Quantitative Finance & Investments
Core Study Manual
Volume I

Spring 2018 Edition

Richard E. Owens, FSA, MAAA, CFA

ACTEX Learning
New Hartford, Connecticut
ACTEX is eager to provide you with helpful study material to assist you in gaining the necessary knowledge to become a successful actuary. In turn we would like your help in evaluating our manuals so we can help you meet that end. We invite you to provide us with a critique of this manual by sending this form to us at your convenience. We appreciate your time and value your input.

Publication:
ACTEX QFI Core Study Manual, Spring 2018 Edition

I found Actex by:  (Check one)
☐ A Professor  ☐ School/Internship Program  ☐ Employer  ☐ Friend  ☐ Facebook/Twitter

In preparing for my exam I found this manual:  (Check one)
☐ Very Good  ☐ Good  ☐ Satisfactory  ☐ Unsatisfactory

I found the following helpful:
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________

I found the following problems:  (Please be specific as to area, i.e., section, specific item, and/or page number.)
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________

To improve this manual I would:
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________
________________________________________________________________________________________

Name: ____________________________  E-mail: ____________________________
Address: ____________________________
Phone: ____________________________

(Please provide this information in case clarification is needed.)

Send to: Stephen Camilli
ACTEX Learning
P.O. Box 715
New Hartford, CT 06057

Or visit our website at www.ActexMadRiver.com to complete the survey on-line. Click on the “Send Us Feedback” link to access the online version. You can also e-mail your comments to Support@ActexMadRiver.com.
# Table of Contents

## Volume I

### 1 SC. Mathematics, Statistics and Stochastic Calculus

<table>
<thead>
<tr>
<th>SOA</th>
<th>Learning Objectives and Learning Outcomes</th>
<th>SC-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text</td>
<td>Quantitative Finance</td>
<td></td>
</tr>
<tr>
<td>Chapter 5</td>
<td>Elementary Stochastic Calculus</td>
<td>SC-3</td>
</tr>
<tr>
<td>Chapter 6</td>
<td>The Black-Scholes Model</td>
<td>SC-15</td>
</tr>
</tbody>
</table>

| Text: An Introduction to the Mathematics of Financial Derivatives, 3rd Edition |
|-----------------------------------------------|---|
| Chapter 1 | Financial Derivatives - A Brief Introduction | SC-23 |
| Chapter 2 | A Primer on the Arbitrage Theorem | SC-31 |
| Chapter 3 | Review of Deterministic Calculus | SC-43 |
| Chapter 4 | Pricing Derivatives: Models and Notation | SC-55 |
| Chapter 5 | Tools in Probability Theory | SC-61 |
| Chapter 6 | Martingale and Martingale Representations | SC-75 |
| Chapter 7 | Differential Equations in Stochastic Environments | SC-93 |
| Chapter 8 | The Wiener Process, Lévy Processes and Rare Events in Financial Markets | SC-101 |
| Chapter 9 | Integration in Stochastic Environments | SC-119 |
| Chapter 10 | Ito's Lemma | SC-131 |
| Chapter 11 | The Dynamics of Derivative Prices | SC-145 |
| Chapter 12 | Pricing Derivative Products: Partial Differential Equations | SC-161 |
| Chapter 13 | PDEs and PIDEs - An Application | SC-171 |
| Chapter 14 | Pricing Derivative Products: Equivalent Martingale Measures | SC-179 |
| Chapter 15 | Equivalent Martingale Measures | SC-195 |

### QFIC-113-17 Frequently Asked Questions in Quantitative Finance

| Question 23 | Jensen's Inequality and Its Role in Finance | SC-205 |
| Question 26 | What Is Girsanov's Theorem | SC-211 |
| Question 36 | What Is Meant By 'Complete' And 'Incomplete' Markets”? | SC-213 |
| Question 37 | Can I Use Real Probabilities To Price Derivatives? | SC-215 |
| Question 60 | What Are The Stupidest Things Said About Risk Neutrality? | SC-217 |

| Text | Problems and Solutions in Mathematical Finance Vol 1 Stochastic Calculus |
|------|-----------------------------|---|
| Chapter 1 |  | SC-219 |
| Chapter 2 |  | SC-221 |
| Chapter 3 |  | SC-226 |
| Chapter 4 |  | SC-236 |
| Chapter 5 |  | SC-249 |
### 2 DH. Derivatives and Hedging

<table>
<thead>
<tr>
<th>SOA Learning Objectives and Learning Outcomes</th>
<th>DH-1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text</strong></td>
<td></td>
</tr>
<tr>
<td>Chapter 2 Derivatives</td>
<td>DH-3</td>
</tr>
<tr>
<td>Chapter 8 The Black-Scholes Formula and the 'Greeks'</td>
<td>DH-11</td>
</tr>
<tr>
<td>Chapter 10 How to Delta Hedge</td>
<td>DH-19</td>
</tr>
<tr>
<td>QFIC-102-13 What does an Option Pricing Model Tell Us about Option Prices</td>
<td>DH-27</td>
</tr>
<tr>
<td>QFIC-103-13 How to Use the Holes in Black-Scholes</td>
<td>DH-31</td>
</tr>
<tr>
<td>QFIC-104-13 Mild vs. Wild Randomness</td>
<td>DH-35</td>
</tr>
<tr>
<td>QFIC-114-17 Frequently Asked Questions in Quantitative Finance</td>
<td></td>
</tr>
<tr>
<td>Question 38 What is volatility?</td>
<td>DH-39</td>
</tr>
<tr>
<td>Question 39 What is the volatility smile?</td>
<td>DH-41</td>
</tr>
<tr>
<td>Question 53 How Robust is the Black-Scholes model?</td>
<td>DH-45</td>
</tr>
<tr>
<td>QFIC-115-17 Delta Hedging, Volatility Arbitrage and Optimal Portfolios</td>
<td>DH-47</td>
</tr>
</tbody>
</table>

