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Fall 2017 Edition | Volume I

Johnny Li, P.h.D., FSA | Andrew Ng, Ph.D., FSA
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Thank you for choosing ACTEX.

A new version of Exam MLC is launched in Spring 2014. The new Exam MLC is significantly different from the old one, most notably in the following aspects:

1. Written-answer questions are introduced and form a major part of the examination.
2. The number of official textbooks is reduced from two to one. The new official textbook, *Actuarial Mathematics for Life Contingent Risks* 2nd edition (AMLCR), contains a lot more technical materials than other textbooks written on the same topic.
3. The level of cognitive skills demanded from candidates is much higher. In particular, the new learning objectives require candidates to not only calculate numerical values but also, for example, interpret the results they obtain.
4. Several new (and more advanced) topics, such as participating insurance, are added to the syllabus.

Because of these major changes, ACTEX have decided to bring you this new study manual, which is written to fit the new exam.

We know very well that you may be worried about written-answer questions. To help you best prepare for the new exam, this manual contains some 150 written-answer questions for you to practice. Eight full-length mock exams, written in exactly the same format as that announced in SoA’s Exam MLC Introductory Note, are also provided. Many of the written-answer questions in our mock exams are highly challenging! We are sorry for giving you a hard time, but we do want you to succeed in the real exam.

The learning outcomes of the new exam syllabus require candidates to be able to interpret a lot of actuarial concepts. This skill is drilled extensively in our written-answer practice problems, which often ask you to interpret a certain actuarial formula or to explain your calculation. Also, as seen in SoA’s Exam MLC Sample Written-Answer Questions (e.g., #9), you may be asked in the new exam to define or describe a certain insurance product or actuarial terminology. To help you prepare for this type of exam problems, we have prepared a special chapter (Chapter 0), which contains definitions and descriptions of various products and terminologies. The special chapter is written in a “fact sheet” style so that you can remember the key points more easily.

Proofs and derivations are another key challenge. In the new exam, you are highly likely to be asked to prove or derive something. This is evidenced by, for example, problem #4 in SoA’s Exam MLC Sample Written-Answer Questions, which demands a mathematical derivation of the Kolmogorov forward differential equations for a certain transition probability. In this new study manual, we do teach (and drill) you how to prove or derive important formulas. This is in stark contrast to some other exam prep products in which proofs and derivations are downplayed, if not omitted.

We have paid special attention to the topics that are newly introduced in the recent two syllabus updates. Seven full-length chapters (Chapters 0, 10, 12 – 16) and two sections (amount to more
than 300 pages) are especially devoted to these topics. Moreover, instead of treating the new topics as “orphans”, we demonstrate, as far as possible, how they can be related to the old topics in an exam setting. This is very important for you, because multiple learning outcomes can be examined in one single exam question.

We have made our best effort to ensure that all topics in the syllabus are explained and practiced in sufficient depth. For your reference, a detailed mapping between this study manual and the official textbook is provided on pages P-10 to P-12.

Besides the topics specified in the exam syllabus, you also need to know a range of numerical techniques in order to succeed. These techniques include, for example, Euler’s method, which is involved in SoA’s Exam MLC Sample Multiple-Choice Question #299. We know that quite a few of you have not even heard of Euler’s method before, so we have prepared a special chapter (Appendix 1, appended to the end of the study manual) to teach you all numerical techniques required for this exam. In addition, whenever a numerical technique is used, we clearly point out which technique it is, letting you follow our examples and exercises more easily.

Other distinguishing features of this study manual include:

− We use graphics extensively. Graphical illustrations are probably the most effective way to explain formulas involved in Exam MLC. The extensive use of graphics can also help you remember various concepts and equations.

− A sleek layout is used. The font size and spacing are chosen to let you feel more comfortable in reading. Important equations are displayed in eye-catching boxes.

− Rather than splitting the manual into tiny units, each of which tells you a couple of formulas only, we have carefully grouped the exam topics into 17 chapters. Such a grouping allows you to more easily identify the linkages between different concepts, which, as we mentioned earlier, are essential for your success.

− Instead of giving you a long list of formulas, we point out which formulas are the most important. Having read this study manual, you will be able to identify the formulas you must remember and the formulas that are just variants of the key ones.

− We do not want to overwhelm you with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.

− We write the practice problems and the mock exams in a similar format as the released exam and sample questions. This will enable you to comprehend questions more quickly in the real exam.

On page P-13, you will find a flow chart showing how different chapters of this manual are connected to one another. You should first study Chapters 0 to 10 in order. Chapter 0 will give you some background factual information; Chapters 1 to 4 will build you a solid foundation; and Chapters 5 to 11 will get you to the core of the exam. You should then study Chapters 12 to 16 in any order you wish. Immediately after reading a chapter, do all practice problems we provide for that chapter. Make sure that you understand every single practice problem. Finally, work on the mock exams.
Before you begin your study, please download the exam syllabus from SoA’s website:

https://www.soa.org/education/exam-req/edu-exam-m-detail.aspx

On the last page of the exam syllabus, you will find a link to Exam MLC Tables, which are frequently used in the exam. You should keep a copy of the tables, as we are going to refer to them from time to time. You should also check the exam home page periodically for updates, corrections or notices.

If you find a possible error in this manual, please let us know at the “Customer Feedback” link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the “Errata & Updates” link.

Enjoy your study!
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2. Life Tables

3. Life Insurances

4. Life Annuities

5. Premium Calculation

6. Net Premium Reserves

7. Insurance Models Including Expenses

A1. Numerical Techniques

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9. Multiple Decrement Models: Applications

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11. Multiple Life Functions

12. Interest Rate Risk

13. Profit Testing

14. Universal Life Insurance

15. Participating Insurance

16. Pension Mathematics

0. Some Factual Information
Chapter 0  Some Factual Information

This chapter serves as a summary of Chapter 1 in AMLCR. It contains descriptions of various life insurance products and pension plans. There is absolutely no mathematics in this chapter.

You should know (and remember) the information presented in this chapter, because in the written answer questions, you may be asked to define or describe a certain pension plan or life insurance policy. Most of the materials in this chapter are presented in a “fact sheet” style so that you can remember the key points more easily.

Many of the policies and plans mentioned in this chapter will be discussed in detail in later parts of this study guide.

0.1 Traditional Life Insurance Contracts

Whole life insurance
A whole life insurance pays a benefit on the death of the policyholder whenever it occurs. The following diagram illustrates a whole life insurance sold to a person age $x$.

![Diagram of whole life insurance](image)

A benefit (the sum insured) is paid here

0  (Age $x$) 

Death occurs  Time from now

The amount of benefit is often referred to as the sum insured. The policyholder, of course, has to pay the “price” of policy. In insurance context, the “price” of a policy is called the premium, which may be payable at the beginning of the policy, or periodically throughout the life time of the policy.
**Term life insurance**
A term life insurance pays a benefit on the death of the policyholder, provided that death occurs before the end of a specified term.

Death occurs here: 
Pay a benefit

Death occurs here: 
Pay nothing

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(Age $x$)

The time point $n$ in the diagram is called the *term* or the *maturity date* of the policy.

**Endowment insurance**
An endowment insurance offers a benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs earlier.

Death occurs here: 
Pay the sum insured on death

Pay the sum insured at time $n$ if the policyholder is alive at time $n$

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(Age $x$)

These three types of traditional life insurance will be discussed in Chapter 3 of this study guide.

**Participating (with profit) insurance**
Any premium collected from the policyholder will be invested, for example, in the bond market. In a participating insurance, the profits earned on the invested premiums are shared with the policyholder. The profit share can take different forms, for example, cash dividends, reduced premiums or increased sum insured. This product type will be discussed in detail in Chapter 15 of this study manual.
0.2 Modern Life Insurance Contracts

Modern life insurance products are usually more flexible and often involve an investment component. The table below summarizes the features of several modern life insurance products.

<table>
<thead>
<tr>
<th>Product</th>
<th>Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universal life insurance</td>
<td>- Combines investment and life insurance</td>
</tr>
<tr>
<td></td>
<td>- Premiums are flexible, as long as the accumulated value of the premiums is enough to cover the cost of insurance</td>
</tr>
<tr>
<td>Unitized with-profit</td>
<td>- Similar to traditional participating insurance</td>
</tr>
<tr>
<td>insurance</td>
<td>- Premiums are used to purchase shares of an investment fund.</td>
</tr>
<tr>
<td></td>
<td>- The income from the investment fund increases the sum insured.</td>
</tr>
<tr>
<td>Equity-linked insurance</td>
<td>- The benefit is linked to the performance of an investment fund.</td>
</tr>
<tr>
<td></td>
<td>- Examples: equity-indexed annuities (EIA), unit-linked policies,</td>
</tr>
<tr>
<td></td>
<td>segregated fund policies, variable annuity contracts</td>
</tr>
<tr>
<td></td>
<td>- Usually, investment guarantees are provided.</td>
</tr>
</tbody>
</table>

In Chapter 14 of this study guide, we will discuss universal life insurance policies in detail.

0.3 Underwriting

Underwriting refers to the process of collecting and evaluating information such as age, gender, smoking habits, occupation and health history. The purposes of this process are:
- To classify potential policyholders into broadly homogeneous risk categories
- To determine if additional premium has to be charged.
The following table summarizes a typical categorization of potential policyholders.

<table>
<thead>
<tr>
<th>Category</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preferred lives</td>
<td>Have very low mortality risk</td>
</tr>
<tr>
<td>Normal lives</td>
<td>Have some risk but no additional premium has to be charged</td>
</tr>
<tr>
<td>Rated lives</td>
<td>Have more risk and additional premium has to be charged</td>
</tr>
<tr>
<td>Uninsurable lives</td>
<td>Have too much risk and therefore not insurable</td>
</tr>
</tbody>
</table>

Underwriting is an important process, because with no (or insufficient) underwriting, there is a risk of adverse selection; that is, the insurance products tend to attract high risk individuals, leading to excessive claims. In Chapter 2, we will introduce the select-and-ultimate table, which is closely related to underwriting.

### 0.4 Life Annuities

A life annuity is a benefit in the form of a regular series of payments, conditional on the survival of the policyholder. There are different types of life annuities.

**Single premium immediate annuity (SPIA)**

The annuity benefit of a SPIA commences as soon as the contract is written. The policyholder pays a single premium at the beginning of the contract.
Single premium deferred annuity (SPDA)

The annuity benefit of a SPDA commences at some future specified date (say \( n \) years from now). The policyholder pays a single premium at the beginning of the contract.

![Diagram showing the annuity benefit begins at time \( n \)]

A single premium is paid at the beginning of the contract

Regular Premium Deferred Annuity (RPDA)

An RPDA is identical to a SPDA except that the premiums are paid periodically over the deferred period (i.e., before time \( n \)).

These three annuity types will be discussed in greater depth in Chapter 4 of this study guide.

Some life annuities are issued to two lives (a husband and wife). These life annuities can be classified as follows.

<table>
<thead>
<tr>
<th>Joint life annuity</th>
<th>The annuity benefit ceases on the first death of the couple.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Last survivor annuity</td>
<td>The annuity benefit ceases on the second death of the couple.</td>
</tr>
<tr>
<td>Reversionary annuity</td>
<td>The annuity benefit begins on the first death of the couple, and ceases on the second death.</td>
</tr>
</tbody>
</table>

These annuities will be discussed in detail in Chapter 11 of this study guide.
0.5 Pensions

A pension provides a lump sum and/or annuity benefit upon an employee’s retirement. In the following table, we summarize a typical classification of pension plans:

<table>
<thead>
<tr>
<th>Defined contribution (DC) plans</th>
<th>The retirement benefit from a DC plan depends on the accumulation of the deposits made by the employ and employee over the employee’s working life time.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defined benefit (DB) plans</td>
<td>The retirement benefit from a DB plan depends on the employee’s service and salary.</td>
</tr>
<tr>
<td></td>
<td><strong>Final salary plan:</strong> the benefit is a function of the employee’s final salary.</td>
</tr>
<tr>
<td></td>
<td><strong>Career average plan:</strong> the benefit is a function of the average salary over the employee’s entire career in the company.</td>
</tr>
</tbody>
</table>

Pension plans will be discussed in detail in Chapter 16 of this study guide.
In Exam FM, you valued cash flows that are paid at some known future times. In Exam MLC, by contrast, you are going to value cash flows that are paid at some unknown future times. Specifically, the timings of the cash flows are dependent on the future lifetime of the underlying individual. These cash flows are called life contingent cash flows, and the study of these cash flows is called life contingencies.

It is obvious that an important part of life contingencies is the modeling of future lifetimes. In this chapter, we are going to study how we can model future lifetimes as random variables. A few simple probability concepts you learnt in Exam P will be used.

1.1 Age-at-death Random Variable

Let us begin with the age-at-death random variable, which is denoted by $T_0$. The definition of $T_0$ can be easily seen from the diagram below.
The age-at-death random variable can take any value within \([0, \infty)\). Sometimes, we assume that no individual can live beyond a certain very high age. We call that age the \textit{limiting age}, and denote it by \(\omega\). If a limiting age is assumed, then \(T_0\) can only take a value within \([0, \omega]\).

We regard \(T_0\) as a \textit{continuous} random variable, because it can, in principle, take any value on the interval \([0, \infty)\) if there is no limiting age or \([0, \omega]\) if a limiting age is assumed. Of course, to model \(T_0\), we need a probability distribution. The following notation is used throughout this study guide (and in the examination).

\(- F_0(t) = \Pr(T_0 \leq t)\) is the (cumulative) distribution function of \(T_0\).

\(- f_0(t) = \frac{d}{dt} F_0(t)\) is the probability density function of \(T_0\). For a small interval \(\Delta t\), the product \(f_0(t) \Delta t\) is the (approximate) probability that the age at death is in between \(t\) and \(t + \Delta t\).

In life contingencies, we often need to calculate the probability that an individual will survive to a certain age. This motivates us to define the survival function:

\[
S_0(t) = \Pr(T_0 > t) = 1 - F_0(t).
\]

Note that the subscript “0” indicates that these functions are specified for the age-at-death random variable (or equivalently, the future lifetime of a person age 0 now).

Not all functions can be regarded as survival functions. A survival function must satisfy the following requirements:

1. \(S_0(0) = 1\). This means every individual can live at least 0 years.
2. \(S_0(\omega) = 0\) or \(\lim_{t \to \infty} S_0(t) = 0\). This means that every individual must die eventually.
3. \(S_0(t)\) is monotonically decreasing. This means that, for example, the probability of surviving to age 80 cannot be greater than that of surviving to age 70.
Summing up, $f_0(t)$, $F_0(t)$ and $S_0(t)$ are related to one another as follows.

\begin{align}
\text{Relations between } f_0(t), F_0(t) \text{ and } S_0(t) \nonumber
\end{align}

\begin{align}
f_0(t) &= \frac{d}{dt} F_0(t) = -\frac{d}{dt} S_0(t), & (1.1) \\
S_0(t) &= \int_a^t f_0(u) du = 1 - \int_0^t f_0(u) du = 1 - F_0(t), & (1.2) \\
Pr(a < T_0 \leq b) &= \int_a^b f_0(u) du = F_0(b) - F_0(a) = S_0(a) - S_0(b). & (1.3)
\end{align}

Note that because $T_0$ is a continuous random variable, Pr($T_0 = c$) = 0 for any constant $c$. Now, let us consider the following example.

\textbf{Example 1.1} [Structural Question]

You are given that $S_0(t) = 1 - t/100 \text{ for } 0 \leq t \leq 100$.

(a) Verify that $S_0(t)$ is a valid survival function.

(b) Find expressions for $F_0(t)$ and $f_0(t)$.

(c) Calculate the probability that $T_0$ is greater than 30 and smaller than 60.

\textbf{Solution}

(a) First, we have $S_0(0) = 1 - 0/100 = 1$.

Second, we have $S_0(100) = 1 - 100/100 = 0$.

Third, the first derivative of $S_0(t)$ is $-1/100$, indicating that $S_0(t)$ is non-increasing.

Hence, $S_0(t)$ is a valid survival function.

(b) We have $F_0(t) = 1 - S_0(t) = t/100$, for $0 \leq t \leq 100$.

Also, we have and $f_0(t) = \frac{d}{dt} F_0(t) = 1/100$, for $0 \leq t \leq 100$.

(c) $Pr(30 < T_0 < 60) = S_0(30) - S_0(60) = (1 - 30/100) - (1 - 60/100) = 0.3$. 

[ END ]
1.2 Future Lifetime Random Variable

Consider an individual who is age \( x \) now. Throughout this text, we use \( (x) \) to represent such an individual. Instead of the entire lifetime of \((x)\), we are often more interested in the future lifetime of \((x)\). We use \( T_x \) to denote the future lifetime random variable for \((x)\). The definition of \( T_x \) can be easily seen from the diagram below.

If there is no limiting age, \( T_x \) can take any value within \([0, \infty)\). If a limiting age is assumed, then \( T_x \) can only take a value within \([0, \omega - x]\). We have to subtract \( x \) because the individual has attained age \( x \) at time 0 already.

We let \( S_x(t) \) be the survival function for the future lifetime random variable. The subscript “\( x \)” here indicates that the survival function is defined for a life who is age \( x \) now. It is important to understand that when modeling the future lifetime of \((x)\), we always know that the individual is alive at age \( x \). Thus, we may evaluate \( S_x(t) \) as a conditional probability:

\[
S_x(t) = \Pr(T_x > t) = \Pr(T_0 > x + t \mid T_0 > x) = \frac{\Pr(T_0 > x + t \cap T_0 > x)}{\Pr(T_0 > x)} = \frac{\Pr(T_0 > x + t)}{\Pr(T_0 > x)} = \frac{S_0(x + t)}{S_0(x)}.
\]

The third step above follows from the equation \( \Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \), which you learnt in Exam P.
With $S_x(t)$, we can obtain $F_x(t)$ and $f_x(t)$ by using
\[ F_x(t) = 1 - S_x(t) \quad \text{and} \quad f_x(t) = \frac{d}{dt} F_x(t), \]
respectively.

---

**Example 1.2 [Structural Question]**

You are given that $S_{0}(t) = 1 - t/100$ for $0 \leq t \leq 100$.

(a) Find expressions for $S_{10}(t)$, $F_{10}(t)$ and $f_{10}(t)$.

(b) Calculate the probability that an individual age 10 now can survive to age 25.

(c) Calculate the probability that an individual age 10 now will die within 15 years.

---

**Solution**

(a) In this part, we are asked to calculate functions for an individual age 10 now (i.e., $x = 10$). Here, $\omega = 100$ and therefore these functions are defined for $0 \leq t \leq 90$ only.

First, we have $S_{10}(t) = \frac{S_{0}(10+t)}{S_{0}(10)} = \frac{1-(10+t)/100}{1-10/100} = 1 - \frac{t}{90}$, for $0 \leq t \leq 90$.

Second, we have $F_{10}(t) = 1 - S_{10}(t) = t/90$, for $0 \leq t \leq 90$.

Finally, we have $f_{10}(t) = \frac{d}{dt} F_{10}(t) = \frac{1}{90}$.

(b) The probability that an individual age 10 now can survive to age 25 is given by
\[ \Pr(T_{10} > 15) = S_{10}(15) = 1 - \frac{15}{90} = \frac{5}{6}. \]

(c) The probability that an individual age 10 now will die within 15 years is given by
\[ \Pr(T_{10} \leq 15) = F_{10}(15) = 1 - S_{10}(15) = \frac{1}{6}. \]
1.3 Actuarial Notation

For convenience, we have designated actuarial notation for various types of death and survival probabilities.

**Notation 1: \( t p_x \)**
We use \( t p_x \) to denote the probability that a life age \( x \) now survives to \( t \) years from now. By definition, we have

\[
 t p_x = \Pr(T_x > t) = S_x(t).
\]

When \( t = 1 \), we can omit the subscript on the left-hand-side; that is, we write \( t p_x \) as \( p_x \).

**Notation 2: \( t q_x \)**
We use \( t q_x \) to denote the probability that a life age \( x \) now dies before attaining age \( x + t \). By definition, we have

\[
 t q_x = \Pr(T_x \leq t) = F_x(t).
\]

When \( t = 1 \), we can omit the subscript on the left-hand-side; that is, we write \( t q_x \) as \( q_x \).

**Notation 3: \( t|u q_x \)**
We use \( t|u q_x \) to denote the probability that a life age \( x \) now dies between ages \( x + t \) and \( x + t + u \).

By definition, we have

\[
 t|u q_x = \Pr(t < T_x \leq t + u) = F_x(t + u) - F_x(t) = S_x(t) - S_x(t + u).
\]

When \( u = 1 \), we can omit the subscript \( u \); that is, we write \( t|1 q_x \) as \( t q_x \).

Note that when we describe survival distributions, “\( p \)” always means a survival probability, while “\( q \)” always means a death probability. The “\( | \)” between \( t \) and \( u \) means that the death probability is deferred by \( t \) years. We read “\( t \mid u \)” as “\( t \) deferred \( u \)”.

It is important to remember the meanings of these three actuarial symbols. Let us study the following example.
Chapter 1: Survival Distributions

Example 1.3

Express the probabilities associated with the following events in actuarial notation.

(a) A new born infant dies no later than age 45.
(b) A person age 20 now survives to age 38.
(c) A person age 57 now survives to age 60 but dies before attaining age 65.

Assuming that \( S_0(t) = e^{-0.0125t} \) for \( t \geq 0 \), evaluate the probabilities.

Solution

(a) The probability that a new born infant dies no later than age 45 can be expressed as \( 45q_0 \).

[Here we have “\( q \)” for a death probability, \( x = 0 \) and \( t = 45 \).

Further, \( 45q_0 = F_0(45) = 1 - S_0(45) = 0.4302 \).

(b) The probability that a person age 20 now survives to age 38 can be expressed as \( 18p_{20} \).

[Here we have “\( p \)” for a survival probability, \( x = 20 \) and \( t = 38 - 20 = 18 \).

Further, we have \( 18p_{20} = S_{20}(18) = \frac{S_0(38)}{S_0(20)} = 0.7985 \).

(c) The probability that a person age 57 now survives to age 60 but dies before attaining age 65 can be expressed as \( 35q_{57} \).

[Here, we have “\( q \)” for a (deferred) death probability, \( x = 57 \), \( t = 60 - 57 = 3 \), and \( u = 65 - 60 = 5 \).

Further, we have \( 35q_{57} = S_{57}(3) - S_{57}(8) = \frac{S_0(60)}{S_0(57)} - \frac{S_0(65)}{S_0(57)} = 0.058357 \).

[ END ]

Other than their meanings, you also need to know how these symbols are related to one another. Here are four equations that you will find very useful.

Equation 1: \( p_x + q_x = 1 \)

This equation arises from the fact that there are only two possible outcomes: dying within \( t \) years or surviving to \( t \) years from now.
**Equation 2:** \( t_u p_x = t p_x \times u p_{x+t} \)

The meaning of this equation can be seen from the following diagram.

![Diagram showing survival distributions](image)

Mathematically, we can prove this equation as follows:

\[
\begin{align*}
  t_u p_x &= \frac{S_o(x+t+u)}{S_o(x)} = \frac{S_o(x+t)}{S_o(x)} \cdot \frac{S_o(x+t+u)}{S_o(x+t)} = S_x(t) S_{x+u}(u) = t p_x \times u p_{x+t}.
\end{align*}
\]

**Equation 3:** \( t_u q_x = t_u p_x - (t+u)p_x \)

This equation arises naturally from the definition of \( t_u q_x \).

We have \( t_u q_x = \Pr(t < T_x \leq t+u) = F_x(t+u) - F_x(t) = t_u q_x = t_u q_x - t q_x \).

Also, \( t_u q_x = \Pr(t < T_x \leq t+u) = S_x(t) - S_x(t+u) = t p_x - t_u p_x \).

**Equation 4:** \( t_u q_x = t p_x \times u q_{x+t} \)

The reasoning behind this equation can be understood from the following diagram:

![Diagram showing survival distributions](image)
Mathematically, we can prove this equation as follows:

\[
\begin{align*}
t_0 q_x &= t p_x - t u p_x \quad \text{(from Equation 3)} \\
&= t p_x - t p_x \times u p_{x+t} \quad \text{(from Equation 2)} \\
&= t p_x (1 - u p_{x+t}) \\
&= t p_x \times u q_{x+t} \quad \text{(from Equation 1)}
\end{align*}
\]

Here is a summary of the equations that we just introduced.

### Relations between \( t p_x \), \( t q_x \) and \( t_0 q_x \)

\[
\begin{align*}
t p_x + t q_x &= 1, \quad \text{(1.5)} \\
(t + u) p_x &= t p_x \times u p_{x+t}, \quad \text{(1.6)} \\
(t_0) q_x &= t_0 q_x - t q_x = t p_x - t_0 p_x = t p_x \times u q_{x+t}. \quad \text{(1.7)}
\end{align*}
\]

Let us go through the following example to see how these equations are applied.

### Example 1.4

You are given:

(i) \( p_x = 0.99 \)

(ii) \( p_{x+1} = 0.985 \)

(iii) \( 3p_{x+1} = 0.95 \)

(iv) \( q_{x+3} = 0.02 \)

Calculate the following:

(a) \( p_{x+3} \)

(b) \( 2p_x \)

(c) \( 2p_{x+1} \)

(d) \( 3p_x \)

(e) \( t_0 q_x \)
### Solution

(a) \( p_{x+3} = 1 - q_{x+3} \)
\[ = 1 - 0.02 = 0.98 \]

(b) \( 2p_x = p_x \times p_{x+1} \)
\[ = 0.99 \times 0.985 = 0.97515 \]

(c) Consider \( 3p_{x+1} = 2p_{x+1} \times p_{x+3} \)
\[ \Rightarrow 0.95 = 2p_{x+1} \times 0.98 \]
\[ \Rightarrow 2p_{x+1} = 0.9694 \]

(d) \( 3p_x = p_x \times 2p_{x+1} \)
\[ = 0.99 \times 0.9694 = 0.9597 \]

(e) \( \frac{1}{2}q_x = p_x \times 2q_{x+1} \)
\[ = p_x (1 - 2p_{x+1}) \]
\[ = 0.99 (1 - 0.9694) = 0.0303 \]

### 1.4 Curtate Future Lifetime Random Variable

In practice, actuaries use Excel extensively, so a discrete version of the future lifetime random variable would be easier to work with. We define

\[ K_x = \lfloor T_x \rfloor, \]

where \( \lfloor y \rfloor \) means the integral part of \( y \). For example, \( \lfloor 1 \rfloor = 1, \lfloor 4.3 \rfloor = 4 \) and \( \lfloor 10.99 \rfloor = 10 \). We call \( K_x \) the curtate future lifetime random variable.

It is obvious that \( K_x \) is a discrete random variable, since it can only take non-negative integral values (i.e., 0, 1, 2, \ldots). The probability mass function for \( K_x \) can be derived as follows:

\[ \Pr(K_x = 0) = \Pr(0 \leq T_x < 1) = q_x, \]
\[ \Pr(K_x = 1) = \Pr(1 \leq T_x < 2) = 1\|q_x, \]
\[ \Pr(K_x = 2) = \Pr(2 \leq T_x < 3) = 2\|q_x, \ldots \]
Inductively, we have

\[ \Pr(K_x = k) = k!q_x, \quad k = 0, 1, 2, \ldots \]  

(1.8)

The cumulative distribution function can be derived as follows:

\[ \Pr(K_x \leq k) = \Pr(T_x < k + 1) = k+1q_x, \quad \text{for} \quad k = 0, 1, 2, \ldots . \]

It is just that simple! Now, let us study the following example, which is taken from a previous SoA Exam.

**Example 1.5 [Course 3 Fall 2003 #28]**

For (x):
(i) \( K \) is the curtate future lifetime random variable.
(ii) \( q_{s+k} = 0.1(k + 1), \quad k = 0, 1, 2, \ldots, 9 \)

Calculate \( \text{Var}(K \wedge 3) \).
(A) 1.1  (B) 1.2  (C) 1.3  (D) 1.4  (E) 1.5

**Solution**

The notation \( \wedge \) means “minimum”. So here \( K \wedge 3 \) means \( \min(K, 3) \). For convenience, we let \( W = \min(K, 3) \). Our job is to calculate \( \text{Var}(W) \). Note that the only possible values that \( W \) can take are 0, 1, 2, and 3.

To accomplish our goal, we need the probability function of \( W \), which is related to that of \( K \). The probability function of \( W \) is derived as follows:

\[ \Pr(W = 0) = \Pr(K = 0) = q_x = 0.1 \]
\[ \Pr(W = 1) = \Pr(K = 1) = 1!q_x = p_x \times q_{s+1} \]
\[(1 - q_x)q_{x+1} = (1 - 0.1) \times 0.2 = 0.18\]

\[
\Pr(W = 2) = \Pr(K = 2) = 2q_x
\]
\[
= 2p_x \times q_{x+2} = p_x \times p_{x+1} \times q_{x+2}
\]
\[
= (1 - q_x)(1 - q_{x+1}) q_{x+2}
\]
\[
= 0.9 \times 0.8 \times 0.3 = 0.216
\]

\[
\Pr(W = 3) = \Pr(K \geq 3) = 1 - \Pr(K = 0) - \Pr(K = 1) - \Pr(K = 2) = 0.504.
\]

From the probability function for \(W\), we obtain \(E(W)\) and \(E(W^2)\) as follows:

\[
E(W) = 0 \times 0.1 + 1 \times 0.18 + 2 \times 0.216 + 3 \times 0.504 = 2.124
\]

\[
E(W^2) = 0^2 \times 0.1 + 1^2 \times 0.18 + 2^2 \times 0.216 + 3^2 \times 0.504 = 5.58
\]

This gives \(\text{Var}(W) = E(W^2) - [E(W)]^2 = 5.58 - 2.124^2 = 1.07\). Hence, the answer is (A).

**1.5 Force of Mortality**

In Exam FM, you learnt a concept called the force of interest, which measures the amount of interest credited in a very small time interval. By using this concept, you valued, for example, annuities that make payouts continuously. In this exam, you will encounter continuous life contingent cash flows. To value such cash flows, you need a function that measures the probability of death over a very small time interval. This function is called the force of mortality.

Consider an individual age \(x\) now. The force of mortality for this individual \(t\) years from now is denoted by \(\mu_{x+t}\) or \(\mu_x(t)\). At time \(t\), the (approximate) probability that this individual dies within a very small period of time \(\Delta t\) is \(\mu_{x+t} \Delta t\). The definition of \(\mu_{x+t}\) can be seen from the following diagram.
From the diagram, we can also tell that \( f_x(t) \Delta t = S_x(t) \mu_{x+t} \Delta t \). It follows that
\[
f_x(t) = S_x(t) \mu_{x+t} = t p_x \mu_{x+t}.
\]
This is an extremely important relation, which will be used throughout this study manual.

Recall that \( \frac{d}{dt} S_x(t) = F_x'(t) = -S_x'(t) \). This yields the following equation:
\[
\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)},
\]
which allows us to find the force of mortality when the survival function is known.

Recall that \( \frac{d}{dx} \ln x = \frac{1}{x} \), and that by chain rule, \( \frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)} \) for a real-valued function \( g \).

We can rewrite the previous equation as follows:
\[
\mu_{x+t} = -\frac{S_x'(t)}{S_x(t)} = -\frac{d[\ln S_x(t)]}{dt} - \mu_{x+t} dt = d[\ln S_x(t)].
\]
Replacing \( t \) by \( u \),
\[
-\int_0^t \mu_{x+u} du = d[\ln S_x(u)]
\]
\[
-\int_0^t \mu_{x+u} du = \int_0^t d[\ln S_x(u)]
\]
\[
-\int_0^t \mu_{x+u} du = \ln S_x(t) - \ln S_x(0)
\]
\[
S_x(t) = \exp\left(-\int_0^t \mu_{x+u} du\right).
\]
This allows us to find the survival function when the force of mortality is known.
Not all functions can be used for the force of mortality. We require the force of mortality to satisfy the following two criteria:

(i) \( \mu_{x+t} \geq 0 \) for all \( x \geq 0 \) and \( t \geq 0 \).

(ii) \( \int_0^\infty \mu_{x+u} \, du = \infty \).

Criterion (i) follows from the fact that \( \mu_{x+t} \Delta t \) is a measure of probability, while Criterion (ii) follows from the fact that \( \lim_{t \to \infty} S_x(t) = 0 \).

Note that the subscript \( x + t \) indicates the age at which death occurs. So you may use \( \mu_x \) to denote the force of mortality at age \( x \). For example, \( \mu_{20} \) refers to the force of mortality at age 20.

The two criteria above can then be written alternatively as follows:

(i) \( \mu_x \geq 0 \) for all \( x \geq 0 \).

(ii) \( \int_0^\infty \mu_x \, dx = \infty \).

The following two specifications of the force of mortality are often used in practice.

**Gompertz’ law**

\[ \mu_x = B c^x \]

**Makeham’s law**

\[ \mu_x = A + B c^x \]

In the above, \( A, B \) and \( c \) are constants such that \( A \geq -B, B > 0 \) and \( c > 1 \).
Let us study a few examples now.

**Example 1.6 [Structural Question]**

For a life age $x$ now, you are given:

$$S_x(t) = \frac{(10-t)^2}{100}, \quad 0 \leq t < 10.$$

(a) Find $\mu_{x+t}$.
(b) Find $f_x(t)$.

--- Solution ---

(a) $\mu_{x+t} = -S_x'(t) = -\frac{2(10-t)}{100} = \frac{2}{10-t}$.

