1.06) = 106$

Year 2: $106 + 0.06(100) = 100 * (1 + 2(0.06)) = 112$

Because the interest earned during the first year is not converted to principal ("compounded") at the end of the first year, it does not earn interest during the second year. The principal is 100 in both years, and the amount of interest earned in each year is 6.

Compound interest is the most widely used method of computing interest, especially for multi-period investments. Simple interest is generally used only for shorter-term investments (usually less than one year). Because it is so widely used, we will begin our study of interest theory with compound interest.

Note: In this manual, amounts of money will generally be given without an indication of what currency is being used. You may want to think of these amounts as U.S. or Canadian dollars ($100, in the case of the above example), or you may just treat them as amounts of money with no specific denomination.

Section 1.2

Present Value and Future Value

The value of an investment today (time 0) is its **present value** [PV], and its value n periods from today is called its **future value** [FV] as of time n. More broadly, if we know the value of an investment as of a particular date and we want to find its value as of an *earlier* date, we are calculating a *present value* as of the earlier date. And if we want to find the value as of a *later* date, then we are calculating a *future value* (or an **accumulated value**) as of that later date. If funds are invested at a compound interest rate of i per period for n periods, the basic relationships are:

(1.1)

$$\boxed{FV = PV(1+i)^n \qquad\qquad PV = \frac{FV}{(1+i)^n}}$$

Example (1.2)

Let $n = 10$ and $i = 0.06$.

a) If $PV = 1{,}000$, then $FV = 1{,}000(1.06)^{10} = 1{,}790.85$

b) If $FV = 1{,}000$, then $PV = \dfrac{1{,}000}{(1.06)^{10}} = 558.39$

Calculation a) demonstrates that if we invest 1,000 today at 6% interest, in 10 years it will have accumulated to a future value of 1,790.85.

Calculation b) shows that if we need 1,000 ten years from now, we can accumulate that amount by investing 558.39 now at 6% interest.

Exercise (1.3)

Using an interest rate of 5% compounded annually, find a) the present value (today) of 20,000 payable in 15 years, and b) the future value 6 years from today of 5,000 deposited today.

Answer: a) 9,620.34 b) 6,700.48

Calculator Note

The BA II Plus calculator has 5 "Time Value of Money" keys:

N	Number of periods
I/Y	Interest rate per period (usually per year)
PV	Present value
PMT	Periodic payment
FV	Future value

In this module we will not look at any problems that involve periodic payments. The PMT key will be used starting in Module 2. Using the other four keys, we can solve compound interest problems like Example **(1.2)**, as we illustrate next.

To begin any new problem, it is wise to clear the Time Value of Money [TVM] registers to erase any entries from prior problems. Note that the legend "CLR TVM" appears above the FV key on the BA II Plus calculator. To clear the TVM registers use the keystrokes 2ND CLR TVM. This sets all 5 of the TVM values to 0.

Before we do the actual calculation, we must choose a sign convention for the values we enter into the calculator as well as the answers we calculate. In this manual, we will use the following convention: **Money that you receive is positive; money that you pay out is negative.** Thus, if you put 1,000 into an account now, you should enter it into the calculator as $-1,000$ to indicate that it is "out of pocket." (You can make an entry negative by using the +|- key.)

In Example **(1.4)**, we will rework Example **(1.2)** using the calculator.

Example (1.4)

We will redo the calculations of Example **(1.2)** using the BA II Plus's TVM functions.

To find the future value of 1,000 in 10 years at 6% compound interest per year, use the following keystrokes:

10 [N] 6 [I/Y] 1,000 [+|-] [PV] [CPT] [FV]

You will see in the display: FV = 1,790.85.
The answer is <u>positive</u>, since this is money that you will <u>receive</u>.
Note that the interest rate is 6% = 0.06, but it is entered into the calculator as 6, not 0.06. The calculator treats your entry as a percentage; that is, the amount entered is divided by 100 when the calculation is done.

To find the present value at 6% compound interest of 1,000 to be received 10 years in the future, use the following keystrokes:

10 [N] 6 [I/Y] 1000 [FV] [CPT] [PV] .