### 3 IRM. Interest Rate Models

<table>
<thead>
<tr>
<th>SOA Learning Objectives and Learning Outcomes</th>
<th>IRM-1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text</strong></td>
<td></td>
</tr>
<tr>
<td>Chapter 16 One-Factor Interest Rate Modeling</td>
<td>IRM-3</td>
</tr>
<tr>
<td>Chapter 17 Yield Curve Fitting</td>
<td>IRM-11</td>
</tr>
<tr>
<td>Chapter 18 Interest Rate Derivatives</td>
<td>IRM-15</td>
</tr>
<tr>
<td>Chapter 19 The Heath, Jarrow &amp; Merton and Brace, Gatarek &amp; Musiela Models</td>
<td>IRM-25</td>
</tr>
<tr>
<td>Text: An Introduction to the Mathematics of Financial Derivatives</td>
<td></td>
</tr>
<tr>
<td>Chapter 16 New Results and Tools for Interest-Sensitive Securities</td>
<td>IRM-35</td>
</tr>
<tr>
<td>Chapter 17 Arbitrage Theorem in a New Setting</td>
<td>IRM-39</td>
</tr>
<tr>
<td>Chapter 18 Modeling Term Structure and Related Concepts</td>
<td>IRM-53</td>
</tr>
<tr>
<td>Chapter 19 Classical and HJM Approaches to Fixed Income</td>
<td>IRM-59</td>
</tr>
<tr>
<td>QFIC-116-17 Low Yield Curves and Absolute/Normal Volatilities</td>
<td>IRM-69</td>
</tr>
</tbody>
</table>

### 4 V. Volatility

<table>
<thead>
<tr>
<th>SOA Learning Objectives and Learning Outcomes</th>
<th>V-1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Text</strong></td>
<td></td>
</tr>
<tr>
<td>Chapter 9 Overview of Volatility Modeling</td>
<td>V-3</td>
</tr>
<tr>
<td>Text: Analysis of Financial Time Series</td>
<td></td>
</tr>
<tr>
<td>Chapter 1 Financial Time Series and Their Characteristics</td>
<td>V-9</td>
</tr>
<tr>
<td>Chapter 2 Linear Time Series Analysis and Its Applications</td>
<td>V-19</td>
</tr>
<tr>
<td>Chapter 3 Conditional Heteroscedastic Models</td>
<td>V-49</td>
</tr>
</tbody>
</table>
Volume II

5 FI.  Fixed Income

SOA  Learning Objectives and Learning Outcomes

Text: Handbook of Fixed Income Investments Chapters
Chapter 1  Overview of the Types and Features
Chapter 2  Risks Associated with Investing
Chapter 9  US Treasury Securities
Chapter 12  Corporate Bonds
Chapter 13  Leveraged Loans
Chapter 18  Inflation-Linked Bonds
Chapter 21  Fixed Income Exchange Traded Funds
Chapter 24  An Overview of Mortgages and the Mortgage Market
Chapter 31  Nonagency Residential Mortgage-Backed Securities
Chapter 32  Commercial Mortgage-Backed Securities

Text: Managing Investment Portfolios Chapters
Chapter 6  Fixed Income Portfolio Management

Text  Quantitative Finance
Chapter 14  FI Products and Analysis: Yield, Duration and Convexity

QFIC-110-15  High-Yield Bond Market Primer

6 E.  Equities

SOA  Learning Objectives and Learning Outcomes

Text: Managing Investment Portfolios Chapters
Chapter 7  Equity Portfolio Management

QFIC-110-15  Liquidity as an Investment

7 IP.  Investment Policy

SOA  Learning Objectives and Learning Outcomes

Text: Managing Investment Portfolios Chapters
Chapter 1  The Portfolio Management Process and the IPS
Chapter 3  Managing Institutional Investor Portfolios

QFIC-108-13  Managing Your Advisor
### 8 AA. Asset Allocation

<table>
<thead>
<tr>
<th>SOA</th>
<th>Learning Objectives and Learning Outcomes</th>
<th>AA-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Text: Managing Investment Portfolios Chapters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chapter 5</td>
<td>Asset Allocation</td>
<td>AA-3</td>
</tr>
<tr>
<td>QFIC-111-16</td>
<td>Stop Playing with Your Optimizer</td>
<td>AA-25</td>
</tr>
<tr>
<td>QFIC-112-16</td>
<td>Risk Factors as Building Blocks for Portfolio Diversification</td>
<td>AA-29</td>
</tr>
</tbody>
</table>

### PP. Practice Problems

<table>
<thead>
<tr>
<th>Practice Problems</th>
<th>PP-1</th>
</tr>
</thead>
</table>

I. Learning Objectives

A. "The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing."

II. Learning Outcomes

A. "The Candidate will be able to:

1. "Understand and apply concepts of probability and statistics important in mathematical finance"

2. "Understand the importance of the no-arbitrage condition in asset pricing"

3. "Understand Ito’s integral and stochastic differential equations"

4. "Understand and apply Ito's Lemma"

5. "Understand and apply Jensen’s Inequality"

6. "Demonstrate an understanding of the option pricing techniques and theory for equity and interest rate derivatives"

7. "Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures."

8. "Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts."

9. "Understand and apply Girsanov’s theorem in changing measures."

10. "Understand the Black Scholes Merton PDE (partial differential equation)."
Quantitative Finance
Paul Wilmott
Chapter 5 Elementary Stochastic Calculus

I. Introduction
   A. Random nature of financial markets requires randomness in mathematical tools, thus stochastic calculus
   B. Author attempts to be not too technical, avoiding excessive rigor
   C. Wants reader to:
      1. Understand technical terms
      2. Be able to use the stochastic techniques

II. A motivating example
   A. Coin toss
      1. Heads you win $1
      2. Tails you pay $1
      3. $i$ is the random amount, +1 or -1, for the $i$-th toss
         a. Expectations of $i$
            i. $E[R_i] = 0$, expected value of any toss is 0
            ii. $E[R_i^2] = 1$, second moment = 1
            iii. $E[R_iR_j] = 0$, each toss is independent, past has no impact on future
      4. $S_i$ is the total amount won up to and including the $i$-th toss
         a. $S_i = \sum_{j=1}^{i} R_j$
         b. Expectations of $S_i$
            i. $E[S_i] = 0$, cumulative expected value of the tosses is 0
            ii. $E[S_i^2] = i$
            iii. Dependent upon what has happened in previous tosses
c. Conditional Expectation

i. \( E[S_i | R_1, \ldots R_{i-1}] = S_{i-1} \)

   a) Expected value of cumulative winnings for the next roll is the same as the current cumulative winnings