(b) You may work directly from $S_x(t)$, but since we have found $\mu_{x+t}$ already, it would be quicker to find $f_x(t)$ as follows:

$$f_x(t) = S_x(t)\mu_{x+t} = \frac{(10-t)^2}{100} \times \frac{2}{10-t} = \frac{10-t}{50}.$$

**Example 1.7 [Structural Question]**

For a life age $x$ now, you are given

$$\mu_{x+t} = 0.002t, \quad t \geq 0.$$

(a) Is $\mu_{x+t}$ a valid function for the force of mortality of $(x)$?
(b) Find $S_x(t)$.
(c) Find $f_x(t)$.

--- Solution ---

(a) First, it is obvious that $\mu_{x+t} \geq 0$ for all $x$ and $t$.

Second, $\int_0^\infty \mu_{x+u} du = \int_0^\infty 0.002udu = 0.002u^2 \bigg|_0^\infty = \infty.$
Hence, it is a valid function for the force of mortality of $(x)$.

(b) $S_x(t) = \exp\left(-\int_0^t \mu_{x+u} \, du\right) = \exp\left(-\int_0^t 0.002u \, du\right) = \exp(-0.001t^2).

(c) $f_x(t) = S_x(t)\mu_{x+t} = 0.002t \exp(-0.001t^2)$.

Solution
First, $R = 1 - S_x(1) = 1 - p_x = q_x$.
Second,

$$S = 1 - \exp\left(-\int_0^t (\mu_{x+t} + k) \, du\right) = 1 - e^{-k} \exp\left(-\int_0^t \mu_{x+t} \, du\right) = 1 - e^{-k} S_x(1) = 1 - e^{-k} p_x.$$

Since $S = 0.75R$, we have
\[ 1 - e^{-k} p_x = 0.75q_x \]
\[ e^{-k} = \frac{1 - 0.75q_x}{p_x} \]
\[ e^k = \frac{p_x}{1 - 0.75q_x}. \]
\[ k = \ln \left( \frac{p_x}{1 - 0.75q_x} \right) = \ln \left( \frac{1 - q_x}{1 - 0.75q_x} \right) \]

Hence, the answer is (A).

---

**Example 1.9**  **[Structural Question]**

(a) Show that when \( \mu_x = Be^x \), we have
\[ p_x = g^{e^x(c^{x-1})}, \]
where \( g \) is a constant that you should identify.

(b) For a mortality table constructed using the above force of mortality, you are given that \( 10p_{50} = 0.861716 \) and \( 20p_{50} = 0.718743 \). Calculate the values of \( B \) and \( c \).

**Solution**

(a) To prove the equation, we should make use of the relationship between the force of mortality and \( p_x \).
\[ p_x = \exp \left( -\int_0^x \mu_s ds \right) = \exp \left( -\int_0^x Be^{s+c} ds \right) = \exp \left( \frac{-B}{\ln c} c^{x+c-1} \right). \]

This gives \( g = \exp(-B/\ln c) \).

(b) From (a), we have \( 0.861786 = g^{e^{50(c^{50-1})}} \) and \( 0.718743 = g^{e^{50(c^{20-1})}} \). This gives
\[ \frac{c^{20} - 1}{c^{10} - 1} = \frac{\ln(0.718743)}{\ln(0.861716)}. \]

Solving this equation, we obtain \( c = 1.02000 \). Substituting back, we obtain \( g = 0.776856 \) and \( B = 0.00500 \).
Now, let us study a longer structural question that integrates different concepts in this chapter.

**Example 1.10  [Structural Question]**

The function

\[
\frac{18000 - 110x - x^2}{18000}
\]

has been proposed for the survival function for a mortality model.

(a) State the implied limiting age \( \omega \).

(b) Verify that the function satisfies the conditions for the survival function \( S_0(x) \).

(c) Calculate \( 20p_0 \).

(d) Calculate the survival function for a life age 20.

(e) Calculate the probability that a life aged 20 will die between ages 30 and 40.

(f) Calculate the force of mortality at age 50.

---

**Solution**

(a) Since

\[
S_0(\omega) = \frac{18000 - 110\omega - \omega^2}{18000} = 0,
\]

We have \( \omega^2 + 110\omega - 18000 = 0 \Rightarrow (\omega - 90)(\omega + 200) = 0 \Rightarrow \omega = 90 \) or \( \omega = -200 \) (rejected).

Hence, the implied limiting age is 90.

(b) We need to check the following three conditions:

(i) \( S_0(0) = \frac{18000 - 110 \times 0 - 0^2}{18000} = 1 \)

(ii) \( S_0(\omega) = \frac{18000 - 110\omega - \omega^2}{18000} = 0 \)

(iii) \( \frac{d}{dx} S_0(x) = - \frac{2x + 110}{18000} < 0 \)

Therefore, the function satisfies the conditions for the survival function \( S_0(x) \).
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(c) \[ 20 p_0 = S_0(20) = \frac{18000 - 110 \times 20 - 20^2}{18000} = 0.85556 \]

(d) \[ S_{20}(x) = \frac{S_0(20 + x)}{S_0(20)} = \frac{\frac{18000}{(90 - 20 - x)(20 + x + 200)}}{\frac{18000}{(90 - 20)(20 + 200)}} \]
\[ = \frac{(70 - x)(x + 220)}{15400} = \frac{15400 - 150x - x^2}{15400}. \]

(e) The required probability is
\[ 10^{10}q_{20} = 10p_{20} - 20p_{20} \]
\[ = \frac{(70 - 10)(10 + 220)}{15400} - \frac{(70 - 20)(20 + 220)}{15400} = 0.89610 - 0.77922 = 0.11688 \]

(f) First, we find an expression for \( \mu_x \).
\[ \mu_x = -\frac{S'_0(x)}{S_0(x)} = -\frac{-110 - 2x}{18000} = \frac{2x + 110}{(90 - x)(x + 200)}. \]

Hence, \( \mu_{50} = \frac{2 \times 50 + 110}{(90 - 50)(50 + 200)} = 0.021. \)

[ END ]

You may be asked to prove some formulas in the structural questions of Exam MLC. Please study the following example, which involves several proofs.
Example 1.11  [Structural Question]

Prove the following equations:

(a) \( \frac{d}{dt} P_x = -p_x \mu_{x+t} \)

(b) \( q_x = \int_0^t p_x \mu_{x+s} \, ds \)

(c) \( \int_0^{\omega-x} t p_x \mu_{x+t} \, dt = 1 \)

Solution

(a) LHS = \( \frac{d}{dt} P_x = \frac{d}{dt} \exp(-\int_0^t \mu_{x+s} \, ds) = \exp(-\int_0^t \mu_{x+s} \, ds) \left( -\frac{d}{dt} \int_0^t \mu_{x+s} \, ds \right) = t p_x (-\mu_{x+t}) = \text{RHS} \)

(b) LHS = \( q_x = \Pr(T_x \leq t) = \int_0^t f_x(s) ds = \int_0^t p_x \mu_{x+s} \, ds = \text{RHS} \)

(c) LHS = \( \int_0^{\omega-x} t p_x \mu_{x+t} \, dt = \int_0^{\omega-x} f_x(t) \, dt = \omega-x q_x = 1 = \text{RHS} \)
Chapter 1: Survival Distributions

Exercise 1

1. [Structural Question] You are given:
   \[ S_0(t) = \frac{1}{1+t}, \quad t \geq 0. \]
   (a) Find \( F_0(t) \).
   (b) Find \( f_0(t) \).
   (c) Find \( S_x(t) \).
   (d) Calculate \( p_{20} \).
   (e) Calculate \( 10.5q_{30} \).

2. You are given:
   \[ f_0(t) = \frac{(30-t)^2}{9000}, \quad 0 \leq t < 30 \]
   Find an expression for \( p_5 \).

3. You are given:
   \[ f_0(t) = \frac{20-t}{200}, \quad 0 \leq t < 20. \]
   Find \( \mu_{10} \).

4. [Structural Question] You are given:
   \[ \mu_x = \frac{1}{100-x}, \quad 0 \leq x < 100. \]
   (a) Find \( S_{20}(t) \) for \( 0 \leq t < 80 \).
   (b) Compute \( 40p_{20} \).
   (c) Find \( f_{20}(t) \) for \( 0 \leq t < 80 \).

5. You are given:
   \[ \mu_x = \frac{2}{100-x}, \quad 0 \leq x < 100. \]
   Find the probability that the age at death is in between 20 and 50.

6. You are given:
   (i) \[ S_0(t) = \left(1 - \frac{t}{\omega}\right)^\alpha, \quad 0 \leq t < \omega, \quad \alpha > 0. \]
   (ii) \( \mu_{40} = 2\mu_{20} \).
   Find \( \omega \).
7. Express the probabilities associated with the following events in actuarial notation.
   (a) A new born infant dies no later than age 35.
   (b) A person age 10 now survives to age 25.
   (c) A person age 40 now survives to age 50 but dies before attaining age 55.
   Assuming that \( S_0(t) = e^{-0.005t} \) for \( t \geq 0 \), evaluate the probabilities.

8. You are given:
   \[
   S_0(t) = \left(1 - \frac{t}{100}\right)^2, \quad 0 \leq t < 100. 
   \]
   Find the probability that a person aged 20 will die between the ages of 50 and 60.

9. You are given:
   (i) \( 2p_x = 0.98 \)
   (ii) \( p_{x+2} = 0.985 \)
   (iii) \( 5q_x = 0.0775 \)
   Calculate the following:
   (a) \( 3p_x \)
   (b) \( 2p_{x+3} \)
   (c) \( 2.3q_x \)

10. You are given:
    \[ q_{x+k} = 0.1(k + 1), \quad k = 0, 1, 2, \ldots, 9. \]
    Calculate the following:
    (a) \( \Pr(K_x = 1) \)
    (b) \( \Pr(K_x \leq 2) \)

11. [Structural Question] You are given \( \mu_x = \mu \) for all \( x \geq 0 \).
    (a) Find an expression for \( \Pr(K_x = k) \), for \( k = 0, 1, 2, \ldots \), in terms of \( \mu \) and \( k \).
    (b) Find an expression for \( \Pr(K_x \leq k) \), for \( k = 0, 1, 2, \ldots \), in terms of \( \mu \) and \( k \).
    Suppose that \( \mu = 0.01 \).
    (c) Find \( \Pr(K_x = 10) \).
    (d) Find \( \Pr(K_x \leq 10) \).
12. Which of the following is equivalent to $\int_0^t p_{x+u} \mu_{x+u} \, du$?

(A) $t p_x$
(B) $t q_x$
(C) $f_x(t)$
(D) $-f_x(t)$
(E) $f_x(t) \mu_{x+t}$

13. Which of the following is equivalent to $\frac{d}{dt} p_x$?

(A) $-p_x \mu_{x+t}$
(B) $\mu_{x+t}$
(C) $f_x(t)$
(D) $-\mu_{x+t}$
(E) $f_x(t) \mu_{x+t}$

14. (2000 Nov #36) Given:
   (i) $\mu_x = F + e^{2x}$, $x \geq 0$
   (ii) $0.4 p_0 = 0.50$
   Calculate $F$.

(A) $-0.20$
(B) $-0.09$
(C) $0.00$
(D) $0.09$
(E) $0.20$

15. (CAS 2004 Fall #7) Which of the following formulas could serve as a force of mortality?

(I) $\mu_x = Bc^x$, $B > 0$, $C > 1$
(II) $\mu_x = a(b + x)^{-1}$, $a > 0$, $b > 0$
(III) $\mu_x = (1 + x)^{-3}$, $x \geq 0$

(A) (I) only
(B) (II) only
(C) (III) only
(D) (I) and (II) only
(E) (I) and (III) only
16. (2002 Nov #1) You are given the survival function $S_0(t)$, where

(i) $S_0(t) = 1, \quad 0 \leq t \leq 1$

(ii) $S_0(t) = 1 - \frac{e^t}{100}, \quad 1 \leq t \leq 4.5$

(iii) $S_0(t) = 0, \quad 4.5 \leq t$

Calculate $\mu_4$.

(A) 0.45
(B) 0.55
(C) 0.80
(D) 1.00
(E) 1.20

17. (CAS 2004 Fall #8) Given $S_0(t) = \left(1 - \frac{t}{100}\right)^{1/2}$, for $0 \leq t \leq 100$, calculate the probability that a life age 36 will die between ages 51 and 64.

(A) Less than 0.15
(B) At least 0.15, but less than 0.20
(C) At least 0.20, but less than 0.25
(D) At least 0.25, but less than 0.30
(E) At least 0.30

18. (2007 May #1) You are given:

(i) $3p_{70} = 0.95$

(ii) $2p_{71} = 0.96$

(iii) $\int_{71}^{75} \mu_x \, dx = 0.107$

Calculate $5p_{70}$.

(A) 0.85
(B) 0.86
(C) 0.87
(D) 0.88
(E) 0.89
19. (2005 May #33) You are given:
\[
\mu_x = \begin{cases} 
0.05 & 50 \leq x < 60 \\
0.04 & 60 \leq x < 70 
\end{cases}
\]
Calculate \(|4|_4 q_{50}\).

(A) 0.38
(B) 0.39
(C) 0.41
(D) 0.43
(E) 0.44

20. (2004 Nov #4) For a population which contains equal numbers of males and females at birth:
   (i) For males, \( \mu^m_x = 0.10, x \geq 0 \)
   (ii) For females, \( \mu^f_x = 0.08, x \geq 0 \)
Calculate \( q_{60} \) for this population.

(A) 0.076
(B) 0.081
(C) 0.086
(D) 0.091
(E) 0.096

21. (2001 May #28) For a population of individuals, you are given:
   (i) Each individual has a constant force of mortality.
   (ii) The forces of mortality are uniformly distributed over the interval (0, 2).
Calculate the probability that an individual drawn at random from this population dies within one year.

(A) 0.37
(B) 0.43
(C) 0.50
(D) 0.57
(E) 0.63
22. [Structural Question] The mortality of a certain population follows the De Moivre’s Law; that is
\[ \mu_x = \frac{1}{\omega - x}, \quad x < \omega. \]
(a) Show that the survival function for the age-at-death random variable \( T_0 \) is
\[ S_0(x) = 1 - \frac{x}{\omega}, \quad 0 \leq x < \omega. \]
(b) Verify that the function in (a) is a valid survival function.
(c) Show that
\[ p_x = 1 - \frac{1}{\omega - x}, \quad 0 \leq t < \omega - x, \quad x < \omega. \]

23. [Structural Question] The probability density function for the future lifetime of a life age 0 is given by
\[ f_0(x) = \frac{\alpha \lambda^x}{(\lambda + x)^{a+1}}, \quad \alpha, \lambda > 0 \]
(a) Show that the survival function for a life age 0, \( S_0(x) \), is \( S_0(x) = \left( \frac{\lambda}{\lambda + x} \right)^a \).
(b) Derive an expression for \( \mu_x \).
(c) Derive an expression for \( S_a(t) \).
(d) Using (b) and (c), or otherwise, find an expression for \( f_a(t) \).

24. [Structural Question] For each of the following equations, determine if it is correct or not. If it is correct, prove it.
(a) \( t\mu q_x = q_x + uq_{x+t} \)
(b) \( t+uq_x = q_x \times uq_{x+t} \)
(c) \( \frac{d}{dx} tP_x = tP_x (\mu_x - \mu_{x+t}) \)
### Solutions to Exercise 1

1. (a) \( F_0(t) = 1 - \frac{1}{1 + t} = \frac{t}{1 + t} \).

   (b) \( f_0(t) = \frac{d}{dt} F_0(t) = \frac{1 + t - t}{(1 + t)^2} = \frac{1}{(1 + t)^2} \).

   (c) \( S_x(t) = \frac{S_0(x + t)}{S_0(x)} = \frac{1 + x + t}{1 + x} \).

   (d) \( p_{20} = S_0(1) = 21/22 \).

   (e) \( 10q_{30} = 10p_{30} - 15p_{30} = S_0(10) - S_0(15) = \frac{1 + 30}{1 + 30 + 10} - \frac{1 + 30}{1 + 30 + 15} = \frac{31}{41} - \frac{31}{46} = 0.0822 \).

2. \( S_0(t) = \int_0^{30} f_0(u)du = \int_0^{30} \frac{(30 - u)^2}{9000} du = \frac{1}{27000} \left[ (30 - u)^3 \right]_0^{30} = \frac{(30 - t)^3}{27000} \).

   If follows that \( t^5 = S_5(t) = \frac{S_0(5 + t)}{S_0(5)} = \frac{(30 - 5 - t)^3}{(30 - 5)^3} = \left( 1 - \frac{t}{25} \right)^3 \).

3. \( S_0(t) = \int_0^{20} f_0(u)du = \int_0^{20} \frac{(20 - u)^2}{200} du = \frac{1}{400} \left[ (20 - u)^3 \right]_0^{20} = \frac{(20 - t)^3}{400} \).

   \( \mu_t = \frac{f_0(t)}{S_0(t)} = \frac{20 - t}{200} - \frac{(20 - t)^2}{400} = \frac{2}{20 - t} \).

   Hence, \( \mu_{10} = 2/(20 - 10) = 0.2 \).

4. (a) First, note that \( \mu_{20+t} = \frac{1}{100 - 20 - t} = \frac{1}{80 - t} \). We have

   \[ S_{200}(t) = \exp \left( -\int_0^t \mu_{20+t} du \right) = \exp \left( -\int_0^t \frac{1}{80 - u} du \right) = \exp \left( \ln (80 - u) \right)_0^t = \exp \left( \frac{80 - t}{80} \right) = 1 - \frac{t}{80} \).

   (b) \( 40p_{20} = S_{20}(40) = 1 - 40/80 = 1/2 \).

   (c) \( f_{20}(t) = S_0(t)\mu_{20+t} = \left( 1 - \frac{t}{80} \right) \left( \frac{1}{80 - t} \right) = \frac{1}{80} \).
5. Our goal is to find \( \Pr(20 < T_0 < 50) = S_0(20) - S_0(50) \).

Given the force of mortality, we can find the survival function as follows:

\[
S_0(t) = \exp\left(-\int_0^t \mu_u \, du\right) = \exp\left(-\int_0^t \frac{2}{100 - u} \, du\right) = \exp(2\ln(100 - u)) = \exp(2\ln\frac{100 - t}{100}) = \left(1 - \frac{t}{100}\right)^2
\]

So, the required probability is \((1 - 20/100)^2 - (1 - 50/100)^2 = 0.8^2 - 0.5^2 = 0.39\).

6. \( \mu_x = -\frac{S'_0(x)}{S_0(x)} = -\frac{-\alpha \left( -\frac{1}{\omega} \right) \left( 1 - \frac{x}{\omega} \right)^{\alpha - 1}}{\left( 1 - \frac{x}{\omega} \right)^\alpha} = \frac{\alpha}{\omega - x} \).

We are given that \( \mu_40 = 2\mu_{20} \). This implies \( \frac{\alpha}{\omega - 40} = \frac{2\alpha}{\omega - 20} \), which gives \( \omega = 60 \).

7. (a) The probability that a new born infant dies no later than age 35 can be expressed as \( 35q_0 \). [Here we have “\( q \)” for a death probability, \( x = 0 \) and \( t = 35 \).]

Further, \( 35q_0 = F_0(35) = 1 - S_0(35) = 0.1605 \).

(b) The probability that a person age 10 now survives to age 25 can be expressed as \( 15p_{10} \). [Here we have “\( p \)” for a survival probability, \( x = 10 \) and \( t = 25 - 10 = 15 \).]

Further, we have \( 15p_{10} = S_{10}(15) = \frac{S_0(25)}{S_0(15)} = 0.9277 \).

(c) The probability that a person age 40 now survives to age 50 but dies before attaining age 55 can be expressed as \( 10|5q_{40} \). [Here, we have “\( q \)” for a (deferred) death probability, \( x = 40 \), \( t = 50 - 40 = 10 \), and \( u = 55 - 50 = 5 \).]

Further, we have \( 10|5q_{40} = S_{40}(10) - S_{40}(15) = \frac{S_0(50)}{S_0(40)} - \frac{S_0(55)}{S_0(40)} = 0.0235 \).

8. The probability that a person aged 20 will die between the ages of 50 and 60 is given by

\[
30|10q_{20} = 30p_{20} - 40p_{20} = S_{20}(30) - S_{20}(40).
\]

\[
S_{20}(t) = S_0(20 + t) = \frac{\left(1 - \frac{20 + t}{100}\right)^2}{\left(1 - \frac{20}{100}\right)^2} = \left(1 - \frac{t}{80}\right)^2.
\]

So, \( S_{20}(30) = \left(1 - \frac{30}{80}\right)^2 = \frac{25}{64}, \quad S_{20}(40) = \left(1 - \frac{40}{80}\right)^2 = \frac{16}{64} \). As a result, \( 30|10q_{20} = 9/64 \).
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9. (a) \( 3p_x = 2p_x \times p_{x+2} = 0.98 \times 0.985 = 0.9653. \)
(b) \( 3p_x \times 2p_{x+2} = 5p_x = 1 - q_x \)
\[ \Rightarrow 2p_{x+3} = \frac{1 - q_x}{5p_x} = \frac{1 - 0.0775}{0.9653} = 0.95566 \]
(c) \( 23q_x = 2p_x - 5p_x = 0.98 - (1 - 0.0775) = 0.0575. \)

10. (a) \( \Pr(K_x = 1) = 1 | q_x = px = (1 - q_x)q_{x+1} = (1 - 0.1) \times 0.2 = 0.18 \)
(b) \( \Pr(K_x = 0) = qx = 0.1 \)
\[ \Pr(K_x = 2) = 2q_x = 2px \times q_{x+2} = px \times p_{x+1} \times q_{x+2} = (1 - q_x)(1 - q_{x+1})q_{x+2} \]
\[ = 0.9 \times 0.8 \times 0.3 = 0.216. \]
Hence, \( \Pr(K_x \leq 2) = 0.1 + 0.18 + 0.216 = 0.496. \)

11. (a) Given that \( \mu_x = \mu \) for all \( x \geq 0 \), we have \( tpx = e^{-\mu t}, px = e^{-\mu} \) and \( qx = 1 - e^{-\mu}. \)
\[ \Pr(K_x = k) = kq_x = kp_x \times q_{x+k} = e^{-k\mu} (1 - e^{-\mu}). \]
(b) \( \Pr(K_x \leq k) = k+1q_x = 1 - k+1p_x = 1 - e^{-(k+1)\mu}. \)
(c) When \( \mu = 0.01 \), \( \Pr(K_x = 10) = e^{-10 \times 0.01} (1 - e^{-0.01}) = 0.0090. \)
(d) When \( \mu = 0.01 \), \( \Pr(K_x \leq 10) = 1 - e^{-(10+1) \times 0.01} = 0.1042. \)

12. First of all, note that \( upx\mu_{x+u} \) in the integral is simply \( f_x(u) \).
\[ \int_0^t px\mu_{x+u} du = \int_0^t f_x(u) du = \Pr(T_x \leq t) = F_x(t) = \mu x. \]
Hence, the answer is (B).

13. Method I: We use \( tpx = 1 - t qx. \) Differentiating both sides with respect to \( t, \)
\[ \frac{d}{dt} px = -\frac{d}{dt} q_x = - \frac{d}{dt} F_x(t) = - f_x(t). \]
Noting that \( f_x(t) = tpx\mu_{x+u}, \) the answer is (A).
Method II: We differentiate \( tpx \) with respect to \( t \) as follows:
\[ \frac{d}{dt} px = \frac{d}{dt} S_x(t) = \frac{d}{dt} \exp \left( - \int_0^t \mu_{x+u} du \right) \]
\[ \quad = \exp \left( - \int_0^t \mu_{x+u} du \right) \frac{d}{dt} \left( - \int_0^t \mu_{x+u} du \right). \]
Recall the fundamental theorem of calculus, which says that \( \frac{d}{dt} \int_c^t g(u) du = g(t). \) Thus
\[ \frac{d}{dt} px = \exp \left( - \int_0^t \mu_{x+u} du \right) (- \mu_{x+1}) = - tpx\mu_{x+1}. \]
Hence, the answer is (A).
14. First, note that

\[ 0.4 P_0 = 0.5 = e^{-\int_0^{0.4} \mu_x \, dx} = e^{-\int_0^{0.4} (F + e^{2u}) \, du}. \]

The exponent in the above is

\[ -\int_0^{0.4} (F + e^{2u}) \, du = - \left( F u + \frac{1}{2} e^{2u} \right)_0^{0.4} \]

\[ = -0.4 F - 1.11277 + 0.5 \]

\[ = -0.4 F - 0.61277 \]

As a result, \[0.5 = e^{-0.4 F - 0.61277},\] which gives \( F = 0.2. \) Hence, the answer is (E).

15. Recall that we require the force of mortality to satisfy the following two criteria:

(i) \( \mu_x \geq 0 \) for all \( x \geq 0, \)

(ii) \( \int_0^\infty \mu_x \, dx = \infty. \)

All three specifications of \( \mu_x \) satisfy Criterion (i). We need to check Criterion (ii).

We have

\[ \int_0^\infty Bc^x \, dx = \frac{Bc^t}{\ln c} \bigg|_0^\infty = \infty, \]

\[ \int_0^\infty \frac{a}{b+x} \, dx = a \ln(b+x) \bigg|_0^\infty = \infty, \]

and

\[ \int_0^\infty \frac{1}{(1+x)^3} \, dx = \frac{-1}{2(1+x)^2} \bigg|_0^\infty = \frac{1}{2}. \]

Only the first two specifications can satisfy Criterion (ii). Hence, the answer is (D).

[Note: \( \mu_x = Bc^x \) is actually the Gompertz’ law. If you knew that you could have identified that \( \mu_x = Bc^x \) can serve as a force of mortality without doing the integration.]

16. Recall that \( \mu_{x+t} = -\frac{S_x'(t)}{S_x(t)}. \)

Since we need \( \mu_4, \) we use the definition of \( S_0(t) \) for \( 1 \leq t \leq 4.5: \)

\[ S_0(t) = 1 - \frac{e^t}{100}, \quad S_0'(t) = \frac{e^t}{100}. \]

As a result, \( \mu_4 = \frac{e^4}{100} = \frac{e^4}{100 - e^4} = 1.203. \) Hence, the answer is (E).
17. The probability that a life age 36 will die between ages 51 and 64 is given by 

\[ S_{36}(15) - S_{36}(28). \]

We have 

\[ S_{36}(t) = \frac{S_t(36+t)}{S_t(36)} = \left(\frac{1 - \frac{36+t}{100}}{1 - \frac{36}{100}}\right)^{1/2} = \left(\frac{64-t}{64}\right)^{1/2} = \frac{\sqrt{64-t}}{8}. \]

This gives 

\[ S_{36}(15) = \frac{7}{8} \text{ and } S_{36}(28) = \frac{6}{8}. \]

As a result, the required probability is 

\[ S_{36}(15) - S_{36}(28) = \frac{1}{8} = 0.125. \]

Hence, the answer is (A).

18. The computation of \( 5p_{70} \) involves three steps.

First, 

\[ 37\text{P}_{70} = 3\text{P}_{71} = 0.95, \quad \frac{271}{0.9896} = 0.96. \]

Second, 

\[ 75\text{P}_{71} = e^{-\int_{71}^{75} \mu du} = e^{-0.107} = 0.8985. \]

Finally, 

\[ 5p_{70} = 0.9896 \times 0.8985 = 0.889. \]

Hence, the answer is (E).

19. 

\[ 4p_{50} = e^{-0.05 \times 4} = 0.8187 \]

\[ 10p_{50} = e^{-0.05 \times 10} = 0.6065 \]

\[ 8p_{60} = e^{-0.04 \times 8} = 0.7261 \]

\[ 18p_{50} = 10p_{50} \times 8p_{60} = 0.6065 \times 0.7261 = 0.4404 \]

Finally, 

\[ 4|14q_{50} = 4p_{50} - 18p_{50} = 0.8187 - 0.4404 = 0.3783. \]

Hence, the answer is (A).

20. For males, we have 

\[ S_0^m(t) = e^{-\int_0^t \mu^m du} = e^{-\int_0^{0.10} du} = e^{-0.10t}. \]

For females, we have 

\[ S_0^f(t) = e^{-\int_0^t \mu^f du} = e^{-\int_0^{0.08} du} = e^{-0.08t}. \]

For the overall population,

\[ S_0(60) = \frac{e^{-0.1 \times 60} + e^{-0.08 \times 60}}{2} = 0.005354, \]

and

\[ S_0(61) = \frac{e^{-0.1 \times 61} + e^{-0.08 \times 61}}{2} = 0.00492. \]
Finally, \( q_{60} = 1 - p_{60} = 1 - \frac{S_0(61)}{S_0(60)} = 0.081 \). Hence, the answer is (B).

21. Let \( M \) be the force of mortality of an individual drawn at random, and \( T \) be the future lifetime of the individual. We are given that \( M \) is uniformly distributed over \((0, 2)\). So the density function for \( M \) is \( f_M(\mu) = \frac{1}{2} \) for \( 0 < \mu < 2 \) and 0 otherwise.

This gives

\[
\Pr(T \leq 1) = E[\Pr(T \leq 1 \mid M)] = \int_0^\infty \Pr(T \leq 1 \mid M = \mu) f_M(\mu) \, d\mu = \int_0^2 (1 - e^{-\mu}) \frac{1}{2} \, d\mu = \frac{1}{2} (2 + e^{-2} - 1) = \frac{1}{2} (1 + e^{-2}) = 0.56767.
\]

Hence, the answer is (D).

22. (a) We have, for \( 0 \leq x < \omega \),

\[
S_0(x) = \exp\left(-\int_0^x \mu_x \, ds\right) = \exp\left(-\int_0^x \frac{1}{\omega - s} \, ds\right) = \exp([\ln(\omega - s)]_0^x) = e^{\ln(1 - \frac{x}{\omega})} = 1 - \frac{x}{\omega}.
\]

(b) We need to check the following three conditions:

(i) \( S_0(0) = 1 - 0/\omega = 1 \)
(ii) \( S_0(\omega) = 1 - \omega/\omega = 0 \)
(iii) \( S'_0(\omega) = -1/\omega < 0 \) for all \( 0 \leq x < \omega \), which implies \( S_0(x) \) is non-increasing.

Hence, the function in (a) is a valid survival function.

(c) \( p_x = \frac{S_0(x + t)}{S_0(x)} = \frac{1 - \frac{x + t}{\omega}}{1 - \frac{x}{\omega}} = 1 - \frac{t}{\omega - x} \), for \( 0 \leq t < \omega - x, \ x < \omega \).

23. (a) \( S_0(x) = 1 - F_0(x) = 1 - \int_0^x f_0(s) \, ds = 1 - \int_0^x \frac{\alpha \lambda^s}{(\lambda + s)^{a+1}} \, ds = \frac{\lambda^x}{(\lambda + x)^a} \).

(b) \( \mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\alpha}{\lambda + x} \).
Chapter 1: Survival Distributions

(c) \[ S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \left( \frac{\lambda}{\lambda + x + t} \right)^\alpha \left( \frac{\lambda}{\lambda + x} \right)^\alpha \].

(d) \[ f_x(t) = S_x(t)\mu_{x+t} = \left( \frac{\lambda + x}{\lambda + x + t} \right)^\alpha \frac{\alpha}{\lambda + x + t} \].

24. (a) No, the equation is not correct. The correct equation should be \( \theta_0 q_x = p_x \times \theta q_{x+t} \).

(b) No, the equation is not correct. The correct equation should be \( t u p_x = p_x \times u p_{x+t} \).

(c) Yes, the equation is correct. The proof is as follows:

\[
\frac{d}{dx} P_x = \frac{d}{dx} \frac{S_0(x+t)}{S_0(x)} = \frac{S_0(x)S'_0(x+ t) - S_0(x+ t)S'_0(x)}{[S_0(x)]^2}
= \frac{S_0(x)(-f_0(x+t)) - S_0(x+ t)(-f_0(x))}{[S_0(x)]^2}
= \frac{-f_0(x+t)S_0(x)}{S_0(x)} + \frac{S_0(x+ t)f_0(x)}{S_0(x)}
= \frac{-f_0(x+t)}{S_0(x)} + \frac{S_0(x+ t)f_0(x)}{S_0(x)}
= \frac{-\mu_{x+t} p_x}{p_x} + t p_x \mu_x
= t p_x (\mu_x - \mu_{x+t})
\]
Chapter 2  Life Tables

OBJECTIVES

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions \( l_x \) and \( d_x \), and explain how they are applied.

\[
\begin{array}{|c|c|c|}
\hline
x & l_x & d_x \\
\hline
0 & 1000 & 16 \\
1 & 984 & 7 \\
2 & 977 & 12 \\
3 & 965 & 75 \\
4 & 890 & 144 \\
\hline
\end{array}
\]
In this hypothetical life table, the value of $l_0$ is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots$, the function $l_x$ stands for the expected number of persons who can survive to age $x$. Given an assumed value of $l_0$, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function $l_x$, we can easily obtain values of $S_0(x)$ for integral values of $x$ using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$ 

Furthermore, we have

$$p_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}}{l_x} = \frac{l_x}{l_0} = \frac{l_{x+t}}{l_x},$$ 

which means that we can calculate $p_x$ for all integral values of $t$ and $x$ from the life table function $l_x$.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by $d_x$. It immediately follows that $d_x = l_x - l_{x+t}$.

