Models for Quantifying Risk

Sixth Edition

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Solutions Manual

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CHAPTER THREE

REVIEW OF MARKOV CHAINS

3-1 (a) Given that the process is in State 0 at age $x+1$, the probability of being in State 1 at age $x+2$ is

$$p_{x+1}^{01} = 1 - p_{x+1}^{00} = 1 - \left(.70 + \frac{.10}{2}\right) = .25.$$

(b) Given that the process is in State 1 at age $x+2$, the probability of being in State 0 at age $x+3$ is

$$p_{x+2}^{10} = 1 - p_{x+2}^{11} = 1 - \left(.60 + \frac{.20}{3}\right) = .3\dot{3}.$$

(c) There are two mutually-exclusive ways to satisfy this probability. The process can remain in State 0 at $t=2$ and remain in State 0 again at $t=3$, or it can transition to State 1 at $t=2$ and then transition back to State 0 at $t=3$. The total probability is

$$_2p_{x+1}^{00} = p_{x+1}^{00} \cdot p_{x+2}^{00} + p_{x+1}^{01} \cdot p_{x+2}^{10}$$

$$= \left(.70 + \frac{.10}{2}\right)\left(.70 + \frac{.10}{3}\right) + (.25)(.3\dot{3})$$

$$= (.25)(.73\dot{3}) + (.25)(.3\dot{3}) = .63\dot{3}.$$

(d) The process is *non-homogeneous*, because the transition probabilities vary by attained age.

3-2 (a) *No*. The probability of moving from State 0 to State 2 over one time interval is zero.

(b) *More*, up to a point. The matrices show that, if endangered at $t=0$, it is not possible to become thriving by $t=1$. But if endangered at $t=1$, it can become thriving by $t=2$ with probability .10; if endangered at $t=2$, it can become thriving by $t=3$ with probability .20; if endangered at $t=3$, it can become thriving by $t=4$ with probability .50. The probability of becoming a thriving species increases the longer it remains endangered, up to a probability of .50, which then remains constant.

(c) The matrix $\mathbf{P}^{(k)}$, for $k = 3,4,5,\cdots$, shows that if the species is thriving, or even endangered, at $t = 3$, it can never thereafter become extinct. Given endangered at $t = 0$, there are three mutually-exclusive paths to extinction, as follows:

State at $t = 0$	**State at $t = 1$**	**State at $t = 2$**	**State at $t = 3$**	**Probability Value**
1	2			.30
1	1	2		$(.70)(.20) = .14$
1	1	1	2	$(.70)(.70)(.10) = .049$

The total probability is $.30 + .14 + .049 = .489$.

(Note that moving from State 0 directly to State 2 is not possible. Also the path State 1 $\rightarrow$ State 0 $\rightarrow$ State 1 $\rightarrow$ State 2 is likewise not possible, because $Pr\left[X_1 = 0 \,|\, X_0 = 1\right] = 0$, as shown in the matrix $\mathbf{P}^{(0)}$.

3-3 (a) The process is *homogeneous*, since the forces of transition are all constant.

(b) Let $i = j = 0$ in Equation (3.14a), so that k takes on the values $1,2,3$ in the summation. Then Equation (3.14a) becomes

$$\frac{d}{dt}\,{}_tp_x^{00} = {}_tp_x^{01} \cdot \mu_{x+t}^{10} - {}_tp_x^{00} \cdot \mu_{x+t}^{01}$$

$$+\,{}_tp_x^{02} \cdot \mu_{x+t}^{20} - {}_tp_x^{00} \cdot \mu_{x+t}^{02}$$

$$+\,{}_tp_x^{03} \cdot \mu_{x+t}^{30} - {}_tp_x^{00} \cdot \mu_{x+t}^{03}.$$

But $\mu_{x+t}^{10} = \mu_{x+t}^{20} = \mu_{x+t}^{30} = 0$ in this model, so Equation (3.14a) reduces to

$$\frac{d}{dt}\,{}_tp_x^{00} = -\,{}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03}\right).$$

In this question,

$$\mu_{x+t}^{01} + \mu_{x+t}^{02} + \mu_{x+t}^{03} = .30 + .50 + .70 = 1.50,$$

so we have

$$\frac{\frac{d}{dt}\,{}_tp_x^{00}}{{}_tp_x^{00}} = \frac{d}{dt} ln\,{}_tp_x^{00} = -1.50.$$

Integrating from $t=0$ to $t=r$, we have

$$\int_0^r d\ ln\, {}_tp_x^{00} = -\int_0^r 1.50\, dt$$

or

$$ln\, {}_tp_x^{00} \Big]_0^r = -1.50r$$

or

$$ln\, {}_rp_x^{00} - ln\, {}_0p_x^{00} = ln\left(\frac{{}_rp_x^{00}}{{}_0p_x^{00}}\right) = -1.50r.$$

But ${}_rp_x^{00} = 1,$ identically, so we finally have ${}_rp_x^{00} = e^{-1.50r}$.