III. The Markov property

A. The Markov property - “The expected value of the random variable \( S_i \) conditional upon all of the past events only depends on the previous value \( S_{i-1} \)”

B. Random walk has no “memory” other than where it is currently

   1. No Memory – it does not matter the path taken to get to \( S_{i-1} \), all that matters is that the random variable is at \( S_{i-1} \), as if the past was forgotten or if there is no memory of how it got to its state at \( S_{i-1} \)

C. Not all random variables have the Markov property

D. Generalization of Markov – If only have information through \( S_j \), \( 1 \leq j < i \), then estimate of \( S_i \) is based only on \( S_j \)

E. Financial models

   1. Most have Markov property

   2. Some have small amount of memory

   3. A few are totally path dependent, all steps to get to current point matter

IV. The martingale property

A. Martingale property – “conditional expectation of your winnings at any time in the future is just the amount you already hold”

   1. \( E[S_i | S_j, j < i] = S_i \)

       a. Here \( j \) is the current time, and \( i \) is some future time

V. Quadratic variation

A. Definition of quadratic variation for the random walk

   1. \( \sum_{j=1}^{i} (S_i - S_{i-1})^2 \)

B. As win/lose 1 after each toss, \( |S_j - S_{j-1}| = 1 \)

C. Therefore, quadratic variation is equal to \( i \)

   1. \( \sum_{j=1}^{i} (S_i - S_{i-1})^2 = i \)
VI. Brownian motion

A. Introduction
   1. Above coin toss is discrete-time random walk
   2. Desire to move to continuous-time random walk

B. Modified coin toss experiment
   1. Restrict time allowed for coin tosses, n tosses in example, in time t
      a. Each toss takes time $t/n$
   2. Change bet from $1$ to $\sqrt{t/n}$
   3. Markov and Martingale properties continue to hold
   4. Quadratic variation $= \sum_{j=1}^{t} (S_{j} - S_{j-1})^2 = n(\sqrt{t/n})^2 = t$

C. Let n get large
   1. Time between tosses decreases by $n^1$
   2. Bet size decreases by $n^{1/2}$

D. Limit of process as $n \to \infty$
   1. That is, time steps go to zero while random walk remains finite
   2. $E[S(t)] = 0$ if $S(0) = 0$
   3. $E[S(t)^2] = t$
   4. Called Brownian motion
   5. Denotes Brownian motion in balance of chapter by $X(t)$ rather than $S(t)$

E. Properties of Brownian motion
   1. Finiteness
      a. The bet, or increment, scales with the square root of the time step, as in above example
      b. Otherwise
         i. Random walk goes to infinity in finite time, or
         ii. Limit has zero motion
2. Continuity
   a. The path has no discontinuities
   b. Brownian motion is the continuous-time random walk, the limit of the discrete-time coin toss example

3. Markov
   a. “the conditional distribution of X(t) given information up until \( \tau < t \) depends only on \( X(\tau) \)”
      i. Information means the values of X previous to and including \( X(\tau) \)

4. Martingale
   a. “Given information up until \( \tau < t \) the conditional expectation of \( X(t) \) is \( X(\tau) \)”

5. Quadratic variation
   a. When time 0 to time \( t \) is divided with \( n+1 \) partition points, \( t_i = i \ t/n \), then
   b. \( \sum_{j=1}^{n} (X(t_j) - X(t_{j-1}))^2 \rightarrow t \)
      i. “almost surely”

6. Normality
   a. With finite increments \( t_{j-1} \) to \( t_j \), \( X(t_j) - X(t_{j-1}) \) is normally distributed
   b. Mean = 0 and variance is time increment \( t_j - t_{j-1} \)

7. Brownian motion is defined by the above six properties
   a. Brownian motion is important in financial models
   b. \( X(t) \) is a random walk process with time steps approaching 0 and is called Brownian motion

VII. Stochastic integration

A. Define stochastic integral as
   \[ W(t) = \int_0^t f(\tau)dX(\tau) = \lim_{n \to \infty} \sum_{j=1}^{n} f(t_{j-1})(X(t_j) - X(t_{j-1})) \]
   a. \( t_j = jt/n \)

B. Important points about the definition
   1. Function \( f(t) \) is evaluated at the left-hand point in the summation, i.e. \( t_{j-1} \)
2. Why? So as to not anticipate the increment in $X(t_j) - X(t_{j-1})$

3. Such integration ensures “that we use not information about the future in our current actions”
   a. Choose a portfolio then find out how it performs

VIII. Stochastic differential equations

A. Stochastic integral $W(t) = \int_0^t f(\tau) dX(\tau)$ \hfill (5.1)

B. Can use “shorthand” for this integral by differentiating (5.1) to get
   1. $dW = f(t)dX$ \hfill (5.2)
   2. $dX$ is an increment of $X$, a normal random variable, mean = 0, standard deviation $dt^{1/2}$

C. “Equations (5.1) and (5.2) are meant to be equivalent” thus (5.2) is “shortcut” for (5.1)
   1. (5.2) looks like ordinary differential equation but unclear what $dX/dt$ is

D. Instead stochastic differential equation $dW = g(t) dt + f(t) dX$ \hfill (5.3)
   1. As shorthand for $W(t) = \int_0^t g(\tau)d\tau + \int_0^t f(\tau)dX(\tau)$

IX. The mean square limit

A. Technical definition $E[(\sum_{j=1}^n (X(t_j) - X(t_{j-1}))^2 - t)^2]$ where $t_j = j\delta t / n$ \hfill (5.4)

B. Practical implication as $n \to \infty$, $\sum_{j=1}^n (X(t_j) - X(t_{j-1}))^2 = t$
   1. Also written as $\int_0^t (dX)^2 = t$

X. Functions of stochastic variables and Itô’s lemma

A. Introduction
   1. Stochastic variable with Brownian motion, $X(t)$ and function $F(X) = X^2$
   2. Ordinary calculus rules do not apply to stochastic functions, so $dF \neq 2X dX$
   3. Balance of section contains a heuristic derivation of Itô’s lemma, important rule of stochastic calculus, based on arbitrary function $F(X)$