You will see in the display: PV = −558.39.
The answer is <u>negative</u>; this is money that you must put <u>into</u> the account.

Note: You may have noticed that there is an asterisk (*) above the value -558.39 in your calculator's display. The asterisk indicates that -558.39 is a <u>computed</u> value, not a number that was entered. As long as the inputs (the values for N, I/Y, PMT, and FV) are not changed, the asterisk will continue to appear over the -558.39 (even if you perform other calculations and then press RCL PV). But if one of the inputs (such as N or FV) is changed, then when you press RCL PV, the -558.39 will appear <u>without</u> the asterisk, indicating that it is <u>not</u> a computed value based on the <u>current</u> entries in the TVM variables.

Exercise (1.5)

Rework Exercise **(1.3)** using the calculator's TVM functions.

Section 1.3

Functions of Investment Growth

An investor might wish to plot the growth of an investment over time. Two functions are commonly used:

> The **accumulation function**, $a(t)$, is the value at time t of an initial investment of 1 made at time 0.
>
> The **amount function**, $A(t)$, is the value at time t of an initial investment of $A(0)$ made at time 0.

For compound interest at a constant annual rate i, these functions are:

(1.6)

Compound interest:

$$a(t) = (1+i)^t \qquad A(t) = A(0)\left(1+i\right)^t$$

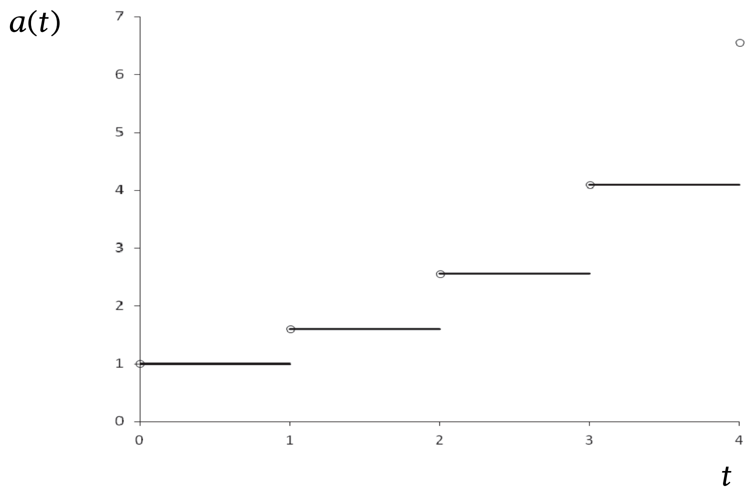
For compound interest at a rate of $i = 60\%$ per year, the values of $a(t)$ at the end of each of the first 4 years are as shown in the following table:

t	0	1	2	3	4
$a(t)$	1	1.60	2.56	4.096	6.5536

Note: An extremely high interest rate (60%) is used here so that the following graphs will clearly show the exponential (and non-linear) form of the accumulation function.

What is the value of $a(t)$ when t is *not* an integer?
For example, what is the value of $a(t)$ for $t = 1.5$?
There can be instances where interest is considered to be earned only at the *end* of each year. In that case, the value of the accumulation function at $t = 1.5$ would be the same as at $t = 1$, that is, it would be 1.60. At the end of the second year, all of the interest for the period from $t = 1$ to $t = 2$ would be credited, and the accumulated value would increase instantaneously from 1.60 to 2.56.

Thus, if interest is considered to be earned only *at the end of each year*, the graph of $a(t)$ is a *step function*:

