Interest Theory

Financial Mathematics
and
Deterministic Valuation

Third Edition

Joe Francis
Fellow of the Society of Actuaries, Chartered Financial Analyst

Chris Ruckman
Fellow of the Society of Actuaries

ActuarialBrew.com

Copyright © 2022 by ActuarialBrew

All rights reserved. No part of this publication may be reproduced, distributed, or transmitted in any form or by any means, including photocopying, recording, or other electronic or mechanical methods, without the prior written permission of the publisher, except in the case of brief quotations for a book review or other noncommercial uses permitted by copyright law. For permission requests, please email info@actuarialbrew.com.

www.ActuarialBrew.com

ISBN 0-9981604-4-3

Table of Contents

Introduction

This textbook strives to move you through the material quickly, highlighting important concepts along the way. The sooner you understand the material, the sooner you can begin the important work of practicing questions. The key to mastering this material is to work as many questions as possible.

Intermediate calculations in this text are not rounded. As you work through the examples and questions, you will match the solutions if you keep the numbers stored in your calculator.

We've formatted this textbook so that our comments, warnings, tips, examples, and Key Concepts are readily recognized.

An italicized section is a side comment from the author.

The friendly owl at left provides helpful comments, often adding context to the surrounding material.

The owl with a book gets into the details. It isn't always necessary to delve into this level of detail, so this icon indicates material that is optional during the first reading of the material.

The owl with an exclamation point sign warns of potential pitfalls or traps. We've identified certain mistakes as being particularly easy to make, and we use this icon to warn you away from them.

The owl with a magic wand alerts you to tips and tricks that can save you time while learning the material and/or working the questions.

Example 0.00

Examples are denoted with double lines in the left margin. The word "Example" appears to the left of the double lines. An example begins with a question.

Solution

An example concludes with a solution. The word "Solution" appears to the left of the double lines.

0.00

Key Concepts

Key Concepts are denoted with a solid dark line in the left margin and a key icon. It is important to understand the Key Concepts thoroughly. In most cases, the Key Concepts should be memorized.

The calculator indicates that the Texas Instruments BA II Plus calculator is used to perform calculations.

If you encounter any errors in this text, please let us know. Send any questions or comments to Info@ActuarialBrew.com.

Full solutions to the practice questions at the end of each chapter can be found online at www.ActuarialBrew.com. Each solution has the following key to indicate the question's degree of difficulty. The more boxes that are filled in, the more difficult the question:

Easy:

Very Difficult:

Chapter 1: Setting the Stage

This chapter introduces basic terminology and lays the groundwork for the subsequent chapters.

1.01 BA II Plus Calculator

This text frequently references the Texas Instruments BA II Plus calculator. The instructions for the BA II Plus are the same as those for the BA II Plus Professional, and either calculator is an excellent choice for use with this text.

Digits Displayed

Although the BA II Plus retains 13 digits of accuracy, it only shows 2 decimal places when at its default setting. The following keystrokes change the calculator to show up to 9 decimal places:

[2nd] [FORMAT] 9 [ENTER] [2nd] [QUIT]

Order of Operations

The calculator uses the chain calculation method when at its default setting. The chain calculation executes the operations as they are entered, so we have:

Chain: $4+2\times 5=30$

The calculation method can be changed to the algebraic operating system (AOS), which uses the standard order of operations:

[2nd] [FORMAT] ↓↓↓↓ [2nd] [SET] [2nd] [QUIT]

Now we have:

AOS: $4+2\times 5=14$

Instructions in this text are based on the AOS calculation method.

Payments per Year and Compounding Periods per Year

The BA II Plus has internal settings for payments per year (P/Y) and compounding periods per year (C/Y). Some BA II Plus calculators default these settings to 12. Changing the P/Y setting to 1 will automatically change the C/Y setting to 1:

[2nd] [P/Y] 1 [ENTER] [2nd] [QUIT]

Instructions in this text are based on both P/Y and C/Y being set equal to 1.

If you take an actuarial exam, the exam administrator will reset your calculator prior to the exam. This means that it will then revert to using the chain calculation method, and it may change the settings for P/Y and C/Y. After it has been reset, you can easily change it back to the settings shown above. If you would like to practice, you can reset the calculator as follows:

[2nd] [RESET] [ENTER] [2nd] [QUIT]

Clearing the Worksheets

Instructions in this text are based on an assumption that that the relevant worksheet's register has been cleared. You can clear the time value of money (TVM) worksheet as follows:

[2nd] [CLR TVM]

To clear any of the other worksheets, you must be in the worksheet. The command for clearing a non-TVM worksheet is:

[2nd] [CLR WORK]

Beginning Vs. End of Period

Unless stated otherwise, instructions in this text are based on an assumption that the calculator is set to treat payments as occurring at the end of each period.

To toggle the calculator between treating payments as occurring at the beginning of each period or the end of each period, use the following key strokes:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

When the calculator is set to treat payments as occurring at the beginning of each period, the letters BGN appear in the upper right of the display.

Unless stated otherwise, the instructions in this text are based on an assumption that BGN does not appear in the upper right of the display.

Basic Equation in TVM Worksheet

The BA II Plus allows us to solve for any one of the following 5 variables if given the other 4: *N*, *I/Y*, *PV*, *PMT*, and *FV*. These variables are components of the two equivalent equations below:

$$PV + PMT \times a_{\overline{N|}I/Y} + FV \times v^{N} = 0$$

$$PV \times (1 + I / Y)^{N} + PMT \times s_{\overline{N|}I/Y} + FV = 0$$

We do not use the expression above until we get to Section 6.02, but it is included here so that we have the useful information about the calculator in one place. Don't worry if the variables and the equations don't make sense to you yet!

Rounding

Although this text uses rounded values when showing intermediate steps, the full values are retained for calculation purposes. For example, the conventions of this text permit the expression on the left but not the one on the right:

YES	NO
$\frac{2}{3} \times 2 = 0.67 \times 2 = 1.33$	$\frac{2}{3} \times 2 = 0.67 \times 2 = 1.34$

1.02 Valuation Date and Payment Date

Valuing assets can be difficult. As anyone who has ever watched *Antiques Road Show* on PBS can attest, the value of an asset isn't always obvious. But valuing cash seems like a much simpler task. What is the value of $100? It's obviously $100.