We can then calculate $q_x$ and $q_{x+t}$ by the following two relations:

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \quad q_{x+t} = \frac{d_{x+t}}{l_x} = \frac{l_x - l_{x+t}}{l_x}.$$

When $t = 1$, we can omit the subscript $t$ and write $d_x$ as $d_x$. By definition, we have

$$d_x = d_x + d_{x+1} + \ldots + d_{x+t-1}.$$ 

Graphically,

$$d_x + d_{x+1} + d_{x+2} + d_{x+3} + \ldots + d_{x+t-1} = d_x = l_x - l_{x+t}.$$ 

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x+1$</th>
<th>$x+2$</th>
<th>$x+3$</th>
<th>$x+4$</th>
<th>$\ldots$</th>
<th>$x+t-1$</th>
<th>$x+t$</th>
<th>$l_x$</th>
<th>$l_{x+t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>age</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Chapter 2: Life Tables

Also, when \( t = 1 \), we have the following relations:

\[
d_x = l_x - l_{x+1}, \quad p_x = \frac{l_{x+1}}{l_x}, \quad \text{and} \quad q_x = \frac{d_x}{l_x}.
\]

Summing up, with the life table functions \( l_x \) and \( d_x \), we can recover survival probabilities \( t p_x \) and death probabilities \( t q_x \) for all integral values of \( t \) and \( x \) easily.

---

**Formula**

**Life Table Functions**

\[
p_x = \frac{l_{x+t}}{l_x} \tag{2.1}
\]

\[
d_x = l_x - l_{x+t} = d_x + d_{x+1} + \ldots + d_{x+t-1} \tag{2.2}
\]

\[
q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} \tag{2.3}
\]

---

Exam questions are often based on the Illustrative Life Table, which is, of course, provided in the examination. To obtain a copy of this table, download the most updated Exam MLC syllabus. On the last page of the syllabus, you will find a link to Exam MLC Tables, which encompass the Illustrative Life Table. You may also download the table directly from


The Illustrative Life Table contains a lot of information. For now, you only need to know and use the first three columns: \( x \), \( l_x \), and \( 1000 q_x \). For example, to obtain the value \( q_{60} \), simply use the column labeled \( 1000 q_x \). You should obtain \( 1000 q_{60} = 13.76 \), which means \( q_{60} = 0.01376 \). It is also possible, but more tedious, to calculate \( q_{60} \) using the column labeled \( l_x \); we have \( q_{60} = 1 - \frac{l_{61}}{l_{60}} = 0.01376 \).

To get values of \( t p_x \) and \( t q_x \) for \( t > 1 \), you should always use the column labeled \( l_x \). For example, we have \( s p_{60} = l_{65} / l_{60} = 7533964 / 8188074 = 0.92011 \) and \( s q_{60} = 1 - s p_{60} = 1 - 0.92011 = \)
0.07989. Here, you should not base your calculations on the column labeled $1000q_x$, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

### Example 2.1

You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$d_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>96178.01</td>
<td>99.0569</td>
</tr>
<tr>
<td>21</td>
<td>96078.95</td>
<td>102.0149</td>
</tr>
<tr>
<td>22</td>
<td>95976.93</td>
<td>105.2582</td>
</tr>
<tr>
<td>23</td>
<td>95871.68</td>
<td>108.8135</td>
</tr>
<tr>
<td>24</td>
<td>95762.86</td>
<td>112.7102</td>
</tr>
<tr>
<td>25</td>
<td>95650.15</td>
<td>116.9802</td>
</tr>
</tbody>
</table>

Calculate the following:

(a) $5p_{20}$

(b) $q_{24}$

(c) $4|1q_{20}$

### Solution

(a) $5p_{20} = \frac{l_{25}}{l_{20}} = \frac{95650.15}{96178.01} = 0.994512$.

(b) $q_{24} = \frac{d_{24}}{l_{24}} = \frac{112.7102}{95762.86} = 0.001177$.

(c) $4|1q_{20} = \frac{1d_{24}}{l_{20}} = \frac{112.7102}{96178.01} = 0.001172$. 
You are given:

(i) \( S_0(x) = 1 - \frac{x}{100}, \quad 0 \leq x \leq 100 \)

(ii) \( l_0 = 100 \)

(a) Find an expression for \( l_x \) for \( 0 \leq x \leq 100 \).

(b) Calculate \( q_2 \).

(c) Calculate \( 3q_2 \).

Solution

(a) \( l_x = l_0 S_0(x) = 100 - x \).

(b) \( q_2 = \frac{l_2 - l_3}{l_2} = \frac{98 - 97}{98} = \frac{1}{98} \).

(c) \( 3q_2 = \frac{l_2 - l_5}{l_2} = \frac{98 - 95}{98} = \frac{3}{98} \).

In Exam MLC, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3

For a certain population of 20 years old, you are given:

(i) \( \frac{2}{3} \) of the population are nonsmokers, and \( \frac{1}{3} \) of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over \([0, 80)\).

(iii) The future lifetime of a smoker is uniformly distributed over \([0, 50)\).

Calculate \( \hat{p}_{40} \) for a life randomly selected from those surviving to age 40.
Solution

The calculation of the required probability involves two steps. First, we need to know the composition of the population at age 20.

- Suppose that there are $l_{20}$ persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}l_{20}$ nonsmokers and $\frac{1}{3}l_{20}$ smokers.

- For nonsmokers, the proportion of individuals who can survive to age 40 is $1 – \frac{20}{80} = \frac{3}{4}$. For smokers, the proportion of individuals who can survive to age 40 is $1 – \frac{20}{50} = \frac{3}{5}$. At age 40, there are $\frac{3 \cdot 2}{4 \cdot 3}l_{20} = 0.5l_{20}$ nonsmokers and $\frac{3 \cdot 1}{5 \cdot 3}l_{20} = 0.2l_{20}$ smokers. Hence, among those who can survive to age 40, $5/7$ are nonsmokers and $2/7$ are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over $[0, 60)$. This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 – \frac{5}{60} = \frac{11}{12}$.

- For a smoker at age 40, the remaining lifetime is uniformly distributed over $[0, 30)$. This means that the probability for a smoker to survive from age 40 to age 45 is $1 – \frac{5}{30} = \frac{5}{6}$.

Finally, for the whole population, we have

$$5 \times \frac{11}{12} + \frac{2 \times 5}{7 \times 6} = \frac{25}{28}.$$  

[END]

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of $t p_x$ and $t q_x$ when both $t$ and $x$ are integers. But what if $t$ and/or $x$ are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.
In Exam MLC, you are required to know two fractional age assumptions:
1. Uniform distribution of death
2. Constant force of mortality
We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam MLC syllabus. The idea behind this assumption is that we use a bridge, denoted by $U$, to connect the (continuous) future lifetime random variable $T_x$ and the (discrete) curtate future lifetime random variable $K_x$. The idea is illustrated diagrammatically as follows:

![Diagram of Uniform Distribution of Death]

It is assumed that $U$ follows a uniform distribution over the interval $[0, 1]$, and that $U$ and $K_x$ are independent. Then, for $0 \leq r < 1$ and an integral value of $x$, we have

\[ r q_x = \Pr(T_x \leq r) \]
\[ = \Pr(U < r \cap K_x = 0) \]
\[ = \Pr(U < r) \Pr(K_x = 0) \]
\[ = rq_x. \]

The second last step follows from the assumption that $U$ and $K_x$ are independent, while the last step follows from the fact that $U$ follows a uniform distribution over $[0, 1]$.

**Key Equation for the UDD Assumption**

\[ r q_x = rq_x, \quad \text{for } 0 \leq r < 1 \] (2.4)
This means that under UDD, we have, for example, $0.4q_{50} = 0.4q_{50}$. The value of $q_{50}$ can be obtained straightforwardly from the life table. To calculate $\varphi_x$, for $0 \leq r < 1$, we use $\varphi_x = 1 - r q_x = 1 - r q_x$. For example, we have $0.1p_{20} = 1 - 0.1q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between $l_x$ and $l_{x+1}$, that is,

$$l_{x+r} = (1-r)l_x + rl_{x+1}.$$ 

**Proof:**

$$\varphi_x = 1 - r q_x = (1-r) + r p_x$$

$$\frac{l_{x+r}}{l_x} = (1-r) + r \frac{l_{x+1}}{l_x}$$

$$l_{x+r} = (1-r)l_x + rl_{x+1}$$

You will find this equation – the interpolation between $l_x$ and $l_{x+1}$ – very useful if you are given a table of $l_x$ (instead of $q_x$).

---

**Application of the UDD Assumption to $l_x$**

$$l_{x+r} = (1-r)l_x + rl_{x+1}, \quad \text{for } 0 \leq r < 1 \quad (2.5)$$

What if the subscript on the left-hand-side of $\varphi_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate $2.5p_{30}$ as follows:

$$2.5p_{30} = 2p_{30} \times 0.5p_{32} = 2p_{30} \times (1 - 0.5q_{32}).$$

The value of $2p_{30}$ and $q_{32}$ can be obtained from the life table straightforwardly.
Chapter 2: Life Tables

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider \(0.1p_{5.7}\) (both subscripts are not integers). The trick is that we multiply this probability with \(0.7p_5\), that is,

\[
0.7p_5 \times 0.1p_{5.7} = 0.8p_5.
\]

This gives \(0.1p_{5.7} = \frac{0.8p_5}{0.7p_5} = \frac{1-0.8q_5}{1-0.7q_5}\). The value of \(q_5\) can be obtained from the life table.

To further illustrate this trick, let us consider \(3.5q_{4.6}\). This probability can be evaluated by first calculating \(3.5p_{4.6}\):

\[
0.6p_4 \times 3.5p_{4.6} = 4.1p_4.
\]

Then, we have \(3.5p_{4.6} = \frac{4.1p_4}{0.6p_4} = \frac{4p_4(1-0.1q_8)}{1-0.6q_4}\), and finally \(3.5q_{4.6} = 1 - \frac{4p_4(1-0.1q_8)}{1-0.6q_4}\).

The values of \(4p_4\), \(q_8\) and \(q_4\) can be obtained from the life table.

Let us study the following example.

**Example 2.4**

You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(l_x)</th>
<th>(d_x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>100000</td>
<td>300</td>
</tr>
<tr>
<td>61</td>
<td>99700</td>
<td>400</td>
</tr>
<tr>
<td>62</td>
<td>99300</td>
<td>500</td>
</tr>
<tr>
<td>63</td>
<td>98800</td>
<td>600</td>
</tr>
<tr>
<td>64</td>
<td>98200</td>
<td>700</td>
</tr>
<tr>
<td>65</td>
<td>97500</td>
<td>800</td>
</tr>
</tbody>
</table>
Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) \(0.26p_{61}\)
(b) \(2.2q_{60}\)
(c) \(0.3q_{62.8}\)

**Solution**

(a) \(0.26p_{61} = 1 - 0.26q_{61} = 1 - 0.26 \times \frac{400}{99700} = 0.998957\).

Alternatively, we can calculate the answer by using a linear interpolation between \(l_{61}\) and \(l_{62}\) as follows:

\[ l_{61.26} = (1 - 0.26)l_{61} + 0.26l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596. \]

It follows that \(0.26p_{61} = l_{61.26} / l_{61} = 99596 / 99700 = 0.998957\).

(b) \(2.2q_{60} = 1 - 2p_{60} = 1 - 2p_{60} \times 2p_{62} = 1 - p_{60} \times (1 - 0.2q_{62})\)

\[ = 1 - \frac{l_{62}}{l_{60}} \left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000} \left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008. \]

Alternatively, we can calculate the answer by using a linear interpolation between \(l_{62}\) and \(l_{63}\) as follows:

\[ l_{62.2} = (1 - 0.2)l_{62} + 0.2l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200. \]

It follows that \(2.2q_{60} = 1 - l_{62.2} / l_{60} = 1 - 99200 / 100000 = 0.008\).

(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute \(0.3p_{62.8}\):

\[
0.8p_{62} \times 0.3p_{62.8} = 1.1p_{62}.
\]

Then, we have

\[
0.3p_{62.8} = \frac{1.1p_{62}}{0.8p_{62}} = \frac{p_{62}e_{0.1}p_{63}}{0.8p_{62}} = \frac{p_{62}(1-0.1q_{63})}{1-0.8q_{62}} = \frac{98800}{99300} \left(1 - 0.1 \times \frac{600}{98800}\right) = 0.998382.
\]

Hence, \(0.3q_{62.8} = 1 - 0.998382 = 0.001618\).
Alternatively, we can calculate the answer by using a linear interpolation between \( l_{62} \) and \( l_{63} \) and another interpolation between \( l_{63} \) and \( l_{64} \):

First,
\[
l_{62.8} = (1 - 0.8)l_{62} + 0.8l_{63} = 0.2 \times 99300 + 0.8 \times 98800 = 98900.
\]

Second,
\[
l_{63.1} = (1 - 0.1)l_{63} + 0.1l_{64} = 0.9 \times 98800 + 0.1 \times 98200 = 98740.
\]

Finally,
\[
0.3q_{62.8} = 1 - 0.3p_{62.8} = 1 - \frac{l_{63.1}}{l_{62.8}} = 1 - 98740 / 98900 = 0.001618.
\]

Sometimes, you may be asked to calculate the density function of \( T_x \) and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

\[
f_x(r) = q_x, \quad 0 \leq r < 1.
\]

**Proof:**
\[
f_x(r) = \frac{d}{dr} F_x(r) = \frac{d}{dr} \Pr(T_x < r) = \frac{d}{dr} q_{r}, q_{r} = \frac{d}{dr} (r q_x) = q_x.
\]

Under UDD, we have the following equation for calculating the force of mortality:

\[
\mu_{x+r} = \frac{q_x}{1 - rq_x}, \quad 0 \leq r < 1.
\]

**Proof:** In general, \( f_x(r) = r p_x \mu_{x+r} \). Under UDD, we have \( f_x(r) = q_x \) and \( p_x = 1 - rq_x \). The result follows.

Let us take a look at the following example.
Chapter 2: Life Tables

Example 2.5 [Course 3 Spring 2000 #12]

For a certain mortality table, you are given:

(i) \( \mu_{80.5} = 0.0202 \)

(ii) \( \mu_{81.5} = 0.0408 \)

(iii) \( \mu_{82.5} = 0.0619 \)

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782  (B) 0.0785  (C) 0.0790  (D) 0.0796  (E) 0.0800

Solution

The probability that a person age 80.5 will die within two years is \( 2q_{80.5} \). We have

\[
0.5p_{80} \times 2p_{80.5} = 2.5p_{80} .
\]

This gives

\[
2p_{80.5} = \frac{2 \cdot p_{80} \cdot 0.5p_{82}}{0.5 \cdot p_{80}} = \frac{p_{80}p_{81}(1-0.5q_{82})}{1-0.5q_{80}} = \frac{(1-q_{80})(1-q_{81})(1-0.5q_{82})}{1-0.5q_{80}} .
\]

We then need to find \( q_{80} \), \( q_{81} \) and \( q_{82} \) from the information given in the question. Using \( \mu_{80.5} \), we have

\[
\mu_{80.5} = \frac{q_{80}}{1-0.5q_{80}} \Rightarrow q_{80} = 0.0200 .
\]

Similarly, by using \( \mu_{81.5} \) and \( \mu_{82.5} \), we obtain \( q_{81} = 0.0400 \) and \( q_{82} = 0.0600 \).

Substituting \( q_{80} \), \( q_{81} \) and \( q_{82} \), we obtain \( 2p_{80.5} = 0.921794 \), and hence \( 2q_{80.5} = 1 - 2p_{80.5} = 0.0782 \).

Hence, the answer is (A).
Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age \( x \), we approximate \( \mu_{x+r} \) for \( 0 \leq r < 1 \) by a constant, which we denote by \( \mu_{x} \). This means

\[
\int_0^1 \mu_{x+u} \, du = \int_0^1 \mu_{x} \, du = \mu_{x},
\]

which implies \( p_x = e^{-\mu_x} \) and \( \mu_x = -\ln(p_x) \).

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued \( x \), we have

\[
p_x = (p_x)^r, \quad 0 \leq r < 1
\]

**Proof:**

\[
p_x = e^{-\int_0^1 \mu_{x+u} \, du} = e^{\int_0^1 \mu_{x} \, du} = e^{-\mu_x} = (e^{-\mu_x})^r = (p_x)^r.
\]

For example, \( 0.3p_{50} = (p_{50})^{0.3} \), and \( 0.4q_{62} = 1 - 0.4p_{62} = 1 - (p_{62})^{0.4} \). We can generalize the equation above to obtain the following key formula.

**Key Equation for the Constant Force of Mortality Assumption**

\[
p_x = (p_x)^r, \quad \text{for } 0 \leq r < 1 \text{ and } r + u \leq 1 \quad (2.6)
\]

**Proof:**

\[
p_x = e^{-\int_0^1 \mu_{x+u} \, du} = e^{\int_0^r \mu_{x} \, du} = e^{-\mu_x} = (p_x)^r. \quad [\text{The second step follows from the fact that given } 0 \leq r < 1, \, u + t \text{ is always less than or equal to } 1 \text{ when } 0 \leq t \leq r.]
\]

Notice that the key equation for the constant force of mortality assumption is based on \( p \), while that for the UDD assumption is based on \( q \).

This key equation means that, for example, \( 0.2p_{30.3} = (p_{30})^{0.2} \). Note that the subscript \( u \) on the right-hand-side does not appear in the result, provided that the condition \( r + u \leq 1 \) is satisfied.
But what if \( r + u > 1? \) The answer is very simple: Split the probability! To illustrate, let us consider \( 0.8p_{30.3}. \) (Here, \( r + u = 0.8 + 0.3 = 1.1 > 1. \)) By using equation (1.6) from Chapter 1, we can split \( 0.8p_{30.3} \) into two parts as follows:

\[
0.8p_{30.3} = 0.7p_{30.3} \times 0.1p_{31}.
\]

We intentionally consider a duration of 0.7 years for the first part, because \( 0.3 + 0.7 = 1, \) which means we can apply the key equation \( p_{x+u} = (p_x)^r \) to it. As a result, we have

\[
0.8p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.
\]

The values of \( p_{30} \) and \( p_{31} \) can be obtained from the life table straightforwardly.

To further illustrate, let us consider \( 5.6p_{40.8}. \) We can split it as follows:

\[
5.6p_{40.8} = 0.2p_{40.8} \times 5.4p_{41} = 0.2p_{40.8} \times sp_{41} \times 0.4p_{46} = (p_{40})^{0.2} \times sp_{41} \times (p_{46})^{0.4}.
\]

The values of \( p_{40}, sp_{41} \) and \( p_{46} \) can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for \( 0 \leq r < 1, \) the value of \( \ln(l_{x+r}) \) can be obtained by a linear interpolation between the values of \( \ln(l_x) \) and \( \ln(l_{x+1}). \)

**Proof:** Setting \( u = 0 \) in equation (2.6), we have

\[
\frac{l_{x+r}}{l_x} = \left(\frac{l_{x+1}}{l_x}\right)^r
\]

\[
\ln(l_{x+r}) - \ln(l_x) = r \ln(l_{x+1}) - r \ln(l_x)
\]

\[
\ln(l_{x+r}) = (1 - r) \ln(l_x) + r \ln(l_{x+1})
\]

You will find this equation – the interpolation between of \( \ln(l_x) \) and \( \ln(l_{x+1}) \) – useful when you are given a table of \( l_x \).
Chapter 2: Life Tables

Application of the Constant Force of Mortality Assumption to $l_x$

$$\ln(l_{x+r}) = (1 - r)\ln(l_x) + r\ln(l_{x+1}), \quad \text{for } 0 \leq r < 1 \quad (2.7)$$

Example 2.6

You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$d_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>100000</td>
<td>300</td>
</tr>
<tr>
<td>61</td>
<td>99700</td>
<td>400</td>
</tr>
<tr>
<td>62</td>
<td>99300</td>
<td>500</td>
</tr>
<tr>
<td>63</td>
<td>98800</td>
<td>600</td>
</tr>
<tr>
<td>64</td>
<td>98200</td>
<td>700</td>
</tr>
<tr>
<td>65</td>
<td>97500</td>
<td>800</td>
</tr>
</tbody>
</table>

Assuming constant force of mortality between integral ages, calculate the following:

(a) $0.26p_{61}$

(b) $2.2q_{60}$

(c) $0.3q_{62.8}$

Solution

(a) $0.26p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955$.

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1 - 0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, $0.26p_{61} = l_{61.26} / l_{61} = 99595.84526 / 99700 = 0.998955$.

(b) $2.2q_{60} = 1 - 2p_{60} = 1 - 2p_{60} \times 0.2p_{62} = 1 - 2p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}} \left( \frac{l_{63}}{l_{62}} \right)^{0.2} = 1 - \frac{99300}{100000} \left( \frac{98800}{99300} \right)^{0.2} = 0.008002.$$
Alternatively, we can calculate the answer by interpolating between \( \ln(l_{62}) \) and \( \ln(l_{63}) \) as follows: \( \ln(l_{62.2}) = (1 - 0.2) \ln(l_{62}) + 0.2 \ln(l_{63}) \), which gives \( l_{62.2} = 99199.79798 \). Hence, \( 2.2q_{60} = 1 - \frac{l_{62.2}}{l_{60}} = 0.008002 \).

(c) First, we consider \( 0.3p_{62.8} \):

\[
0.3p_{62.8} = 0.2p_{62.8} \times 0.1p_{63} = (p_{62})^{0.2} (p_{63})^{0.1}.
\]

Hence,

\[
0.3q_{62.8} = 1 - (p_{62})^{0.2} (p_{63})^{0.1} = 1 - \left( \frac{l_{63}}{l_{62}} \right)^{0.2} \left( \frac{l_{64}}{l_{63}} \right)^{0.1} = 1 - \left( \frac{98800}{99300} \right)^{0.2} \left( \frac{98200}{98800} \right)^{0.1} = 0.001617.
\]

Alternatively, we can calculate the answer by an interpolation between \( \ln(l_{62}) \) and \( \ln(l_{63}) \) and another interpolation between \( \ln(l_{63}) \) and \( \ln(l_{64}) \).

First, \( \ln(l_{62.8}) = (1 - 0.8) \ln(l_{62}) + 0.8 \ln(l_{63}) \), which gives \( l_{62.8} = 98899.79818 \).

Second, \( \ln(l_{63.1}) = (1 - 0.1) \ln(l_{63}) + 0.1 \ln(l_{64}) \), which gives \( l_{63.1} = 98739.8354 \).

Finally, \( 0.3q_{62.8} = 1 - \frac{l_{63.1}}{l_{62.8}} = 0.001617 \).

Example 2.7 [Structural Question]

You are given the following life table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( l_x )</th>
<th>( d_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>1000</td>
<td>50</td>
</tr>
<tr>
<td>91</td>
<td>950</td>
<td>50</td>
</tr>
<tr>
<td>92</td>
<td>900</td>
<td>60</td>
</tr>
<tr>
<td>93</td>
<td>840</td>
<td>( c_1 )</td>
</tr>
<tr>
<td>94</td>
<td>( c_2 )</td>
<td>70</td>
</tr>
<tr>
<td>95</td>
<td>700</td>
<td>80</td>
</tr>
</tbody>
</table>

(a) Find the values of \( c_1 \) and \( c_2 \)

(b) Calculate \( 1.4p_{90} \), assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.
Solution

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

\[
1.4 p_{90} = p_{90} \times 0.4 p_{91} \\
= p_{90} (1 - 0.4q_{91}) \\
= \frac{l_{91}}{l_{90}} \left(1 - 0.4 \frac{d_{91}}{l_{91}}\right) \\
= \frac{950}{1000} \left(1 - 0.4 \times \frac{50}{950}\right) \\
= 0.93
\]

Alternatively, you can compute the answer by interpolating between $l_{92}$ and $l_{91}$:

\[
1.4 p_{90} = p_{90} \times 0.4 p_{91} \\
= \frac{l_{91}}{l_{90}} \left(0.4l_{92} + 0.6l_{91}\right) \\
= \frac{0.4 \times 900 + 0.6 \times 950}{1000} \\
= 0.93
\]

(c) Assuming constant force of mortality between integer ages, we have

\[
1.4 p_{90} = p_{90} \times 0.4 p_{91} \\
= p_{90} \times (p_{91})^{0.4} \\
= \frac{950}{1000} \left(\frac{900}{950}\right)^{0.4} \\
= 0.92968
\]

[END]
Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

<table>
<thead>
<tr>
<th></th>
<th>UDD</th>
<th>Constant force</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r p_x )</td>
<td>( 1 - r q_x )</td>
<td>( (p_x)^r )</td>
</tr>
<tr>
<td>( r q_x )</td>
<td>( r q_x )</td>
<td>( 1 - (p_x)^r )</td>
</tr>
<tr>
<td>( \mu_{x+r} )</td>
<td>( \frac{q_x}{1 - r q_x} )</td>
<td>( -\ln(p_x) )</td>
</tr>
</tbody>
</table>

In the table, \( x \) is an integer and \( 0 \leq r < 1 \). The shaded formulas are the key formulas that you must remember for the examination.

### 2.3 Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called “guaranteed issue” which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.
- \([x]\) indicates the age at selection (i.e., the age at which the underwriting was done).
- \([x] + t\) indicates a person currently age \( x + t \) and was selected at age \( x \) (i.e., the underwriting was done at age \( x \)). This implies that the insurance contract has elapsed for \( t \) years.
For example, we have the following select probabilities:
- \( q_x \) is the probability that a life age \( x \) now dies before age \( x + 1 \), given that the life is selected at age \( x \).
- \( q_{x+t} \) is the probability that a life age \( x + t \) now dies before age \( x + t + 1 \), given that the life was selected at age \( x \).

Due to the effect of underwriting, a select death probability \( q_{x+t} \) must be no greater than the corresponding ordinary death probability \( q_{x+t} \). However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is \( n \) years, we have

\[
q_{x+t} < q_{x+t}, \quad \text{for} \ t < n,
\]
\[
q_{x+t} = q_{x+t}, \quad \text{for} \ t \geq n.
\]

The ordinary death probability \( q_{x+t} \) is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( q_x )</th>
<th>( q_{x+1} )</th>
<th>( q_{x+2} )</th>
<th>( x + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.04</td>
<td>0.06</td>
<td>0.08</td>
<td>42</td>
</tr>
<tr>
<td>41</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
<td>43</td>
</tr>
<tr>
<td>42</td>
<td>0.06</td>
<td>0.08</td>
<td>0.10</td>
<td>44</td>
</tr>
<tr>
<td>43</td>
<td>0.07</td>
<td>0.09</td>
<td>0.11</td>
<td>45</td>
</tr>
</tbody>
</table>

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: \( q_{41} = 0.05 \)
Age 42: \( q_{41} + 1 = 0.07 \)
Age 43: \( q_{41} + 2 = q_{43} = 0.09 \)
Age 44: \( q_{41} + 3 = q_{44} = 0.10 \)
Age 45: \( q_{41} + 4 = q_{45} = 0.11 \)
As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{40+1} = 0.06$
Age 42: $q_{40+2} = q_{42} = 0.08$
Age 43: $q_{40+3} = q_{43} = 0.09$
Age 44: $q_{40+4} = q_{44} = 0.10$
Age 45: $q_{40+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x, k) = 1 - \frac{q_{x+k}}{q_{x+k}}.$$  

For example, on the basis of the preceding table, $I(41,1) = 1 - q_{41+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{x+k}$ would be small compared to $q_{x+k}$, and therefore $I(x, k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{x+k}$ would be close to $q_{x+k}$, and therefore $I(x, k)$ would be close to zero.
Let us first go through the following example, which involves a table of $q_{x}$.

**Example 2.8**  
**[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

<table>
<thead>
<tr>
<th></th>
<th>$q_{x}$</th>
<th>$q_{x+1}$</th>
<th>$q_{x+2}$</th>
<th>$q_{x+3}$</th>
<th>$x + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.09</td>
<td>0.11</td>
<td>0.13</td>
<td>0.15</td>
<td>63</td>
</tr>
<tr>
<td>61</td>
<td>0.10</td>
<td>0.12</td>
<td>0.14</td>
<td>0.16</td>
<td>64</td>
</tr>
<tr>
<td>62</td>
<td>0.11</td>
<td>0.13</td>
<td>0.15</td>
<td>0.17</td>
<td>65</td>
</tr>
<tr>
<td>63</td>
<td>0.12</td>
<td>0.14</td>
<td>0.16</td>
<td>0.18</td>
<td>66</td>
</tr>
<tr>
<td>64</td>
<td>0.13</td>
<td>0.15</td>
<td>0.17</td>
<td>0.19</td>
<td>67</td>
</tr>
</tbody>
</table>

(ii) White was a newly selected life on 01/01/2000.

(iii) White’s age on 01/01/2001 is 61.

(iv) $P$ is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate $P$.

(A) $0 \leq P < 0.43$

(B) $0.43 \leq P < 0.45$

(C) $0.45 \leq P < 0.47$

(D) $0.47 \leq P < 0.49$

(E) $0.49 \leq P < 1.00$

**Solution**

White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = s p_{[60]+1}$. We have

$$P = s p_{[60]+1}$$

$$= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5}$$

$$= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65}$$

$$= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65})$$

$$= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17)$$

$$= 0.4589.$$ 

Hence, the answer is (C).

[ END ]
In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

**Example 2.9**  
[MLC Spring 2012 #13]

Lorie’s Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

<table>
<thead>
<tr>
<th>Limousine age (x)</th>
<th>$q_x$</th>
<th>$q_{x+1}$</th>
<th>$q_{x+2}$</th>
<th>$x+2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.100</td>
<td>0.167</td>
<td>0.333</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0.100</td>
<td>0.333</td>
<td>0.500</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0.150</td>
<td>0.400</td>
<td>1.000</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.250</td>
<td>0.750</td>
<td>1.000</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0.500</td>
<td>1.000</td>
<td>1.000</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>7</td>
</tr>
</tbody>
</table>

Lorie’s Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie’s Lorries fleet immediately after the purchase of this year’s limousines.

(A) 93  (B) 94  (C) 95  (D) 96  (E) 97

**Solution**

Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

For the 20 brand new limousines, their “age at selection” is 0. As such, the sequence of “death” probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, $q_2$, $q_3$, $q_4$, $q_5$, ... Note that $q_4 = q_5 = ... = 1$, which implies that these limousines can last for at most four years since the time
of purchase. For these 20 brand new limousines, the expected number of “survivors” limousines in each future year can be calculated as follows:

\[
\begin{array}{cccc}
1 - q_0 & 1 - q_1 & 1 - q_2 & 1 - q_3 \\
20 & 18 & 15 & 10 \\
0 & 1 & 2 & 3 & 4
\end{array}
\]

Expected number of surviving limousines
Time since purchase

For the 10 1-year-old limousines, their “age at selection” is 1. As such, the sequence of “death” probabilities applicable to these 10 1-year-old limousines are \(q_1, q_{1+1}, q_3, q_4, \ldots\). Note that \(q_4 = q_5 = \ldots = 1\), which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of “surviving” limousines in each future year can be calculated as follows:

\[
\begin{array}{cccc}
1 - q_1 & 1 - q_2 & 1 - q_3 & 1 - q_4 \\
10 & 9 & 6 & 3 & 0 \\
0 & 1 & 2 & 3 & 4
\end{array}
\]

Expected number of surviving limousines
Time since purchase

Considering the entire purchase of 30 limousines, we have the following:

\[
\begin{array}{cccc}
30 & 27 & 21 & 13 & 5 \\
0 & 1 & 2 & 3 & 4
\end{array}
\]

Expected number of surviving limousines
Time since purchase

Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie’s fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:
Sometimes, you may be given a select-and-ultimate table that contains the life table function $l_x$. In this case, you can calculate survival and death probabilities by using the following equations:

$$s_P_{x+r} = \frac{l_{x+r}}{l_x}$$

$$s_q_{x+r} = 1 - \frac{l_{x+r}}{l_x}$$

Let us study the following two examples.

**Example 2.10**

You are given the following select-and-ultimate table with a 2-year select period:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$l_{x+1}$</th>
<th>$l_{x+2}$</th>
<th>$x + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>9907</td>
<td>9905</td>
<td>9901</td>
<td>32</td>
</tr>
<tr>
<td>31</td>
<td>9903</td>
<td>9900</td>
<td>9897</td>
<td>33</td>
</tr>
<tr>
<td>32</td>
<td>9899</td>
<td>9896</td>
<td>9892</td>
<td>34</td>
</tr>
<tr>
<td>33</td>
<td>9894</td>
<td>9891</td>
<td>9887</td>
<td>35</td>
</tr>
</tbody>
</table>

Calculate the following:

(a) $2q_{[31]}$
(b) $2p_{[30]+1}$
(c) $1/2q_{[31]+1}$
Solution

(a) \( q_{[31]} = 1 - \frac{l_{[31]+2}}{l_{[31]+1}} = 1 - \frac{9987}{9903} = 0.000606 \).

(b) \( q_{[30]+1} = l_{[30]+2} - l_{[30]+1} = \frac{9897}{9905} = 0.999192 \).

(c) \( q_{[31]+1} = l_{[31]+1} - l_{[31]+1+2} = \frac{9897}{9900} - 9887 = 0.001010 \).

Exam questions such as the following may involve both \( q_x \) and \( l_x \).

Example 2.11 [MLC Spring 2012 #1]

For a 2-year select and ultimate mortality model, you are given:

(i) \( q_{[x]+1} = 0.95q_{x+1} \)

(ii) \( l_{76} = 98,153 \)

(iii) \( l_{77} = 96,124 \)

Calculate \( l_{[75]+1} \).

(A) 96,150  (B) 96,780  (C) 97,420  (D) 98,050  (E) 98,690

Solution

From (ii) and (iii), we know that \( q_{76} = 1 - \frac{96124}{98153} = 0.020672 \).

From (i), we know that \( q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638 \).

Since

\[ l_{[75]+2} = l_{[75]+1} (1 - q_{[75]+1}), \]

and \( l_{[75]+2} = l_{77} \), we have \( l_{[75]+1} = \frac{96124}{(1 - 0.019638)} = 98049.5 \). The answer is (D).