B. Derivation
   1. Time scale, very small, $h = \delta t / n$
2. Approximate $F(X(t+h))$ with Taylor series
   
a. \[ F(X(t+h)) - F(X(t)) = (X(t+h) - X(t)) \frac{dF}{dX} X(t) + \frac{1}{2} (X(t+h) - X(t))^2 \frac{d^2F}{dX^2} X(t) + \cdots \]

b. Sum the series of time steps differences to time $t + nh$
   
c. \[ F(X(t+h)) - F(X(t)) + F(X(t+2h)) - F(X(t+h)) + \cdots + F(X(t+nh)) - F(X(t + (n-1)h)) = \sum_{j=1}^{n} (X(t + jh) - X(t + (j-1)h)) \frac{dF}{dX} X(t + (j-1)h) + \frac{1}{2} \frac{d^2F}{dX^2} (X(t) \sum_{j=1}^{n} (X(t + jh) - X(t + (j-1)h))^2 + \cdots \]

i. Above uses approximation good enough for accuracy desired of
   
a) \[ \frac{d^2F}{dX^2} (X(t + (j-1)h)) = \frac{d^2F}{dX^2} (X(t)) \]
   
ii. RHS of c. simplifies to $F(X(t + nh)) - F(X(t)) = F(X(t + \delta t)) - F(X(t))$
   
iii. First term of LHS of c. is $\int_{t}^{t+\delta t} \frac{dF}{dX} dX$
   
iv. Second term of LHS of c. is $\frac{1}{2} \frac{d^2F}{dX^2} (X(t)) \delta t$

   d. Result is \[ F(X(t + \delta t)) - F(X(t)) = \int_{t}^{t+\delta t} \frac{dF}{dX} (X(\tau)) dX(\tau) + \frac{1}{2} \int_{t}^{t+\delta t} \frac{d^2F}{dX^2} (X(\tau)) d\tau \]

3. Extend result from time 0 to time $t$
   
a. \[ F(X(t)) = F(X(0)) + \int_{0}^{t} \frac{dF}{dX} (X(\tau)) dX(\tau) + \frac{1}{2} \int_{0}^{t} \frac{d^2F}{dX^2} (X(\tau)) d\tau \]
   
i. Integral version of Itô’s Lemma
   
   b. Usually written as a stochastic differential equation
       
i. Itô’s Lemma \[ dF = \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2F}{dX^2} dt \] (5.5)

   a) More general results of Itô’s Lemma are shown below

4. Example of Use of Itô’s Lemma
   
   a. $F = X^2$
   
   b. $dF/dX = 2X$ and $d^2F/dX^2 = 2$, assume standard derivatives
   
   c. Result by Itô’s Lemma, \[ dF = 2X \, dX + (1/2)2 dt = 2X \, dX + dt \]
   
i. Not the same as if $F$ were deterministic
Quantitative Finance Chapter 5

d. Integral form \( X^2 = F(X) = F(0) + \int_0^t 2X \, dX + \int_0^t 1 \, d\tau = \int_0^t 2X \, dX + t \)

i. Rearranging, \( \int_0^t X \, dX = \frac{1}{2} X^2 - \frac{1}{2} t \)

XI. Interpretation of Itô’s Lemma

A. Lemma important in pricing options

B. Figure 5.4 shows random stock price movement and option prices based on that stock price movement

C. Stock price follows stochastic differential equation \( dS = \mu \, S \, dt + \sigma \, S \, dX \)

D. Maybe option value, \( V(S, t) \) follows a stochastic differential equation \( dV = ____ \, dt + ____ \, dX? \)

E. Itô will help fill in the blanks

XII. Itô and Taylor

A. Seeking intuition behind Itô’s lemma

B. Simple Taylor series expansion of \( F \) with small change in \( X, dX \)

1. \( F(X + dX) = F(X) + \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2 F}{dX^2} (dX)^2 \)

2. \( F(X + dx) - F(X) \) is change in \( F, \) or \( dF, \) and

3. \( dF = \frac{dF}{dX} (dX) + \frac{1}{2} \frac{d^2 F}{dX^2} (dX)^2 \)

   a. Very similar to (5.5) and Itô

   b. Difference, (5.5) has “dt” while above has “\( dX^2 \)”

4. And without “rigor”, \( dX^2 = dt \) \hspace{1cm} (5.6)

C. Generalize the stochastic differential equation

1. \( dS = a(S) \, dt + b(S) \, dX \) \hspace{1cm} (5.7)

   a. \( a(S), b(S) \) functions of \( S, \) although in some formulas Wilmott uses simply \( a, b, \)
      dropping (\( S \))

   b. \( dX \) the Brownian motion increment

   c. \( S \) thus has deterministic time element, \( dt, \) and a random element, \( dX \)
2. Another function of S, V(S), what is dV?
   a. “Cheat by using Taylor series with dX^2 = dt”, replacing “F” with “V” and “X” with “S”
      i. \[ dV = \frac{dV}{dS} dS + \frac{1}{2} b^2 \frac{d^2V}{dS^2} dt \] (5.7a)
      ii. Random element in this equation is not shown directly, i.e. no dX term, but is inside of the dS term.
   
   b. Equation using the pure random term dX
      i. \[ dV = \left( a(S) \frac{dV}{dS} + \frac{1}{2} b(S)^2 \frac{d^2V}{dS^2} \right) dt + b(S) \frac{dV}{dS} dX \]
      ii. Derive from 2.a.i and using equation (5.7), gathering the dt terms

XIII. Itô in higher dimensions

A. In finance, many functions of deterministic t and stochastic S, and the coefficients of stochastic differential equation of S are functions of both S and t, a(S, t) and b(S, t). The derivative V(S, t) a function of both S and t.