Valuing cash is a simple proposition when the valuation date and the payment date are the same. Below are some simple examples where the valuation date is equal to the payment date:

- If someone gave you $50 last month, then it was worth $50 at that time.
- The value now of a payment of $100 now is $100.
- The value in one year of a payment of $200 at that time is then $200.

When the payment date does not match the valuation date, however, valuation can become more complicated.

What is the value now of $100 to be received in one year? As a starting point, observe that receiving $100 in one year is less desirable than receiving $100 now. This is because if we have $100 now, then we can deposit that $100 in a bank account and the balance will grow to more than $100 in one year. The difference between the $100 deposited and the bank balance in one year is known as **interest**.

The **time value of money** is the value at a particular point in time of a payment or set of payments. As noted above, a payment of $100 to be made one year from now has a value of less than $100 now, and its value in one year is $100. The time value of money is also known as the **current value** at a particular point in time.

An **equation of value** equates the time value of a payment or set of payments with another payment or set of payments, where all of the payments are valued at a particular point in time. For example, suppose that \$100 deposited in a bank account now will grow to \$103 at the end of one year. In that case, an equation of value for time 0 would tell us that the **present value** today of the \$103 is \$100:

$$\text{Time 0 equation of value: } PV_0 = 100$$

where:

$$PV_t = \begin{array}{c}\text{Present value at time } t \text{ of a payment} \\ \text{or payments occuring on or after time } t\end{array}$$

The present value of a future payment is the amount that must be lent in order to receive that payment later. In subsequent chapters, we see that the present value function depends on the type of interest rate used.

We can also set up an equation of value for time 1 that tells us that the **future value**, or **accumulated value**, in one year of \$100 payable now is \$103:

$$\text{Time 1 equation of value: } AV_1 = 103$$

where:

$$AV_t = \begin{array}{c}\text{Accumulated value at time } t \text{ of a payment} \\ \text{or payments occuring on or before time } t\end{array}$$

The accumulated value of a payment is the amount that a lender receives for lending that payment earlier. In subsequent chapters, we see that the accumulated value function depends on the type of interest rate used.

*When the accumulated value is expressed as a function of the time elapsed, the function is sometimes referred to as the **amount function**.*

1.03 Current Values

The current value of a set of payments is the sum of the accumulated value of the payments that occur before the valuation date, the value of any payment made on the valuation date, and the present value of the payments made after the valuation date:

$$CV_t = AV_t(\text{payments occurring before } t) + Pmt_t + PV_t(\text{payments occurring after } t)$$

where:

$$CV_t = \text{Current value at time } t \text{ of a set of payments}$$

$$Pmt_t = \text{Payment made at time } t$$

If an entire set of payments is made on or before the valuation date t, then the time-t current value is an accumulated value. If an entire set of payments is made on or after the valuation date t, then the time-t current value is a present value. Although accumulated values and present values are current values, the converse is not necessarily the case.

1.04 Loans Repaid with a Single Payment

Consider a transaction between two parties, in which the first party receives an amount of money now from the second party in exchange for agreeing to make a future, larger payment to the second party at some point in the future. This transaction is a **loan**.

The amount of money exchanged now is called the **principal**. The amount by which the future payment exceeds the principal is called the **interest**:

$$\text{Future Payment} - \text{Principal} = \text{Interest}$$

The party that accepts the principal now is the **borrower**. The borrower agrees to repay the principal plus interest in the future. The present value of the loan is the principal:

$$\text{Present Value} = \text{Principal}$$

The party that provides the principal now is the **lender**. The lender receives the principal plus interest in the future. The accumulated value of the loan is the future payment:

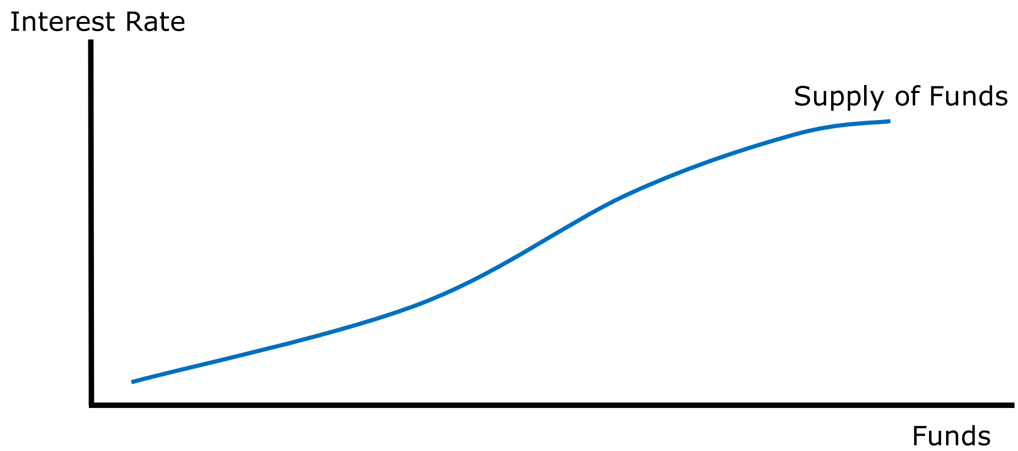
$$\text{Accumulated Value} = \text{Future Payment}$$

The accumulated value is also known as the future value.

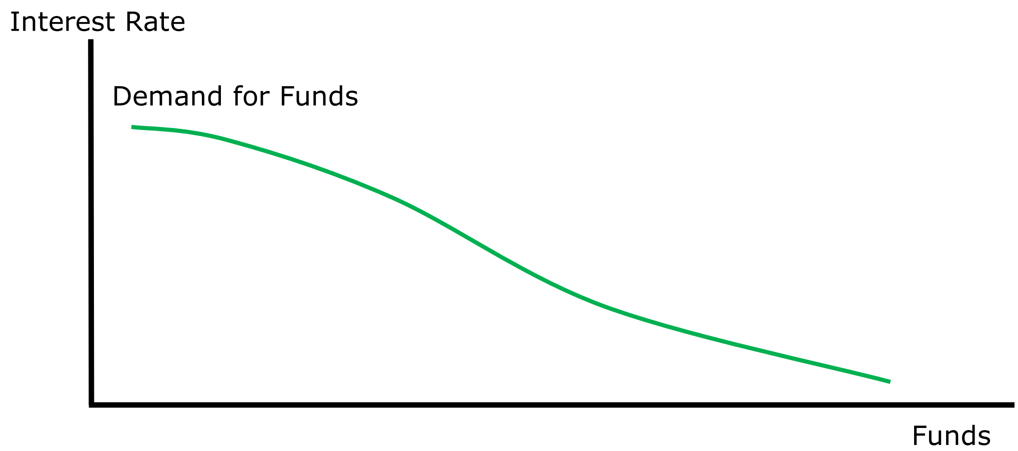
1.05 Supply and Demand Curves for Funds

An interest rate is used to calculate the amount of interest that is paid on a loan. There are many different ways of expressing interest rates, and this text explores those ways in the upcoming chapters. For now, however, it is sufficient to know that higher interest rates lead to higher interest payments.