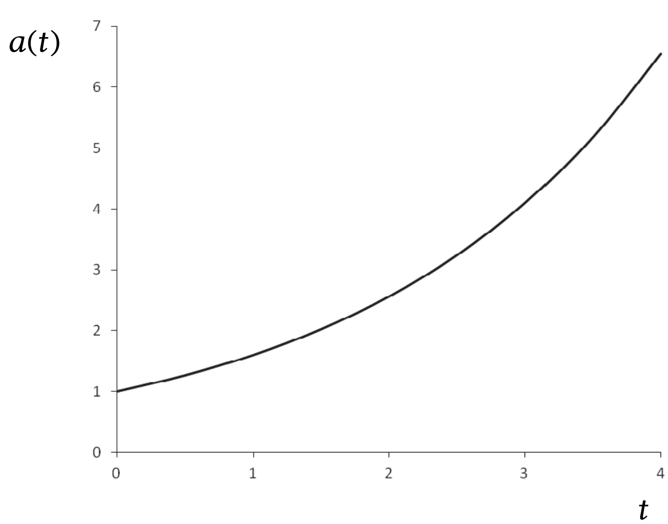


Real-world contracts that involve interest should specify how interest for partial periods will be calculated. For our purposes in studying interest theory, we will generally assume that interest is earned *continuously*. When interest is earned continuously, the formulas $a(t)=(1+i)^t$ and $A(t)=A(0)\left(1+i\right)^t$ are valid for *all* values of t, not just *integer* values, and the accumulation and amount functions are continuous functions (not step functions). Unless an exam question specifies that interest is credited only at the end of each year (or the end of each month or each quarter, etc.), you should assume that interest is earned continuously.

In the current example, the value of the accumulation function at time 1.5 is:

$$a(1.5)=(1.60)^{1.5}=2.0239$$

If interest is *earned continuously*, the graph of $a(t)$ is a smooth, continuous function:



For *simple* interest at a constant annual rate i, the accumulation and amount functions are:

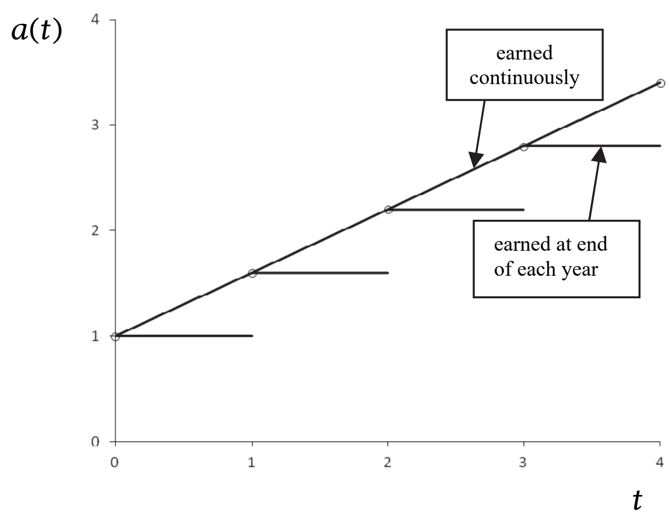
(1.7)

Simple interest:
$$a(t) = (1 + i \cdot t) \qquad A(t) = A(0)\left(1 + i \cdot t\right)$$

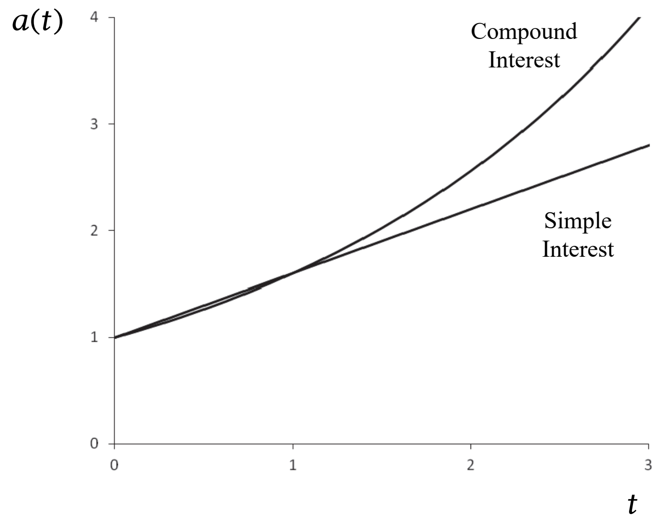
For simple interest at a rate i = 60% per year, the values of $a(t)$ at the end of each of the first 4 years are:

t	0	1	2	3	4
$a(t)$	1	1.60	2.20	2.80	3.40

Again, $a(t)$ can be a step function if interest is considered to be earned only at the end of each year. However, as with compound interest, we will generally treat simple interest as being earned continuously, so that $a(t)$ is a continuous function. The following graph includes plots for both the step function and the continuous function.