[ END ]
It is also possible to set questions to examine your knowledge on select-and-ultimate tables and fractional age assumptions at the same time. The next example involves a select-and-ultimate table and the UDD assumption.

**Example 2.12  [Course 3 Fall 2000 #10]**

You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$l_{x+1}$</th>
<th>$l_{x+2}$</th>
<th>$x+2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>80625</td>
<td>79954</td>
<td>78839</td>
<td>62</td>
</tr>
<tr>
<td>61</td>
<td>79137</td>
<td>78402</td>
<td>77252</td>
<td>63</td>
</tr>
<tr>
<td>62</td>
<td>77575</td>
<td>76770</td>
<td>75578</td>
<td>64</td>
</tr>
</tbody>
</table>

Assume that deaths are uniformly distributed between integral ages.

Calculate $0.9q_{60.6}$.

(A) 0.0102 (B) 0.0103 (C) 0.0104 (D) 0.0105 (E) 0.0106

--- **Solution**

We illustrate two methods:

1. **Interpolation**
   
   The live age $q_{60.6}$ is originally selected at age [60]. So, we can use $l_{[60]} = 80625$, $l_{[60]+1} = 79954$, $l_{[60]+2} = l_{62} = 78839$ and so on to calculate mortality rate.

   $0.9q_{[60]+0.6} = 1 - \frac{l_{[60]+1.5}}{l_{[60]+0.6}}$

   $l_{[60]+0.6} = 0.4l_{[60]} + 0.6l_{[60]+1} = 0.4 \times 80625 + 0.6 \times 79954 = 80222.4$

   $l_{[60]+1.5} = 0.5l_{[60]+1} + 0.5l_{[60]+2} = 0.5 \times 79954 + 0.6 \times 78839 = 79396.5$

   The death probability is $1 - \frac{79396.5}{80222.4} = 0.010295$.

2. **The trick we have introduced to shift the fractional age to integral age**

   Recall that when UDD is assumed and the subscript on the right-hand-side is not an integer, we will need to use the trick. We first calculate $0.9p_{[60]+0.6}$. Using the trick, we have
Then, we have

\[
0.6 P_{[60]} \times 0.9 P_{[60]+0.6} = 1.5 P_{[60]}.
\]

As a result, \(0.9 q_{[60]+0.6} = 1 - 0.989705 = 0.0103\). Hence, the answer is (B).

Alternatively, you can make use of the fact that the UDD assumption is equivalent to a linear interpolation between \(l_x\) and \(l_{x+1}\). This means that

\[
l_{[60]+0.6} = 0.6 \times 79954 + 0.4 \times 80625 = 80222.4,
\]

and that

\[
l_{[60]+1.5} = 0.5 \times 79954 + 0.5 \times 78839 = 79396.5.
\]

As a result,

\[
0.9 q_{[60]+0.6} = \frac{80222.4 - 79396.5}{80222.4} = 0.0103.
\]

The following example involves a select-and-ultimate table and the constant force of mortality assumption.
Example 2.13  [MLC Fall 2012 #2]

You are given:

(i) An excerpt from a select and ultimate life table with a select period of 3 years.

<table>
<thead>
<tr>
<th>x</th>
<th>l_x</th>
<th>l_{x+1}</th>
<th>l_{x+2}</th>
<th>l_{x+3}</th>
<th>x + 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>80,000</td>
<td>79,000</td>
<td>77,000</td>
<td>74,000</td>
<td>63</td>
</tr>
<tr>
<td>61</td>
<td>78,000</td>
<td>76,000</td>
<td>73,000</td>
<td>70,000</td>
<td>64</td>
</tr>
<tr>
<td>62</td>
<td>75,000</td>
<td>72,000</td>
<td>69,000</td>
<td>67,000</td>
<td>65</td>
</tr>
<tr>
<td>63</td>
<td>71,000</td>
<td>68,000</td>
<td>66,000</td>
<td>65,000</td>
<td>66</td>
</tr>
</tbody>
</table>

(ii) Deaths follow a constant force of mortality over each year of age.

Calculate $1000 \cdot 2^3 q_{60+0.75}$.

(A) 104  (B) 117  (C) 122  (D) 135  (E) 142

--- Solution ---

As discussed in Section 2.2, there are two methods for solving such a problem.

Method 1: Interpolation

The probability required is

$$2^3 q_{60+0.75} = \frac{l_{60+2.75} - l_{60+5.75}}{l_{60+0.75}} = \frac{l_{60+2.75} - l_{65.75}}{l_{60+0.75}}.$$

Under the constant force of mortality assumption, we have

$$\ln(l_{60+0.75}) = 0.25\ln(l_{60}) + 0.75\ln(l_{60+1}) = 0.25\ln(80000) + 0.75\ln(79000)$$

$\Rightarrow l_{60+0.75} = \exp(11.28035) = 79248.82$

$$\ln(l_{60+2.75}) = 0.25\ln(l_{60+2}) + 0.75\ln(l_{63}) = 0.25\ln(77000) + 0.75\ln(74000)$$

$\Rightarrow l_{60+2.75} = \exp(11.22176) = 74738.86$

$$\ln(l_{65.75}) = 0.25\ln(l_{65}) + 0.75\ln(l_{66}) = 0.25\ln(67000) + 0.75\ln(65000)$$

$\Rightarrow l_{63.75} = \exp(11.08972) = 65494.33$

As a result, $2^3 q_{60+0.75} = (74738.86 - 65494.33) / 79248.82 = 0.11665$.

Method 2: Working on the survival probabilities

The probability required is
Chapter 2: Life Tables

Both methods imply 1000 $2q_{60|0.75} = 116.65$, which corresponds to option (B).

2.4 Moments of Future Lifetime Random Variables

In Exam P, you learnt how to calculate the moments of a random variable.

- If $W$ is a discrete random variable, then $E(W) = \sum_w w \Pr(W = w)$.

- If $W$ is a continuous random variable, then $E(W) = \int_{-\infty}^{\infty} w f_W(w) dw$, where $f_W(w)$ is the density function for $W$.

- To calculate variance, we can always use the identity $\text{Var}(W) = E(W^2) - [E(W)]^2$.

First, let us focus on the moments of the future lifetime random variable $T_x$. We call $E(T_x)$ the complete expectation of life at age $x$, and denote it by $\overset{\circ}{e}_x$. We have

$$\overset{\circ}{e}_x = \int_0^\infty t f_x(t) dt = \int_0^\infty t p_x \mu_x + dt .$$

By rewriting the integral as $-\int_0^\infty t dS_x(t)$ and using integration by parts, we can show that the above formula can be simplified to

$$\overset{\circ}{e}_x = \int_0^\infty p_x dt .$$

Note that if there is a limiting age, we replace $\infty$ with $\omega - x$. 

\[ \text{END} \]
The second moment of $T_x$ can be expressed as

$$E(T_x^2) = \int_0^\infty t^2 f_x(t)dt.$$  

Using integration by parts, we can show that the above formula can be rewritten as

$$E(T_x^2) = 2 \int_0^\infty t p_x dt,$$

which is generally easier to apply. Again, if there is a limiting age, we replace $\infty$ with $\omega - x$.

In the exam, you may also be asked to calculate $E(T_x \wedge n) = E[\min(T_x, n)]$. This expectation is known as the $n$-year temporary complete expectation of life at age $x$, and is denoted by $\overset{\circ}{e}_{x\mid n}$. It can be shown that

$$\overset{\circ}{e}_{x\mid n} = \int_0^n t p_x dt.$$

The following is a summary of the formulas for the moments of $T_x$.

---

**FORMULA**

**Moments of $T_x$**

$$\overset{\circ}{e}_x = \int_0^\infty t p_x dt \quad \text{(2.8)}$$

$$E(T_x^2) = 2 \int_0^\infty t p_x dt \quad \text{(2.9)}$$

$$\overset{\circ}{e}_{x\mid n} = \int_0^n t p_x dt \quad \text{(2.10)}$$

---

**Example 2.14 [Structural Question]**

You are given $\mu_x = 0.01$ for all $x \geq 0$. Calculate the following:

(a) $\overset{\circ}{e}_x$

(b) $\text{Var}(T_x)$
Chapter 2: Life Tables

Solution

(a) First of all, we have \( t \, p_x = e^{-0.01t} \). Then,

\[
\hat{e}_x = \int_0^\infty t \, p_x \, dt = \int_0^\infty e^{-0.01t} \, dt = \frac{1}{-0.01} \left[ e^{-0.01t} \right]_0^\infty = \frac{1}{0.01} = 100.
\]

(b) We first calculate the second moment of \( T_x \) as follows:

\[
E(T_x^2) = 2 \int_0^\infty t^2 e^{-0.01t} \, dt
= \frac{-2}{0.01} \left( \left. te^{-0.01t} \right|_0^\infty - \int_0^\infty e^{-0.01t} \, dt \right)
= \frac{2}{0.01} \int_0^\infty e^{-0.01t} \, dt
= \frac{2}{0.01^2} = 20000.
\]

Then, the variance of \( T_x \) can be calculated as:

\[
\text{Var}(T_x) = E(T_x^2) - [E(T_x)]^2
= 20000 - 100^2
= 10000.
\]

Example 2.15 [Course 3 Fall 2001 #1]

You are given:

\[
\mu_x = \begin{cases} 
0.04, & 0 < x < 40 \\
0.05, & x > 40 
\end{cases}
\]

Calculate \( \hat{e}_{25|x} \).

(A) 14.0  (B) 14.4  (C) 14.8  (D) 15.2  (E) 15.6

Solution

First, we need to find \( t \, p_x \). Because the value of \( \mu_x \) changes when \( x \) reaches 40, the derivation of \( t \, p_x \) is not as straightforward as that in the previous example.

For \( 0 < t < 15 \), \( \mu_{25+t} \) is always 0.04, and therefore
For \( t > 15 \), \( \mu_{25+t} \) becomes 0.05, and therefore
\[
_iP_{25} = e^{-\int_{0}^{t} 0.04 \, dt} = e^{-0.04t}.
\]

Given the expressions for \( _iP_{25} \), we can calculate \( \hat{e}_{25:25} \) as follows:
\[
\hat{e}_{25:25} = \int_{0}^{15} _iP_{25} \, dt + \int_{15}^{25} _iP_{25} \, dt
= \int_{0}^{15} e^{-0.04t} \, dt + \int_{15}^{25} e^{-0.05t+0.15} \, dt
= \frac{e^{-0.04t}}{-0.04} \bigg|_{0}^{15} + e^{0.15} \left[ \frac{e^{-0.05t}}{-0.05} \right]_{15}^{25}
= 15.60.
\]

Hence, the answer is (E).

Example 2.16  [Structural Question]  SoA Sample #1

You are given the following survival function for a newborn:
\[
S_0(t) = \frac{(121-t)^{1/2}}{k}, \quad 0 \leq t \leq \omega.
\]

(a) Show that \( k \) must be 11 for \( S_0(t) \) to be a valid survival function.

(b) Show that the limiting age, \( \omega \), for this survival model is 121.

(c) Calculate \( \hat{e}_q \) for this survival model.

(d) Derive an expression for \( \mu_x \) for this survival model, simplifying the expression as much as possible.

(e) Calculate the probability, using the above survival model, that (57) dies between the ages of 84 and 100.
Solution

(a) Recall that the first criterion for a valid survival function is that \( S_0(0) = 1 \). This implies that
\[
\frac{(121 - 0)^{1/2}}{k} = 1
\]
\[
(121)^{1/2} = k
\]
\[
k = 11
\]

(b) At the limiting age, the value of the survival function must be zero. Therefore,
\[
S_0(\omega) = 0
\]
\[
\frac{(121 - \omega)^{1/2}}{k} = 0
\]
\[
\omega = 121
\]

(c) Using formula (2.8) with \( x = 0 \), we have
\[
e^x = \int_0^{\omega-0} p_0 dt
\]
\[
= \int_0^{121} \frac{(121 - t)^{1/2}}{11} dt
\]
\[
= \frac{1}{11} \left[ -\frac{2}{3} (121 - t)^{3/2} \right]_0^{121}
\]
\[
= 80.6667
\]

(d) This part involves the relationship between the \( \mu_x \) and \( S_0(x) \), which was taught in Chapter 1:
\[
\mu_x = -\frac{d}{dx} \left( \frac{S'_0(x)}{S_0(x)} \right) = -\frac{d}{dx} \left( \frac{(121 - x)^{1/2}}{k} \right)
\]
\[
= -\frac{d}{dx} \left( \frac{1}{2}(121 - x)^{-1/2} \right) = \frac{1}{2(121 - x)}
\]

(e) First, we derive an expression for \( S_{57}(t) \) as follows:
\[
S_{57}(t) = \frac{S_0(57 + t)}{S_0(57)} = \frac{(121 - (57 + t))^{1/2}}{(121 - 57)^{1/2}} = \frac{64 - t}{11}
\]
The required probability is $27|16q_{57}$, which can be calculated as follows:

$$27|16q_{57} = S_{57}(27) - S_{57}(43) = \sqrt{64 - 27 \over 64} - \sqrt{64 - 43 \over 64} = 0.1875$$

[ END ]

Now, we focus on the moments of the curtate future lifetime random variable $K_x$. The first moment of $K_x$ is called the curtate expectation of life at age $x$, and is denoted by $e_x$. The formula for calculating $e_x$ is derived as follows:

$$e_x = E(K_x) = \sum_{k=0}^{\infty} k \Pr(K_x = k) = \sum_{k=0}^{\infty} k k! q_x = 0 \times q_x + 1 \times 1! q_x + 2 \times 2! q_x + 3 \times 3! q_x + \ldots = (p_x - 2 p_x) + 2(2 p_x - 3 p_x) + 3(3 p_x - 4 p_x) + \ldots = p_x + 2 p_x + 3 p_x + \ldots = \sum_{k=1}^{\infty} k P(x)$$

If there is a limiting age, we replace $\infty$ with $\omega - x$.

The formula for calculating the second moment of $K_x$ can be derived as follows:

$$E(K_x^2) = \sum_{k=0}^{\infty} k^2 \Pr(K_x = k) = \sum_{k=0}^{\infty} k^2 k! q_x = 0^2 \times q_x + 1^2 \times 1! q_x + 2^2 \times 2! q_x + 3^2 \times 3! q_x + \ldots = (p_x - 2 p_x) + 4(2 p_x - 3 p_x) + 9(3 p_x - 4 p_x) + \ldots = p_x + 3 p_x + 5 p_x + \ldots = \sum_{k=1}^{\infty} (2k - 1) k p_x$$

Again, if there is a limiting age, we replace $\infty$ with $\omega - x$. Given the two formulas above, we can easily obtain $\text{Var}(K_x)$. 
In the exam, you may also be asked to calculate $E(K_x \land n) = E[\min(K_x, n)]$. This is called the $n$-year temporary curtate expectation of life at age $x$, and is denoted by $e_{x|n}$. It can be shown that

$$e_{x|n} = \sum_{k=1}^{n} k p_x,$$

that is, instead of summing to infinity, we just sum to $n$.

There are two other equations that you need to know. First, you need to know that $e_x$ and $e_{x+1}$ are related to each other as follows:

$$e_x = p_x(1 + e_{x+1}).$$

Formulas of this form are called recursion formulas. We will further discuss recursion formulas in Chapters 3 and 4.

Second, assuming UDD holds, we have $T_x = K_x + U$, where $U$ follows a uniform distribution over the interval $[0, 1]$. Taking expectation on both sides, we have the following relation:

$$e_x = e_x + \frac{1}{2}. $$

The following is a summary of the key equations for the moments of $K_x$.

**FORMULA**

**Moments of $K_x$**

\[
\begin{align*}
\text{Moments of } K_x & \\

\hat{e}_x &= \sum_{k=1}^{\infty} k p_x, \\
E(K_x^2) &= \sum_{k=1}^{\infty} (2k - 1) k p_x, \\
e_{x|n} &= \sum_{k=1}^{n} k p_x, \\
\hat{e}_x &= p_x(1 + e_{x+1}).
\end{align*}
\]

Under UDD, $\hat{e}_x = e_x + \frac{1}{2}$
You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>x</th>
<th>95</th>
<th>96</th>
<th>97</th>
<th>98</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_x )</td>
<td>400</td>
<td>300</td>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Calculate the following:

(a) \( e_{95} \)
(b) \( \text{Var}(K_{95}) \)
(c) \( e_{95\vert 1} \)
(d) \( \hat{e}_{95} \), assuming UDD
(e) \( e_{96} \), using the recursion formula

**Solution**

(a) \( e_{95} = \sum_{k=1}^{3} k \cdot p_{95} = \frac{l_{96}}{l_{95}} + \frac{l_{97}}{l_{95}} + \frac{l_{98}}{l_{95}} = \frac{300}{400} + \frac{100}{400} + \frac{0}{400} = 1 \)

(b) We have

\[
E(K^{2}_{95}) = \sum_{k=1}^{3} (2k - 1) \cdot k \cdot p_{95} = \frac{l_{96}}{l_{95}} + 3 \cdot \frac{l_{97}}{l_{95}} + 5 \cdot \frac{l_{98}}{l_{95}} = \frac{300}{400} + 3 \cdot \frac{100}{400} + 5 \cdot \frac{0}{400} = 1.5.
\]

Hence, \( \text{Var}(K_{95}) = 1.5 - 1^2 = 0.5 \).

(c) \( e_{95\vert 1} = \sum_{k=1}^{1} k \cdot p_{95} = \frac{l_{96}}{l_{95}} \cdot \frac{300}{400} = 0.75 \).

(d) Assuming UDD, \( \hat{e}_{95} = e_{95} + 0.5 = 1 + 0.5 = 1.5 \).

(e) Using the recursion formula, \( e_{95} = p_{95}(1 + e_{96}) = 0.75(1 + e_{96}) \). Therefore,

\[
e_{96} = 1/0.75 - 1 = 0.3333.
\]
You are given:

(i) \( S_0(t) = \left(1 - \frac{t}{\omega}\right)^{1/4} \), for \( 0 \leq t \leq \omega \)

(ii) \( \mu_{65} = 1 / 180 \)

Calculate \( \varepsilon_{106} \), the curtate expectation of life at age 106.

(A) 2.2 (B) 2.5 (C) 2.7 (D) 3.0 (E) 3.2

**Solution**

From statement (i), we know that there is a limiting age \( \omega \). Our first step is to compute the value of \( \omega \), using the information given.

Since \( \mu_x = -\frac{d}{dx} \ln S_0(x) = \frac{1}{4(\omega - x)} \), by statement (ii) we have \( \frac{1}{4(\omega - 65)} = \frac{1}{180} \), or \( \omega = 110 \).

Then, using formula (2.9), we can calculate \( \varepsilon_{106} \) as follows:

\[
\varepsilon_{106} = p_{106} + 2p_{106} + 3p_{106} + 4p_{106} + \ldots
= \frac{S_0(107) + S_0(108) + S_0(109) + S_0(110) + \ldots}{S_0(106)}
= \frac{0.02727^{1/4} + 0.01818^{1/4} + 0.00909^{1/4} + 0 + \ldots}{0.03636^{1/4}}
= 2.4786
\]

The answer is (B).
Example 2.19 [MLC Spring 2012 #2]

You are given:
(i) $p_x = 0.97$
(ii) $p_{x+1} = 0.95$
(iii) $e_{x+1.75} = 18.5$
(iv) Deaths are uniformly distributed between ages $x$ and $x + 1$.
(v) The force of mortality is constant between ages $x + 1$ and $x + 2$.

Calculate $e_{x+0.75}$.

(A) 18.6 (B) 18.8 (C) 19.0 (D) 19.2 (E) 19.4

Solution

Our goal is to calculate $e_{x+0.75}$. Since we are given the value of $e_{x+1.75}$, it is quite obvious that we should use the following recursive relation:

$$e_{x+0.75} = p_{x+0.75}(1 + e_{x+1.75}).$$

All then that remains is to calculate $p_{x+0.75}$. As shown in the following diagram, this survival probability covers part of the interval $[x, x + 1)$ and part of the interval $[x + 1, x + 2)$.

We shall apply fractional age assumptions accordingly. Decomposing $p_{x+0.75}$, we have

$$p_{x+0.75} = 0.25p_{x+0.75} \times 0.75p_{x+1}.$$ 

According to statement (iv), the value of $0.25p_{x+0.75}$ should be calculated by assuming UDD. Under this assumption, we have

$$0.25p_{x+0.75} = \frac{p_x}{0.75p_x} = \frac{0.97}{1 - 0.75(1 - 0.97)} = 0.992327366.$$
According to statement (v), the value of \(0.75p_{x+1}\) should be calculated by assuming constant force of mortality over each year of age. Under this assumption, we have
\[
0.75p_{x+1} = (p_{x+1})^{0.75} = (0.95)^{0.75} = 0.9622606.
\]
It follows that \(p_{x+0.75} = 0.992327366 \times 0.9622606 = 0.954878\).
Finally,
\[
e_{x+0.75} = 0.954878 \times (1 + 18.5) = 18.620.
\]
The answer is (A).

\[\text{END}\]

2.5 Useful Shortcuts

**Constant Force of Mortality for All Ages**

Very often, you are given that \(\mu_x = \mu\) for all \(x \geq 0\). In this case, we can easily find that
\[
dP_x = e^{-\mu t}, \quad F_x(t) = 1 - e^{-\mu t}, \quad f_x(t) = \mu e^{-\mu t}.
\]
From the density function, you can tell that in this case \(T_x\) follows an exponential distribution with parameter \(\mu\). By using the properties of an exponential distribution, we have
\[
\bar{e}_x = E(T_x) = 1/\mu, \quad \text{Var}(T_x) = 1/\mu^2 \quad \text{for all } x.
\]
These shortcuts can save you a lot of time on doing integration. For instance, had you known these shortcuts, you could complete Example 2.13 in a blink!

**De Moivre’s Law**

De Moivre’s law refers to the situation when
\[
l_x = \omega - x \text{ for } 0 \leq x < \omega,
\]
or equivalently
\[
\mu_x = \frac{1}{\omega - x}.
\]
De Moivre’s law implies that the age at death random variable \((T_0)\) is uniformly distributed over the interval \([0, \omega)\). It also implies that the future lifetime random variable \((T_x)\) is uniformly distributed over the interval \([0, \omega - x)\), that is, for \(0 \leq t < \omega - x\),

\[
\begin{align*}
_i p_x &= 1 - \frac{t}{\omega - x}, \\
F_i(t) &= \frac{t}{\omega - x}, \\
f_i(t) &= \frac{1}{\omega - x}, \\
\mu_{x+t} &= \frac{1}{\omega - x - t}.
\end{align*}
\]

By using the properties of uniform distributions, we can immediately obtain

\[
e_x^* = \frac{\omega - x}{2}, \quad \text{Var}(T_x) = \frac{(\omega - x)^2}{12}.
\]

The useful shortcuts are summarized in the following table.

<table>
<thead>
<tr>
<th>Assumption</th>
<th>(\mu_{x+t})</th>
<th>(i p_x)</th>
<th>(F_i(t))</th>
<th>(f_i(t))</th>
<th>(e_x^*)</th>
<th>(\text{Var}(T_x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant force</td>
<td>(\mu)</td>
<td>(e^{-\mu t})</td>
<td>(1 - e^{-\mu t})</td>
<td>(\mu e^{-\mu t})</td>
<td>(1/\mu)</td>
<td>(1/\mu^2)</td>
</tr>
<tr>
<td>for all ages</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>De Moivre’s law</td>
<td>(\frac{1}{\omega - x - t})</td>
<td>(1 - \frac{t}{\omega - x})</td>
<td>(\frac{t}{\omega - x})</td>
<td>(\frac{1}{\omega - x})</td>
<td>(\frac{\omega - x}{2})</td>
<td>(\frac{(\omega - x)^2}{12})</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 2.20**

You are given:

\(l_x = 100 - x, \ 0 \leq x < 100\).

Calculate the following:

(a) \(25p_{25}\)

(b) \(q_{25}\)

(c) \(\mu_{50}\)

(d) \(\overset{50}{e}\)
Solution

First of all, note that \( l_x = 100 - x \) for \( 0 \leq x < 100 \) means mortality follows De Moivre’s law with \( \omega = 100 \).

(a) \( 25 \, p_{25} = 1 - \frac{25}{100 - 25} = \frac{2}{3} \).

(b) \( q_{25} = 1 - \frac{l_{26}}{l_{25}} = 1 - \frac{74/75}{75} = 1/75 \).

Alternatively, you can obtain the answer by using the fact that \( T_{25} \) is uniformly distributed over the interval \([0, 75)\). It immediately follows that the probability that (25) dies within one year is 1/75.

(c) \( \mu_{50} = \frac{1}{100 - 50} = 0.02 \).

(d) \( \varepsilon_{50} = \frac{100 - 50}{2} = 25 \).

Example 2.21  [Structural Question]

The survival function for the age-at-death random variable is given by

\[ S_\omega(t) = 1 - \frac{t}{\omega}, \quad t < \omega. \]

(a) Find an expression for \( S_x(t) \), for \( x < \omega \) and \( t < \omega - x \).

(b) Show that \( \mu_x = \frac{1}{\omega - x} \), for \( x < \omega \).

(c) Assuming \( \hat{\varepsilon}_o = 25 \), show that \( \omega = 50 \).

Solution

(a) The survival function implies that the age-at-death random variable is uniformly distributed over \([0, \omega]\). De Moivre’s law applies here, so can immediately write down the expression for \( S_x(t) \) as follows:

\[ S_x(t) = 1 - \frac{t}{\omega - x}. \]
(b) From Section 2.5, we know the expression for $\mu_x$. However, since we are asked to prove the relation, we should show the steps involved:

$$\mu_x = -\frac{S'_0(x)}{S_0(x)} = -\frac{1}{\omega} \frac{1}{1 - \frac{x}{\omega}} = \frac{1}{\omega - x}$$

(c) Under De Moivre’s law, $\hat{\omega}_0 = \frac{\omega}{2}$. Hence, we have $\frac{\omega}{2} = 25$, which gives $\omega = 50$.  

Example 2.22  [Structural Question]

Given $\mu_k = \mu$ for all $x \geq 0$.

(a) Show that $\hat{e}_{x+1} = \frac{1-e^{-\mu}}{\mu}$.

(b) Explain verbally why $\hat{e}_{x+1}$ does not depend on $x$ when we assume $\mu_k = \mu$ for all $x \geq 0$.

(c) State the value of $\hat{e}_{x+1}$ when $\mu$ tends to zero. Explain your answer.

Solution

(a) Since the force of mortality is constant for all ages, we have $\dot{\mu}_x = e^{-\mu}$. Then,

$$\hat{e}_{x+1} = \int_0^1 p_x \, dt = \int_0^n e^{-\mu} \, dt = \frac{1-e^{-\mu n}}{\mu}.$$  

(b) The assumption “$\mu_k = \mu$ for all $x \geq 0$” means that the future lifetime random variable is exponentially distributed. By the memoryless property of an exponential distribution, the expectation should be independent of the history (i.e., how long the life has survived).

(c) When $\mu$ tends to zero, $\hat{e}_{x+1}$ tends to $n$. This is because when $\mu$ tends to zero, the underlying lives become immortal (i.e., the lives will live forever). As a result, the average number of years survived from age $x$ to age $x + n$ (i.e., from time 0 to time $n$) must be $n$.  

[ END ]
Exercise 2

1. You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100,000</td>
</tr>
<tr>
<td>51</td>
<td>99,900</td>
</tr>
<tr>
<td>52</td>
<td>99,700</td>
</tr>
<tr>
<td>53</td>
<td>99,500</td>
</tr>
<tr>
<td>54</td>
<td>99,100</td>
</tr>
<tr>
<td>55</td>
<td>98,500</td>
</tr>
</tbody>
</table>

Calculate the following:
(a) $2d_{52}$
(b) $3q_{50}$

2. You are given:

$$l_x = 10000e^{-0.05x}, \quad x \geq 0.$$ 

Find $5|15q_{10}$.

3. You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>10000</td>
</tr>
<tr>
<td>41</td>
<td>9900</td>
</tr>
<tr>
<td>42</td>
<td>9700</td>
</tr>
<tr>
<td>43</td>
<td>9400</td>
</tr>
<tr>
<td>44</td>
<td>9000</td>
</tr>
<tr>
<td>45</td>
<td>8500</td>
</tr>
</tbody>
</table>

Assuming uniform distribution of deaths between integral ages, calculate the following:
(a) $0.2p_{42}$
(b) $2.6q_{41}$
(c) $1.6q_{40.9}$

4. Repeat Question 3 by assuming constant force of mortality between integral ages.

5. You are given:

(i) $l_{40} = 9,313,166$
(ii) $l_{41} = 9,287,264$
(iii) $l_{42} = 9,259,571$

Assuming uniform distribution of deaths between integral ages, find $1.4q_{40.3}$.
6. You are given:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>60500</td>
</tr>
<tr>
<td>50</td>
<td>55800</td>
</tr>
<tr>
<td>60</td>
<td>50200</td>
</tr>
<tr>
<td>70</td>
<td>44000</td>
</tr>
<tr>
<td>80</td>
<td>36700</td>
</tr>
</tbody>
</table>

Assuming that deaths are uniformly distributed over each 10-year interval, find $15|20q_{40}$.

7. You are given the following select-and-ultimate table with a select period of 2 years:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q_x$</th>
<th>$q_{x+1}$</th>
<th>$q_{x+2}$</th>
<th>$x + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.02</td>
<td>0.04</td>
<td>0.06</td>
<td>52</td>
</tr>
<tr>
<td>51</td>
<td>0.03</td>
<td>0.05</td>
<td>0.07</td>
<td>53</td>
</tr>
<tr>
<td>52</td>
<td>0.04</td>
<td>0.06</td>
<td>0.08</td>
<td>54</td>
</tr>
</tbody>
</table>

Find $22q_{50}$.

8. [Structural Question] You are given the following select-and-ultimate table with a select period of 2 years:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$l_{x+1}$</th>
<th>$l_{x+2}$</th>
<th>$x + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>22507</td>
<td>22200</td>
<td>21722</td>
<td>72</td>
</tr>
<tr>
<td>71</td>
<td>21500</td>
<td>21188</td>
<td>20696</td>
<td>73</td>
</tr>
<tr>
<td>72</td>
<td>20443</td>
<td>20126</td>
<td>19624</td>
<td>74</td>
</tr>
<tr>
<td>73</td>
<td>19339</td>
<td>19019</td>
<td>18508</td>
<td>75</td>
</tr>
<tr>
<td>74</td>
<td>18192</td>
<td>17871</td>
<td>17355</td>
<td>76</td>
</tr>
</tbody>
</table>

(a) Compute $3p_{73}$.

(b) Compute the probability that a life age 71 dies between ages 75 and 76, given that the life was selected at age 70.

(c) Assuming uniform distribution of deaths between integral ages, calculate $0.5p_{[70],0.7}$.

(d) Assuming constant force of mortality between integral ages, calculate $0.5p_{[70],0.7}$.

9. You are given:

$$f_x(t) = \frac{20-t}{200}, \quad 0 \leq t < 20.$$ 

Find $\hat{e}_3$.
Chapter 2: Life Tables

10. For a certain individual, you are given:

\[ S_0(t) = \begin{cases} 
1 - \frac{t}{100}, & 0 \leq t < 30 \\
0.7e^{-0.02(t-30)}, & t \geq 30 
\end{cases} \]

Calculate \( E(T_0) \) for the individual.

11. You are given:

\[ \mu_x = \frac{2x}{400 - x^2}, \quad 0 \leq x < 20. \]

Find \( \text{Var}(T_0) \).

12. You are given:

(i) \( \mu_x = \frac{1}{x - \omega}, \quad 0 \leq x < \omega. \)

(ii) \( \text{Var}(T_0) = 468.75. \)

Find \( \omega. \)

13. You are given:

(i) \( \mu_x = \mu \) for all \( x \geq 0. \)

(ii) \( e_{30} = 40. \)

Find \( s_p_{20}. \)

14. You are given:

\( l_x = 10000 - x^2, \quad 0 \leq x \leq 100. \)

Find \( \text{Var}(T_0). \)

15. You are given:

\( \mu_x = 0.02, \quad x \geq 0. \)

Find \( e_{10.50}. \)

16. You are given:

(i) \( S_0(t) = 1 - \frac{t}{\omega}, \quad 0 \leq t < \omega. \)

(ii) \( e_{20:30} = 22.5. \)

Calculate \( \text{Var}(T_{30}). \)
17. You are given:

\[ l_x = 80 - x, \quad 0 \leq x \leq 80. \]

Find \( \hat{e}_{435} \).