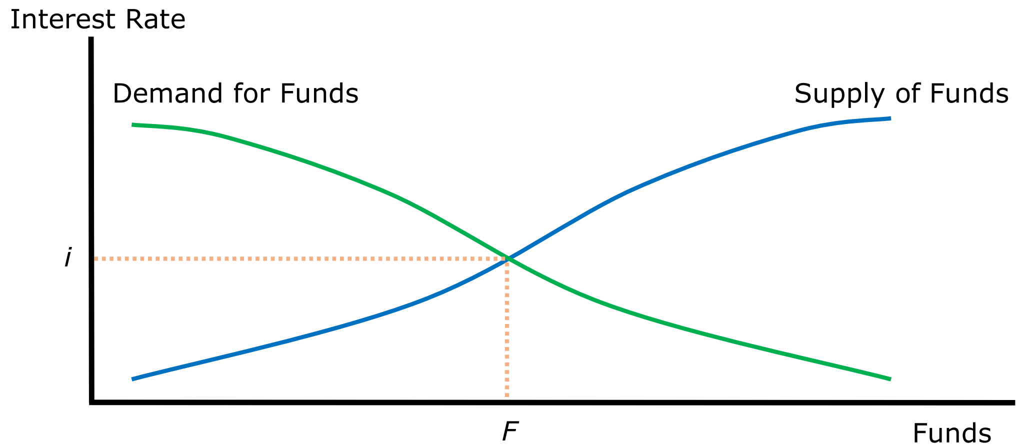
A lender is motivated by the opportunity to earn interest. In order to earn interest, the lender must forgo current consumption because the lender is unable to use the loaned funds to consume goods and services until the loan is repaid. Interest is therefore compensation for deferred consumption. A higher interest rate encourages lenders to lend more. If the interest rate is relatively low, then lenders makes fewer loans. The quantity of funds made available by lenders increases with the interest rate, as shown below in the graph of a supply curve for funds.



A borrower has the opposite response to interest rates. A borrower wishes to consume now and is willing to pay a higher price in the future in order to consume immediately. Interest, from the borrower's perspective, is the cost to consume immediately when resources are not currently available. The higher the interest rate, the higher the future price of immediate consumption because the future price is the principal plus interest. Therefore, a higher interest rate causes the borrower to borrow less, while a lower interest rate causes the borrower to borrow more. This can be observed in the graph below of a demand curve for funds.



The equilibrium interest rate, *i*, can be observed where the supply and demand curves intersect.



Suppose the interest rate is less than *i*. In that case, the quantity of funds demanded is greater than *F* and the quantity of funds supplied is less than *F*. Some borrowers are therefore unable to obtain loans. These borrowers contact potential lenders and offer to pay more than *i*, thereby enticing lenders to make more funds available. This continues until the interest rate increases to *i*.

If the interest rate is greater than *i*, then the quantity of funds supplied is greater than *F* and the quantity of funds demanded is less than *F*. Some lenders are unable to place their funds with a borrower. These lenders contact borrowers and offer to lend at less than *i*, thereby enticing borrowers to take out more loans. This continues until the interest rate falls to *i*.

The discussion above describes how the supply and demand for loanable funds determines the interest rate, but to fully understand the interest rates observed in the real world, we must take many other considerations into account, including the following:

- Term – the length of time until the loan matures
- Credit quality – the likelihood that the borrower will make the required payments
- Forecasts – the predicted levels of future interest rates and other macroeconomic variables
- Liquidity – the ability of a lender to sell a loan with little to no transaction costs
- Inflation – the loss of purchasing power of a currency over time

Furthermore, decisions regarding saving versus lending can be affected by lifestyle decisions and demographics. For example, a large number of young adults may choose to borrow in order to buy houses. Conversely, a large number of older adults may choose to increase their savings as they anticipate retirement. As a result, the supply and demand curves shift and change shape over time.

1.06 Day Count Convention

Unless otherwise specified, this text is based on the following assumptions:

- There are 30 days in a month.
- There 365 days in a year.
- There are 52 weeks in a year.
- There are 12 months in a year.

Only the final assumption is indisputable. In fact, the other assumptions aren't even internally consistent. They suffice, however, when only one assumption is required.

Example 1.01 Ann lends money for 5 days. For how many years does Ann lend the money?

Solution The number of years that Ann lends the money is:

$$\frac{5}{365} = \mathbf{0.0137}$$

Example 1.02 Ann lends money for 5 days. For how many months does Ann lend the money?

Solution The number of months that Ann lends the money is:

$$\frac{5}{30} = \mathbf{0.1667}$$

Example 1.03 Ann lends money for 2 weeks. For how many years does Ann lend the money?

Solution The number of years that Ann lends the money is:

$$\frac{2}{52} = \mathbf{0.0385}$$

1.07 Useful Formulas

The formulas below are provided for convenient reference.

In the following formulas for derivatives, u and v are functions of x, and a is a constant:

Product rule: $(uv)' = u'v + uv'$

Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Exponential rule: $\left(a^x\right)' = a^x \ln(a)$

Logarithm rule: $\left(\ln(x)\right)' = \frac{1}{x}$

The sum of a geometric series is the first term of the series minus the term that would come next, divided by 1 minus the factor:

$$\sum_{t=0}^{n-1} ar^t = \frac{a - ar^n}{1 - r}$$

If a quadratic equation does not factor easily, then the quadratic formula can be useful:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.08 Questions

Question 1.01

Consider the following statements:

I. The future value of a set of payments is also known as the accumulated value of the payments.

II. The current value of a set of payments is always the present value of payments that will occur in the future.

III. When a loan is to be repaid by a single payment in the future, the interest is equal to the difference between that payment and the principal.

IV. A lender pays interest to receive funds now.

Which of the statements is TRUE?

A I only B II only C I and III only D II and IV only E IV only

Question 1.02

Which of the statements regarding the market for loanable funds is FALSE?