The following graph compares the growth of 2 investments. In each case, an amount of 1 is invested at time 0. One investment earns *simple* interest at a 60% annual rate; the other earns *compound* interest at a 60% annual rate.



Both investments have a value of 1.60 at the end of 1 year. After the first year, the one earning compound interest grows much faster, as it earns "interest on interest." The simple-interest investment earns interest on only the original principal of 1, so its growth (its slope) is constant at 0.60 per year. Note, however, that the investment at simple interest has a *larger* value between time 0 and time 1 than the investment earning compound interest, since $1+t\cdot i>(1+i)^t$ for values of t between 0 and 1.

The simple-interest investment is growing faster (has a steeper slope) than the compound-interest investment at the beginning of the first year. But the compound-interest investment grows faster and faster as the year progresses (because it earns interest on a larger and larger principal). By the end of the first year, the compound-interest investment has caught up with the simple-interest investment, and thereafter exceeds it.

Section 1.4

Effective Rate of Interest

We will now use the amount function to define the **effective rate of interest** for any specified time period. For the one-year time period $[t, t+1]$, the beginning amount is $A(t)$, the ending amount is $A(t+1)$, and the amount of interest earned over the interval is $A(t+1) - A(t)$.

The effective rate of interest for this period is defined as:

$$\boxed{i_{[t,t+1]} = \frac{\text{interest earned between } t \text{ and } t\text{+1}}{\text{value of investment at time } t} = \frac{A(t+1) - A(t)}{A(t)} = \frac{a(t+1) - a(t)}{a(t)}}$$

(1.8)

Note: The notation $i_{[t_1,t_2]}$ will be used in this module to represent an effective interest rate for the period $[t_1, t_2]$. This is <u>not</u> standard actuarial notation, and it will not be used in the other modules of this manual.

Example (1.9)

Let the interest rate be 6% and the time interval be [1,2].

For compound interest:

$$i_{[1,2]} = \frac{a(2) - a(1)}{a(1)} = \frac{1.1236 - 1.06}{1.06} = 0.06$$

For simple interest:

$$i_{[1,2]} = \frac{a(2) - a(1)}{a(1)} = \frac{1.12 - 1.06}{1.06} = 0.0566$$

Exercise (1.10)

Let the interest rate be 6% and the time interval be [2,3].
Find $i_{[2,3]}$ for a) compound interest at 6%, and b) simple interest at 6%.

Answers: a) 0.06 b) 0.0536

Note that over multi-year periods a compound interest rate of 6% per year gives a constant effective rate of 6% for each one-year period, while a simple interest rate of 6% leads to declining effective rates over time. This is because the investment at compound interest always earns 6% on the entire beginning-of-year balance, but the investment at simple interest earns 6% on only the original principal.

The above discussion involves effective rates over a 1-year period of time. These are called **annual effective rates**. We will usually express rates of compound interest as annual effective rates, but we can also calculate an effective rate for a shorter or longer time period (e.g., a quarterly effective rate, or a 2-year effective rate). In each case, the effective rate equals the amount of interest earned during the period (e.g., a 3-month period or a 2-year period), divided by the value of the investment at the beginning of the period.

Example (1.11)

An investment earns a 6% annual interest rate.
We will calculate the quarterly (3-month) effective rates for the periods [0.25,0.50] and [1.25,1.50]. We will do each of these calculations based on a 6% rate of compound interest, and also for a 6% rate of simple interest.