18. (CAS, 2003 Fall #5) You are given:
   (i) Mortality follows De Moivre’s Law.
   (ii) \( \hat{e}_{20} = 30 \).
   Calculate \( q_{20} \).
   (A) \( \frac{1}{60} \)
   (B) \( \frac{1}{70} \)
   (C) \( \frac{1}{80} \)
   (D) \( \frac{1}{90} \)
   (E) \( \frac{1}{100} \)

19. (2005 Nov #32) For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:
   (i) For non-smokers, \( \mu_x^{\text{n}} = 0.08, x \geq 30 \).
   (ii) For smokers, \( \mu_x^{\text{s}} = 0.16, x \geq 30 \).
   Calculate \( q_{80} \) for a life randomly selected from those surviving to age 80.
   (A) 0.078
   (B) 0.086
   (C) 0.095
   (D) 0.104
   (E) 0.112

20. (2004 Nov #4) For a population which contains equal numbers of males and females at birth:
   (i) For males: \( \mu_x^{\text{m}} = 0.10, x \geq 0 \).
   (ii) For females: \( \mu_x^{\text{f}} = 0.08, x \geq 0 \).
   Calculate \( q_{60} \) for this population.
   (A) 0.076
   (B) 0.081
   (C) 0.086
21. (2000 May #1) You are given:
   (i) \( \hat{e}_0 = 25 \)
   (ii) \( l_x = \omega - x, \ 0 \leq x \leq \omega \).
   (iii) \( T_x \) is the future lifetime random variable.
   Calculate \( \text{Var}(T_{10}) \).
   (A) 65
   (B) 93
   (C) 133
   (D) 178
   (E) 333

22. (2005 May #21) You are given:
   (i) \( \hat{e}_{30:40} = 27.692 \)
   (ii) \( S_0(t) = 1 - t/\omega, \ 0 \leq t \leq \omega \)
   (iii) \( T_x \) is the future lifetime random variable for \( x \).
   Calculate \( \text{Var}(T_{30}) \).
   (A) 332
   (B) 352
   (C) 372
   (D) 392
   (E) 412

23. (2005 Nov #13) The actuarial department for the SharpPoint Corporation models the
    lifetime of pencil sharpeners from purchase using a generalized DeMoivre model with \( S_0(t) = (1 - t/\omega)^\alpha \), for \( \alpha > 0 \) and \( 0 \leq t \leq \omega \).

    A senior actuary examining mortality tables for pencil sharpeners has determined that the
    original value of \( \alpha \) must change. You are given:
    (i) The new complete expectation of life at purchase is half what it was previously.
    (ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of
        mortality for all durations.
    (iii) \( \omega \) remains the same.

    Calculate the original value of \( \alpha \).
24. (2000 Nov #25) You are given:
(i) Superscripts $M$ and $N$ identify two forces of mortality and the curtate expectations of life calculated from them.
(ii) \[
\mu_{25+t}^N = \begin{cases} 
\mu_{25+t}^M + 0.10(1-t), & 0 \leq t \leq 1 \\
\mu_{25+t}^M, & t > 1 
\end{cases}
\]
(iii) $e_{25}^M = 10.0$
Calculate $e_{25}^N$.
(A) 9.2
(B) 9.3
(C) 9.4
(D) 9.5
(E) 9.6

25. (2003 Nov #17) $T_0$, the future lifetime of (0), has a spliced distribution:
(i) $f^a(t)$ follows the Illustrative Life Table.
(ii) $f^b(t)$ follows De Moivre’s law with $\omega = 100$.
(iii) The density function of $T_0$ is $f_0(t) = \begin{cases} 
k^a(t), & 0 \leq t \leq 50 \\
1.2f^b(t), & t > 50 
\end{cases}$
Calculate $10p_{40}$.
(A) 0.81
(B) 0.85
(C) 0.88
(D) 0.92
(E) 0.96
Chapter 2: Life Tables

26. [Structural Question]
   (a) Show that
   \[ e_x = px(1 + e_{x+1}) \]
   (b) Show that if deaths are uniformly distributed between integer ages, then
   \[ \dot{e}_x = e_x + \frac{1}{2} \]
   (c) For a life table with a one-year select period, you are given:

   \[
   \begin{array}{|c|c|c|c|c|}
   \hline
   x & l_x & d_x & l_{x+1} & \dot{e}_x \\
   \hline
   80 & 1000 & 90 & - & 8.5 \\
   81 & 920 & 90 & - & - \\
   \hline
   \end{array}
   \]
   (i) Find \( l_{81} \) and \( l_{82} \).
   (ii) Assuming deaths are uniformly distributed over each year of age, \( \dot{e}_{[81]} \).

27. [Structural Question] For a certain group of individuals, you are given:
   \[ F_0(t) = 1 - e^{-0.02t}, \quad t \geq 0. \]
   (a) Show that \( S_x(t) = e^{-0.02t} \) for \( x, t \geq 0 \).
   (b) Show that \( \mu_x = 0.02 \) for \( x \geq 0 \).
   (c) Calculate \( \dot{e}_{[10]} \).
   (d) Calculate \( e_{10} \).

28. [Structural Question] Consider the curtate future lifetime random variable, \( K_x \).
   (a) Explain verbally why \( \Pr(K_x = k) = kq_x \) for \( k = 0, 1, \ldots \).
   (b) Show that \( e_{x|x} = \sum_{k=1}^{n} k p_x \).

29. [Structural Question] A mortality table is defined such that
   \[ p_x = \left(1 - \frac{t}{100 - x}\right)^{0.5} \]
   for \( 0 \leq x < 100 \) and \( 0 \leq t < 100 - x \); and \( p_x = 0 \) for \( t \geq 100 - x \).
   (a) State the limiting age, \( \omega \).
   (b) Calculate \( \dot{e}_{40} \)
   (c) Calculate \( \text{Var}(T_{40}) \)
30. **[Structural Question]**

(a) Define ‘selection effect’.

(b) You are given the following two quotations for a 10-year term life insurance:

<table>
<thead>
<tr>
<th>Company</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Policyholder</td>
<td>Age 28, non-smoker</td>
<td>Age 28, non-smoker</td>
</tr>
<tr>
<td>Medical exam required?</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Annual premium</td>
<td>$120.00</td>
<td>$138.00</td>
</tr>
</tbody>
</table>

(i) Explain the difference between the two premiums in laymen’s terms.

(ii) Explain the difference between the two premiums in actuarial terms.

(c) You are given the following select-and-ultimate life table:

<table>
<thead>
<tr>
<th>x</th>
<th>q_x</th>
<th>q_{x+1}</th>
<th>q_{x+2}</th>
<th>x + 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>0.01</td>
<td>0.04</td>
<td>0.07</td>
<td>67</td>
</tr>
<tr>
<td>66</td>
<td>0.03</td>
<td>0.06</td>
<td>0.09</td>
<td>68</td>
</tr>
<tr>
<td>67</td>
<td>0.05</td>
<td>0.08</td>
<td>0.12</td>
<td>69</td>
</tr>
</tbody>
</table>

(i) State the select period.

(ii) Calculate $1[q_{[65]+1}$

(iii) Calculate $0.4p_{[66]+0.3}$, assuming constant force of mortality between integer ages.

31. **[Structural Question]** You are given the following life table:

<table>
<thead>
<tr>
<th>x</th>
<th>l_x</th>
<th>x</th>
<th>l_x</th>
<th>x</th>
<th>l_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>91</td>
<td>27</td>
<td>94</td>
<td>12</td>
<td>97</td>
<td>3</td>
</tr>
<tr>
<td>92</td>
<td>21</td>
<td>95</td>
<td>8</td>
<td>98</td>
<td>1</td>
</tr>
<tr>
<td>93</td>
<td>16</td>
<td>96</td>
<td>5</td>
<td>99</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Calculate $e_{91}$.

(b) Calculate $\hat{e}_{91}$, assuming uniform distribution of deaths between integer ages.

(c) Calculate $\hat{e}_{91}$, assuming De Moivre’s law with $\omega = 99$.

32. **[Structural Question]** You are given the following 4-year select-and-ultimate life table:

<table>
<thead>
<tr>
<th>[x]</th>
<th>q_{[x]}</th>
<th>q_{[x]+1}</th>
<th>q_{[x]+2}</th>
<th>q_{[x]+3}</th>
<th>q_{x+4}</th>
<th>x + 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.00101</td>
<td>0.00175</td>
<td>0.00205</td>
<td>0.00233</td>
<td>0.00257</td>
<td>44</td>
</tr>
<tr>
<td>41</td>
<td>0.00113</td>
<td>0.00188</td>
<td>0.00220</td>
<td>0.00252</td>
<td>0.00293</td>
<td>45</td>
</tr>
<tr>
<td>42</td>
<td>0.00127</td>
<td>0.00204</td>
<td>0.00240</td>
<td>0.00280</td>
<td>0.00337</td>
<td>46</td>
</tr>
<tr>
<td>43</td>
<td>0.00142</td>
<td>0.00220</td>
<td>0.00262</td>
<td>0.00316</td>
<td>0.00384</td>
<td>47</td>
</tr>
<tr>
<td>44</td>
<td>0.00157</td>
<td>0.00240</td>
<td>0.00301</td>
<td>0.00367</td>
<td>0.00445</td>
<td>48</td>
</tr>
</tbody>
</table>
(a) Calculate the index of selection at age 44, \( l(44, k) \) for \( k = 0, 1, 2, 3 \).
(b) Construct the table of \( l_{x+t} \), for \( x = 40, 41, 42 \) and for all \( t \). Use \( l_{40} = 10,000 \).
(c) Calculate the following probabilities:
   (i) \( 2p_{[42]} \)
   (ii) \( 3q_{[41]}+1 \)
   (iii) \( 3q_{[41]} \)

33. [Structural Question] You are given the following excerpt of a life table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( l_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100,000</td>
</tr>
<tr>
<td>51</td>
<td>99,900</td>
</tr>
<tr>
<td>52</td>
<td>99,700</td>
</tr>
<tr>
<td>53</td>
<td>99,500</td>
</tr>
<tr>
<td>54</td>
<td>99,100</td>
</tr>
<tr>
<td>55</td>
<td>98,500</td>
</tr>
</tbody>
</table>

(a) Calculate \( d_{52} \)
(b) Calculate \( 2q_{50} \)
(c) Assuming uniform distribution of deaths between integer ages, calculate the value of \( 4.3p_{50.4} \)
(d) Assuming constant force of mortality between integer ages, calculate the value of \( 4.3p_{50.4} \).
Solutions to Exercise 2

1. (a) \( 2d_{52} = l_{52} - l_{54} = 99700 - 99100 = 600. \)

   (b) \( 3q_{50} = \frac{d_{53}}{l_{50}} = \frac{l_{53} - l_{54}}{l_{50}} = \frac{99500 - 99100}{100000} = 0.004. \)

2. Expressing \( 5|_{15} q_{10} \) in terms of \( l_x \), we have
   \[
   5|_{15} q_{10} = \frac{l_{15} - l_{30}}{l_{10}} = \frac{10000e^{-0.05 \times 15} - 10000e^{-0.05 \times 10}}{10000e^{-0.05 \times 10}} = 0.4109.
   \]

3. (a) \( 0.2p_{42} = 1 - 0.2q_{42} = 1 - 0.2 \times (1 - 9400/9700) = 0.993814. \)

   (b) \( 2.6q_{41} = 1 - 2.6p_{41} = 1 - 2p_{41} \times 0.6p_{43} = 1 - 2p_{41} \times (1 - 0.6q_{43}) \)
   \[
   = 1 - l_{43} \left( 1 - 0.6 \times \frac{l_{43} - l_{44}}{l_{43}} \right) = 1 - \frac{9400}{9900} \left( 1 - 0.6 \times \frac{9400 - 9000}{9400} \right) = 0.074747.
   \]

   (c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute \( 1.6p_{40.9} \):
   \[
   0.9p_{40} \times 1.6p_{40.9} = 2.5p_{40}.
   \]

   Then, we have
   \[
   1.6p_{40.9} = \frac{2.5p_{40}}{0.9p_{40}} = \frac{2p_{40} \cdot 0.5p_{42}}{1 - 0.9q_{40}} = \frac{9700 \cdot \left( 1 - 0.5 \times \frac{300}{9700} \right)}{1 - 0.9 \times \frac{100}{10000}} = 0.963673.
   \]

   Hence, \( 1.6p_{40.9} = 1 - 0.963673 = 0.036327. \)

4. (a) \( 0.2p_{42} = (p_{42})^{0.2} = (9400/9700)^{0.2} = 0.993736. \)

   (b) \( 2.6q_{41} = 1 - 2.6p_{41} = 1 - 2p_{41} \times 0.6p_{43} = 1 - 2p_{41} \times (p_{43})^{0.6} \)
   \[
   = 1 - l_{43} \left( 1 - 0.6 \times \frac{l_{43} - l_{44}}{l_{43}} \right)^{0.6} = 1 - \frac{9400}{9900} \left( \frac{9000}{9400} \right)^{0.6} = 0.074958.
   \]

   (c) First, we consider \( 1.6p_{40.9} \):
   \[
   1.6p_{40.9} = 0.1p_{40.9} \times 1.5p_{41} = 0.1p_{40.9} \times p_{41} \times 0.5p_{42} = (p_{40})^{0.1} \times p_{41} \times (p_{42})^{0.5}
   \]

   Hence,
\[ 1.6q_{40.9} = 1 - \left(p_{40}\right)^{0.1} \left(p_{41}\right) \left(p_{42}\right)^{0.5} = 1 - \left(\frac{l_{41}}{l_{40}}\right)^{0.1} \left(\frac{l_{42}}{l_{41}}\right) \left(\frac{l_{43}}{l_{42}}\right)^{0.5} \]
\[ = 1 - \left(\frac{9900}{10000}\right)^{0.1} \left(\frac{9700}{9900}\right) \left(\frac{9400}{9700}\right)^{0.5} = 0.036441 \]

5. Consider \(1.4p_{40.3}\). We have

\[ 0.3p_{40} \times 1.4p_{40.3} = 1.7p_{40} \]

This gives

\[ 1.4p_{40.3} = \frac{p_{40}}{0.3} p_{40} \left(1 - 0.7q_{41}\right) \]
\[ = \frac{p_{40}}{1 - 0.3q_{40}} \left(1 - 0.7 \left(1 - \frac{l_{42}}{l_{41}}\right)\right) \]
\[ = \frac{l_{41}}{l_{40}} \left(1 - 0.7 \left(1 - \frac{l_{42}}{l_{41}}\right)\right) \]
\[ = 1 - 0.3 \left(1 - \frac{l_{41}}{l_{40}}\right) \]
\[ = \frac{9287264}{9313166} \left(1 - 0.7 \left(1 - \frac{9259571}{9287264}\right)\right) \]
\[ = \frac{9287264}{9313166} \left(1 - 0.7 \left(1 - \frac{9287264}{9313166}\right)\right) \]
\[ = 0.995968. \]

Therefore, \(1.4q_{40.3} = 1 - 0.995968 = 0.004032\).

6. Expressing \(15|20q_{40}\) in terms of \(l_x\), we have \(15|20q_{40} = \frac{l_{55} - l_{75}}{l_{40}}\).

From the table, we have \(l_{40} = 60500\). Since deaths are uniformly distributed over each 10-year span, we have

\[ l_{55} = \frac{l_{50} + l_{60}}{2} = \frac{55800 + 50200}{2} = 53000 \]
\[ l_{75} = \frac{l_{70} + l_{80}}{2} = \frac{44000 + 36700}{2} = 40350 \]
\[ 15|20q_{40} = \frac{53000 - 40350}{60500} = 0.2091. \]
7. \(2q_{50} = 2p_{50} - 4p_{50}\).

\[2p_{50} = p_{50} 	imes p_{50+1} = 0.98 \times 0.96 = 0.9408.
\]

\[4p_{50} = p_{50} 	imes p_{50+1} \times p_{52} \times p_{53} = 0.98 \times 0.96 \times 0.94 \times 0.93 = 0.8224.
\]

Hence, \(2q_{50} = 0.9408 - 0.8224 = 0.1184\).

8. (a) \(3p_{73} = \frac{l_{76}}{l_{73}} = \frac{17355}{20696} = 0.838568\).

(b) \[ q_{[70]+1} = \frac{l_{70+5} - l_{70+6}}{l_{70+1}} = \frac{l_{75} - l_{76}}{l_{70+1}} = \frac{18508 - 17355}{2200} = 0.05194 .
\]

(c) Here, both subscripts are non-integers, so we need to use the trick.

\[0.7p_{70} \times 0.5p_{70+0.7} = 1.2p_{70}.
\]

Then, we have
\[0.5p_{70+0.7} = \frac{1.2p_{70}}{0.7p_{70}} = \frac{p_{70} \times 0.2p_{70+1}}{0.7p_{70}} = \frac{l_{70+1}}{l_{70}} \left( 1 - 0.2 \left( 1 - \frac{l_{72}}{l_{70+1}} \right) \right) = 1 - 0.7 \left( 1 - \frac{l_{70+1}}{l_{70}} \right) \]
\[= \frac{22200 \left( 1 - 0.2 \left( 1 - \frac{21722}{22200} \right) \right)}{22507 \left( 1 - 0.7 \times \frac{22200}{22507} \right)} = 0.991580 .
\]

(d) \(0.5p_{70+0.7} = 0.3p_{70+0.7} \times 0.2p_{70+1} = (p_{70})^{0.3} (p_{70+1})^{0.2} = \left( \frac{1}{l_{70}} \right)^{0.3} \left( \frac{l_{72}}{l_{70+1}} \right)^{0.2} = 0.991562 .
\]

9. We begin with finding \(S_0(t)\) for \(0 \leq t \leq 20:\)
\[S_0(t) = \int_0^t f_0(u)du = \int_0^t (20 - u)du = \left[ 20u - \frac{u^2}{2} \right]_0^t = \frac{(20 - t)^2}{200} = \frac{(20 - t)^2}{400}.
\]

\[p_5 = \frac{S_0(t+5)}{S_0(5)} = \frac{(15 - t)^2}{15^2} = \left( 1 - \frac{t}{15} \right)^2 , \text{ for } 0 \leq t \leq 15.
\]

\[e^5 = \int_0^5 p_s dt = \int_0^5 \left( 1 - \frac{t}{15} \right)^2 dt = -\frac{15}{3} \left[ \left( 1 - \frac{t}{15} \right)^3 \right]_0^5 = 5 .
\]
10. Since the survival function changes at \( t = 30 \), we need to decompose the integral into two parts.

\[
E(T_0) = \int_0^\infty S_0(t)\,dt \\
= \int_0^{30} \left(1 - \frac{t}{100}\right)\,dt + \int_3^{\infty} 0.7e^{-0.02(t-30)}\,dt \\
= \left[ t - \frac{t^2}{200}\right]_0^{30} + 0.7\int_0^\infty e^{-0.02u}\,du \\
= 25.5 + \frac{0.7}{0.02} \\
= 60.5
\]

11. We begin with the calculation of \( t \cdot p_0 \):

\[
t \cdot p_0 = S_0(t) = e^{-\int_0^t \mu_u\,du} = e^{-\int_0^{400-u} \mu_u\,du} = e^{\ln(400-u)} = e^{\frac{400-t^2}{400}} = 1 - \frac{t^2}{400}
\]

\[
E(T_0) = \int_0^{20} t \cdot p_0\,dt = \int_0^{20} \left(1 - \frac{t^2}{400}\right)\,dt = \left[ t - \frac{t^3}{1200}\right]_0^{20} = \frac{40}{3}
\]

\[
E(T_0^2) = 2\int_0^{20} t^2 \cdot p_0\,dt = 2\int_0^{20} \left(t - \frac{t^3}{400}\right)\,dt = 2\left[ \frac{t^2}{2} - \frac{t^4}{1600}\right]_0^{20} = 200
\]

\[
\text{Var}(T_0) = E(T_0^2) - [E(T_0)]^2 = 200 - \left(\frac{40}{3}\right)^2 = 22.22.
\]

12. Here, the lifetime follows De Moivre’s law (i.e., a uniform distribution). By using the properties of uniform distributions, we immediately obtain

\[
\text{Var}(T_0) = 468.75 = \frac{\omega^2}{12} \\
\omega^2 = 5625 \\
\omega = 75
\]

13. When \( \mu_x = \mu \) for all \( x \geq 0 \), the lifetime follows an exponential distribution. Using the properties of exponential distributions, we immediately obtain

\[
e_30 = 40 = \frac{1}{\mu} \\
\mu = \frac{1}{40} = 0.025
\]

Also, we know that when \( \mu_x = \mu \) for all \( x \geq 0 \), \( t \cdot p_x = e^{-\mu t} \). Hence,
\[ sP_{20} = e^{-0.025 \times 5} = 0.8825. \]

14. First, we calculate \( S_0(t) \):

\[
S_0(t) = \frac{l_t}{l_0} = \frac{10000 - t^2}{10000} = 1 - \frac{t^2}{10000}.
\]

Then,

\[
E(T_0) = \int_0^{100} S_0(t) \, dt = \int_0^{100} \left(1 - \frac{t^2}{10000}\right) \, dt = \left[t - \frac{t^3}{30000}\right]_0^{100} = 66.6667,
\]

and

\[
E(T_0^2) = 2 \int_0^{100} tS_0(t) \, dt = 2 \int_0^{100} \left(t - \frac{t^3}{10000}\right) \, dt = 2 \left[\frac{t^2}{2} - \frac{t^4}{40000}\right]_0^{100} = 5000.
\]

Hence, \( \text{Var}(T_0) = E(T_0^2) - [E(T_0)]^2 = 5000 - 66.6667^2 = 555.6. \)

15. Since \( \mu_x = 0.02 \) for all \( x \geq 0 \), we immediately have \( p_x = e^{-0.02t} \). Then,

\[
e^{10e_{10}} = \int_0^{10} t P_{10} \, dt = \int_0^{10} e^{-0.02t} \, dt = -\frac{1}{0.02} e^{-0.02t} \bigg|_0^{10} = 9.063.
\]

16. First, we obtain \( P_{20} \) as follows:

\[
P_{20} = \frac{S_0(20 + t)}{S_0(20)} = 1 - \frac{t}{\omega - 20}.
\]

Then, we have

\[
e^{20e_{20,30}} = \int_0^{30} tP_{20} \, dt
\]

\[
= \int_0^{30} \left(1 - \frac{t}{\omega - 20}\right) \, dt
\]

\[
= \left[t - \frac{t^2}{2(\omega - 20)}\right]_0^{30}
\]

\[
= 30 - \frac{450}{\omega - 20} = 22.5.
\]

This gives \( \omega = 80. \)

Note that the underlying lifetime follows De Moivre’s law. This implies that \( T_{30} \) is uniformly distributed over the interval \([0, \omega - 30]\), that is, \([0, 50]\). Using the properties of uniform distributions, we have \( \text{Var}(T_{30}) = 50^2 / 12 = 208.33. \)
17. First, we compute $t^5_p$:

$$t^5_p = \frac{l_{5+t}}{l_5} = \frac{80-(5+t)}{80-5} = 1 - \frac{t}{75}.$$ 

Hence,

$$e_{5:15} = \int_0^{15} \left(1 - \frac{t}{75}\right) dt = \left(t - \frac{t^2}{150}\right) \bigg|_0^{15} = 13.5.$$  

18. Since mortality follows De Moivre’s law, for 20 year olds, future lifetime follows a uniform distribution over $[0, \omega - 20)$. We have

$$e_{20} = 30 = \frac{\omega - 20}{2},$$

which gives $\omega = 80$. Since death occurs uniformly over $[0, 60)$, we have $q_{20} = 1/60$. Hence, the answer is (A).

19. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 80.

- Suppose that there are $l_{30}$ persons in the entire population initially. At time 0 (i.e., at age 30), there are $0.5l_{30}$ nonsmokers and $0.5l_{30}$ smokers.

- For nonsmokers, the proportion of individuals who can survive to age 80 is $e^{-0.08 \times 50} = e^{-4}$. For smokers, the proportion of individuals who can survive to age 80 is $e^{-0.16 \times 50} = e^{-8}$. As a result, at age 80, there are $0.5l_{30}e^{-4}$ nonsmokers and $0.5l_{30}e^{-8}$ smokers. Hence, among those who can survive to age 80,

$$\frac{0.5l_{30}e^{-4}}{0.5l_{30}e^{-4} + 0.5l_{30}e^{-8}} = \frac{1}{1 + e^{-4}} = 0.982014$$

are nonsmokers and $1 - 0.982014 = 0.017986$ are smokers.

Second, we need to calculate $q_{80}$ for both smokers and nonsmokers.

- For a nonsmoker at age 80, $q_{80}^n = 1 - e^{-0.08}$.

- For a smoker at age 80, $q_{80}^s = 1 - e^{-0.16}$.

Finally, for the whole population, we have

$$q_{80} = 0.982014(1 - e^{-0.08}) + 0.017986(1 - e^{-0.16}) = 0.07816.$$ 

Hence, the answer is (A).

20. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 60.
Suppose that there are \( l_0 \) persons in the entire population initially. At time 0 (i.e., at age 0), there are 0.5\( l_0 \) males and 0.5\( l_0 \) females.

For males, the proportion of individuals who can survive to age 60 is \( e^{-0.10 \times 60} = e^{-6} \). For females, the proportion of individuals who can survive to age 60 is \( e^{-0.08 \times 60} = e^{-4.8} \). As a result, at age 60, there are 0.5\( l_0 e^{-6} \) nonsmokers and 0.5\( l_0 e^{-4.8} \) smokers. Hence, among those who can survive to age 60,

\[
\frac{0.5 l_0 e^{-6}}{0.5 l_0 e^{-6} + 0.5 l_0 e^{-4.8}} = \frac{1}{1 + e^{1.2}} = 0.231475
\]

are males and \( 1 - 0.231475 = 0.768525 \) are females.

Second, we need to calculate \( q_{60} \) for both males and females.

For a male at age 60, \( q_{60}^m = 1 - e^{-0.10} \).

For a female at age 60, \( q_{60}^f = 1 - e^{-0.08} \).

Finally, for the whole population, we have

\[
q_{60} = 0.231475(1 - e^{-0.10}) + 0.768525(1 - e^{-0.08}) = 0.0811.
\]

Hence, the answer is (B).

21. From Statement (ii), we know that the underlying lifetime follows De Moivre’s law. By using the properties of uniform distributions, we immediately have

\[
e_0 = \frac{\omega}{2} = 25,
\]

which gives \( \omega = 50 \).

Under De Moivre’s law, \( T_{10} \) is uniformly distributed over the interval \([0, \omega - 10)\), that is, \([0, 40)\). By using the properties of uniform distributions, we immediately obtain

\[
\text{Var}(T_{10}) = \frac{40^2}{12} = 133.3.
\]

Hence, the answer is (C).

22. From the given survival function, we know that the underlying lifetime follows De Moivre’s law. First, we find \( \nu_{30} \):

\[
\nu_{30} = \frac{S_0(30 + t)}{S_0(30)} = 1 - \frac{t}{\omega - 30}.
\]

We then use Statement (i) to find \( \omega \):

\[
\nu_{30-30} = \int_{0}^{40} \nu_{30} \, dt = \int_{0}^{40} \left(1 - \frac{t}{\omega - 30}\right) \, dt = \left[t - \frac{t^2}{2(\omega - 30)}\right]_{0}^{40} = 40 - \frac{40^2}{2(\omega - 30)} = 27.692.
\]

This gives \( \omega = 95 \).
Under De Moivre’s law, \( T_{30} \) is uniformly distributed over \([0, \omega - 30)\), that is \([0, 65)\). By using the properties of uniform distributions, we immediately obtain

\[
\text{Var}(T_{30}) = \frac{65^2}{12} = 352.1.
\]

Hence the answer is (B).

23. For the original model, \( S_0(t) = (1 - t/\omega)\alpha \). This gives

\[
E(T_0) = \int_0^\omega S_0(t) \, dt = \int_0^\omega \left(1 - \frac{t}{\omega}\right)^\alpha \, dt = -\frac{\omega}{\alpha+1}\left(1 - \frac{t}{\omega}\right)^{\alpha+1}\bigg|_0^\omega = \frac{\omega}{\alpha+1},
\]

and

\[
\mu_x = -\frac{S_0'(x)}{S_0(x)} = \frac{\alpha}{\omega} \left(1 - \frac{x}{\omega}\right)^{\alpha-1} = \frac{\alpha}{\omega - x}.
\]

Let \( \alpha \) and \( \alpha^* \) be the original and new values of \( \alpha \), respectively. Since the new complete expectation of life is half what it was previously, we have

\[
\frac{\omega}{\alpha^* + 1} = \frac{1}{2} \left(\frac{\omega}{\alpha + 1}\right), \quad \text{or} \quad 2(\alpha + 1) = \alpha^* + 1.
\]

Also, since the new force of mortality is 2.25 times the previous force of mortality for all durations, we have

\[
\frac{\alpha^*}{\omega - x} = 2.25 \frac{\alpha}{\omega - x},
\]

or \( \alpha^* = 2.25 \alpha \). Solving \( 2(\alpha + 1) = 2.25 \alpha + 1 \), we obtain \( \alpha = 4 \). Hence, the answer is (D).

24. The primary objective of this question is to examine your knowledge on the recursion formula \( e_x = p_x(1 + e_{x+1}) \).

Note that \( M \) and \( N \) have the same force of mortality from age 26. This means that

\[
k \, p_{26}^M = k \, p_{26}^N, \quad k = 1, 2, 3, \ldots,
\]

and consequently that

\[
e_{26}^M = e_{26}^N.
\]

Using the identity above, we have

\[
e_{25}^N = p_{25}^N (1 + e_{25}^N) = p_{25}^N (1 + e_{26}^M).
\]

We can find \( p_{25}^N \) using the force of mortality given:
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This implies that
\[ e_{25}^N = e^{-0.05} p_{25}^M (1 + e_{26}^M) = e^{-0.05} e_{25}^M = 0.951 \times 10 = 9.51. \]

Hence, the answer is (D).

25. Splicing two functions \( h(x) \) and \( g(x) \) on an interval \([a, b]\) means that we break up the interval into two smaller intervals \([a, c]\) and \((c, d]\) and define the spliced function to equal \( h(x) \) on \([a, c]\) and \( g(x) \) on \((c, d]\). In this case, we are breaking up \([0, 100]\) into \([0, 50]\) and \((50, 100]\). Our new function will equal \( k f^a(t) \) on \([0, 50]\) and \( 1.2 f^b(t) \) \((50, 100]\). The spliced function needs to be a density function on \([0, 100]\), so we need to find the value of \( k \) that makes the total area under the curve equal 1.

We will start by looking at \( 1.2 f^b(t) \). For a De Moivre’s model with \( \omega = 100, \) \( f^b(t) = 1/100, \) which means \( 1.2 f^b(t) = 1.2/100. \) Thus, the area under the curve \((50, 100]\) is
\[ \int_{50}^{100} \frac{1.2}{100} \, dt = \frac{1.2(100 - 50)}{100} = 0.6. \]

This means that the area under the curve on \([0, 50]\) must be 0.4. So,
\[ k \int_0^{50} f^a(t) \, dt = k \left( \frac{L_0 - L_{50}}{L_0} \right) = k \frac{1049099}{1000000} = 0.4, \]
where \( L_t \) is the life function that corresponds to the Illustrative Life Table. This gives \( k = 3.8128. \)

Let \( q_0^* \) be death probabilities that corresponds to the Illustrative Life Table. Then
\[ 10 \ p_{40} = \frac{50 \ p_0}{40 \ p_0} = \frac{1 - 50 q_0}{1 - 40 q_0} = \frac{1 - \int_0^{50} k f^a(t) \, dt}{1 - \int_0^{40} k f^a(t) \, dt} = \frac{1 - k_{40} q_0^*}{1 - k_{40} q_0^*} = \frac{1 - 0.4}{1 - k} = \frac{0.6}{0.738124} = 0.8129 \]

Hence, the answer is (A).
26. (a) The proof is as follows:

\[ e_x = \sum_{k=1}^{\infty} k \, p_x = p_x + \sum_{k=2}^{\infty} k \, p_x \]
\[ = p_x + p_x \sum_{k=2}^{\infty} k \, p_{x+1} = p_x + p_x \sum_{j=1}^{\infty} j \, p_{x+1} \]
\[ = p_x + p_x e_{x+1} = p_x (1 + e_{x+1}) \]

(b) Under UDD, \( T_x = K_x + U \), where \( U \) follows a uniform distribution over the interval \([0,1]\). Taking expectation on both sides, we have \( E(T_x) = E(K_x) + E(U) \), which implies

\[ e_x = e_x + \frac{1}{2}. \]

(c) This is a difficult question. To answer this question, you need to use the following three facts:

- For a one-year select period, \( l_{[x+1]} = l_{[x]} - d_{[x]} \) and \( e_{[x+1]} = e_{x+1} \).
- \( e_x = p_x(1 + e_{x+1}) \)
- Under UDD, \( e_x = e_x + \frac{1}{2} \)

We can complete the second last column of the table by using \( l_{[x+1]} = l_{[x]} - d_{[x]} \):

\[ l_{81} = l_{[80]} - d_{[80]} = 1000 - 90 = 910, \]
\[ l_{82} = l_{[81]} - d_{[81]} = 920 - 90 = 830. \]

Under UDD, we have \( e_{[80]} = 8.5 - 0.5 = 8 \).

We then apply the recursion formula as follows:

\[ e_{[80]} = p_{[80]}(1 + e_{[80]+1}) = p_{[80]}(1 + e_{81}) = \frac{910}{1000} (1 + e_{81}). \]

This gives \( e_{81} = 7.791208791 \).

Also, we have

\[ e_{[81]} = p_{[81]}(1 + e_{82}), \]
\[ e_{81} = p_{81}(1 + e_{82}). \]

This means

\[ \frac{p_{[81]}}{p_{81}} = \frac{e_{81}}{e_{[81]}} = \frac{l_{[81]+1}}{l_{[81]}} = \frac{e_{81}}{l_{82}/l_{81}} = \frac{7.791208791 \times 910}{920} = 7.7065. \]

Finally, assuming UDD, \( e_{[81]} = 7.7065 + 0.5 = 8.2065 \).
27. (a) \( S_x(t) = \frac{S_0(x+t)}{S_0(x)} = e^{-0.02(x+t)} / e^{-0.02x} = e^{-0.02t} \)

(b) \( \mu_x = \frac{-S'_0(x)}{S_0(x)} = \frac{0.02e^{-0.02x}}{e^{-0.02x}} = 0.02 \)

(c) \( \hat{\epsilon}^{0\rightarrow0}_{10\rightarrow10} = \int_0^{10} p_{t0} dt = \left[ \int_0^{10} e^{-0.02t} dt \right]^{10}_0 = 9.06346 \)

(d) Since \( \epsilon_x = \epsilon_x, \) we have
\[
e_x = \sum_{k=1}^{\infty} k \epsilon_x = \sum_{k=1}^{\infty} e^{-0.02k} = \frac{e^{-0.02}}{1-e^{-0.02}} = 49.5017
\]

28. (a) The event \( K_x = k \) is the same as \( k \leq T_x < k + 1 \), which means the individual cannot die within the first \( k \) years and must die during the subsequent year. The probability associated with this event must be \( kq_x \), the \( k \)-year deferred one-year death probability.