A Interest compensates the lender for deferring consumption.

B The supply of funds available for lending increases with the interest rate.

C The quantity of funds demanded by borrowers decreases as the interest rate increases.

D Interest is the cost that a borrower pays to consume immediately when resources are not currently available.

E Supply and demand curves do not change over time.

Chapter 2: Simple Interest and Discount

2.01 Simple Interest Rates

Simple interest rates can be used to compute the interest for a very simple transaction. If the simple interest rate is i per unit of time, then the interest per unit of time is the principal times the interest rate:

$$\text{Interest per unit of time} = \text{Principal} \times i$$

We often use one year as our unit of time, in which case i is an annual interest rate.

Using simple interest, the amount of interest is calculated by multiplying the simple interest rate times the principal times the amount of time that the loan is in effect. If the loan lasts for t units of time then:

$$\text{Interest} = \text{Principal} \times i \times t$$

Unless otherwise specified, we assume that the unit of time is one year.

Example 2.01 A loan of $1,000 is made now. The simple interest rate is 5% per year. At the end of 5 years, what is the amount of interest earned over the 5 years?

Solution The amount of interest is:

$$\text{Interest} = \text{Principal} \times i \times t = 1{,}000 \times 0.05 \times 5 = \mathbf{250}$$

The present value of the loan is equal to the principal:

$$\text{Present Value} = \text{Principal}$$

The accumulated value is equal to the present value plus the interest:

$$\begin{aligned}\text{Accumulated Value} &= \text{Present Value} + \text{Interest} \\ &= \text{Present Value} + (\text{Present Value}) \times i \times t \\ &= (\text{Present Value})(1 + it)\end{aligned}$$

The ratio of the accumulated value to the present value is known as the **accumulation function**, and it is the accumulated value of an initial investment of 1. It is denoted by $a(t)$, and under simple interest, the accumulation function is:

$$a(t) = \frac{\text{Accumulated Value}}{\text{Present Value}} = 1 + it$$

2.01

Present Value and Accumulated Value Under Simple Interest

The accumulated value of a payment is the present value of the payment times the accumulation function:

$$AV_t = PV_0(1 + it)$$

where:

$AV_t = $ Accumulated value at time t

$PV_0 = $ Present value at time 0

$i = $ Simple interest rate per unit of time

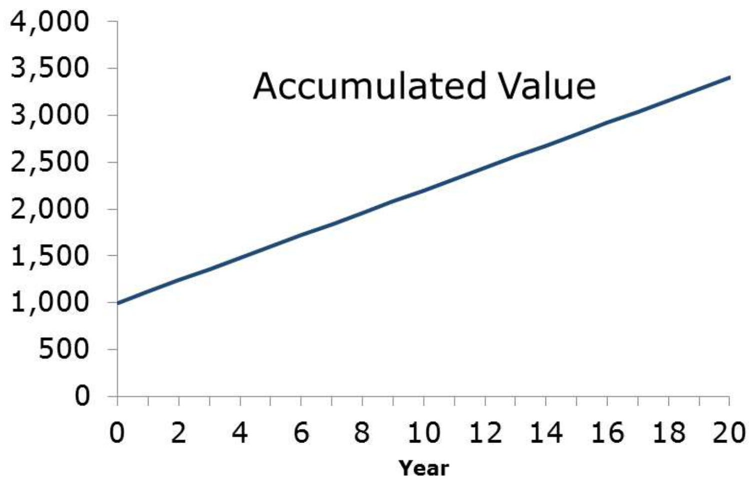
A more general form of the formula in the Key Concept above is given by:

$$AV_{t_2} = PV_{t_1}\left[1 + i(t_2 - t_1)\right] \qquad \text{where: } t_1 < t_2$$

When the accumulated value is the future value of a single initial amount, the accumulated value can be referred to as the **amount function**, and it is the initial value times the accumulation function:

$$AV_t = PV_0 \times a(t) = PV_0(1 + it)$$

The graph below uses a simple interest rate of 12% to show how $1,000 accumulates over 20 years. Under simple interest, the accumulated value is a straight line.



We can rearrange the formula above to find the present value of an accumulated value:

$$AV_t = PV_0(1 + it)$$

$$PV_0 = \frac{AV_t}{(1 + it)}$$

Example 2.02 The simple interest rate is 4% per year. How much money would Jenny need to deposit now to have $12,870 in 8 years?

Solution The amount to be deposited now is the present value of $12,870:

$$PV_0 = \frac{AV_8}{(1 + 8i)} = \frac{12,870}{1 + 8 \times 0.04} = \frac{12,870}{1.32} = \mathbf{9,750}$$

The next example verifies the result from the preceding example.

Example 2.03 The simple interest rate is 4% per year. If Jenny deposits $9,750 now, what is her accumulated value at the end of 8 years?

Solution The accumulated value is:

$$AV_8 = PV_0(1 + 8i) = 9,750(1 + 8 \times 0.04) = 9,750(1.32) = \mathbf{12,870}$$

If the time interval is not an integer, then interest is credited for the partial unit of time.

Example 2.04 The simple interest rate is 4% per year. If Jenny deposits $10,000 now, what is her accumulated value at the end of 18 months?

Solution Since the simple interest rate is expressed as an annual value, we use years as our unit of time. 18 months is equal to 1.5 years, and her accumulated value at the end of 1.5 years is:

$$AV_{1.5} = PV_0(1 + 1.5i) = 10,000(1 + 1.5 \times 0.04) = 10,000(1.06) = \mathbf{10,600}$$

2.02 Simple Discount Rates

As we saw above, a simple interest rate is multiplied times the principal to obtain the amount of interest per unit of time. A simple discount rate, on the other hand, is multiplied times the accumulated value to obtain the interest per unit of time, which in this context is often called the discount per unit of time:

If the simple discount rate is d per unit of time, then the discount per unit of time is:

$$\text{Discount per unit of time} = (\text{Accumulated value}) \times d$$

Using simple discount, the amount of discount is calculated by multiplying the simple discount rate times the accumulated value times the amount of time that the loan is in effect. If the loan lasts for t units of time then:

$$\text{Discount} = \text{Interest} = (\text{Accumulated Value}) \times d \times t$$