1. \[ dS = a(S, t) dt + b(S, t) dX \]
2. \[ dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} dt, \text{ the general version of Itô with a single random variable} \]

B. Function of two or more random variables V(S1, S2, t) with correlated random increments dX1, dX2

1. \[ dS_1 = a_1(S_1, t) dt + b_1(S_1, t) dX_1 \]
2. \[ dS_2 = a_2(S_2, t) dt + b_2(S_2, t) dX_2 \]
3. Rules of thumb, \[ dX_1^2 = dt, dX_2^2 = dt, dX_1 dX_2 = \rho dt \]
4. Itô’s Lemma result
   a. \[ dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2 + \frac{1}{2} b_1^2 \frac{\partial^2 V}{\partial S_1^2} dt + b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} dt + \frac{1}{2} b_2^2 \frac{\partial^2 V}{\partial S_2^2} dt \] (5.9)

XIV. Some pertinent examples

A. Section Beginning

1. General form of stochastic differential equation of interest, dS = ____dt + ____dX
   a. Deterministic piece with dt and random piece with dX
b. What functions to use with each piece? Different functions for different financial instruments

B. Brownian motion with drift

1. \( dS = \mu \, dt + \sigma \, dX \)
   
   a. \( \mu \) is the drift, or movement away from a mean of 0

   b. Can go negative, shown in Figure 5.5, not good for some financial values such as equity values or interest rates (text written before the 2007-09 financial crisis and central banks causing interest rates to go negative)

   c. Integrate SDE to get \( S(t) = S(0) + \mu t + \sigma (X(t) - X(0)) \)

C. The lognormal random walk

1. \( dS = \mu S \, dt + \sigma S \, dX \) \hspace{1cm} (5.10)

   a. Both drift and randomness multiplied by \( S \)

   b. With \( S(0) > 0 \), \( S(t) \) cannot be negative

2. Define \( F(S) = \log S \) then

   a. \( dF = \frac{dF}{dS} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{d^2F}{dS^2} \, dt = \frac{1}{S} (\mu S \, dt + \sigma S \, dX) - \frac{1}{2} \sigma^2 \, dt = (\mu - \frac{1}{2} \sigma^2) \, dt + \sigma \, dX \)

      i. As \( \frac{dF}{dS} = \frac{1}{S} \) and \( \frac{d^2F}{dS^2} = -\frac{1}{S^2} \)

   b. \( -\infty < \log S < \infty \), but cannot reach the limit in finite time, and \( S \) cannot reach either 0 or \( \infty \)

3. Integral form: \( S(t) = S(0) e^{(\mu - \frac{1}{2} \sigma^2)t + \sigma (X(t) - X(0))} \)

4. For \( V(S, t) \), with a lognormal random walk, using Itô

   a. \( dV = \frac{\partial V}{\partial t} \, dt + \frac{\partial V}{\partial S} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \, dt \) \hspace{1cm} (5.11)

D. A mean-reverting random walk

1. \( dS = (\nu - \mu S) \, dt + \sigma \, dX \)

   a. Negative coefficient of \( S \) implies large \( S \) will tend to move down, small \( S \) will tend to move up, “reverting to the mean”

   b. \( S \) does not have to stay positive

2. Mean reverting often used for interest rates
3. Vasicek model for short-term interest rates, use “r” rather than “S”

E. And another mean-reverting random walk

1. Adjust the random term of the prior SDE3e
2. \( dS = (ν – μS)dt + σS^{1/2}dX \)
   a. When \( S \) gets close to 0, randomness decreases
3. Define \( F = S^{1/2} \), find SDE for \( F \)
   a. Itô’s Lemma result \( dF = \frac{dF}{dS}dS + \frac{1}{2}σ^2FdS^2dt \)
      i. Why \( σ^2S \) in the \( dt \) term? It is the square of the coefficient of the random \( dX \) term in the SDE for \( S \)
      ii. Note similar usage in the formula at the bottom of page 135
   b. Derivatives of \( F \)
      i. \( \frac{dF}{dS} = \frac{1}{2}S^{-1/2}, \frac{d^2F}{dS^2} = -\frac{1}{4}S^{-3/2} \)
   c. Substitute derivatives and \( dS \) into Itô
      i. \( dF = \frac{1}{2}S^{-1/2}[(ν – μS)dt + σS^{1/2}dX] + \frac{1}{2}σ^2S[-\frac{1}{4}S^{-2}]dt \)
      ii. \( dF = \frac{1}{2}νdt - \frac{1}{2}μS^2dt + \frac{1}{2}σdX - \frac{σ^2}{8F}dt \)
      iii. \( dF = (\frac{4ν - σ^2}{8F} - \frac{1}{2}μF)dt + \frac{1}{2}σdX \)
         a) Constant random term, messy drift term
         b) Drift term is problematic at \( F = S = 0 \)
4. Find \( F(S) \) such that drift term is 0
   a. Need \( (ν – μS)\frac{dF}{dS} + \frac{1}{2}σ^2S\frac{d^2F}{dS^2} = 0 \)
      i. Applied Itô to \( dS \) above
   b. Resulting equation \( \frac{dF}{dS} = A S^{-\frac{2ν}{σ^2}} e^{\frac{2μS}{σ^2}} \)
      i. \( A \) is any constant
      ii. If \( ν \) is large enough, random walk avoids a value of 0
XV. Summary

A. With Itô’s Lemma, can handle functions of a random variable

B. Asset S with a stochastic differential equation, then can price derivatives of S

C. Need to be able to use the Itô tool

XVI. Time Out – Learn by Using

A. “Stochastic differential equations are like recipes for generating random walks”

B. “If you have some quantity, let’s call it S, that follows such a random walk, then any function of S is also going to follow a random walk.”

C. “The question then becomes ‘What is the random walk for this function of S?’ That is, what is its recipe, or what is its stochastic differential equation?”

D. “Applying something very like Taylor series but with two tricks”

1. One – in Taylor expansion, “only keep terms of size dt or bigger (dt^{1/2})”

2. Two – Replace dX^2 with dt
Quantitative Finance
Paul Wilmott
Chapter 6 The Black-Scholes Model

I. Introduction
   A. "Most important chapter in the book"
   B. Building blocks of derivative theory
      1. Delta hedging
      2. No arbitrage
   C. While Black-Scholes (B-S) assumptions can be incorrect, model is important in practice and theory

II. A Very Special Portfolio
   A. Notation: Value of an option:
      1. \( V(S, t; \sigma, \mu; E, T; r) \)
         a. \( S \) and \( t \) are variables
            i. \( S \) spot price of underlying asset
            ii. \( t \) current time or today's date
         b. \( \sigma, \mu \) are parameters of the asset price
            i. \( \sigma \): annualized volatility or standard deviation
            ii. \( \mu \): annualized growth rate, however we know this does not impact the price of the option
         c. \( E \) and \( T \) are details of the option contract
            i. \( E \) strike price (note different notation than other authors)
            ii. \( T \) time of expiry of contract
         d. \( r \) is parameter of the currency of the option
         e. Generally, will use \( V(S, t) \) or \( V \) if others clear from context
   B. Notation: Value of Portfolio of one long option and a short position of \( \Delta \) in underlying:
      1. \( \Pi = V(S, t) - \Delta S \quad (6.1) \)
C. Assume price of underlying asset follows lognormal random walk