(b) The proof is as follows:
\[
e_{x|n} = \sum_{k=0}^{n-1} k \Pr(K_x = k) + \sum_{k=n}^{\infty} n \Pr(K_x = k)
\]
\[
= \sum_{k=0}^{n-1} k \times q_x + \sum_{k=n}^{\infty} n \times q_x
\]
\[
= 0 + 1 \times q_x + 2 \times q_x + \ldots + (n-1) \times q_x + n \times q_x + \ldots
\]
\[
= (p_x - 2p_x) + 2(2p_x - 3p_x) + \ldots + (n-1)(n-1)p_x - np_x) + \ldots
\]
\[
= p_x + 2p_x + 3p_x + \ldots + np_x
\]
\[
= \sum_{k=0}^{n} k \epsilon_x
\]

29. (a) \( \omega = 100 \)

(b) \( \hat{\epsilon} = \int_0^{100-40} p_{40} dt = \int_0^{60} \left(1 - \frac{t}{60}\right)^{0.5} dt = \left[ -\frac{60}{1.5} \left(1 - \frac{t}{60}\right)^{1.5}\right]^{60}_0 = 0 + 60/1.5 = 40 \)

(c) \( E(T_{40}^{2}) = 2 \int_0^{60} t \left(1 - \frac{t}{60}\right)^{0.5} dt \). Let \( y = 1 - t/60 \). We have
\[
E(T_{40}^{2}) = 2 \int_0^{60} 60(1-y)y^{0.5}(-60)\,dy
\]
\[
= 7200 \int_0^{1} y^{1/2} - y^{3/2} \,dy
\]
\[
= 7200 \left[ \frac{2}{3} y^{3/2} - \frac{2}{5} y^{5/2} \right]_0^1 = 1920
\]
Hence, \( \text{Var}(T_{40}) = 1920 - 40^2 = 320. \)

30. (a) The effect of medical (or other) evidence at the inception of an insurance contract.

(b) (i) In laymen’s terms:
- Company Y requires no medical examination, so it is taking more risk.
- Company Y has a higher change of adverse selection.
- Company Y has to charge more premium to compensate for the additional risk.

(ii) In actuarial terms:
- Company X requires a medical examination, which means there is a stronger effect of selection.
- The index of selection is higher (closer to 1).
- The death probabilities used to price the policy are lower. This means the premium charged by Company X is lower than that charged by Company Y.

(c) (i) 2 years.

(ii) \( 12q_{65+1} = p_{65+1} \times 2q_{65+2} = p_{65+1} \times 2q_{67} \)
\( = p_{65+1}(1 - (1 - q_{67})(1 - q_{68})) \)
\( = (1 - 0.04)(1 - (1 - 0.07)(1 - 0.09)) = 0.147552. \)

(iii) \( 0.4p_{66+0.3} = (p_{66})^{0.4} = (1 - 0.03)^{0.4} = 0.987890. \)

31. (a) \( e_{91} = p_{91} + 2p_{91} + 3p_{91} + \ldots = (l_{92} + l_{93} + \ldots + l_{99})/l_{91} = 2.44 \) years.

(b) Under UDD \( \hat{e}_{91} = e_{91} + 0.5 = 2.94. \)

(c) Under De Moivre’s law, \( q_{41} = 1 - \int_0^8 (1 - t^2/8) \, dt = \left( t - \frac{t^3}{16} \right)_0^8 = 8 - \frac{64}{16} = 4. \)

32. (a) \( I(44, 0) = 1 - \frac{q_{44}}{q_{44}} = 1 - \frac{0.00157}{0.00257} = 0.38911 \)
\( I(44, 1) = 1 - \frac{q_{44+1}}{q_{44+1}} = 1 - \frac{0.00240}{0.00293} = 0.18089 \)
\( I(44, 2) = 1 - \frac{q_{44+2}}{q_{44+2}} = 1 - \frac{0.00301}{0.00337} = 0.10682 \)
\( I(44, 3) = 1 - \frac{q_{44+3}}{q_{44+3}} = 1 - \frac{0.00367}{0.00384} = 0.04427 \)

Comment: The index of selection reduces as the value of \( k \) increases. This agrees with the fact that as duration increases, selection effect tapers off.

(b) The table is calculated as follows:
Chapter 2: Life Tables

<table>
<thead>
<tr>
<th>$[x]$</th>
<th>$l_x$</th>
<th>$l_{x+1}$</th>
<th>$l_{x+2}$</th>
<th>$l_{x+3}$</th>
<th>$l_{x+4}$</th>
<th>$x + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>10000</td>
<td>9989.9</td>
<td>9972.4</td>
<td>9952.0</td>
<td>9928.8</td>
<td>44</td>
</tr>
<tr>
<td>41</td>
<td>9980.2</td>
<td>9969.0</td>
<td>9950.2</td>
<td>9928.3</td>
<td>9903.3</td>
<td>45</td>
</tr>
<tr>
<td>42</td>
<td>9958.8</td>
<td>9946.1</td>
<td>9925.8</td>
<td>9902.0</td>
<td>9878.3</td>
<td>46</td>
</tr>
</tbody>
</table>

(c) (i) $2p_{[42]} = l_{[42]+2}/l_{[42]} = 0.99669.$

(ii) $3q_{[41]+1} = (l_{[41]+1} - l_{[41]+4})/l_{[41]+1} = 0.00659$

(iii) $3/2q_{[41]} = (l_{[41]+3} - l_{[41]+5})/l_{[41]} = 0.00542$

33. (a) $d_{52} = l_{52} - l_{53} = 99700 - 99500 = 200$

(b) $2q_{50} = 2p_{50} q_{52} = \frac{l_{52} d_{52}}{l_{50}} = \frac{200}{100000} = 0.002$

(c) Since $0.4p_{50} 4.3p_{50.4} = 4.7p_{50}$, we have

$$4.3p_{50.4} = \frac{4.7p_{50}}{0.4p_{50}} = \frac{4p_{50} \times 0.7p_{54}}{0.4p_{50} \times 1 - 0.4q_{50}} = \frac{l_{54} \left(1 - 0.7d_{54} \right)}{l_{50} \left(1 - 0.4d_{50} \right)}.$$ 

Substituting, we obtain $4.3p_{50.4} = 0.987195$.

(d) $4.3p_{50.4} = 0.6p_{50} \times 4.3p_{51} = 0.6p_{50.4} \times 3p_{51} \times 0.7p_{54} = (p_{50})^{0.6} 3p_{51}(p_{54})^{0.7}$

$$= \left(\frac{99900}{100000}\right)^{0.6} \left(\frac{99100}{99900}\right) \left(\frac{98500}{99100}\right)^{0.7} = 0.987191.$$
Chapter 3  Life Insurances

OBJECTIVES

1. To understand the specifications of some simple life insurance contracts
2. To calculate actuarial present values and variances of present value random variables
3. To apply recursion formulas for discrete life insurances
4. To relate continuous, discrete and monthly life insurances

In Exam FM, you learnt how to value the present value of a payment that is made at a fixed future time. For example, if one guarantees to make a payment of $10,000 in 10 years, then the present value of that payment today at an annual effective interest rate of $i$ is

$$\frac{10,000}{(1+i)^{10}}.$$

By contrast, insurance payments are contingent upon one or more future events. For example, if an insurance policy sold to ($x$) promises to pay an amount of $10,000 at the moment of the policyholder’s death, then the present value of that life-contingent payment today at an effective interest rate of $i$ is

$$\frac{10,000}{(1+i)^{T_x}}.$$

Note that the present value is random, because it depends on a random quantity $T_x$. We call it the present value random variable for the insurance policy. In this chapter, we will study in great depth the present value random variables for several important life insurance policies.
3.1 Continuous Life Insurances

A few important life insurance policies are covered in this section. First, let us talk about the simplest policy, level benefit whole life insurance.

**Level Benefit Whole Life Insurance**

A whole life insurance pays a benefit at the moment of death of the policyholder, whenever it occurs. Throughout this section, we use $b_{x}$ to denote the benefit function, that is, the amount of benefit as a function of $T_{x}$.

For now we focus on level benefit policies. A level benefit policy pays the same amount of benefit, regardless of the time at which the benefit is paid. Furthermore, to simplify calculations, we assume for now that the benefit is $1. The diagram below illustrates the mechanism of a unit-benefit continuous whole life insurance:

![Diagram of a unit-benefit continuous whole life insurance](image)

Now, let us describe this insurance contract mathematically. It is obvious that the benefit function is

$$b_{x} = 1, \text{ for all } T_{x} \geq 0.$$ 

Multiplying the benefit function with the discount factor for a period of $T_{x}$ years, we obtain the present value random variable

$$Z = v^{T_{x}}, \text{ for all } T_{x} \geq 0.$$
Throughout this study guide, we use $Z$ to denote the present value random variable for a unit-benefit insurance policy.

We are particularly interested in knowing the expected value and the variance of $Z$. To calculate the expected value, we simply integrate out the product of $Z$ and the density function of the underlying random variable $T_x$, that is,

$$E(Z) = E(v^{T_x})$$
$$= \int_0^\infty v^t f_x(t) dt$$
$$= \int_0^\infty v^t p_x \mu_{x+t} dt$$
$$= \int_0^\infty e^{-\delta t} p_x \mu_{x+t} dt.$$

Of course, if there is a limiting age, we replace $\infty$ with $\omega - x$.

We call $E(Z)$ the expected present value or more commonly the actuarial present value (APV). We also say that $E(Z)$ is the net single premium for the policy. By net we mean that the premium is not loaded with any expenses, and by single we mean that the premium is paid as a single lump sum at the inception of the contract.

We denote $E(Z)$ for a continuous unit-benefit whole life insurance by $\bar{A}_x$. In the symbol, the $A$ stands for the APV of a life insurance, the $x$ indicates the age of the policyholder at the beginning of the contract, and the bar above $A$ indicates that it is a continuous insurance (one with a benefit payable at the moment of death).

To calculate the variance of $Z$, we can make use of the identity

$$\text{Var}(Z) = E(Z^2) - [E(Z)]^2.$$ 

We have calculated $E(Z)$ already, so all then that remains is to calculate $E(Z^2)$. We have

$$E(Z^2) = \int_0^\infty (v^t)^2 p_x \mu_{x+t} dt$$
$$= \int_0^\infty e^{-2\delta t} p_x \mu_{x+t} dt.$$
We observe that the formula for \( E(Z^2) \) is the same as that for \( E(Z) \), except that \( \delta \) is replaced with \( 2\delta \). This important fact means that we do not have to calculate \( E(Z^2) \) from scratch. To obtain \( E(Z^2) \), we simply evaluate \( E(Z) \) at two times the original force of interest. We denote \( E(Z^2) \) for a continuous unit-benefit whole life insurance by \( ^2A_x \). The superscript 2, as you can guess, indicates that the quantity is \( E(Z) \) calculated at \( 2\delta \).

The result that \( E(Z^2) \) is \( E(Z) \) evaluated at \( 2\delta \) is applicable to not only a unit-benefit whole life insurance, but also any insurance policy with \( b_{xT} = 0 \) or 1 for all \( T_x \). We will apply this important property again in subsequent discussions.

**Example 3.1 [Structural Question]**

You are given that \( \mu_x = \mu \) for all \( x \geq 0 \).

Let \( Z \) be the present value random variable for a whole life insurance with a level benefit of $1 payable at the moment of death.

(a) Derive expressions for \( A_x \) and \( \text{Var}(Z) \) in terms of \( \mu \) and \( \delta \).

(b) Assume that \( \mu = 0.01 \) and \( \delta = 0.02 \). Find the probability that the net single premium is insufficient to cover the benefit.

**Solution**

(a) Recall that if \( \mu_x = \mu \) for all \( x \geq 0 \), then \( i_p = e^{-\mu t} \). Hence,

\[
A_x = \int_0^\infty e^{-\delta t} e^{-\mu t} \mu dt = \mu \int_0^\infty e^{-(\delta+\mu)t} dt
\]

\[
= \mu \left[ 0 \right]^{\infty \atop \infty} = \frac{\mu}{\mu + \delta}. 
\]

Also, \( ^2A_x = \frac{\mu}{\mu + 2\delta} \). [\( E(Z) \) evaluated at \( 2\delta \)]

Therefore, \( \text{Var}(Z) = ^2A_x - A_x^2 = \frac{\mu}{\mu + 2\delta} \left( \frac{\mu}{\mu + \delta} \right)^2 \).
(b) First, at the given values of \( \mu \) and \( \delta \), \( \bar{A}_x = \frac{1}{3} \). The required probability can be calculated as follows:

\[
\Pr(Z > \bar{A}_x) = \Pr\left( v^{r_x} > \frac{1}{3} \right) = \Pr\left( e^{-0.02r_x} > \frac{1}{3} \right) \\
= \Pr\left( T_x < \frac{\ln(1/3)}{-0.02} \right) = 1 - \exp\left( -0.01 \left( \frac{\ln(1/3)}{-0.02} \right) \right). \\
= 1 - \left( \frac{1}{3} \right)^{1/2} = 0.4226.
\]

[ END ]

**Example 3.2  [Structural Question]**

You are given that \( S_0(x) = \exp(-\mu x) \), for \( x > 0 \).

(a) Write down \( Z \), the present value random variable for a whole life insurance on \( (x) \), with a death benefit of $1 payable at the moment of death.

(b) Sketch the relationship between \( Z \) and \( T_x \).

(c) Let \( \mu = 0.01 \) and \( \delta = 0.02 \). Compute the 90\(^{th}\) percentile of \( Z \).

---

**Solution**

(a) For this insurance policy, \( b_{r_x} = 1 \) for all \( T_x \geq 0 \). Hence, the present value random variable is simply \( Z = v^{r_x} \) for all \( T_x \geq 0 \).

(b)
Note:
- The vertical intercept is 1, because when $T_x = 0$ (i.e., the policyholder dies immediately after the policy starts), the death benefit of $1$ is payable immediately, given a present value of exactly $1$.
- As $T_x$ tends to infinity, $Z$ tends to $0$. This is because if the policyholder is immortal, a death benefit will never be paid, which means the present value is zero.
- It is entirely possible that you will be asked to sketch a graph in the real exam. The SoA has the following message to candidates concerning graph sketching:

“Some problems may require candidates to sketch a graph. In these problems, candidates should clearly label and mark both axes, show the general form of the function being graphed, and indicate any limiting values, extrema, asymptotes, and discontinuities. The exact shape of the function is not required for full credit.”

(c) Let $\gamma$ be the 90th percentile of $Z$. By definition, we have $\Pr(Z \leq \gamma) = 0.9$, from which we can solve for $\gamma$.

$$
\frac{\ln(v^T)}{-\delta} = \ln(\gamma) \\
\Pr(T_x > \ln(\gamma) - \delta) = 0.9
$$

Flip the inequality sign here as $\ln(v) = -\delta$ is negative!

$\exp\left(-\frac{\ln(\gamma)}{-\delta}\right) = 0.9$

$\gamma = 0.9^2 = 0.81$

The use of unit benefits allows us to define standardized actuarial notation, such as $\overline{A}_x$ and $\overline{2}A_x$. In practice, insurance companies sell policies for other amounts. Consider a whole life insurance with a benefit of $k$ dollars payable at the moment of death. The present value random variable for this policy is $kZ$. The actuarial present value of this policy is given by

$$
E(kZ) = kE(Z) = k \overline{A}_x.
$$
The variance of the present value random variable for this policy is

\[
\text{Var}(kZ) = k^2 \text{Var}(Z) = k^2 (\overline{A}_x - \overline{A}_x^2).
\]

This technique is applicable to other policies as well.

**Level Benefit Term Life Insurance**

An \(n\)-year term life insurance pays a benefit at the moment of death only if death occurs in the following \(n\) years. Because it does not cover the entire lifetime, it is cheaper than a whole life insurance with the same amount of benefit.

For now, we focus on level benefit \(n\)-year term life insurances. Further, to standardize notation, we assume that the benefit amount is $1. The payout from a unit-benefit \(n\)-year term life insurance is illustrated in the following diagram.

![Diagram showing level benefit term life insurance payout](image)

Mathematically, the benefit function is given by

\[
b_{rx} = \begin{cases} 
1, & T_x \leq n \\
0, & T_x > n.
\end{cases}
\]

Multiplying the benefit function with the discount factor, we obtain the present value random variable:

\[
Z = \begin{cases} 
v^{\tau_x}, & T_x \leq n \\
0, & T_x > n.
\end{cases}
\]

The actuarial present value, i.e., \(E(Z)\), can be calculated as
\[
E(Z) = \int_0^n v^t \, d\mu_{x+e} dt.
\]

We denote \(E(Z)\) for a unit-benefit \(n\)-year term life insurance by \(\overline{A}_{x\, \mid \, n}\). In this symbol, the 1 above \(x\) indicates that the policy is an \(n\)-year term life insurance. (It does not carry any numerical meaning.) The subscript \(\mid\) indicates that coverage is provided for at most \(n\) years.

This policy satisfies the criterion that \(b_{ix} = 0\) or 1 for all \(T_x\). Hence, we have \(E(Z^2) = 2\overline{A}_{x\, \mid \, n}\) (i.e., \(\overline{A}_{x\, \mid \, n}\) evaluated at \(2\delta\)), and \(\text{Var}(Z) = 2\overline{A}_{x\, \mid \, n} - \overline{A}_{x\, \mid \, n}^2\).

**Pure Endowment**

An \(n\)-year pure endowment provides a payment at the end of \(n\) years if the policyholder survives, but makes no payment if the policyholder dies within \(n\) years. As before, we assume the benefit amount is $1. The payout from a unit benefit \(n\)-year pure endowment is illustrated in the diagram below.

Mathematically, the benefit function is given by

\[
b_{ix} = \begin{cases} 
0, & T_x \leq n \\
1, & T_x > n
\end{cases}
\]

Unlike the previous two policies, the benefit from an \(n\)-year pure endowment is paid at time \(n\) (instead of time \(T_x\)). Hence, the appropriate discount factor is \(v^n\), and the present value random variable is given by

\[
Z = \begin{cases} 
0, & T_x \leq n \\
v^n, & T_x > n
\end{cases}
\]
It is easy to see that \( \text{E}(Z) = v^n \Pr(T_x > n) = v^n q_{x \overline{n}} \). We denote \( \text{E}(Z) \) for a unit-benefit pure endowment by \( A_{x \overline{n}} \). The \( \overline{n} \) indicates that the policy is a pure endowment. (It does not carry any numerical meaning.) We do not write a bar above \( A \), because the payment is made at time \( n \) instead of time \( T_x \).

It is also easy to see that \( \text{E}(Z^2) = 2 A_{x \overline{n}} = v^{2n} p_x \). As a result,

\[
\text{Var}(Z) = \text{E}(Z^2) - [\text{E}(Z)]^2 = v^{2n} p_x - (v^n q_{x \overline{n}})^2 = v^{2n} p_x q_{x \overline{n}}.
\]

**Level Benefit Endowment Insurance**

An \( n \)-year endowment insurance provides an amount to be paid at the moment of death if death occurs in the next \( n \) years or at the end of year \( n \) if the policyholder survives to that time. As before, to standardize notation, we assume that the benefit amount is $1. The payout from a unit-benefit \( n \)-year endowment insurance is illustrated in the diagram below.

Mathematically, the benefit function is \( b_{T_x} = 1 \) for all \( T_x \). Multiplying it with appropriate discount factors, we obtain the present value random variable:

\[
Z = \begin{cases} 
v^{T_x}, & T_x \leq n \\
v^n, & T_x > n. \end{cases}
\]

By comparing the previous three diagrams, you can see that an \( n \)-year endowment insurance is simply a combination of an \( n \)-year term insurance and an \( n \)-year pure endowment. It follows that \( \text{E}(Z) \) for an \( n \)-year endowment insurance can be expressed as

\[
\text{E}(Z) = \overline{A}_{x \overline{n}} + A_{x \overline{n}}.
\]
We denote \( E(Z) \) by \( \bar{A}_{x|n} \). Note that there is no 1 above either \( x \) or \( n \).

Furthermore, this policy satisfies the criterion that \( b_{Tx} = 0 \) or 1 for all \( T_x \). It follows that \( E(Z^2) = 2\bar{A}_{x|n} \) and that \( \text{Var}(Z) = 2\bar{A}_{x|n} - \bar{A}_{x|n}^2 \).

### Example 3.3 [Structural Question]

You are given that \( \mu_x = \mu \) for all \( x \geq 0 \).

(a) Derive an expression for \( \bar{A}_{x|n} \) in terms of \( \mu \) and \( \delta \).

(b) Assume that \( \mu = 0.01 \) and \( \delta = 0.02 \). Calculate \( \bar{A}_{30|5} \).

---

**Solution**

(a) We split \( \bar{A}_{x|n} \) into two components: \( \bar{A}_{x|n} = \bar{A}_{x|n}^1 + \bar{A}_{x|n}^\perp \).

First, we have

\[
\bar{A}_{x|n}^1 = \int_0^n e^{-\delta t} e^{-\mu t} \mu dt = \mu \int_0^n e^{-(\mu+\delta)t} dt = \left. \frac{-\mu e^{-(\mu+\delta)t}}{\mu+\delta} \right|_0^n = \frac{\mu}{\mu+\delta} (1-e^{-(\mu+\delta)n}) .
\]

Second, we have \( \bar{A}_{x|n}^\perp = v^n \mu P_x = e^{-(\delta+\mu)n} = e^{-(\delta+\mu)n} \). As a result,

\[
\bar{A}_{x|n} = \frac{\mu}{\mu+\delta} (1-e^{-(\mu+\delta)n}) + e^{-(\delta+\mu)n} .
\]

(b) Substituting \( \mu = 0.01 \), \( \delta = 0.02 \) and \( n = 5 \) into the formula derived in part (a), we obtain

\[
\bar{A}_{30|5} = \frac{0.01}{0.01+0.02} (1-e^{-(0.01+0.02)\times5}) + e^{-(0.01+0.02)\times5} = 0.9072 .
\]
**Level Benefit Deferred Whole Life Insurance**

An $n$-year deferred whole life insurance pays a benefit at the moment of death if the policyholder has lived at least $n$ years. As before, for notational convenience, we assume that the benefit amount is $1$. The payout from a unit-benefit $n$-year deferred whole life insurance is illustrated in the diagram below.

Mathematically, the benefit function is given by

$$b_{x\mid n} = \begin{cases} 0, & T_x \leq n \\ 1, & T_x > n \end{cases}.$$ 

Multiplying the benefit function with an appropriate discount factor, we obtain the present value random variable

$$Z = \begin{cases} 0, & T_x \leq n \\ v^{T_x}, & T_x > n \end{cases}.$$ 

We have $E(Z) = \int_n^\infty v^t p_x \mu_x \, dt$. We denote $E(Z)$ for a unit-benefit $n$-year deferred whole life insurance as $n\overline{A}_x$. The subscript $n\mid$ on the left-hand side of the symbol indicates that the coverage is deferred by $n$ years.

Also, this policy satisfies the criterion that $b_{x\mid n} = 0$ or 1 for all $T_x$. It follows that $E(Z^2) = 2n\overline{A}_x$ and that $\text{Var}(Z) = 2n\overline{A}_x - n\overline{A}_x^2$. 

(Age $x$)
The following table summarizes the formulas for level-benefit insurances.

<table>
<thead>
<tr>
<th>Policy</th>
<th>Notation for E(Z)</th>
<th>Formula for E(Z)</th>
<th>E(Z^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life</td>
<td>( \bar{A}_x )</td>
<td>( \frac{x}{x} )</td>
<td>( ^2 \bar{A}_x )</td>
</tr>
<tr>
<td>( n )-year term life</td>
<td>( A_{x+n}^{1} )</td>
<td>( \int_0^\infty v^{-t} p_x \mu_x dt )</td>
<td>( ^2 A_{x+n}^{1} )</td>
</tr>
<tr>
<td>( n )-year pure endowment</td>
<td>( A_{x+n}^{1} )</td>
<td>( v^n p_x )</td>
<td>( ^2 A_{x+n}^{1} )</td>
</tr>
<tr>
<td>( n )-year endowment</td>
<td>( \bar{A}_{x+n} )</td>
<td>( \bar{A}<em>{x+n}^{1} + A</em>{x+n}^{1} )</td>
<td>( ^2 \bar{A}_{x+n} )</td>
</tr>
<tr>
<td>( n )-year deferred whole life</td>
<td>( a_{n} \bar{A}_x )</td>
<td>( \int_0^\infty v^{-t} p_x \mu_x dt )</td>
<td>( ^2 a_{n} \bar{A}_x )</td>
</tr>
</tbody>
</table>

Now, let us consider insurance policies which have non-level benefits.

**Continuously Increasing Whole Life Insurance**

First, we consider a whole life insurance with a benefit that increases continuously with time.

The benefit function is \( b_x = T_x \), and the present value random variable is \( T_x v^{-T} \). Hence the actuarial present value can be calculated as
\[ (\overline{IA})_x = \int_0^\infty t v^i \cdot p_x \mu_{x+i} \, dt. \]

The \( I \) in the symbol indicates that the policy has an increasing benefit, and the bar above \( I \) indicates that the increase in benefit is continuous.

**Annually increasing whole life insurance**

Alternatively, the benefit may increase annually instead of continuously.

The benefit function is \( b_x = \lfloor T_x + 1 \rfloor \), and the present value random variable is \( \lfloor T_x + 1 \rfloor v^x \). The actuarial present value can be calculated as follows:

\[ (\overline{IA})_x = \int_0^\infty \lfloor t + 1 \rfloor v^t \cdot p_x \mu_{x+t} \, dt. \]

There is no bar above the \( I \) in the symbol, because the benefit is not increasing continuously.

Note that for non-level benefit insurances, the criterion that \( b_x = 0 \) or \( 1 \) for all \( T_x \) is not met. Hence, \( E(Z^2) \) is not \( E(Z) \) evaluated at \( 2\delta \). To calculate \( E(Z^2) \) for a non-level benefit insurance, the only way is to perform an integration:

\[ \int_0^\infty (b_x v^t)^2 \cdot p_x \mu_{x+t} \, dt. \]
Chapter 3: Life Insurances

For a continuously increasing whole life insurance on \(x\), you are given:

(i) The force of mortality is constant.
(ii) \(\delta = 0.06\).
(iii) \(\overline{A}_{x} = 0.25\).

Calculate \(\overline{I}(\overline{A})_{x}\).

(A) 2.889  (B) 3.125  (C) 4.000  (D) 4.667  (E) 5.500

--- Solution ---

In Example 3.1, we have demonstrated that when \(\mu_{x} = \mu\) for all \(\mu \geq 0\), then

\[
\overline{A}_{x} = \frac{\mu}{\mu + \delta} \quad \text{and} \quad \overline{2A}_{x} = \frac{\mu}{\mu + 2\delta}.
\]

Hence, we have

\[
0.25 = \frac{\mu}{\mu + 2\delta} = \frac{\mu}{\mu + 2 \times 0.06},
\]

which gives \(\mu = 0.04\).

Then, we calculate \((\overline{I}(\overline{A})_{x}\) as follows:

\[
(\overline{I}(\overline{A})_{x} = \int_{0}^{\infty} t v^{'} P_{x} \mu_{x+t} dt = \int_{0}^{\infty} t e^{-0.06t} e^{-0.04t} 0.04 dt
\]
\[
= 0.04 \int_{0}^{\infty} t e^{-0.14t} dt = -0.4([te^{-0.14t}]_{0}^{\infty} - \int_{0}^{\infty} e^{-0.14t} dt)
\]
\[
= -0.4 \int_{0}^{\infty} e^{-0.14t} dt = \frac{0.4}{0.1} = 4
\]

Similarly, we can define other non-level benefit insurance policies, for example, a continuously (or annually) increasing \(n\)-year term life insurance and a continuously (or annually) increasing \(n\)-year endowment insurance. We can also prescribe a decreasing benefit function to create policies such as a continuously decreasing \(n\)-year term life insurance. The following table summarizes the formulas for various non-level-benefit insurances.
### Non-Level-Benefit Insurances (Continuous)

<table>
<thead>
<tr>
<th>Policy</th>
<th>Actuarial present value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Notation</td>
</tr>
<tr>
<td>Continuously increasing whole life</td>
<td>$(\overline{I}A)_x$</td>
</tr>
<tr>
<td>Annually increasing whole life</td>
<td>$(\overline{A})_x$</td>
</tr>
<tr>
<td>Continuously increasing $n$-year term</td>
<td>$(\overline{I}A)^1_{x:\overline{n}}$</td>
</tr>
<tr>
<td>Annually increasing $n$-year term</td>
<td>$(\overline{A})^1_{x:\overline{n}}$</td>
</tr>
<tr>
<td>Continuously increasing $n$-year endowment</td>
<td>$(\overline{I}A)_{x:\overline{n}}$</td>
</tr>
<tr>
<td>Annually increasing $n$-year endowment</td>
<td>$(\overline{A})_{x:\overline{n}}$</td>
</tr>
<tr>
<td>Continuously decreasing $n$-year term</td>
<td>$(\overline{D}A)^1_{x:\overline{n}}$</td>
</tr>
<tr>
<td>Annually decreasing $n$-year term</td>
<td>$(\overline{D}A)_{x:\overline{n}}$</td>
</tr>
</tbody>
</table>

### Example 3.5 [Structural Question]

You are given:

(i) The remaining lifetime for $(x)$ is uniformly distributed over $[0, 2]$.

(ii) $\delta = 0.05$.

Calculate the following:

(a) $(\overline{I}A)_x$

(b) $(\overline{A})_x$

(c) $(\overline{D}A)^1_{x:2}$
Solution

Statement (i) means \( f_x(t) = \frac{1}{2} \) for \( 0 < t < 2 \).

(a) The underlying policy pays a benefit of \( t \) for death at time \( t \). We have

\[
(I\overline{A})_x = \int_0^2 t e^{-0.05t} \frac{1}{2} \, dt = \frac{1}{2} \left[ \frac{te^{-0.05t}}{0.05} - \frac{e^{-0.05t}}{0.05^2} \right]_0^2 = 0.9358
\]

[The second step involves integration by parts.]

(b) The underlying policy pays a benefit of $1 if death occurs within the first year, and $2 if death occurs within the second year. We have

\[
(I\overline{A})_x = \int_0^1 e^{-0.05t} \frac{1}{2} \, dt + \int_1^2 2e^{-0.05t} \frac{1}{2} \, dt = \left[ \frac{e^{-0.05t}}{0.1} \right]_0^1 + \left[ \frac{e^{-0.05t}}{0.05} \right]_1^2 = 0.48771 + 0.92784 = 1.4156
\]

(c) The underlying policy pays a benefit of $2 if death occurs within the first year, and $1 if death occurs within the second year. We have

\[
(D\overline{A})^{1}_{x\overline{3}} = \int_0^1 2e^{-0.05t} \frac{1}{2} \, dt + \int_1^2 1e^{-0.05t} \frac{1}{2} \, dt = 2(0.48771) + 0.5(0.92784) = 1.4393
\]

[ END ]
3.2 Discrete Life Insurances

The only difference between discrete life insurances and continuous life insurances is that the death benefit (if any) is payable at the end of the year of death (rather than precisely at the moment of death). The timing of the death benefit is illustrated by the following diagram.

Since the death benefit is payable at time $K_x + 1$, we should use a discount factor $v^{K_x+1}$ to discount it.

**Level Benefit Whole Life Insurance**

As before, to standardize notation, we assume that the benefit is $1. The present value random variable is simply $Z = v^{K_x+1}$, for $K_x = 0, 1, 2, \ldots$. Here, $Z$ depends on a discrete random variable ($K_x$). This means that $E(Z)$ should be calculated as a summation instead of an integral, and it should based on the probability function of $K_x$. We have

$$E(Z) = \sum_{k=0}^{\infty} v^{k+1} kP_x q_{x+k}.$$ 

We denote $E(Z)$ by $A_x$. As usual, the subscript $x$ indicates the age at inception. We do not place a bar above $A$, because here the benefit is not paid precisely at the moment of death.
If there is a limiting age, we replace $\infty$ in the formula above with $\omega - x - 1$. This is because if no one can live to age $\omega - x$ (where $\omega$ is an integer), then $T_x$ must be strictly smaller than $\omega - x$, which means that the largest possible value of $K_x$ (the integral part of $T_x$) would be $\omega - x - 1$.

For discrete life insurances, we also have the useful property that $E(Z^2)$ is the same as $E(Z)$ calculated at $2\delta$, provided that the benefit is either 0 or 1. Hence, for a discrete unit-benefit whole life insurance, we have $E(Z^2) = 2Ax$ and $\text{Var}(Z) = 2Ax - Ax^2$.