At 6% compound interest

For the period [0.25,0.50]:

$$i_{[0.25,0.50]} = \frac{a(0.50)-a(0.25)}{a(0.25)} = \frac{1.06^{0.5}-1.06^{0.25}}{1.06^{0.25}} = 0.01467$$

For the period [1.25,1.50]:

$$i_{[1.25,1.50]} = \frac{a(1.50)-a(1.25)}{a(1.25)} = \frac{1.06^{1.50}-1.06^{1.25}}{1.06^{1.25}} = 0.01467$$

At 6% simple interest

For the period [0.25,0.50]:

$$i_{[0.25,0.50]} = \frac{a(0.50)-a(0.25)}{a(0.25)} = \frac{\left[1+(0.5)0.06\right]-\left[1+(0.25)0.06\right]}{1+(0.25)0.06} = 0.01478$$

For the period [1.25,1.50]:

$$i_{[1.25,1.50]} = \frac{a(1.50)-a(1.25)}{a(1.25)} = \frac{\left[1+(1.50)0.06\right]-\left[1+(1.25)0.06\right]}{1+(1.25)0.06} = 0.01395$$

During the period [0.25,0.50], 6% simple interest generated a higher quarterly effective rate than 6% compound interest. As we noted previously, early in the first year simple interest produces faster growth than compound interest at the same numerical interest rate (but compound interest catches up at the end of the first year).

During the period [1.25,1.50], of course, compound interest produces a higher effective rate than simple interest. The quarterly effective rate for compound interest during this period is 0.01467, the same as it was for the period [0.25,0.50]. But the quarterly effective rate for simple interest has decreased from 0.01478 to 0.01395.

Exercise (1.12)

An investment earns a 6% annual interest rate. Calculate the quarterly (3-month) effective rates for the periods [0.50,0.75] and [1.50,1.75]:

a) at 6% compound interest, and
b) at 6% simple interest

Are the rates calculated in a) (at compound interest) higher or lower than the rates calculated in b) (at simple interest)?

Answer: a) 0.01467 for each period b) 0.01456 and 0.01376;
Compound interest produces a higher effective rate in each period.

Section 1.5

Nominal Rates of Interest

In many instances where payments (such as loan payments) are made more frequently than once a year (e.g., monthly, quarterly, or semi-annually), the interest rate is expressed as a **nominal annual rate**. A nominal annual rate of interest is equal to the effective interest rate *per period* multiplied by the number of periods per year.

For example, if an investment is earning interest at a 2% quarterly effective rate, you could multiply 2% by 4 and refer to this as a "nominal annual rate of 8%, convertible quarterly" (or "compounded quarterly"). This gives us a simple way of referring to the interest rate on an *annual scale*, but 8% is *not* the rate you *actually* earn each year. In this example, you actually earn *more* than 8%. One dollar accumulates to $(1.02)^4 = 1.0824$ in one year, so a nominal annual rate of 8% convertible quarterly is *equivalent* to an annual effective rate of 8.24%.

Many students find this confusing, so we will go over it again for reinforcement:

1. **The effective rate per period is your starting point**
   Example: 2% per quarter (a 2% "quarterly effective rate")

2. **Calculate the nominal annual rate.**
   Nominal Rate = (Rate/period) × (Number of periods per year)
   Example $2\% \times 4 = 8\%$
   (an 8% "nominal annual rate convertible quarterly")

3. **The annual effective rate is the annual rate you *actually* earn with compounding of interest**
   Example. End-of-year accumulated value is $(1.02)^4 = 1.0824$
   The annual effective rate is 8.24%.

A nominal rate is an artificial rate that gives you a way of *talking about* a periodic rate (such as a quarterly or monthly effective rate) in familiar *annual* terms. The annual effective rate is *not* artificial. It is the rate you *actually* earn in a year. Similarly, a quarterly effective rate is the rate you *actually* earn in one quarter.

It is important to understand that interest calculations are *always* done using effective rates (whether annual, quarterly, monthly, etc.) Nominal rates are *not* used in calculations; a nominal rate must first be converted to an effective rate, and then calculations are performed using the effective rate.

Exercise (1.13)

Suppose you earn interest at a rate of 1% per month, compounded monthly (i.e., a 1% monthly effective rate).

a) What is your nominal annual rate?

b) What is your annual effective rate?