1. \( dS = \mu S \, dt + \sigma S \, dX \)
   a. Change in price of the underlying is mean return on asset for time period of \( dt \) plus the volatility of the underlying time the value of the underlying times \( dX \)
   b. \( X(t) \) is a Brownian motion
      i. Properties of Brownian motion p121-122 (these pages are not on syllabus but they can assist understanding)
         a) Finiteness - increment scales with square root of time step
         b) Continuity - Brownian motion is continuous-time limit of discrete time random walk
         c) Markov - Conditional \( X(t) \) given prior info until \( \tau < t \) depends only on \( X(\tau) \)
         d) Martingale - Given info until \( \tau < t \), conditional expectation of \( X(t) \) is \( X(\tau) \)
         e) Quadratic Variation - sum of squares of differences of \( X(t) \), of \([X(t_i) - X(t_{i-1})]\)
            approach \( t \), on a 0 to \( t \) time range
         f) Normality - differences of \([X(t_i) - X(t_{i-1})]\) are normally distributed with mean zero and variance \( t_{j-t_{j-1}} \)

D. Change in value of portfolio

1. \( d\Pi = dV - \Delta dS \)

2. From Itô’s Lemma
   a. \( dV = \frac{\partial V}{\partial t} \, dt + \frac{\partial V}{\partial S} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \, dt \)
   b. For review of Itô’s Lemma, see Neftci Chapter 10

3. Then change in value of portfolio is
   a. \( d\Pi = \frac{\partial V}{\partial t} \, dt + \frac{\partial V}{\partial S} \, dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \, dt - \Delta dS \) \hspace{1cm} (6.2)
      i. Deterministic terms of \( dt \)
      ii. Random or stochastic terms of \( dS \), the risk in the portfolio

III. Elimination of Risk: Delta Hedging

A. If random term \((\frac{\partial V}{\partial S} - \Delta)dS = 0\), then risk eliminated

B. If \( \Delta = \frac{\partial V}{\partial S} \), then risk eliminated \hspace{1cm} (6.3)
C. Hedging - "any reduction in randomness"

D. Delta-hedging - "exploiting correlation between two instruments"

E. Dynamic hedging strategy - as delta changes with time, the delta-hedge must be adjusted or rebalanced, theoretically continuously.

IV. No Arbitrage

A. Given \( \Delta \) above, then \[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \] (6.4)

1. No risk in \( d\Pi \) as no \( dS \) term in above equation

2. No risk implies portfolio should earn risk free rate using no-arbitrage principle

3. \( d\Pi = r\Pi dt \) (6.5)

V. The Black-Scholes Equation

A. Given

1. \[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \]

2. \( d\Pi = r\Pi dt \)

3. \( \Pi = V(S, t) - \Delta S \)

4. \( \Delta = \frac{\partial V}{\partial S} \),

B. Then, substituting 4 into 3, and the result into right hand side of 2, equating 1 and 2, dividing all than by \( dt \) and rearranging,

1. \[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \] (6.6)

C. Above equation is the B-S equation

1. A linear parabolic partial differential equation
   a. Linear - If have two solutions, then the sum of the two solutions is also a solution
   b. Parabolic - related to diffusion equation
   c. Relatively easy to solve numerically

D. Why no "drift" term or mean or expected return parameter in B-S?

1. Mathematical argument - \( \mu \) dropped out with \( dS \) term

2. Economic argument - If can hedge risk of option with underlying, there should be no reward for such a portfolio, it should earn the risk free rate
3. Option can be replicated with cash and buying/selling the underlying

4. Complete market - one which an option can be replicated with the underlying
   a. Transaction costs lead to incomplete markets

5. Time Out
   a. At a time-step before expiry, $\delta t$, the value of the option appears to be the
      "moneyness" of the option plus, if in the money, a time value that looks like an
      interest earning on the strike price for that time-step

VI. The Black-Scholes Assumptions
   A. Wilmott lists most important assumptions and discusses some generalizations
   B. Underlying follows lognormal random walk
      1. To find solutions
         a. Random term must be proportional to $S$
         b. $\sigma$ does not need to be constant but needs to be proportional to time
         c. Elimination of arbitrage eliminates $\mu$ term from solution
   C. Risk-free interest rate is a known function of time
      1. Restriction helps find solutions
      2. If constant formula easier
      3. Reality - risk-free rate is not known and it is stochastic
   D. No dividends on the underlying
      1. Restriction to be dropped shortly
   E. No transaction costs
      1. Not realistic
      2. Trade-off between transaction costs and frequency of rebalancing
   F. No arbitrage opportunities
      1. Exist in reality
      2. Rule out model-dependent arbitrage
VII. Final Conditions

A. B-S equation independent on type of option, strike and expiry

B. These distinctions handled by "final conditions" or the payoff at expiry

1. Heaviside function, (using Greek capital eta as I cannot find the symbol Wilmott uses)
   a. \( \eta(\cdot) \)
      i. Zero when argument is zero
      ii. One when argument is positive

2. Examples of Final Conditions
   a. Call option \( V(S, T) = \max(S - E, 0) \)
   b. Put option \( V(S, T) = \max(E - S, 0) \)
   c. Binary Call \( V(S, T) = \eta(S - E) \)
   d. Binary Put \( V(S, T) = \eta(E - S) \)

C. Observe that \( S \) and \( e^t \) satisfy the B-S equation

1. Let \( V = S \),
   a. Then
      i. \( \frac{\partial V}{\partial S} = 1 \)
      ii. \( \frac{\partial^2 V}{\partial S^2} = 0 \)
      iii. \( \frac{\partial V}{\partial t} = 0 \)
   b. Substituting into B-S, \( rS - rS = 0 \) satisfying the equation

2. Similar process letting \( V = e^t \)

VIII. Options on Dividend-Paying Equities

A. Simple modification of non-dividend paying B-S equation

1. Let \( D = \) continuous dividend rate

2. \( D S \ dt = \) dividend paid in time \( dt \)