Under both simple interest and simple discount, the accumulated value is the present value plus the interest:

$$\text{Accumulated Value} = \text{Present Value} + \text{Interest}$$

$$\text{Accumulated Value} = \text{Present Value} + \text{Discount}$$

Rearranging the final expression above, the present value is equal to the accumulated value minus the discount:

$$\text{Present Value} = \text{Accumulated Value} - \text{Discount}$$

$$\text{Present Value} = \text{Accumulated Value} - (\text{Accumulated Value}) \times d \times t$$

$$\text{Present Value} = (\text{Accumulated Value})(1 - dt)$$

If t becomes too large, then the present value becomes negative. Therefore, simple discount only makes sense when t is less than the inverse of d:

$$t < \frac{1}{d}$$

Example 2.05

The simple discount rate is 6% per year. Determine the present value of $1,000 to be paid in:

a. 1 year
b. 10 years
c. 20 years

Solution

The present value is the accumulated value minus the discount:

a. $PV_0 = 1{,}000(1 - 0.06 \times 1) = 1{,}000 \times 0.94 = \mathbf{940}$

b. $PV_0 = 1{,}000(1 - 0.06 \times 10) = 1{,}000 \times 0.4 = \mathbf{400}$

c. $PV_0 = 1{,}000(1 - 0.06 \times 20) = 1{,}000 \times (-0.20) = -200$

But the inverse of d is less than 20:

$$\frac{1}{0.06} = 16.67$$

Therefore, we cannot use the simple discount rate of 6% to find the present value of a payment to be made in 20 years, and **there is no solution**.

The present value of a future value can be found by multiplying the future value by a **discount factor**. Under simple discount, the discount factor is:

$$\text{Discount factor} = 1 - dt$$

2.02

Present Value and Accumulated Value Under Simple Discount

The present value is the accumulated value times a discount factor:

$$PV_0 = AV_t(1 - dt)$$

where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

d = Simple discount rate per unit of time

A more general form of the formula in the Key Concept above is given by:

$$PV_{t_1} = AV_{t_2}\left[1 - d(t_2 - t_1)\right] \qquad \text{where: } t_1 < t_2$$

We can rearrange the formula in the Key Concept above to find the accumulated value of a present value:

$$PV_0 = AV_t(1 - dt)$$

$$AV_t = \frac{PV_0}{(1 - dt)}$$

Example 2.06

The simple discount rate is 4% per year. How much money would Adam need to deposit now to have $12,870 in 8 years?

Solution

The amount to be deposited now is the present value of $10,000:

$$PV_0 = AV_t(1 - dt) = 12,870(1 - 0.04 \times 8) = 12,870 \times 0.68 = \mathbf{8,751.60}$$

The next example verifies the result from the preceding example.

Example 2.07

The simple discount rate is 4% per year. If Adam deposits $8,751.60 now, what is his accumulated value at the end of 8 years?

Solution

The accumulated value is:

$$AV_8 = \frac{PV_0}{(1 - 8d)} = \frac{8,751.60}{1 - 8 \times 0.04} = \frac{8,751.60}{0.68} = \mathbf{12,870}$$

If the time interval is not an integer, then discount is credited for the partial unit of time.

Example 2.08

The simple discount rate is 4% per year. Adam will receive $8,000 in 30 months. What is the present value of the payment?

Solution

Since the simple discount rate is expressed as an annual value, we use years as our unit of time. 30 months is equal to 2.5 years, and the present value is:

$$PV_0 = AV_t(1 - dt) = 8,000(1 - 0.04 \times 2.5) = \mathbf{7,200}$$

2.03 Equivalent Simple Interest and Discount Rates

For a given time interval of length t, we can find equivalent simple interest and discount rates. Let's set the accumulated value under simple interest equal to the accumulated value under simple discount:

$$PV_0(1 + it) = \frac{PV_0}{(1 - dt)}$$

$$1 + it = \frac{1}{(1 - dt)}$$

2.03

Equivalent Simple Interest and Simple Discount Rates

If a loan is to be repaid with a single payment at time t, then the equivalent simple interest and discount rates satisfy:

$$1 + it = \frac{1}{(1 - dt)}$$

Example 2.09

Kate borrows $1,000 now and will repay the loan with a payment of $2,000 in 20 years.

a. What is the simple interest rate?

b. What is the simple discount rate?

Solution

The simple interest rate is greater than the simple discount rate because the interest rate applies to the beginning balance (i.e., the present value), while the discount rate applies to the larger ending balance (i.e., the accumulated value):

a. The simple interest rate is found below:

$$AV_t = PV_0(1 + it)$$
$$2,000 = 1,000(1 + 20i)$$
$$i = \mathbf{0.05}$$

b. The simple discount rate is found below:

$$PV_0 = AV_t(1 - dt)$$
$$1,000 = 2,000(1 - 20d)$$
$$d = 0.025$$

Alternatively, we can convert the simple interest rate into the equivalent simple discount rate for a 20-year loan using the Key Concept above:

$$1 + it = \frac{1}{(1 - dt)}$$
$$1 + 0.05 \times 20 = \frac{1}{1 - 20d}$$
$$d = \mathbf{0.025}$$

The example below shows that to convert a simple interest rate into a simple discount rate, we must know the term of the loan.

Example 2.10

Alan borrows $1,000 now and will repay the loan with a payment of $1,500 in 10 years.

a. What is the simple interest rate?

b. What is the simple discount rate?

Solution

Although the simple interest rate is 5%, as in the example above, the discount rate is not the same as in the example above because the term of the loan is not the same:

a. The simple interest rate is found below:

$$AV_t = PV_0(1 + it)$$
$$1,500 = 1,000(1 + 10i)$$
$$i = \mathbf{0.05}$$

b. The simple discount rate is found below:

$$PV_0 = AV_t(1 - dt)$$
$$1,000 = 1,500(1 - 10d)$$
$$d = 0.033$$

Alternatively, we can convert the simple interest rate into the equivalent simple discount rate for a 10-year loan using the Key Concept above:

$$1 + it = \frac{1}{(1 - dt)}$$

$$1 + 0.05 \times 10 = \frac{1}{1 - 10d}$$

$$d = \mathbf{0.033}$$

In Section 3.04, we will see that under compound interest, we do not need to know the term of the loan to find a discount rate that is equivalent to the interest rate.

2.04 Equations of Value under Simple Interest and Simple Discount

To determine the value of a payment (also known as a **cash flow**), we need to know when it occurs and when it is being valued. Using the timing of a cash flow and the applicable interest or discount rate to find its value is known as taking the time value of money into account.