(c) Since transition out of State 2 is not possible (i.e., State 2 is an absorbing state), then the event $\{X(1)=2\,|\,X(0)=0\}$ is satisfied if the process transitions from State 0 to State 2 at any time within the interval $(0,1]$. The density for transition from State 0 to State 2 at time r is ${}_rp_x^{00} \cdot \mu_{x+r}^{02}$, the probability of remaining in State 0 to time r times the force of transition from State 0 to State 2 at that time, which is $\mu_{x+r}^{02} = .50$. Then we have

$$Pr\big[X(1)=2\,|\,X(0)=0\big] = \int_0^1 {}_rp_x^{00} \cdot \mu_{x+r}^{02}\, dr$$

$$= .50\int_0^1 e^{-1.50r}\, dr = .50\left[\frac{e^{-1.50r}}{-1.50}\right]_0^1 = .25896.$$

(Note: Using Kolmogorov to solve for ${}_rp_x^{00}$ is the only method available to us at this stage of our knowledge. The reader will become much more familiar with these results after studying later chapters in the text.)

3-4 (a) The expected value of the payments is found by summing the products of each payment times the probability of the payment being made. (Since each payment is amount 1, we need only to sum the probability values.) The matrices show that if the process is in State 1, it then moves to State 2 with probability 1, and if it is in State 2, it remains in State 2 with probability 1. Further, starting at $t=2$, if the process is in State 0 it cannot remain in State 0. Clearly the probability of making a payment at $t=0$ is 1. A payment is made at $t=1$ if in State 0 or State 1, the probability of which is $.60+.30=.90$. The probability of being in either State 0 or 1 at $t=2$ is

$$(.60)(.60)+(.60)(.30) = .54.$$

It is not possible to be in State 0 at $t=3$. (If the process leaves State 0 before $t=2$, it is not possible to return; if the process is in State 0 at $t=2$, it is not possible to remain.) Therefore a payment is made at $t=3$ only if the process is in State 1, the probability of which is

$$(.60)(.60)(.30) = .108,$$

by moving from State 0 (at $t=0$) to State 0 (at $t=1$) to State 0 (at $t=2$) to State 1 (at $t=3$). Once the process leaves State 1 it must move to State 2, so it cannot be in either State 0 or State 1 at $t \geq 4$. Therefore the expected amount of payment is

$$1+.90+.54+.108=2.548.$$

(b) As in part (a), the process cannot be in State 1 at $t \geq 4$. The probability of being in State 1 at $t=1$ is .30. The probability of being in State 2 at $t=1$ is $(.60)(.30)=.18$. From part (a), the probability of being in State 1 at $t=3$ is .108. Then the total expected value of the payments is

$$4(.30+.18+.108)=2.352.$$

(Note: In actuarial applications, the payments would be discounted back to time 0 at some interest rate, as the reader will discover in Chapters 7 and later in the text.)

CHAPTER FOUR

CHARACTERISTICS OF INSURANCE AND PENSIONS

Note: These answers are possible solutions. Given the open-ended nature of many of these questions, many possible variants would be acceptable.

4-1 Prior to the advent of modern computing, the practical issues and complexities of computing premiums, cash values and interest accruals had to be taken into account during product development. With the practical difficulties of calculation eliminated, insurance companies were free to be more creative with their development of new products and offer more complex variations.

4-2 Term life insurance offers lower-cost protection against the risks of early death. There is little or no cash value accumulation or investment component, and a lesser tax benefit to this type of insurance. Whole life insurance also offers protection against the risks of early death, but also includes an investment component, offering cash-value accumulation and certain tax and estate-planning benefits that vary based upon the income level and tax situation of the insured. For comparative coverage whole life insurance costs significantly more.

Similar to the financial risk of early death, there is a significant financial risk involved with disability leading to an inability to work. Some portion of this risk is covered by governmental programs; however, they only cover the most severe disabilities and at a certain income level, replace only a small percentage of income. This risk can be mitigated and future financial health protected by paying a reasonable premium for disability insurance.

The cost of long-term care is another financial risk that mainly afflicts individuals in the later years of their lives and can lead to a complete depletion of savings and assets. This adversely affects the financial situation of immediate family and any potential inheritors. Long-term care insurance spreads out this risk among many insureds through the mechanism of insurance and allows the insured to make the best medical decision without undue concern for financial impact.