3. Holding \( \Delta \) shares, dividend is \( D \Delta S \ dt \)

B. Generalized equation is

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0 \quad (6.7)
\]
IX. Currency Options

A. Modification of the dividend-paying B-S equation where the "dividend rate" is the interest rate on the foreign currency, \( r_f \)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - r_f) S \frac{\partial V}{\partial S} - r V = 0
\]  \hspace{1cm} (6.8)

X. Commodity Options

A. Modification of the dividend-paying B-S equation where the "dividend rate" is negative, and it is the cost to carry the commodity, \( q \)

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r + q) S \frac{\partial V}{\partial S} - r V = 0
\]  \hspace{1cm} (6.9)

2. Cost of carry, example, costs to store the commodity

XI. Expectations and Black-Scholes

A. "The fair value of an option is the present value of the expected payoff at expiry under a risk-neutral random walk for the underlying"

1. Option value = \( e^{-r(T-t)} \mathbb{E}[\text{payoff}(S)] \)

B. Real Random Walk

1. "actual random walk as realized with volatility \( \sigma \) and drift rate \( \mu \)"

C. Risk-Neutral Random Walk

1. Artificial random walk with volatility \( \sigma \) and drift rate \( r \)
   a. Volatility same as real random walk
   b. Drift is risk-free rate

2. Use risk-neutral valuation only when hedging can be used to eliminate all risk

XII. Some Other Ways of Deriving the Black-Scholes Equation

A. The Martingale Approach

1. Uses hedging, no-arbitrage and risk-neutral valuation

2. But requires probabilistic description of financial world

B. The Binomial Approach

1. B-S equation is limit of binomial as time-steps approaches 0
C. CAPM/Utility
   1. Uses risk and reward (return) to optimally allocate portfolio
   2. Including options in framework does not increase risk/reward as options are simply functions of the underlyings
   3. Market completeness

XIII. No Arbitrage in the Binomial, Black-Scholes and 'Other' Worlds
   A. With binomial and B-S, proper choice of $\Delta$ can eliminate risk
   B. Not so with other models
      1. Trinomial model example
         a. Effectively trying to solve two equations with the one unknown, $\Delta$
   C. Footnote 4, page 150, Wilmott makes an important point that these are models, not reality. However, they can be a good starting point, not an ending point as models can be "wrong"

XIV. Forwards and Futures
   A. Forwards
      1. $\bar{S}$ is fixed delivery price
      2. Portfolio $\Pi = V(S, t) - \Delta S$, then as above, choose $\Delta = \partial V / \partial S$, eliminate risk, apply no arbitrage, result is B-S equation
         a. At expiry, $V(S, T) = S - \bar{S}$
         b. Solution to PDE is $V(S, t) = S(t) - \bar{S} e^{-r(T-t)}$
         c. At initiation, $V(S, 0) = 0$, no price is needed to enter a forward, then $\bar{S} = S e^{rT}$
      3. Verify B-S holds, Let $V = S(t) - \bar{S} e^{r(T-t)}$
         a. Then
            i. $\partial V / \partial S = 1$
            ii. $\partial^2 V / \partial S^2 = 0$
            iii. $\partial V / \partial t = -r \bar{S} e^{-r(T-t)}$
         b. Substituting into B-S, $-r \bar{S} e^{-r(T-t)} + rS - r(S - \bar{S} e^{-r(T-t)}) = 0$ satisfying the equation
XV. Futures Contracts

A. $F(S, t)$ notation for futures price
   
   1. $F(S, t) = 0$ for all $t$ due to daily settlement of contract
   
   2. Portfolio long future and short $\Delta$ shares of stock
      
      a. $\Pi = 0 - \Delta S$
      
      b. $d\Pi = dF - \Delta dS$, with $dF$ being the continuous settlement cash flow
      
      c. $d\Pi = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} dt - \Delta dS$
      
      d. Make $dS$ terms go away by setting $\Delta = \frac{\partial F}{\partial S}$
      
      e. Eliminate risk by $d\Pi = r\Pi dt$
   
   3. Result is not a B-S form, $\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} = 0$
      
      a. Final condition, $F(S, T) = S$ as futures price and underlying must have the same value at maturity
      
      b. General solution $F(S, t) = Se^{(T-t)}$
   
   4. Note, when interest rates are a known function of time, forward and futures prices are the same
      
      a. Differences when rates are stochastic

XVI. Options on Futures

A. Futures price
   
   1. $F = e^{r(T-t)} S$

B. $V(S, t) = \mathcal{V}(F, t)$

   1. $\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 \mathcal{V}}{\partial F^2} - r\mathcal{V} = 0$ (6.10)
An Introduction to the Mathematics of Financial Derivatives, 3rd Edition
Ali Hirsa, Salih Neftci
Chapter 1 Financial Derivatives - A Brief Introduction

I. Introduction
   A. Book discusses logic behind asset pricing
   B. Chapter discusses two building blocks of financial derivatives, options and forwards
   C. Complicated derivatives can be decomposed into simpler derivatives
   D. Study manual author note: Much of the material in this chapter is a review from Exams FM and MFE

II. Definition
   A. Derivative security or contingent claim: "A financial contract is a derivative security, or a contingent claim, if its value at expiration date T is determined exactly by the market price of the underlying cash instrument at time T"
   B. Notation
      1. T: expiration date of derivative
         a. After this date the derivative contract no longer exists
      2. F(T): price of derivative asset
      3. S_T: price of the relevant cash instrument, called the "underlying asset", at time T
      4. F(t) and F(S_t, t): price of derivative on underlying S_t at time t
      5. d_t: Yield of payout on a derivative

III. Types of Derivatives
   A. Introduction
      1. Three general types of derivatives
         a. Futures and forwards: basic building block
         b. Options: basic building block
         c. Swap: hybrid security that can be decomposed into set of forwards and options
2. Five main groups of underlying assets
   a. Stocks
      i. Claims on returns from production of goods/services
   b. Currencies
      i. Not direct claims on real assets
   c. Interest Rates
      i. Not really assets, so a notional asset needs to be determined
   d. Indexes
      i. E. G. S&P 500, FT-SE100
      ii. Need notional amount
   e. Commodities
      i. Not financial assets, can be physically purchased and stored