What are cash flows? A cash flow is a payment of cash. Payments are positive cash flows for the recipient and negative cash flows for the payer. For example, if Larry lends $1,000 to Mike today, and Mike repays Larry with $800 in one year and $400 in two years, then the cash flows to Larry and Mike are as follows:

Time	*Larry's Cash Flows*	*Mike's Cash Flows*
0	*–1,000*	*1,000*
1	*800*	*–800*
2	*400*	*–400*

Example 2.11 On January 31, 2022, $100 is deposited into a bank account earning simple interest at a rate of 8% per year. What is the value of the deposit on January 31, 2023?

Solution The value of the deposit after one year is found using an equation of value as of January 31, 2023. We can treat January 31, 2022 as time 0, and since January 31, 2023 is one year later, it is time 1:

$$AV_1 = PV_0 \times (1 + it) = 100 \times (1 + 0.08 \times 1) = \mathbf{108}$$

Don't be confused by the conversion of the calendar date into a numeric time. We are just assigning the earlier date to time 0 and counting the number of units of time (in this case, years) between the two calendar dates.

In the example above, the depositor can have $100 now or $108 in one year. At a simple interest rate of 8% per year, the two choices are equivalent.

An equation of value allows us to equate one set of cash flows with another at a particular point in time. In the example above, we used the simple interest rate to equate the January 31, 2022 deposit with the January 31, 2023 accumulated value. The equation of value equated the values on January 31, 2023.

Example 2.12 Evan will receive $$X$ in 2 years. Celeste will receive $400 in 1 year. At a simple rate of interest of 6% per year, the present values of their payments are equal. Determine X.

Solution An equation of value at time 0 equates the present value of Evan's payment with the present value of Celeste's payment:

$$\frac{X}{1+0.06\times 2}=\frac{400}{1+0.06\times 1}$$

$$\frac{X}{1.12}=377.36$$

$$X=377.36\times 1.12$$

$$X=\mathbf{422.64}$$

The given information above tells us that the present values are equal, so we know that an equation of value holds at time 0. Under simple interest, however, this does not imply that an equation of value holds for other times. For example, if Celeste lends her $400 for one year at a simple interest rate of 6% per year, then at time 2 she has:

$$400\times 1.06=424$$

This is clearly not equal to the payment of $422.64 that Evan receives at time 2. Therefore, even though we have an equation of value equating the present value of Celeste's payment and Evan's payment at time 0, the equation of value doesn't hold at time 2.

This is in contrast to compound interest, which is discussed in the next chapter. Under compound interest, if an equation of value can be used to equate two sets of payments at one point in time, then an equation of value can be used to equate the two sets of payments at any point in time.

2.05 Questions

Question 2.01

The simple interest rate is 7% per year. If James deposits \$10,000 at the end of 18 months, what is the accumulated value at the end of 5 years?

A 2,450 B 3,500 C 10,000 D 12,450 E 13,500

Question 2.02

The simple interest rate is 7% per year. Kevin makes a deposit of \$$X$ now, which accumulates to \$10,000 at the end of 8 years. Calculate X.

A 4,400 B 5,596 C 5,600 D 6,410 E 6,944

Question 2.03

The simple discount rate is 7% per year. Kevin makes a deposit of \$$X$ now, which accumulates to \$10,000 at the end of 8 years. Calculate X.

A 4,400 B 5,596 C 5,600 D 6,410 E 6,944

Question 2.04

Larry receives a medical bill that is due to be paid in full within 45 days.

If Larry pays within 15 days, he receives a 3% discount.

Larry has two choices:

1. Pay the bill in 15 days for 97% of the amount due in 45 days.
2. In 15 days, deposit 97% of the amount due in 45 days. The deposit will earn an annual simple discount rate of d for 30 days, at which time Larry will use the proceeds to pay the bill in full.

There are 365 days in a year. The amount of time elapsed in years is therefore equal to the number of days elapsed divided by 365.

What is the minimum value of d that will cause Larry to choose the second option?

A 36.00% B 36.50% C 37.11% D 37.63% E 42.58%

Question 2.05

Consider the following five accumulated values:

- A deposit of \$1,000 at an annual simple interest rate of 5% accumulates to A at the end of 5 years.
- A deposit of \$900 at an annual simple discount rate of 5% accumulates to B at the end of 5 years.
- A deposit of \$900 at an annual simple discount rate of 4% accumulates to C at the end of 6 years.
- A deposit of \$1,000 at an annual simple interest rate of 6% accumulates to D at the end of 4 years.
- A deposit of \$1,000 at an annual simple discount rate of 3% accumulates to E at the end of 8 years.

Which of the following accumulated values is highest: A, B, C, D, or E?

A Amount A B Amount B C Amount C D Amount D E Amount E

Question 2.06

Eric deposits $100 for *t* years at an annual simple interest rate of 5%.

Judy deposits $100 for *t* years at an annual simple discount rate of 4%.

At the end of *t* years, Eric's accumulated value is equal to *X*, and Judy's accumulated value is also equal to *X*. You are given that $t > 0$.

Calculate *X*.

A 105 B 115 C 125 D 150 E 175

Question 2.07

Diane provides a loan to Mike. The amount paid to repay the loan at time *t* is the same regardless of whether the loan accrues at an annual simple interest rate of *i* or an annual simple discount rate of *d*. Which of the following is a valid expression of *t*?

I. $t = \dfrac{i-d}{id}$

II. $t = \dfrac{i+d}{id}$

III. $t = \dfrac{i-d+id}{id}$

A I only B II only C III only D I and III only E None is valid

Question 2.08

Ann deposits $800 into a fund earning an annual simple interest rate of 6%.

Mike deposits $*X* into a fund earning an annual simple interest rate of 9%.

Each year, Ann earns twice the interest that Mike does. Calculate *X*.

A 133.33 B 266.67 C 400.00 D 533.33 E 1,066.67

Question 2.09

Tyler and Haley both invest 1,000 for a period of length *T* at an annual rate of *x*.

Tyler's account earns an annual simple interest rate of *x*.

Haley's account earns an annual simple discount rate of *x*.

You are given:

- $0 < x < 0.10$
- $0 < T < 10$

Which of the following statements about the final account values is true?

A Haley's account value is certain to be greater than Tyler's account value.

B Tyler's account value is certain to be greater than Haley's account value.

C If $x < \dfrac{1}{T}$, then Tyler's account value is certain to be greater than Haley's account value.

D If $x > \dfrac{0.5}{T}$, then Tyler's account value is certain to be greater than Haley's account value.