**Level Benefit Term Life Insurance**

An $n$-year term life insurance covers only the first $n$ years from now. We should expect $E(Z)$ for a term life insurance to be the same as that for a whole life insurance, except that the upper limit of the summation is no longer $\infty$.

\[
\begin{align*}
K_x = 0 & \quad K_x = n - 1 \\
0 & \quad 1 & \quad n - 1 & \quad n & \quad n + 1 \\
\text{(Age } x) & \quad \text{Coverage provided} & \quad \text{Time from now}
\end{align*}
\]

From the diagram above, you can see that the summation should run from $k = 0$ to $k = n - 1$. Hence, $E(Z)$ for a unit-benefit term life insurance is given by

\[
E(Z) = \sum_{k=0}^{n-1} v^{k+1} q_x q_{x+k}.
\]

We denote $E(Z)$ by $A^{1}_{x\bar{n}}$. As before, $\bar{n}$ indicates that this policy will last for at most $n$ years, and the 1 above $x$ indicates that the policy is a term life insurance. There is no bar above $A$, because the policy is discrete rather than continuous. Also, as the criterion that the benefit is either 0 or 1 is satisfied, $E(Z^2)$ is the same as $E(Z)$ evaluated at $2\delta$, and $\text{Var}(Z) = 2A^{1}_{x\bar{n}} - A^{1}_{x\bar{n}}^2$. 
Level Benefit Endowment Insurance

Recall that an $n$-year endowment insurance is just a combination of an $n$-year term life insurance and an $n$-year pure endowment. Hence, we have

$$E(Z) = A^1_{x|n} + A^1_{x|n}. $$

We denote $E(Z)$ by $A_{x|n}$. The criterion that the benefit is either 0 or 1 is satisfied, so $E(Z^2)$ is the same as $E(Z)$ evaluated at $2\delta$, and $\text{Var}(Z) = A^2_{x|n} - A_{x|n}^2$.

Level Benefit Deferred Whole Life Insurance

An $n$-year deferred whole life insurance does not cover the first $n$ years from now, but will provide coverage for life thereafter. It is not difficult to see that $E(Z)$ for a unit-benefit $n$-year deferred whole life insurance is given by

$$E(Z) = \sum_{k=n}^{\infty} v^{k+1} kP_x q_{x+k}. $$

We denote $E(Z)$ by $A_{n|n}$. As before, the subscript $n|n$ indicates the coverage is deferred by $n$ years. As with the previous two policies, the criterion that the benefit is either 0 or 1 is satisfied, so $E(Z^2)$ is the same as $E(Z)$ evaluated at $2\delta$, and $\text{Var}(Z) = A^2_{n|n} - A_{n|n}^2$.

Non-Level-Benefit Insurances

We can also construct discrete non-level-benefit insurances. For example, we can construct a whole life insurance policy with a benefit that is initially $1$ and increases by an amount of $1$ annually. Assume that the benefit is payable at the end of the year of death, the actuarial present value of the benefit is given by

$$(IA)_x = \sum_{k=0}^{\infty} (k+1)v^{k+1} kP_x q_{x+k}. $$
In the symbol, there is no bar above $I$, because the benefit is increasing annually rather than continuously; there is no bar above $A$, because the policy is discrete, i.e., the benefit is not payable precisely at the moment of death.

The following table summarizes the formulas for various discrete life insurances.

### Discrete Insurances

<table>
<thead>
<tr>
<th>Policy</th>
<th>Actuarial present value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Notation</td>
</tr>
<tr>
<td>Whole life</td>
<td>$A_x$</td>
</tr>
<tr>
<td>$n$-year term life</td>
<td>$A_{x\mid n}$</td>
</tr>
<tr>
<td>$n$-year endowment</td>
<td>$A_{x\mid n}$</td>
</tr>
<tr>
<td>$n$-year deferred whole life</td>
<td>$n/A_x$</td>
</tr>
<tr>
<td>Annually increasing whole life</td>
<td>$(IA)_x$</td>
</tr>
<tr>
<td>Annually increasing $n$-year term life</td>
<td>$(IA)_{x\mid n}$</td>
</tr>
<tr>
<td>Annually increasing $n$-year endowment</td>
<td>$(IA)_{x\mid n}$</td>
</tr>
<tr>
<td>Annually decreasing $n$-year term life</td>
<td>$(DA)_{x\mid n}$</td>
</tr>
</tbody>
</table>
Example 3.6  [Structural Question]

You are given:

(i) The following life table:

<table>
<thead>
<tr>
<th></th>
<th>90</th>
<th>91</th>
<th>92</th>
<th>93</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>90</td>
<td>91</td>
<td>92</td>
<td>93</td>
</tr>
<tr>
<td>$l_x$</td>
<td>100</td>
<td>72</td>
<td>39</td>
<td>0</td>
</tr>
<tr>
<td>$d_x$</td>
<td>28</td>
<td>33</td>
<td>39</td>
<td>-</td>
</tr>
</tbody>
</table>

(ii) $i = 0.06$

Calculate the following:

(a) $A_{90}$

(b) $A_{90:T_1}^1$

(c) $A_{90:T_1}$

(d) $v_{90}$

(e) $(IA)_{90}$

---

**Solution**

(a) First of all, note that $\omega = 93$, since $l_{92} > 0$ and $l_{93} = 0$. We have

\[
A_{90} = \sum_{k=0}^{93-90-1} v^{k+1} \cdot kP_{90} q_{90+k} = \sum_{k=0}^{2} v^{k+1} \cdot \frac{d_{90+k}}{l_{90}} = \frac{1}{1.06^{100}} \cdot 28 + \frac{1}{1.06^{2}} \cdot 33 + \frac{1}{1.06^{3}} \cdot 39 = 0.885301.\]

(b) There is only one term in the expression for $A_{90:T_1}^1$:

\[
A_{90:T_1}^1 = \frac{1}{1.06^{100}} \cdot 28 = 0.264151.\]

(c) We use the equation $A_{90:T_1} = A_{90:T_1}^1 + A_{90:T_1}^1$. From (b), we have $A_{90:T_1}^1 = 0.264151$.

Also, we have $A_{90:T_1}^1 = v_{90} = \frac{1}{1.06^{100}} \cdot 72 = 0.679245$.

As a result, $A_{90:T_1} = 0.943396$. 

---
(d) We have

\[
\mathbf{l}_0^k A_{90} = \sum_{k=1}^{93-90-1} v^{k+1} k P_{90} q_{90+k}
\]

\[
= \sum_{k=1}^{2} v^{k+1} \frac{d_{90+k}}{l_{90}}
\]

\[
= \frac{1}{1.06^2} + \frac{1}{1.06^3} = 0.621150.
\]

(e) \( (LA)_{90} = \sum_{k=0}^{2} (k+1)v^{k+1} \frac{d_{90+k}}{l_{90}} = \frac{1}{1.06} + \frac{2}{1.06^2} + \frac{3}{1.06^3} = 1.83390 \).

Sometimes, you may be asked to work on a “special” policy, of which the death benefit has an irregular pattern. The following two examples demonstrate how problems involving “special” policies can be solved.

**Example 3.7 [Course 3 Spring 2001 #17]**

For a special 3-year term life insurance on \((x)\), you are given:

(i) \( Z \) is the present value random variable for the death benefits.

(ii) \( q_{x+k} = 0.02(k + 1), k = 0, 1, 2 \)

(iii) The following death benefits, payable at the end of the year of death:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( b_{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>300,000</td>
</tr>
<tr>
<td>1</td>
<td>350,000</td>
</tr>
<tr>
<td>2</td>
<td>400,000</td>
</tr>
</tbody>
</table>

(iv) \( i = 0.06 \)

Calculate \( E(Z) \).

(A) 36,800  (B) 39,100  (C) 41,400  (D) 43,700  (E) 46,000

**Solution**

This is not a standard policy, but we can still solve this problem by using the basic concepts. This is a 3-year term life insurance, which means that there is no payout if death occurs after three years from now. Hence, we need to consider three cases only.
If death occurs during the first year, the death benefit is 300,000, and the associated probability is \( q_x = 0.02 \).

If death occurs during the second year, the death benefit is 350,000, and the associated probability is \( p_x q_{x+1} = (1 - 0.02)(0.04) = 0.0392 \).

If death occurs during the third year, the death benefit is 400,000, and the associated probability is \( 2p_x q_{x+2} = (1 - 0.02)(1 - 0.04)(0.06) = 0.056448 \).

As a result, the actuarial present value is given by

\[
E(Z) = \frac{300000}{1.06} \times 0.02 + \frac{350000}{1.06^2} \times 0.0392 + \frac{400000}{1.06^3} \times 0.056448 = 36829.06
\]

For a special whole life insurance policy issued on (40), you are given:

(i) Death benefits are payable at the end of the year of death.

(ii) The amount of death benefit is 2 if death occurs within the first 20 years and is 1 thereafter.

(iii) \( Z \) is the present value random variable for the payments under this insurance.

(iv) \( i = 0.03 \)

(v)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( A_x )</th>
<th>( 20E_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.36987</td>
<td>0.51276</td>
</tr>
<tr>
<td>60</td>
<td>0.62567</td>
<td>0.17878</td>
</tr>
</tbody>
</table>

(vi) \( E[Z^2] = 0.24954 \)

Calculate the standard deviation of \( Z \).

(A) 0.27  (B) 0.32  (C) 0.37  (D) 0.42  (E) 0.47

**Solution**

The standard deviation of \( Z \) is the square root of \( \text{Var}(Z) \). We can always calculate \( \text{Var}(Z) \) by using the formula \( \text{Var}(Z) = E(Z^2) - [E(Z)]^2 \). Statement (vi) gives us the value of \( E(Z^2) \), so all then that remains is to compute \( E(Z) \).
According to statement (ii), for this special whole life insurance, the amount of death benefit is 2 if death occurs within the first 20 years and is 1 thereafter. We can therefore decompose this special policy as a level-benefit whole life insurance with a benefit of 2 less a level-benefit 20-year deferred whole life insurance with a benefit of 1. It follows that

$$E(Z) = 2A_{40} - 20A_{40} = 2 \times 0.36987 - 0.51276 \times 0.62567 = 0.41892.$$ 

This gives

$$\text{Var}(Z) = 0.24954 - 0.41892^2 = 0.074046.$$ 

Finally, the standard deviation of $Z$ is given by $\sqrt{0.074046} = 0.27211$, which corresponds to option (A).

Although the question does not require you to write down the specification of $Z$, we would like to point out that the correct specification of $Z$ is given by

$$Z = \begin{cases} 
2v^{K_{40}+1} & K_{40} = 0, 1, \ldots, 19 \\
v^{K_{40}+1} & K_{40} = 20, 21, \ldots 
\end{cases}$$

Example 3.9 [Structural Question]

Assume $i = 0.06$. You are given the following extract from a life table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$d_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>10169</td>
<td>1897</td>
</tr>
<tr>
<td>71</td>
<td>8272</td>
<td>1654</td>
</tr>
<tr>
<td>72</td>
<td>6618</td>
<td>1414</td>
</tr>
<tr>
<td>73</td>
<td>5204</td>
<td>1185</td>
</tr>
<tr>
<td>74</td>
<td>4019</td>
<td>972</td>
</tr>
<tr>
<td>75</td>
<td>3047</td>
<td>780</td>
</tr>
</tbody>
</table>

(a) Calculate $A_{70:4}$.

(b) Calculate $(IA)_{70:4}$.

(c) The standard deviation of the present value of a 4-year term life insurance, issued to $(70)$, with sum insured $\$10000$ payable at the end of the year of death.

(d) The probability that the present value of the term insurance in (c) is greater than 0.85.
### Solution

(a) 

\[
A_{70\bar{4}} = A_{70\bar{3}} + A_{70\bar{4}}
\]

\[
= \sum_{k=0}^{3} v^{k+1} q_{70} + v^{4} p_{70}
\]

\[
= \sum_{k=0}^{3} v^{k+1} \frac{d_{70+k}}{l_{70}} + v^{4} \frac{l_{74}}{l_{70}}
\]

\[= 0.52980 + 0.31305 = 0.84285.\]

(b) First, we compute the increasing term life insurance component:

\[
(IA)_{70\bar{3}} = \sum_{k=0}^{3} (k+1) v^{k+1} q_{70} = \sum_{k=0}^{3} (k+1) v^{k+1} \frac{d_{70+k}}{l_{70}} = 1.18497.
\]

Then we include the endowment component. Note that by the time when the endowment
benefit is paid, the benefit amount will have increased to 4.

\[
(IA)_{70\bar{4}} = (IA)_{70\bar{3}} + 4A_{70\bar{4}}
\]

\[= 1.18497 + 4 \times 0.31305 = 2.43717.\]

(c) Let \(Z\) be the present value random variable for a 4-year term life insurance of \$1\ issued to
(70). We have

\[
\text{Var}(Z) = 2A_{70\bar{3}} + (A_{70\bar{4}})^{2} = \sum_{k=0}^{3} v^{2(k+1)} q_{70} - 0.52980^{2} = 0.18531.
\]

[Note that at two times the original force of interest, the one period discount factor is
squared.]

Hence, the standard deviation of the present value of a 4-year term life insurance, issued to
(70), with sum insured \$10000\ payable at the end of the year of death is given by

\[10000\sqrt{0.18531} = 4304.8.\]

(d) 

\[
\text{Pr}(10000v^{K_{70}+1} > 8500)
\]

\[= \text{Pr}(\ln v(K_{70} + 1) > \ln 0.85)
\]

\[= \text{Pr}(K_{70} < 1.7891)
\]

\[= \text{Pr}(K_{70} = 0) + \text{Pr}(K_{70} = 1)
\]

\[= \frac{d_{70} + d_{71}}{l_{70}} = 0.34920.\]

There are only two possible values of \(K_{70}\) that are smaller than 1.7891.
3.3 mthly Life Insurances

Instead of assuming that the death benefit is payable at the end of the year of death, we may assume that it is payable at the end of the month of death or even the week of death. To achieve this goal, we require the following random variable:

\[ K_{x}^{(m)} = \frac{1}{m} \left[ mT_{x} \right] \cdot \]

This special random variable means that we divide one year into \( m \) equal fractions, and round the future lifetime random variable down to the nearest \( 1/m \) of a year. Consider the following example:

In this example, \( K_{x} = 21 \), \( K_{x}^{(2)} = 21 \frac{1}{2} \) and \( K_{x}^{(4)} = 21 \frac{3}{4} \). Of course, \( K_{x}^{(m)} \) is a discrete random variable. The probability function for \( K_{x}^{(m)} \) is given by

\[ \Pr(K_{x}^{(m)} = k) = \Pr \left( k \leq T_{x} < k + \frac{1}{m} \right) = q_{\frac{k}{m}}, \quad \text{for } k = 0, \frac{1}{m}, \frac{2}{m}, \ldots. \]

Let us consider an \( m \)thly whole life insurance of $1 on \( x \). The payout from the policy is illustrated diagrammatically as follows.

The benefit is payable at time \( K_{x}^{(m)} + \frac{1}{m} \). So, the present value random variable is \( Z = v^{K_{x}^{(m)} + \frac{1}{m}} \).

The actuarial present value \( E(Z) \) is denoted by \( A_{x}^{(m)} \), and can be calculated as follows:
Similarly, the APV of a unit-benefit $m$thly $n$-year term life insurance can be expressed as

$$A_{x}^{m(m)} = \sum_{k=0}^{mn-1} \frac{q_{x}^{m}}{v^{km}} + \sum_{k=0}^{mn-1} \frac{q_{x+1}^{m}}{v^{km}} + \ldots$$

Note that the upper limit of the summation is $mn - 1$, where $mn$ is the total number of time intervals (each has a length of $1/m$) encompassed by the duration of the policy.

Likewise, the APV of a unit-benefit $m$thly $n$-year endowment insurance can be calculated using the following formula:

$$A_{x;m}^{1(m)} = \sum_{k=0}^{mn-1} \frac{q_{x}^{m}}{v^{km}} + \sum_{k=0}^{mn-1} \frac{q_{x+1}^{m}}{v^{km}} + q_{x}^{n} p_{x}.$$ 

Let us study the following example.

**Example 3.10 [MLC Fall 2012 #29]**

For a special 2-year term insurance policy on $(x)$, you are given:

(i) Death benefits are payable at the end of the half-year of death.

(ii) The amount of death benefit is 300,000 for the first half-year and increases by 30,000 per half-year thereafter.

(iii) $q_{x} = 0.16$ and $q_{x+1} = 0.23$.

(iv) $i^{(2)} = 0.18$

(v) Deaths are assumed to follow a constant force of mortality between integral ages.

(vi) $Z$ is the present value random variable for this insurance.

Calculate $\Pr(Z > 277,000)$.

(A) 0.08  (B) 0.11  (C) 0.14  (D) 0.18  (E) 0.21
Using the terminologies defined earlier, this is a special $m$thly 2-year term life insurance, with $m = 2$.

Obviously, the value of $Z$ depends on the underlying random variable, $K_x^{(2)}$. When $K_x^{(2)} = 0, 0.5, 1$ or $1.5$, a death benefit will be paid and $Z$ will take a non-zero value; otherwise, $Z = 0$.

The required probabilities are computed as follows:

\[
\begin{align*}
\text{Pr}(K_x^{(2)} = 0) &= 0.5q_x = 1 - 0.5p_x = 1 - 0.84^{1/2} = 0.083485, \\
\text{Pr}(K_x^{(2)} = 0.5) &= 0.50.5q_x = q_x - 0.5q_x = 0.16 - 0.083485 = 0.076515, \\
\text{Pr}(K_x^{(2)} = 1) &= 1.050.5q_x = p_x0.5q_{x+1} = 0.84 \times (1 - 0.77^{1/2}) = 0.102903, \\
\text{Pr}(K_x^{(2)} = 1.5) &= 1.50.5q_x = 1.5p_x - 2p_x = 0.84 \times 0.77^{1/2} - 0.84 \times 0.77 = 0.090297, \\
\text{Pr}(K_x^{(2)} \geq 2) &= 2p_x = 0.84 \times 0.77 = 0.6468.
\end{align*}
\]

The value of $Z$ and the corresponding probabilities are shown in the following table:

<table>
<thead>
<tr>
<th>$K_x^{(2)}$</th>
<th>$Z$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>300000 / 1.09 = 275229.4</td>
<td>0.083485</td>
</tr>
<tr>
<td>0.5</td>
<td>330000 / 1.092 = 277754.4</td>
<td>0.076515</td>
</tr>
<tr>
<td>1</td>
<td>360000 / 1.093 = 277986.1</td>
<td>0.102903</td>
</tr>
<tr>
<td>1.5</td>
<td>390000 / 1.094 = 276285.8</td>
<td>0.090297</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>0</td>
<td>0.6468</td>
</tr>
</tbody>
</table>

From the table, it is clear that

\[
\text{Pr}(Z > 277000) = 0.076515 + 0.102903 = 0.179418.
\]

The answer is (D).

Just like discrete life insurances, we have the useful property that $E(Z^2)$ is the same as $E(Z)$ calculated at $2\delta$, provided that the benefit is either 0 or 1. Hence, for an $m$thly payable unit-benefit whole life insurance, we have $E(Z^2) = A_x^{(m)}$ and $\text{Var}(Z) = A_x^{(m)} - (A_x^{(m)})^2$. 

[ END ]
Chapter 3: Life Insurances

3.4 Relating Different Policies

It is important to know how various level-benefit policies are related to one another. There are three equations that you need to remember.

Equation 1

By definition, an \( n \)-year endowment insurance is a combination of an \( n \)-year term life insurance (which provides a death benefit) and an \( n \)-year pure endowment (which provides a survival benefit). Hence, we have the following equation:

\[
\overline{A}_{x\mid n} = \overline{A}_{x\mid n}^{\dagger} + \overline{a}_{x\mid n}^{\dagger}.
\]

[You have seen this equation in previous sections.]

Equation 2

Let us consider two policies.

(i) An \( n \)-year term life insurance:

(ii) An \( n \)-year deferred whole life insurance:

The combination of these two policies gives us the following coverage:
Chapter 3: Life Insurances

The above is precisely the coverage provided by a whole life insurance. As a result, we have the following equation:

$$
\overline{A}_x = \overline{A}_{x \mid \text{if}} + n \overline{A}_x .
$$

**Equation 3**

Suppose that you are now $x$ years old and that you want your life insurance coverage to begin $n$ years from now. You have two options.

- The first option is to purchase at time 0 an $n$-year deferred whole life insurance for $\overline{A}_{n \mid x \mid}$ amount of money.
- The second option is to do nothing now, and then purchase a whole life insurance at time $n$ (if you survive to time $n$). The amount that you need to pay at time $n$ will be $\overline{A}_{x+n}$, because at that time your age will be $x + n$. Therefore, at time 0, the expected present value of the cost associated with this option is $v^n p_x x + \overline{A}_{x+n}$.

Since the two options give you exactly the same coverage, we have the following equation:

$$
\overline{A}_{n \mid x} = v^n p_x \overline{A}_{x+n} .
$$

The logics behind the three equations are valid no matter if the underlying policies are continuous, discrete or mthly. So, we have parallel equations for discrete insurances and mthly insurances. The equations are summarized in the following table.

<table>
<thead>
<tr>
<th>Relations between Level-Benefit Policies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Continuous</strong></td>
</tr>
<tr>
<td>Equation 1</td>
</tr>
<tr>
<td>Equation 2</td>
</tr>
<tr>
<td>Equation 3</td>
</tr>
</tbody>
</table>
We call $v^n_p\mathbb{X}_x$ the actuarial discount factor, and denote it by $nE_x$. It discounts a cash flow over a period of $n$ years, incorporating both survivorship and the time value of money. In Section 3.1, we demonstrated that $A_x^{\perp \bar{\pi}}$, the APV of a pure endowment, is also $v^n_p\mathbb{X}_x$. Hence, we have the following triangular relationship:

You will find this triangular relationship extremely useful. To help you better understand how this triangular relation can be applied, we are going to mark all calculations involving this relation with the marker $\triangle$.

**Example 3.11 [Structural Question]**

Consider the following two present value random variables:

$$Z_1 = \begin{cases} 
10000v^{T_x}, & T_x \leq 15 \\
20000v^{15}, & T_x > 15 
\end{cases}$$

$$Z_2 = \begin{cases} 
0, & T_x \leq 5 \\
10000v^{T_x}, & 5 < T_x \leq 15 \\
10000v^{15}, & T_x > 15 
\end{cases}$$

(a) Describe the insurance policies represented by $Z_1$ and $Z_2$.

(b) Express $E(Z_1)$ and $E(Z_2)$ using actuarial symbols.

(c) You are given that, at an effective rate of interest of 6% per year, $\bar{A}_x = 0.166117$, $\bar{A}_{x+5} = 0.20718$, $\bar{A}_{x+15} = 0.314208$.

You are also given that $l_x = 93132$, $l_{x+5} = 91641$ and $l_{x+15} = 86409$.

Calculate $E(Z_1)$ and $E(Z_2)$. 

Solution

(a) $Z_1$: The insurance policy pays a death benefit of 10,000 at the moment of death if death occurs within the first 15 years, and a survival benefit of 20,000 at the end of year 15 if the policyholder survives to that time.

$Z_2$: The insurance policy pays nothing if death occurs within the first 5 years, a death benefit of 10,000 at the moment of death if death occurs between ages $x + 5$ and $x + 15$, and a survival benefit of 10,000 at age $x + 15$ if the policyholder survives to age $x + 15$.

(b) For $Z_1$, the term life insurance component has an APV of $10000 \overline{A}_{x+15}^1$, while the pure endowment component has an APV of $20000 \overline{A}_{x+15}^1$. Therefore,

$$E(Z_1) = 10000 \overline{A}_{x+15}^1 + 20000 \overline{A}_{x+15}^1.$$ 

For $Z_2$, the (deferred) term life insurance component has an APV of $10000(\overline{A}_{x+15}^1 - \overline{A}_{x+5}^1)$. You need to subtract $\overline{A}_{x+5}^1$ because there is no coverage for the first five years. The pure endowment component, on the other hand, has an APV of $10000 \overline{A}_{x+15}^1$. As a result,

$$E(Z_2) = 10000(\overline{A}_{x+15}^1 - \overline{A}_{x+5}^1 + \overline{A}_{x+15}^1).$$

[Note: You may denote the deferred term life insurance component by a single notation as $\overline{A}_{x+10}^1$.]

(c) First, we have

$$\overline{A}_{x+15}^1 = v_{x+5}^{15} p_x = v_{x+5}^{15} l_{x+15} = \frac{1}{1.06^{15}} \frac{86409}{98132} = 0.387144.$$ 

Second, we have

$$\overline{A}_{x+15}^1 - \overline{A}_{x+5}^1 = \overline{A}_x - v_{x+5}^{15} p_x \overline{A}_{x+5} = 0.166117 - 0.387144 \times 0.314208 = 0.044473.$$ 

Third, we have

$$\overline{A}_{x+5} = \overline{A}_x - 5\overline{A}_x = \overline{A}_x - v^5 p_x \overline{A}_{x+5}$$

$$= 0.166117 - \frac{1}{1.06^5} \frac{91641}{93132} = 0.01378.$$ 

This gives $E(Z_1) = 8187.61$ and $E(Z_2) = 4178.38$. 

[END]
Example 3.12

You are given:
(i) \( \overline{A}_{20} = 0.35 \)
(ii) \( \overline{A}_{40} = 0.55 \)
(iii) \( \overline{A}_{20:20} = 0.485 \)

Find \( A_{20:20} \) and \( A_{20:20} \).

--- Solution ---

First, we have
\[
\overline{A}_{20} = \overline{A}_{20:20} + 20 \overline{A}_{20} = \overline{A}_{20:20} + v^{20}_20 p_{20} A_{40} .
\]

Since \( \overline{A}_{20} = 0.35 \), \( \overline{A}_{40} = 0.55 \) and \( A_{20:20} = v^{20}_20 p_{20} \), we have
\[
0.35 = \overline{A}_{20:20} + 0.55 A_{20:20}.
\]

Second, we have \( \overline{A}_{20:20} = 0.485 = \overline{A}_{20:20} + A_{20:20} \).

Solving the two equations simultaneously, we obtain \( A_{20:20} = 0.3 \) and \( A_{20:20} = 0.185 \).

--- END ---

Example 3.13 [Structural Question SoA Sample #7]

For a special deferred term insurance on (40) with death benefits payable at the end of the year of death, you are given:

(i) The death benefit is 0 in years 1-10; 1000 in years 11-20; 2000 in years 21-30; 0 otherwise.
(ii) Mortality follows the Illustrative Life Table.
(iii) \( i = 0.06 \)
(iv) The random variable \( Z \) is the present value, at age 40, of the death benefits.
(a) Write an expression for $Z$ in terms of $K_{40}$, the curtate-time-until-death random variable.
(b) Calculate $\Pr(Z = 0)$.
(c) Calculate $\Pr(Z > 400)$.
You are also given that $E(Z) = 107$.
(d) Show that $\text{Var}(Z) = 36,000$ to the nearest 1,000.
(e) (i) Using the normal approximation without continuity correction, calculate $\Pr(Z > 400)$.
(ii) Explain why your answer to (c) is quite different from your answer to (e part i).

**Solution**

(a) If the life dies in the first 10 years (i.e., $K_{40} < 10$), then the benefit is 0.
   If the life dies in year 11 to 20 (i.e., $10 \leq K_{40} < 20$), then the benefit is 1,000.
   If the life dies in year 21 to 30 (i.e., $20 \leq K_{40} < 30$), then the benefit is 2,000.
   If the life dies after year 30 (i.e., $K_{40} \geq 30$), then the benefit is 0.
   Hence, we can write down $Z$ as follows:
   \[
   Z = \begin{cases} 
   0 & K_{40} < 10 \\
   1000/1.06^{K_{40}+1} & 10 \leq K_{40} < 20 \\
   2000/1.06^{K_{40}+1} & 20 \leq K_{40} < 30 \\
   0 & K_{40} \geq 30 
   \end{cases}
   \]

(b) From (a), we know that $Z = 0$ when $K_{40} < 10$ or $K_{40} \geq 30$. Therefore,
   \[
   \Pr(Z = 0) = \Pr(K_{40} < 10) + \Pr(K_{40} \geq 30)
   \]
   \[
   = \frac{l_{40}}{l_{40}} - \frac{l_{50}}{l_{40}} + \frac{l_{70}}{l_{40}}
   \]
   \[
   = \frac{9,313,166 - 8,950,901}{9,313,166} + \frac{6,616,155}{9,313,166} \approx 0.75
   \]

(c) This is a difficult part. First, note that it is impossible to have $Z > 400$ when $K_{40} < 10$ or $K_{40} \geq 30$. Hence, let us focus on the remaining two cases.

Case I: $10 \leq K_{40} < 20$
   \[
   Z > 400 \iff 1000/1.06^{K_{40}+1} > 400 \iff K_{40} < 14.73 \iff K_{40} \leq 14
   \]
   This means that in this case, $Z > 400$ when $10 \leq K_{40} \leq 14$. 
Case II: $20 \leq K_{40} < 30$

$$Z > 400 \iff 2000/1.06^{K_{40}+1} > 400 \iff K_{40} < 26.62 \iff K_{40} \leq 26$$

This means that in this case, $Z > 400$ when $20 \leq K_{40} \leq 26$

Overall, we have

$$\Pr(Z > 400) = \Pr(10 \leq K_{40} \leq 14) + \Pr(20 \leq K_{40} \leq 26)$$

$$= \frac{l_{50} - l_{55}}{l_{40}} + \frac{l_{60} - l_{67}}{l_{40}}$$

$$= \frac{8,950,901 - 8,640,861}{9,313,166} + \frac{8,188,074 - 7,021,365}{9,313,166} \approx 0.1392$$

(d) This is a special policy. To calculate variance, we need to use first principles.

Since we are given $E(Z) = 107$, what we need is to calculate $E(Z^2)$.

It follows from part (a) that $Z^2$ is given by

$$Z^2 = \begin{cases} 
0 & K_{40} < 10 \\
1000^2 / 1.06^{2(K_{40}+1)} & 10 \leq K_{40} < 20 \\
2000^2 / 1.06^{2(K_{40}+1)} & 20 \leq K_{40} < 30 \\
0 & K_{40} \geq 30 
\end{cases}$$

We can regard $Z^2$ as the present value random variable for the combination of a 10-year deferred 10-year term life insurance with a benefit of $1000^2$ and a 20-year deferred 10-year term life insurance with a benefit of $2000^2$. Both policies are evaluated at two times the original force of interest.

For the 10-year deferred 10-year term life insurance with a benefit of $1000^2$:

$$\text{APV} = 1000^2 \left(10 A_{40}^{\mathsfit{10}} - 20 A_{40}^{\mathsfit{20}}\right)$$

$$= 1000^2 \left(10 E_{40} v^{10} A_{50}^{\mathsfit{10}} - 20 E_{40} v^{20} A_{60}^{\mathsfit{20}}\right)$$

$$= 1000^2 \left(0.53667 \times 1.06^{-10} \times 0.99476 - 0.27414 \times 1.06^{-20} \times 0.17741\right)$$

$$= 13232.40$$

A 10-year deferred 10-year term life insurance can be decomposed into the difference between a 10-year deferred whole life insurance and a 20-year whole life insurance. Draw a diagram yourself to verify!

When doubling the force of interest, $v^{10}$ becomes $v^{20}$ and $10 E_{40}$ becomes $10 E_{40} v^{10}$. 
For the 20-year deferred 10-year term life insurance with a benefit of $2000^2$:

\[
\text{APV} = 2000^2 (20^{2} A_{40} \cdot 30^{2} A_{40})
\]

\[
= 2000^2 (20^{2} E_{40} \times 30^{3} A_{40})
\]

\[
= 2000^2 (0.27414 \times 1.06^{-20} \times 0.17741 - 0.27414 \times 0.45120 \times 1.06^{-30} \times 0.30642)
\]

\[
= 34262.44
\]

Therefore, \( E(Z^2) = 13232.4 + 34262.44 = 47494.84 \), and finally

\[
\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = 47494.84 - 107^2 = 36045.84 \approx 36000.
\]

(e) (i) Using a normal approximation, \( \frac{Z - E(Z)}{\sqrt{\text{Var}(Z)}} = \frac{Z - 107}{\sqrt{36045.84}} \) follows a standard normal distribution. Now, \( Z > 400 \) means

\[
\frac{Z - 107}{\sqrt{36045.84}} > \frac{400 - 107}{\sqrt{36045.84}} = 1.543.
\]

Hence, the required probability is \( 1 - \Phi(1.54) = 1 - 0.9382 = 0.0618 \).

(ii) The normal approximation is almost never a good approximation for a single observation, unless the underlying distribution is close to normal. Here, the underlying distribution, with 75% probability of 0, is far from normal.

[ END ]

### 3.5 Recursions

Recursions are examined extensively in Exam MLC. First of all, let us consider recursions for level-benefit insurances. The recursion for a discrete whole life insurance is given by

\[
A_x = v q_x + v p_x A_{x+1}.
\]

The meaning behind this formula can be seen from the following diagram:
The insured could die during the first year (the interval from age $x$ to age $x + 1$) with probability $q_x$. A death benefit $1 would be made at year end. The APV at time 0 in this case is $v q_x$.

The insured could survive the first year with probability $p_x$. At time 1 (i.e., age $x + 1$), the insured would still have a whole life policy with a value of $A_{x+1}$. The APV at time 0 in this case is $v p_x A_{x+1}$.