B. Cash-and-Carry Markets
   1. Cash-and-Carry are an alternative to holding a forward/futures contract on commodity
      a. Buy directly in cash markets or buy indirectly using forward/futures
         i. At expiration, in either case, the long position holds the commodity
   2. Examples: gold, silver, currencies, T-bonds
   3. Features of Cash-and-Carry Market
      a. Borrow at risk-free rates by collateralizing the underlying
      b. Buy and store the product
      c. Insure it until expiration of derivative contract
   4. Additional property
      a. Information about future conditions, supply/demand, etc., should not change the spread between cash and futures prices
         i. New information changes the price of both by about the same amount
C. Price-Discovery Markets

1. Differences from Cash-and-Carry
   a. Commodity is perishable or
   b. Commodity does not yet exist, e.g. spring wheat

2. Features
   a. Not physically possible to buy commodity and store until expiration date of derivative
   b. No cash market yet exists, e.g. crop has not yet been harvested

3. Impact of Information
   a. Future supply/demand cannot influence cash price
   b. Such information will influence, and thus can be discovered, in the futures market

D. Expiration Date

1. Value of futures contract at expiry should be equal to the value of the underlying
   a. \[ F(T) = S_T \] (1.1)

2. Value of \( F(t) \), \( t < T \), is not necessarily equal to \( S_t \)

IV. Forwards and Futures

A. Forward

1. Forward: "a forward contract is an obligation to buy (sell) an underlying asset at a specified forward price on a known date"

2. Review of payoff diagrams, pages 4, 6

B. Futures

1. Similar to forwards in that they are both obligations for settlement at a specified future date of an underlying

2. Differences between forwards and futures
   a. Where Traded
      i. Futures: on formal, public exchanges
      ii. Forwards: private contracts in the OTC market (over the counter)
b. Credit Risk
   i. Futures: cleared through an exchange designed to reduce default risk
   ii. Forwards: private contracts so risk depends on the credit quality of the counterparty

c. Mark-to-Market
   i. Futures: Daily, effectively settled daily with a new contract created
      a) Daily profit/loss recorded
   ii. Forwards: no mark-to-market

C. Repos, Reverse Repos, and Flexible Repos

1. Repos
   a. Definition
      i. "A repurchase agreement, also known as a repo, is a transaction in which one party sells securities to another party in return for cash, with an agreement to repurchase equivalent securities at an agreed upon price and on an agreed upon future date."
         a) You could think of this as a loan with the difference between the cash amount and the "agreed upon price" as interest on the loan.
            i) Securities are the collateral for the loan
            ii) This interest amount creates the "repo rate"
         b) Alternatively, think of a repo as a spot sale and a forward contract
      ii. For seller, called repo
      iii. For buyer of security, called reverse repo
      iv. Classified as money market instruments

b. Types of repos
   i. Overnight: one-day maturity
   ii. Term: specified maturity date other than one-day
   iii. Open: no end date

c. Forms of Repo Transactions
   i. Specified Delivery
ii. Tri-party

iii. Hold-in-custody
   a) Selling party holds security during repo term

2. Flexible Repos
   a. A repo with a flexible withdrawal schedule, both in terms of timing and amount
   b. Differences from traditional repo
      i. "Convexity due to cash withdrawals
      ii. "Formal written auction like trade
      iii. Additional documentation is necessary for credit protection
      iv. Counterparties are typically muni bond issuers
   c. Types of flexible repos
      i. Secured
         a) Municipality receives collateral
            i) Treasuries, GNMA, agency MBS, etc.
            ii) Collateral comes from a reverse repo
         b) Average size 10 - 20 million
      ii. Unsecured
         a) No collateral but likely higher rate
         b) Average size - smaller than secured

V. Options

A. General
   1. The right, but not an obligation, to do something
   2. Call: "A European-type call option on a security $S_t$ is the right to buy the security at a
      preset strike price $K$. This right may be exercised at the expiration date $T$ of the Option. The call option
      can be purchased for a price of $C_t$ dollars, the call premium, at time $t < T"
   3. Put: right to sell
4. Exercise Dates
   a. European options: only at expiry
   b. American options: exercised at any time, not just the expiration date

5. Reasons traders need to know the value of $C_t$
   a. Estimate of price to trade, especially if traded infrequently
   b. Evaluate risk
   c. Determine any mispricing for an arbitrage opportunity

B. Some Notation

1. Best to find a formula, or closed form solution, for $C_t$ as a function the underlying

2. Only known formula for $C_t$ is at $T$
   a. Out of the money expiry, $S_T < K$  
      i. Implies $C_T = 0$  
   b. In-the-money expiry, $S_T > K$  
      i. Implies $C_T = S_T - K$  
   c. $C_T = \max(S_T - K, 0)$  
      i. Formula for $C_T$ shows options are non-linear

3. Figures 1.3 and 1.4, page 8, graph values of call options at times before expiry

VI. Swaps

A. General

1. Swap: "A swap is the simultaneous selling and purchasing of cash flows involving various currencies, interest rates, and a number of other financial assets"

2. Method to price swaps is to decompose swap into forwards and options, price the forwards and options, sum these prices to get price of swap

B. A Simple Interest Rate Swap

1. Notional principals
2. Two counterparties for interest rate swap
   a. Party A to pay fixed on notional, receive floating on notional
   b. Party B to received fixed on notional, pay floating on notional
   c. Each period, payments are netted, net payment is essentially the interest rate differential times the notional principal
   d. Based on comparative advantage, each party should secure lower rates
   e. Swap dealer earns a fee for bringing parties together

3. Basket of forward contracts will replicate the cash flow helping to value the swap

C. Cancelable Swaps

1. Definition
   a. "A cancelable swap is a swap where one or both parties has the right but not the obligation to cancel the swap before its maturity."
      i. Dates to cancel the swap specified in the contract

2. Types of Cancelable Swaps
   a. Callable Swap
      i. Payer of fixed rate has the option
         a) On specified dates
         b) At cancelation, payer
            i) Pays present value of future payments
            ii) Receives a premium
   b. Puttable Swap
      i. Receiver of fixed rate has the option
         a) On specified dates
   c. Vanilla Swap can be "closed out"
      i. By paying net present value of future payments

3. Uses of Cancelable Swaps
   a. Use with callable bonds
b. Use as asset/liability hedges
   i. Specifically, when liabilities have prepayment options