E None of the above

Question 2.10

A bank loans Jeff \$1,000 at an annual simple discount rate of 5%. The bank requires that the loan must be repaid within 10 years.

He puts the \$1,000 into a fund that earns an annual simple interest rate of 6%.

At time t, he withdraws the accumulated value in the fund and repays the loan, leaving him with \$$X$:

$$X = \begin{pmatrix} \text{Accumulated value} \\ \text{in fund at time } t \end{pmatrix} - \begin{pmatrix} \text{Accumulated value} \\ \text{of loan at time } t \end{pmatrix}$$

Jeff chooses t to maximize X.

Calculate X.

A 1.74 B 4.39 C 9.11 D 104.55 E 4,390.89

Chapter 15: Asset-Liability Management

15.01 Interest Rate Risk

Assets produce cash inflows, which are called asset cash flows. Liabilities produce cash outflows, which are called liability cash flows. The surplus of the present value of a firm's asset cash flows over the present value of the firm's liability cash flows is often simply called the firm's **surplus**:

$$S(y) = PV_A - PV_L$$

where:

$S(y) =$ Surplus when the yield is y

$PV_A =$ Present value of asset cash flows

$PV_L =$ Present value of liability cash flows

Example 15.01

A firm must pay \$100 at the end of 5 years. The firm has a 1-year zero-coupon bond that matures for \$66.15 and a 20-year zero-coupon bond that matures for \$31.72.

Calculate the surplus of the firm, using an annual effective interest rate of 8% to value the cash flows.

Solution

The present value of the liability cash flow is:

$$PV_L = \frac{100}{1.08^5} = 68.06$$

The present value of the asset cash flows is:

$$PV_A = \frac{66.15}{1.08} + \frac{31.72}{1.08^{20}} = 68.06$$

The surplus of the firm is:

$$S(0.08) = PV_A - PV_L = 68.06 - 68.06 = \mathbf{0.00}$$

If the yield decreases, then the firm in the example above faces **reinvestment risk** because the 1-year bond matures prior to the liability payment and must be reinvested at the new, lower yield. Reinvestment risk is the risk of earning a relatively low interest rate when investing future cash flows.

Example 15.02

A firm must pay \$100 at the end of 5 years. The firm has a 1-year zero-coupon bond that matures for \$66.15 and a 20-year zero-coupon bond that matures for \$31.72.

Both bonds were purchased when the annual effective interest rate was 8%.

Immediately after the firm purchases the bonds, the annual effective interest rate falls to 6%. Calculate the new surplus of the firm.

Solution

The present value of the liability cash flow is higher than before because the new interest rate of 6% is lower than the previous interest rate of 8%:

$$PV_L = \frac{100}{1.06^5} = 74.73$$

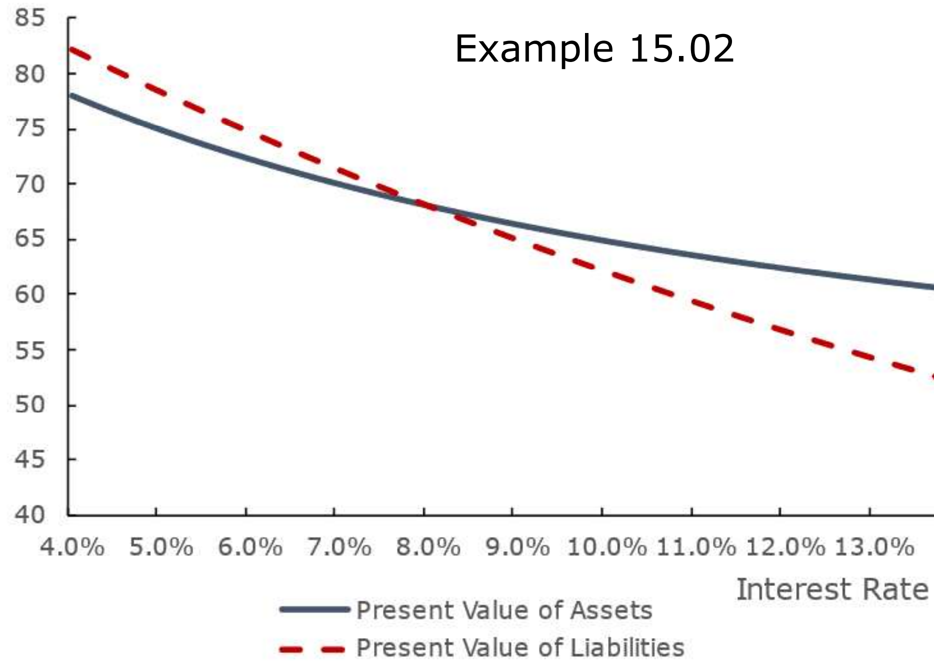
The present value of the asset cash flows is also higher before because the new interest rate of 6% is lower than the previous interest rate of 8%:

$$PV_A = \frac{66.15}{1.06} + \frac{31.72}{1.06^{20}} = 72.30$$

The surplus of the firm is now negative because the present value of the liabilities increased more than the present value of the assets:

$$S(0.06) = PV_A - PV_L = 72.30 - 74.73 = \mathbf{-2.43}$$

The graph below shows that when the interest rate decreases, the value of the asset cash flows described in the example above increases by less than the value of the liability cash flow.



To reduce reinvestment risk, the firm in the above example could reduce the amount invested in the shorter bond and increase the amount invested in the longer bond. This strategy exposes the firm to **price risk** though. Price risk is the risk that assets to be sold in the future could sell for a relatively low price if the yield on the assets increases.

Example 15.03

A firm must pay \$100 at the end of 5 years. Observing the potential for reinvestment risk, the firm purchases less of the 1-year bond and more of the 20-year bond than shown in the previous example. The firm purchases a 1-year zero-coupon bond that matures for \$36.75 and a 20-year zero-coupon bond that matures for \$158.61. Both bonds were purchased when the annual effective interest rate was 8%.

a. Calculate the surplus of the firm when the annual effective interest rate is 8%.

b. Calculate the surplus of the firm if the annual effective interest rate immediately increases to 10%.