Answer: a) 12% convertible monthly b) 12.6825%

In the general case of m conversion periods per year, we denote the nominal rate by $i^{(m)}$. The effective interest rate per period is $\frac{i^{(m)}}{m}$, and the annual effective rate is:

(1.14)

$$\boxed{i=\left(1+\frac{i^{(m)}}{m}\right)^{m}-1}$$

This has the important consequence that:

(1.15)

$$\boxed{1+i=\left(1+\frac{i^{(m)}}{m}\right)^{m}}$$

You will often see the statement that interest is "**convertible**" or "**compounded**" ***m* times per year.** This means that the interest earned during each period (of length $1/m$ years) is "compounded" (converted to principal) at the end of that period and earns interest during the following period.

Example (1.16)

Suppose interest is convertible monthly and the nominal rate is $i^{(12)}=0.09$. Then the annual effective rate is:

$$\left(1+\frac{0.09}{12}\right)^{12}-1=1.0075^{12}-1=0.0938 \qquad \text{(or 9.38\%)}$$

This process can easily be reversed to find the nominal rate if we are given the effective rate.

Example (1.17)

Interest is convertible semi-annually and results in an annual effective rate of 10.25%. Find the nominal annual rate convertible semi-annually.

Solution.
$m = 2$, so we need to find $i^{(2)}$.
By **(1.15)**,

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.1025$$

$$\left(1+\frac{i^{(2)}}{2}\right) = \sqrt{1.1025} = 1.05$$

$$i^{(2)} = 2(1.05-1) = 10\%$$

Thus the nominal annual rate is 10% convertible semi-annually, and the semi-annual effective rate is 5%.

Note that you can derive a formula that solves for $i^{(m)}$ given i and m. It is:

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

It is not necessary to memorize this formula. Formula **(1.15)** is intuitive and easy to remember, and we can always substitute the given values for i and m into it to solve for $i^{(m)}$. This is the approach we used in Example **(1.17)**.

Calculator Note

The BA II Plus calculator has an interest conversion worksheet that can be used to solve these problems. The legend above the $\boxed{2}$ key is ICONV, which stands for "interest conversion." You can activate the worksheet by using the keystrokes. $\boxed{\text{2ND}}$ ICONV. The worksheet has three variables:

NOM for nominal annual rate
EFF for annual effective rate
C/Y for number of conversion periods per year

You can scroll among these variables using the ↑ and ↓ keys.

In Example **(1.16)** we found the effective rate corresponding to a nominal annual rate of 9% convertible monthly. To do this on the BA II Plus calculator, enter the ICONV worksheet and scroll to the line for NOM. Key in 9 and hit the $\boxed{\text{Enter}}$ key. Then scroll ↑ to the line for C/Y and key in 12 and hit the $\boxed{\text{Enter}}$ key. Then scroll ↑ to the line for EFF and use the $\boxed{\text{CPT}}$ key to compute the effective rate. The rate displayed is EFF = 9.38 (to two decimal places).This means 9.38%, so the rate is 0.0938.

In **(1.17)** we found the nominal rate corresponding to an effective rate of 10.25% convertible semi-annually. To do this on the BA II Plus calculator, enter the ICONV worksheet and scroll to the line for EFF. Key in 10.25 and hit the $\boxed{\text{Enter}}$ key. Then scroll ↓ to the line for C/Y and key in 2 and hit the $\boxed{\text{Enter}}$ key. Then scroll ↓ to the line for NOM and use the $\boxed{\text{CPT}}$ key to compute the effective rate. The rate displayed is NOM = 10 (that is, 10%, or 0.10).

The ICONV worksheet can be used to calculate EFF or NOM, but not C/Y (the number of conversion periods per year.)

To exit the ICONV worksheet, press the $\boxed{\text{C/E}}$ key. This key will also allow you to exit any other BAII Plus worksheet. (Sometimes you will have to press $\boxed{\text{C/E}}$ more than once.)

Exercise (1.18)

a) Given $i^{(12)} = 6\%$, find the annual effective rate i.
b) Given an annual effective rate of $i = 5\%$, find $i^{(12)}$.

Answers: a) 6.168% b) 4.889%