4-3 Insurers underwrite life insurance policies to ensure that the premium they charge reflects the risk they are taking on when insuring a specific life. Given the fact that insureds will always know more about themselves than an insurance company, underwriting expands the knowledge base of the insurance company to help them make a more informed decision regarding a specific insured and charge the most appropriate premium rate given the insured's risk profile.

The question of fairness of differentiated insurance premiums is one that ultimately must be answered by society and has been responded to in various ways in different countries and for different types of insurance. Some countries or states mandate unisex rates for some types of insurance and not for others. The passage of the Affordable Care Act in the U.S. severely restricted the ability of insurers to charge differentiated premiums for health insurance. There are fewer restrictions on differential premiums for life insurance. When differentiation is restricted, individuals with a lower expected cost subsidize individuals with a higher expected cost. In many other instances, such as taxes, and the postage for a basic letter, society judges it fair for some groups to subsidize others. For a private insurance system to be viable, the restriction on differentiation of premiums must not be so great as to make profitability unattainable.

4-4 The answer to this question largely depends on one's "risk profile" and many specifics regarding the insureds and underwriting that are not answered. With an annual premium of $50, and disregarding expenses, the company can withstand a mortality rate of one in every twenty insureds, or 5%, a level much higher than the societal mortality rates for all ages up to approximately 80 years old. I would implement underwriting to eliminate insureds with severely impaired mortality; I would also set down limits on the age groups that could purchase insurance (in this case there is no differentiation of premium by age). Following these precautions, I would feel comfortable selling a significant number of policies given the high margin in the mortality rates.

Regarding the necessary capital to hold, I would estimate the expected number of deaths and run simulations to estimate the probability of a number of deaths that would exhaust my capital. I would seek a level of capital that would ensure 99% probability of capital adequacy.

The problem does not specify the conditions for renewal of the policy or if the insurance company is obligated to renew the policies each year. Assuming that the insurance company is required to renew the policies, I would set aside a certain amount of premium as a reserve for worsening mortality. For end-of-year accounting, the company's profit would be equal to premium plus interest income minus claims minus expenses minus the reserve.

4-5 This is a much debated topic within different societies. The benefits associated with charging unisex rates include equality of treatment of males and females, thus avoiding a certain type of discrimination. It can also minimally simplify policy administration due to the elimination of one rating variable.

Women in general have a lower mortality rate than men and thus their insurance cost, theoretically, is lower. In this way, female insurance premiums must subsidize male insurance premiums. However, insurance companies then must price the insurance estimating ahead of time what percentage of their policyholders will be men and women. This adds an extra level of uncertainty and risk to the pricing process for insurance companies, and can lead to the need for extra margin by the insurance company to compensate for the added risk. The cost of life insurance will go up for women and down for men.

4-6 Defined benefit plans offer the greatest reward to employees that stay with the same employer for a significant length of time and are thus also more expensive to the employer should they have a significant percentage of long-term employees. A defined benefit plan would provide an incentive for employees to stay longer, but could seem unattractive to employees seeking a more short-term tenure with the company; thus, this would seem to fulfill the company's goal of attempting to build a base of stable employees. In the case of a defined benefit plan, the company has greater long-term uncertainty regarding the ultimate cost of their retirement plan.

A defined contribution plan is portable, and attractive to employees who are uncertain regarding how long they will stay, and provides no additional financial benefit to an employee for staying with an employer long-term. This plan would not provide the motivations that this company is seeking. The benefits to the employer would be that they have a certain and defined cost to their retirement plan, thus avoiding the risks of underfunding and underestimation of future liabilities associated with defined benefit plans.

4-7 Given the fact that someone purchasing an annuity is paid a fixed amount for the remainder of the annuitant's life, an insurer is concerned that they would live beyond their expected lifespan. The insurance company should take into account expected mortality improvements and the fact that those applying for an annuity generally consider themselves to be in better-than-average health. If a company chose to underwrite an applicant, they would be seeking to understand the expected lifespan of the potential insured given their current health status and risk profile.

4-8 Given the low level of capitalization, this is a high number of policies for the company to take on. At the same time, there is a high level of risk concentration since all of the company's policies would be at one geographical location. Any large accident at the worksite would be catastrophic for New Beginnings, Inc. In order to lessen the risk, New Beginnings could seek partners to take on some of the risk, reinsurance (insurance for insurance companies), raise more capital, or wait to write this policy until it had other business to diversify and spread out the risk.