Solution

a. The present value of the liability cash flow is:

$$PV_L = \frac{100}{1.08^5} = 68.06$$

The present value of the asset cash flows is:

$$PV_A = \frac{36.75}{1.08} + \frac{158.61}{1.08^{20}} = 68.06$$

The surplus of the firm is:

$$S(0.08) = PV_A - PV_L = 68.06 - 68.06 = \mathbf{0.00}$$

b. The present value of the liability cash flow is lower than before because the new interest rate of 10% is higher than the previous interest rate of 8%:

$$PV_L = \frac{100}{1.10^5} = 62.09$$

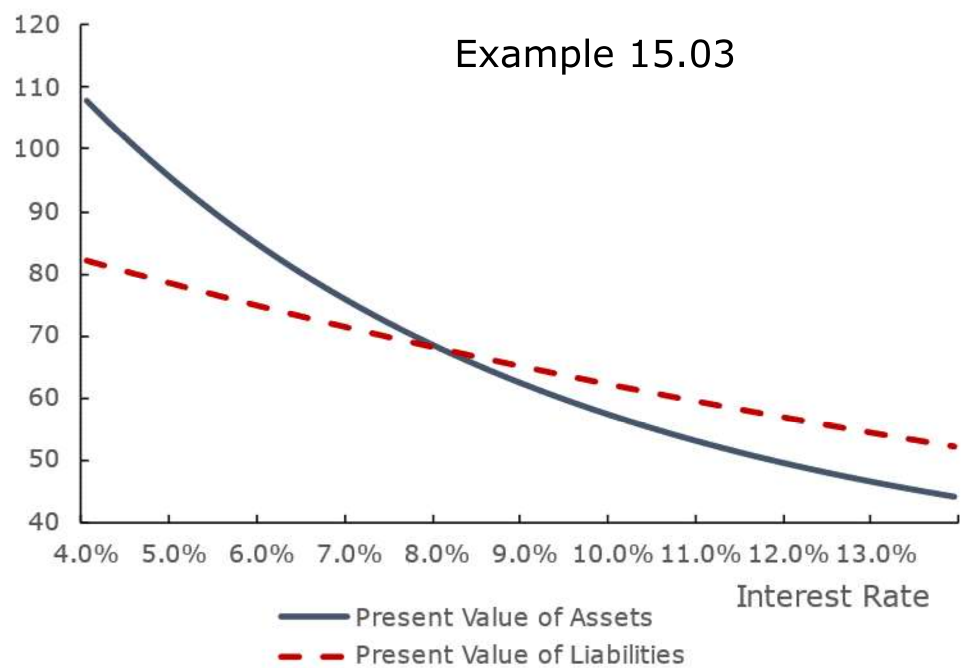
The present value of the asset cash flows is also lower than before because the new interest rate of 10% is higher than the previous interest rate of 8%:

$$PV_A = \frac{36.75}{1.10} + \frac{158.61}{1.10^{20}} = 56.99$$

The surplus of the firm is negative because the present value of the liabilities increased more than the present value of the assets:

$$S(0.10) = PV_A - PV_L = 56.99 - 62.09 = -\mathbf{5.11}$$

The graph below shows that when the interest rate increases, the value of the asset cash flows described in the example above falls below the value of the liability cash flow.



Reinvestment risk and price risk are examples of **interest rate risk**. As interest rates (or yields) change, the surplus of the firm changes.

15.02 Cash Flow Matching

One way to avoid interest rate risk is to match the asset cash flows with the liability cash flows exactly. **Cash flow matching** eliminates reinvestment risk because no funds are reinvested after the initial investment. Cash flow matching avoids price risk because it is not necessary to sell any assets to fund the liabilities. Cash flow matching is also known as **dedication** because it dedicates each asset cash flow to offset a specified liability cash flow.

An easy way to do this is to purchase zero-coupon bonds that provide cash flows that match the liability cash flows exactly. The example below shows that when a position is cash flow matched, the surplus does not change as the interest rate changes.

Example 15.04

A firm must pay \$100 at the end of 5 years. The firm implements a cash flow matching strategy by purchasing a zero-coupon bond that matures for \$100 at the end of 5 years.

The annual effective interest rate is 8%.

a. Calculate the surplus of the firm when the annual effective interest rate is 8%.

b. Calculate the surplus of the firm if the annual effective interest rate immediately decreases to 6%.

c. Calculate the surplus of the firm if the annual effective interest rate immediately increases to 10%.

Download full solutions from ActuarialBrew.com.

Answer Key

Question	Answer	Topic	Section	Difficulty
1.01	C	Terminology	1.04	1
1.02	E	Supply and Demand for Funds	1.05	1
2.01	D	Simple Interest	2.01	1
2.02	D	Simple Interest	2.01	1
2.03	A	Simple Discount	2.02	1
2.04	B	Simple Discount	2.02	2
2.05	E	Simple Interest & Discount	2.02	2
2.06	C	Simple Interest & Discount	2.03	2
2.07	A	Simple Interest & Discount	2.03	1
2.08	B	Simple Interest	2.04	1
2.09	A	Simple Interest & Discount	2.04	3
2.10	C	Simple Interest & Discount	2.04	5
3.01	D	Compound Interest	3.01	1
3.02	D	Compound Interest	3.01	2
3.03	C	Compound Interest	3.01	1
3.04	E	Compound Interest	3.01	2
3.05	B	Compound Interest	3.01	2
3.06	E	Simple Interest & Compound Interest	3.01	1
3.07	A	Compound Interest	3.01	2
3.08	C	Compound Interest	3.01	3
3.09	A	Compound Interest	3.01	3
3.10	B	Compound Interest	3.01	3
3.11	C	Compound Discount	3.03	1
3.12	E	Compound Discount	3.03	1
3.13	B	Compound Interest and Discount	3.03	1
3.14	B	Compound Discount	3.03	2
3.15	B	Compound Discount	3.03	3
3.16	E	Equivalent Compound Interest and Discount	3.04	2
3.17	D	Interest Rate Conversions	3.05	1
3.18	E	Interest Rate Conversions	3.05	2
3.19	A	Simple Interest & Compound Interest	3.05	2
3.20	C	Interest Rate Conversions	3.05	1
3.21	D	Nominal Interest Rates	3.05	2
3.22	D	Nominal Interest Rates	3.05	3
3.23	E	Nominal Interest Rates	3.05	3
3.24	D	Nominal Interest Rates	3.05	4
3.25	B	Nominal Interest Rates	3.05	4
3.26	D	Nominal Interest and Discount Rates	3.06	2
3.27	D	Nominal Discount Rates	3.06	4
3.28	C	Interest Rate Conversions	3.07	1
3.29	C	Interest Rate Conversions	3.07	1