Using a similar reasoning, we can obtain recursions for an $n$-year term life insurance,

$$A^{(n)}_{x,n/m} = v q_x + v p_x A^{(1)}_{x+1,n/m}$$

and for an $n$-year endowment insurance,

$$A^{(n)}_{x,n/m} = v q_x + v p_x A^{(1)}_{x+1,n/m}$$

Recall that if the benefit is either 0 or 1, then $E(Z^2)$ is just $E(Z)$ evaluated at $2\delta$. Further, at $2\delta$, the one-period discount factor becomes $e^{-2\delta} = v^2$. Hence, we have the following recursions:

$$A^{(2)}_x = v^2 q_x + v^2 p_x (A^{(2)}_{x+1})$$

$$A^{(2)}_{x,n/m} = v^2 q_x + v^2 p_x (A^{(2)}_{x+1,n/m})$$

For mthly insurances, each time step is $1/m$ of a year. We can modify the above recursions accordingly to obtain recursions for mthly insurances. For example, we have

$$A^{(m)}_x = \frac{1}{m} q_x + \frac{1}{m} p_x A^{(m)}_{x+1}$$
Example 3.14

For a whole life insurance of $1 on (41) with death benefit payable at the end of year of death, you are given:

(i) \( i = 0.05 \)

(ii) \( p_{40} = 0.9972 \)

(iii) \( A_{41} - A_{40} = 0.00822 \)

(iv) \( 2A_{41} - 2A_{40} = 0.00433 \)

(v) \( Z \) is the present value random variable for this insurance.

Calculate \( \text{Var}(Z) \).

**Solution**

First, we have

\[
A_{40} = vq_{40} + vp_{40}A_{41}
\]

\[
\Rightarrow 0.00822 = A_{41} - \left( \frac{0.0028}{1.05} + \frac{0.9972}{1.05} \times A_{41} \right)
\]

\[
\Rightarrow A_{41} = 0.21650.
\]

Second, we have

\[
2A_{40} = v^2q_{40} + v^2p_{40}(2A_{41})
\]

\[
\Rightarrow 0.00433 = 2A_{41} - \left( \frac{0.0028}{1.05^2} + \frac{0.9972}{1.05^2} \times 2A_{41} \right)
\]

\[
\Rightarrow 2A_{41} = 0.07193.
\]

As a result,

\[
\text{Var}(Z) = 2A_{41} - (A_{41})^2 = 0.07193 - 0.21650^2 = 0.02505.
\]

[ END ]
Next, we focus on recursions for non-level benefit insurances. The recursion for an annually decreasing $n$-year term life insurance is given by

$$(DA)_{x+n}^1 = v n q_x + v p_x (DA)_{x+1+n-1}^1.$$ 

To illustrate, let us consider an annually decreasing 5-year term life insurance. The benefit function for this insurance is as follows:

- The insured could die during the first year with probability $q_x$. A death benefit of $5$ would be made at year end. The APV at time 0 in this case is $5v q_x$.
- The insured could survive the first year with probability $p_x$. At time 1 (i.e., age $x + 1$), the insured would still have an annually decreasing 4-year term life insurance with a value of $(DA)_{x+1+4}^1$. The APV at time 0 in this case is $v p_x (DA)_{x+1+3}^1$.
- Overall, we have the relation $(DA)_{x+5}^1 = 5v q_x + v p_x (DA)_{x+1+3}^1$. 

![Diagram showing the benefit function for a 5-year term life insurance with a decreasing benefit over time.](image-url)
The recursion for an annually increasing \( n \)-year term life insurance is given by

\[
(IA)_{x\mid n}^1 = vq_x + vp_x [(IA)_{x+1\mid n-1}^1 + A_{x+1\mid n-1}^1].
\]

To illustrate, let us consider an annually increasing 5-year term life insurance. The benefit function of this insurance is as follows:

- The insured could die during the first year with probability \( q_x \). A death benefit of $1 would be made at year end. The APV at time 0 in this case is \( vq_x \).

- The insured could survive the first year with probability \( p_x \). At time 1 (i.e., age \( x + 1 \)), the insured would have an annually increasing 4-year term life insurance with a value of \( (IA)_{x+1\mid 4}^1 \) plus a unit-benefit 4-year term life insurance with a value of \( A_{x+1\mid 4}^1 \). The APV at time 0 in this case is \( vp_x (A_{x+1\mid 4}^1 + (IA)_{x+1\mid 4}^1) \).

- Overall, we have the relation \( (IA)_{x\mid 5}^1 = vq_x + vp_x ((IA)_{x+1\mid 4}^1 + A_{x+1\mid 4}^1) \).

By using a similar reasoning, we can obtain the following recursion for an annually increasing whole life insurance:

\[
(IA)_x = vq_x + vp_x ((IA)_{x+1} + A_{x+1}).
\]
The following table summarizes the recursions for various insurance policies:

<table>
<thead>
<tr>
<th>Policy</th>
<th>Recursion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level-benefit whole life</td>
<td>$A_x = \nu q_x + \nu p_x A_{x+1}$</td>
</tr>
<tr>
<td>Level-benefit $n$-year term life</td>
<td>$A_{x\mid n}^1 = \nu q_x + \nu p_x A_{x+1\mid n}$</td>
</tr>
<tr>
<td>Level-benefit $n$-year endowment</td>
<td>$A_{x\mid n} = \nu q_x + \nu p_x A_{x+1\mid n}$</td>
</tr>
<tr>
<td>Annually increasing whole life</td>
<td>$(IA)<em>x = \nu q_x + \nu p_x (IA)</em>{x+1} + A_{x+1}$</td>
</tr>
<tr>
<td>Annually increasing $n$-year term life</td>
<td>$(IA)<em>{x\mid n}^1 = \nu q_x + \nu p_x ((IA)</em>{x+1\mid n} + A_{x+1})$</td>
</tr>
<tr>
<td>Annually decreasing $n$-year term life</td>
<td>$(DA)<em>{x\mid n}^1 = \nu q_x + \nu p_x (DA)</em>{x+1\mid n}$</td>
</tr>
</tbody>
</table>

Example 3.15 [Course 3 Fall 2000 #28]

A decreasing term life insurance on $(80)$ pays $(20 - k)$ at the end of the year of death if $(80)$ dies in year $k + 1$ for $k = 0, 1, 2, \ldots, 19$. You are given:

(i) $i = 0.06$

(ii) For a certain mortality table with $q_{80} = 0.2$, the single benefit premium for this insurance is 13.

(iii) For this same mortality table except that $q_{80} = 0.1$, the single benefit premium is $P$.

Calculate $P$.

(A) 11.1  (B) 11.4  (C) 11.7  (D) 12.0  (E) 12.3
Solution

Note that the single benefit premium is the same thing as the actuarial present value.

From Statement (ii), we know that \((DA)_{80:20}^{1} = 13\) for the original table. What we need to do is to find out how this APV is changed if we change \(q_{80}\) from 0.2 to 0.1 (other things equal). There is no information given for a direct calculation, so we use the only other thing we know about decreasing insurances: the recursion relation \((DA)_{80:20}^{1} = 20vq_{80} + vp_{80}(DA)_{81:19}^{1}\).

Note that the term \((DA)_{81:19}^{1}\) is not affected by the change in \(q_{80}\). We will use the recursion for both the original and changed tables.

For the original table, we have

\[
(DA)_{80:20}^{1} = 20vq_{80} + vp_{80}(DA)_{81:19}^{1} \\
13 = 20 \times \frac{0.2}{1.06} + \frac{0.8}{1.06}(DA)_{81:19}^{1}.
\]

This gives \((DA)_{81:19}^{1} = 12.225\).

For the changed table, the value of \((DA)_{81:19}^{1}\) is unchanged. What did change was \(q_{80}\). Now we re-state the recursion, but make the change \(q_{80} = 0.1\). This gives

\[
(DA)_{80:20}^{1} = 20vq_{80} + vp_{80}(DA)_{81:19}^{1} \\
= 20 \times \frac{0.1}{1.06} + \frac{0.9}{1.06} \times 12.225 \\
= 12.267.
\]

[ END ]
3.6 Relating continuous, discrete and monthly insurances

We can relate continuous, discrete and monthly insurances by using a simple adjustment factor. To apply the adjustment factor, the following criteria must be satisfied:
1. There is no endowment component (i.e., survival benefit) in the policy.
2. The benefit within each year is non-varying.

There are two methods of adjustment.
- Using the uniform distribution of death (UDD) assumption, the APV of an monthly insurance can be obtained by multiplying the corresponding discrete (annual) insurance with the adjustment factor $\frac{i}{i^{(m)}}$. [Recall that $i^{(m)} = m[(1 + i)^{1/m} - 1].$]
- Using the accelerated claims approach, the APV of an monthly insurance can be obtained by multiplying the corresponding discrete (annual) insurance with the adjustment factor $\frac{m-1}{(1+i)^{2m}}$.

This leads to the following collection of equations.

<table>
<thead>
<tr>
<th>Formula Table</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Relating monthly and discrete (annual) insurances</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Policy</th>
<th>UDD</th>
<th>Accelerated Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level benefit whole life</td>
<td>$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$</td>
<td>$A_x^{(m)} = (1+i)^{\frac{m-1}{2m}} A_x$</td>
</tr>
<tr>
<td>Level benefit $n$-year term life</td>
<td>$A_{x\mid n}^{(m)} = \frac{i}{i^{(m)}} A_{x\mid n}^{1}$</td>
<td>$A_{x\mid n}^{(m)} = (1+i)^{\frac{m-1}{2m}} A_{x\mid n}^{1}$</td>
</tr>
<tr>
<td>Annually increasing whole life</td>
<td>$(IA^{(m)})_x = \frac{i}{i^{(m)}} (IA)_x$</td>
<td>$(IA^{(m)})_x = (1+i)^{\frac{m-1}{2m}} (IA)_x$</td>
</tr>
<tr>
<td>Annually increasing $n$-year term</td>
<td>$(IA^{(m)})<em>{x\mid n}^1 = \frac{i}{i^{(m)}} (IA)</em>{x\mid n}^1$</td>
<td>$(IA^{(m)})<em>{x\mid n}^1 = (1+i)^{\frac{m-1}{2m}} (IA)</em>{x\mid n}^1$</td>
</tr>
</tbody>
</table>
When \( m \to \infty \), \( i/(m) \to i/\delta \), \( (1 + i)^{m-1} \to (1 + i)^{1/2} \), and an mthly insurance becomes a continuous insurance. Hence, we have the following for continuous insurances:

- Using the UDD assumption, the APV of a continuous insurance can be obtained by multiplying the corresponding discrete (annual) insurance with the adjustment factor \( i/\delta \).
- Using the accelerated claims approach, the APV of a continuous insurance can be obtained by multiplying the corresponding discrete (annual) insurance with the adjustment factor \( (1 + i)^{1/2} \).

This leads to the following collection of equations.

**Formula**

<table>
<thead>
<tr>
<th>Policy</th>
<th>UDD</th>
<th>Accelerated Claims</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level benefit whole life</td>
<td>( \overline{A}_x = \frac{i}{\delta} A_x )</td>
<td>( \overline{A}_x = (1 + i)^{1/2} A_x )</td>
</tr>
<tr>
<td>Level benefit n-year term life</td>
<td>( \overline{A}<em>{x:n} = \frac{i}{\delta} A</em>{x:n} )</td>
<td>( \overline{A}<em>{x:n} = (1 + i)^{1/2} A</em>{x:n} )</td>
</tr>
<tr>
<td>Annually increasing whole life</td>
<td>( (IA)_x = \frac{i}{\delta} (IA)_x )</td>
<td>( (IA)_x = (1 + i)^{1/2} (IA)_x )</td>
</tr>
<tr>
<td>Annually increasing n-year term</td>
<td>( (IA)<em>{x:n} = \frac{i}{\delta} (IA)</em>{x:n} )</td>
<td>( (IA)<em>{x:n} = (1 + i)^{1/2} (IA)</em>{x:n} )</td>
</tr>
</tbody>
</table>

It is important to know that the adjustment factors are not applicable to any insurance with an endowment component. For example, \( \overline{A}_{x:n} \neq \frac{i}{\delta} A_{x:n} \) under UDD. We can find \( \overline{A}_{x:n} \) in two steps as follows:

\[
\overline{A}_{x:n} = \overline{A}_{x} + A_{x:n} \\
\overline{A}_{x:n} = \frac{i}{\delta} A_{x:n} + A_{x:n} \text{ (under UDD).}
\]
Similarly, we find \((\overline{IA})_{x\overline{\uparrow}}\) as follows:

\[
(\overline{IA})_{x\overline{\uparrow}} = (\overline{IA})_{x\overline{\uparrow}}^1 + nA_{x\overline{\uparrow}}^1 \\
= \frac{i}{\delta} (\overline{IA})_{x\overline{\uparrow}}^1 + nA_{x\overline{\uparrow}}^1 \text{ (under UDD)}.
\]

**Example 3.16**

You are given:

(i) \(i = 0.01\)

(ii) \(q_x = 0.05\)

(iii) \(q_{x+1} = 0.08\)

Calculate \(\overline{A}_{x\overline{\uparrow}}\), assuming uniform distribution of deaths for fractional ages.

**Solution**

This is an endowment insurance, so we need to split it into two components (term life and pure endowment) before we can apply the adjustment factor. We have

\[
\overline{A}_{x\overline{\uparrow}} = \overline{A}_{x\overline{\uparrow}}^1 + A_{x\overline{\uparrow}}^1 \\
= \frac{i}{\delta} \overline{A}_{x\overline{\uparrow}}^1 + A_{x\overline{\uparrow}}^1.
\]

First,

\[
A_{x\overline{\uparrow}}^1 = vq_x + v^2 p_x q_{x+1} \\
= \frac{1}{1.01} \times 0.05 + \frac{1}{1.01^2} \times 0.95 \times 0.08 \\
= 0.12400745.
\]

Second,

\[
A_{x\overline{\uparrow}}^1 = v^2 p_x \\
= \frac{1}{1.01^2} \times 0.95 \times 0.92 \\
= 0.856778747.
\]

Third, \(\delta = \ln(1 + i) = 0.009950331\).

Substituting, we obtain \(\overline{A}_{x\overline{\uparrow}} = 0.981405\).
### 3.7 Useful Shortcuts

**Constant Force of Mortality**

If the force of mortality is constant beyond age \( x \), then \( \overline{A}_x = \frac{\mu}{\mu + \delta} \) and \( nE_x = e^{-(\mu + \delta)n} \).

We suggest that you remember the formula for \( \overline{A}_x \) only. Formulas for other APVs can be derived straightforwardly without using integration. For example,

\[
\overline{A}_x \mid_{n} = v^n p_x \overline{A}_{x+n} = e^{-\delta n} e^{-\mu n} \frac{\mu}{\mu + \delta} = \frac{\mu e^{-(\delta + \mu)n}}{\mu + \delta}
\]

and

\[
\overline{A}_x \mid_{x=n} = \overline{A}_x - n\overline{A}_x = \frac{\mu}{\mu + \delta} (1 - e^{-(\delta + \mu)n}).
\]

**De Moivre’s Law**

If \( l_x = \omega - x \) or equivalently \( \mu_x = \frac{1}{\omega - x} \) for \( 0 \leq x < \omega \), then \( \overline{A}_x = \frac{\overline{A}_x}{\omega - x} \) and \( nE_x = \left(1 - \frac{n}{\omega - x}\right)v^n \).

Similarly, \( A_x = \frac{A_x}{\omega - x} \). Also, for \( n \leq \omega - x \), \( \overline{A}_x \mid_{x=n} = \frac{\overline{A}_n}{\omega - x} \), and for if \( n \) is an integer, \( A_x \mid_{x=n} = \frac{A_n}{\omega - x} \).

**The Illustrative Life Table**

Questions in Exam MLC are often based on the Illustrative Life Table. It is therefore important to know how to use it efficiently. Below we have reproduced a portion of the Illustrative Life Table. The actual table has more columns and gives more information, but this portion is all we need for now. The table assumes an interest rate of 6%.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( l_x )</th>
<th>( 1000q_x )</th>
<th>( 1000A_x )</th>
<th>( 1000(\overline{A}_x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>7,533,964</td>
<td>21.32</td>
<td>439.80</td>
<td>236.03</td>
</tr>
<tr>
<td>66</td>
<td>7,373,338</td>
<td>23.29</td>
<td>454.56</td>
<td>249.20</td>
</tr>
<tr>
<td>67</td>
<td>7,201,635</td>
<td>25.44</td>
<td>469.47</td>
<td>262.83</td>
</tr>
<tr>
<td>68</td>
<td>7,018,432</td>
<td>27.79</td>
<td>484.53</td>
<td>276.92</td>
</tr>
</tbody>
</table>
Chapter 3: Life Insurances

The APVs for whole life insurances are provided in the table. For example, we have $1000A_{65} = 439.80$, which means $A_{65} = 0.4398$. However, the APVs for term life insurances are not given. We now illustrate how they can be calculated. As an example, we consider $A^i_{65:3}$.

In principle, we can calculate $A^i_{65:3}$ by using a summation-type formula:

$$A^i_{65:3} = \sum_{k=0}^{3} v^{k+1} k p_{65} q_{65+k}$$

$$= vq_{65} + v^2 p_{65} q_{66} + v^3 p_{65} q_{67}.$$

We are given the values of $q_x$ that we need. The necessary values of $p_x$ are $p_{65} = 1 - q_{65} = 1 - 0.02132 = 0.97868$, and $2p_{65} = p_{65} p_{66} = 0.97868 \times 0.97868 \times 0.02329 = 0.95589$. This gives

$$A^i_{65:3} = \frac{0.02132}{1.06} + \frac{0.97868 \times 0.02329}{1.06^2} + \frac{0.95589 \times 0.02544}{1.06^3} = 0.06082.$$

The use of a summation-type formula is usually slow. To obtain the answer more quickly, we can make use of the relationship

$$A^i_{65:3} = A_{65} - 3A_{65} = A_{65} - v^3 p_{65} A_{68}.$$  

From the table, we have $3p_{65} = \frac{l_{68}}{l_{65}} = \frac{7,018,432}{7,533,964} = 0.93157$. Hence,

$$A^i_{65:3} = 0.43980 - \frac{0.93157}{1.06^3} \times 0.48453 = 0.06082.$$

We have given a short term for this insurance so that the problem can be done in both ways. If the term had been 20, the time saved in calculation would have been more obvious.

To calculate the APV of an endowment insurance with the Illustrative Life Table, we calculate separately the APV of the term life insurance component and the APV for the pure endowment component. For example,

$$A_{65:3} = A^i_{65:3} + A^i_{65:3} = A^i_{65:3} + v^3 p_{65} = 0.06082 + \frac{0.93157}{1.06^3} = 0.84298.$$
Exercise 3

1. You are given:
   (i) \( l_x = 95 - x \) for \( 0 \leq x < 95 \).
   (ii) \( \delta = 0.05 \)

   Find \( \overline{A}_{30} \).

2. You are given:
   (i) \( \mu_x = 0.06 \) for all \( x \geq 0 \)
   (ii) \( \delta = 0.08 \)
   (iii) \( Z^* \) is the present value random variable for a whole life insurance of $1,000 on (25).

   Find \( \text{Var}(Z^*) \).

3. You are given:
   (i) \( \mu_x = \frac{1}{100 - x} \) for \( 0 \leq x < 100 \).
   (ii) \( \delta = 0.06 \)

   Find \( \overline{A}^1_{20:30|} \).

4. You are given:
   (i) \( \mu_x = \frac{1}{110 - x} \) for \( 0 \leq x < 110 \).
   (ii) \( \delta = 0.05 \)

   Find \( A^1_{10:20} \).

5. Using the Illustrative Life Table with \( i = 0.06 \), find \( A^1_{50:50} \).

6. You are given:
   (i) \( p_x = 0.95 \) for \( x = 50, 51, 52 \)
   (ii) \( i = 0.05 \)

   Find \( A^1_{50:51} \).

7. You are given:
   (i) \( A_{60} = 0.630 \)
Chapter 3: Life Insurances

(ii) \( p_{60} = p_{61} = 0.9 \)

(iii) \( i = 0.05 \)

Find \( A_{62} \).

8. Using the Illustrative Life Table with \( i = 0.06 \), find \( A_{40:20} \).

9. For a special whole life insurance policy on (20), you are given:
   (i) The death benefit is payable at the moment of death.
   (ii) The death benefit is \( 1000e^{0.02t} \) if death occurs at time \( t \) from now.
   (iii) \( \mu_x = 0.06 \) for all \( x \geq 0 \).
   (iv) \( \delta = 0.06 \)
   (v) \( Z \) is the present value random variable for this insurance.

Find \( \text{Var}(Z) \).

10. You are given:
   (i) \( (DA)_{40:10}^1 = 5.8 \)
   (ii) \( p_{40} = 0.9 \)
   (iii) \( i = 0.05 \)

Find \( (DA)_{41:5}^1 \).

11. You are given:
   (i) \( \delta_t = \begin{cases} 0.03, & 0 \leq t \leq 10 \\ 0.06, & t > 10 \end{cases} \)
   (ii) \( \mu_{x+t} = \begin{cases} 0.05, & 0 \leq t \leq 10 \\ 0.07, & t > 10 \end{cases} \)

Find \( 1000 \bar{A}_x \).

12. You are given:
   (i) \( A_{60} = 0.585 \)
   (ii) \( A_{61} = 0.605 \)
   (iii) \( q_{60} = q_{61} \)
   (iv) \( i = 0.05 \)

Find \( A_{62} \).
13. You are given:
   (i) \( \mu_x = \mu \) for all \( x \geq 0 \)
   (ii) The force of interest is constant.
   (iii) \( \dot{e}_x = 25 \)
   (iv) \( A_{\infty} = 0.4 \)

Find \( A_{x+\infty} \).

14. (2002 Nov #39) For a whole life insurance of 1 on \((x)\), you are given:
   (i) The force of mortality is \( \mu_{x+t} \).
   (ii) The benefits are payable at the moment of death.
   (iii) \( \delta = 0.06 \)
   (iv) \( A_{x} = 0.60 \)

Calculate the revised actuarial present value of this insurance assuming \( \mu_{x+t} \) is increased by 0.03 for all \( t \) and \( \delta \) is decreased by 0.03.

   (A) 0.5  
   (B) 0.6  
   (C) 0.7  
   (D) 0.8  
   (E) 0.9  

15. (2004 Nov #1) For a special whole life insurance on \((x)\) payable at the moment of death, you are given:
   (i) \( \mu_{x+t} = 0.05 \)
   (ii) \( \delta = 0.08 \)
   (iii) The death benefit at time \( t \) is \( b_t = e^{0.06t}, \ t > 0 \).
   (iv) \( Z \) is the present value random variable for this insurance at issue.

Calculate \( \text{Var}(Z) \).

   (A) 0.038  
   (B) 0.041  
   (C) 0.043  
   (D) 0.045  
   (E) 0.048
16. (CAS Fall 2003 #7) You are given:
   (i) \( i = 5\% \)
   (ii) The force of mortality is constant.
   (iii) \( \overset{\circ}{e}_x = 16 \)

   Calculate \( 20A_x \).
   (A) Less than 0.050
   (B) At least 0.050, but less than 0.075
   (C) At least 0.075, but less than 0.100
   (D) At least 0.100, but less than 0.125
   (E) At least 0.125

17. (2003 Nov #2) For a whole life insurance of 1000 on \((x)\) with benefits payable at the moment of death:
   (i) \( \delta_t = \begin{cases} 
   0.04, & 0 < t \leq 10 \\
   0.05, & t > 10 
\end{cases} \)
   (ii) \( \mu_{x+t} = \begin{cases} 
   0.06, & 0 < t \leq 10 \\
   0.07, & t > 10 
\end{cases} \)

   Calculate the single benefit premium for this insurance.
   (A) 379
   (B) 411
   (C) 444
   (D) 519
   (E) 594

18. (2004 Nov #2) For a group of individuals all age \(x\), you are given:
   (i) 25% are smokers (s); 75% are nonsmokers (ns).
   (ii) \( k \) \( q^{s}_{x+k} \) \( q^{as}_{x+k} \)

   \[
   \begin{array}{c|cc}
   k & q^{s}_{x+k} & q^{as}_{x+k} \\
   \hline
   0 & 0.10 & 0.05 \\
   1 & 0.20 & 0.10 \\
   2 & 0.30 & 0.15 \\
   \end{array}
   \]
   (iii) \( i = 0.02 \)

   Calculate 10,000 \( A^{i}_{x-n} \) for an individual chosen at random from this group.
   (A) 1690
19. (2004 Nov #37) Z is the present value random variable for a 15-year pure endowment of 1 on (x):
   (i) The force of mortality is constant over the 15-year period.
   (ii) \( v = 0.9 \)
   (iii) \( \text{Var}(Z) = 0.065E(Z) \)

Calculate \( q_x \).
   (A) 0.020
   (B) 0.025
   (C) 0.030
   (D) 0.035
   (E) 0.040

20. (2001 May #34) Lee, age 63, considers the purchase of a single premium whole life insurance of 10,000 with death benefit payable at the end of the year of death. The company calculates benefit premiums using:
   (i) mortality based on the Illustrative Life Table,
   (ii) \( i = 0.05 \)

The company calculates contract premiums as 112% of benefit premiums.

The single contract premium at age 63 is 5233. Lee decides to delay the purchase for two years and invests the 5233.

Calculate the minimum annual rate of return that the investment must earn to accumulate to an amount equal to the single contract premium at age 65.
   (A) 0.030
   (B) 0.035
   (C) 0.040
   (D) 0.045
   (E) 0.050

21. (2002 Nov #40) A maintenance contract on a hotel promises to replace burned out light bulbs at the end of each year for three years. The hotel has 10,000 light bulbs. The light bulbs are all new. If a replacement bulb burns out, it too will be replaced with a new bulb.

You are given:
Chapter 3: Life Insurances

(i) For new light bulbs, \( q_0 = 0.10, q_1 = 0.30 \) and \( q_2 = 0.50 \).
(ii) Each light bulb costs 1.
(iii) \( i = 0.05 \)
Calculate the actuarial present value of the contract.
(A) 6700
(B) 7000
(C) 7300
(D) 7600
(E) 8000

22. (CAS Nov 2003 #2) For a special fully discrete life insurance on (45), you are given:
   (i) \( i = 6\% \)
   (ii) Mortality follows the Illustrative Life Table.
   (iii) The death benefit is 1000 until age 65, and 500 thereafter.
   (iv) Benefit premiums of 12.5 are payable at the beginning of each year for 20 years.
Calculate the actuarial present value of the benefit payment.
(A) Less than 100
(B) At least 100, but less than 150
(C) At least 150, but less than 200
(D) At least 200, but less than 250
(E) At least 250

23. (2003 Nov #10) For a sequence, \( u(k) \) is defined by the following recursion formula
   \[ u(k) = \alpha(k) + \beta(k) \times u(k - 1) \text{ for } k = 1, 2, 3, \ldots \]
(i) \( \alpha(k) = -\left(\frac{q_{k-1}}{p_{k-1}}\right) \)
(ii) \( \beta(k) = \frac{1+i}{p_{k-1}} \)
(iii) \( u(70) = 1.0 \)
Which of the following is equal to \( u(40) \)?
(A) \( A_{30} \)
(B) \( A_{40} \)
(C) \( A_{40:30} \)
24. (2000 May #36) A new insurance salesperson has 10 friends, each of whom is considering buying a policy.

(i) Each policy is a whole life insurance of 1000, payable at the end of the year of death.
(ii) The friends are all age 22 and make their purchase decisions independently.
(iii) Each friend has a probability of 0.10 of buying a policy.
(iv) The 10 future lifetimes are independent.
(v) $S$ is the random variable for the present value at issue of the total payments to those who purchase the insurance.
(vi) Mortality follows the Illustrative Life Table.
(vii) $i = 0.06$

Calculate the variance of $S$.

(A) 9,200
(B) 10,800
(C) 12,300
(D) 13,800
(E) 15,400

25. (2005 May #7) $Z$ is the present-value random variable for a whole life insurance of $b$ payable at the moment of death of $(x)$. You are given:

(i) $\delta = 0.04$
(ii) $\mu_{x+t} = 0.02$, $t \geq 0$
(iii) The single benefit premium for this insurance is equal to $\text{Var}(Z)$.

Calculate $b$.

(A) 2.75
(B) 3.00
(C) 3.25
(D) 3.50
(E) 3.75
26. (2005 May #15) For an increasing 10-year term insurance, you are given:
   (i) \( b_{k+1} = 100,000(1+k), \; k = 0, 1, \ldots, 9 \)
   (ii) Benefits are payable at the end of the year of death.
   (iii) Mortality follows the Illustrative Life Table.
   (iv) \( i = 0.06 \)
   (v) The single benefit premium for this insurance on (41) is 16,736.

Calculate the single benefit premium for this insurance on (40).

(A) 12,700  
(B) 13,600  
(C) 14,500  
(D) 15,500  
(E) 16,300

27. (2005 May #38) A group of 1000 lives each age 30 sets up a fund to pay 1000 at the end of the first year for each member who dies in the first year, and 500 at the end of the second year for each member who dies in the second year. Each member pays into the fund an amount equal to the single benefit premium for a special 2-year term insurance, with:
   (i) Benefits:
   \[
   \begin{array}{c|c}
   k & b_{k+1} \\
   \hline
   0 & 1,000 \\
   1 & 500 \\
   \end{array}
   \]
   (ii) Mortality follows the Illustrative Life Table.
   (iii) \( i = 0.06 \)

The actual experience of the fund is as follows:

<table>
<thead>
<tr>
<th>( k )</th>
<th>Interest Rate Earned</th>
<th>Number of Deaths</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.070</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.069</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the difference, at the end of the second year, between the expected size of the fund as projected at time 0 and the actual fund.

(A) 840  
(B) 870  
(C) 900  
(D) 930  
(E) 960
28. For a special 10-year term insurance on \((x)\), you are given:

(i) \(Z\) is the present value random variable for this insurance.

(ii) Death benefits are paid at the moment of death.

(iii) \(\mu_{x+t} = 0.02, \ t \geq 0\)

(iv) \(\delta = 0.08\)

(v) \(b_t = e^{0.03t}\)

Calculate \(\text{Var}(Z)\).

29. (2005 Nov #25) For a special 3-year term insurance on \((x)\), you are given:

(i) \(Z\) is the present value random variable for this insurance.

(ii) \(q_{x+k} = 0.02(k + 1), \ k = 0, 1, 2\)

(iii) The following benefits are payable at the end of the year of death:

<table>
<thead>
<tr>
<th>(k)</th>
<th>(b_{k+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>300</td>
</tr>
<tr>
<td>1</td>
<td>350</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
</tr>
</tbody>
</table>

(iv) \(i = 0.06\)

Calculate \(\text{Var}(Z)\).

(A) 9,600

(B) 10,000

(C) 10,400

(D) 10,800

(E) 11,200

30. (MLC Sample #286) You are given:

(i) The force of mortality follows Gompertz’s law with \(B = 0.000005\) and \(c = 1.2\).

(ii) The annual effective rate of interest is 3%.

Calculate \(A_{0.25}^1\).

(A) 0.1024

(B) 0.1018

(C) 0.1009

(D) 0.1000

(E) 0.0994
31. You are given:
   (i) $X$ is the present value random variable for a fully continuous 35-year term insurance of 7 on (35).
   (ii) $Y$ is the present value random variable for a 25-year deferred, 10-year fully continuous term insurance of 4 on the same life.
   (iii) $E(X) = 2.80$ and $E(Y) = 0.12$
   (iv) $Var(X) = 5.76$ and $Var(Y) = 0.1$
   Calculate $Var(X + Y)$.

32. You are given:
   (i) $i = 0.10$
   (ii) $q_x = 0.04$ and $q_{x+1} = 0.08$
   (iii) Deaths are uniformly distributed over each year of age.
   Calculate $A_{x:2}^{(12)}$.
   (A) 0.104
   (B) 0.108
   (C) 0.112
   (D) 0.115
   (E) 0.119

33. [Structural Question] You are given:
   (i) Deaths are uniformly distributed over each year of age.
   (ii) $i = 0.12$
   (iii) $q_x = 0.1$ and $q_{x+1} = 0.2$
   (a) Calculate $A_{x:2}^{(1)}$.
   (b) Calculate $A_{x:2}^{(3)}$.
   (c) Let $Z = \begin{cases} v^{K_{x}^{Y}+1/3} & T_x < 2 \\ v^2 & T_x \geq 2 \end{cases}$. Calculate the standard deviation of $Z$. 
34. **[Structural Question]** An engineering firm installs a tunnel-surrounded steam pipe system on the campus of ACTEX University and guarantees it for 10 years against a major failure. In the event of failure during this period the contract calls for a 2 million dollar payment to the university. Let $T$ denote the time after installation until a major failure occurs in the system. You are given.

(i) $\delta = 0.05$

(ii) The conditional probability of failure in $(t, t + \Delta t)$ given the survival at time $t$ (for small $\Delta t$) is proportional to $\frac{\Delta t}{60 - t}$, for $0 < t < 60$.

(iii) $E(T) = 20$

Let $Z$ denote the random present value of the possible guarantee payment.

(a) Derive the survival function of $T$.

(b) Calculate the probability that $Z$ is zero.

(c) Calculate the expected present value of the guarantee.

(d) Calculate the 81st percentile of $Z$.

35. **[Structural Question]** The following table gives the survival probabilities of a certain population:

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{x+t}$</td>
<td>0.8</td>
<td>0.9</td>
<td>0.95</td>
<td>0.96</td>
<td>0.95</td>
<td>0.95</td>
<td>0.9</td>
</tr>
</tbody>
</table>

Assume $i = 4\%$.

(a) Calculate the actuarial present value of a term life insurance of $1$ issued to $(x)$, payable at the end of the first policy year if death occurs in the first policy year.

(b) Calculate the actuarial present value of a term life insurance of $3$ issued to $(x + 1)$, payable at the end of the third policy year if death occurs in the third policy year.

(c) An endowment insurance issued to $(x + 2)$, which pays $1$ at the end of the year of death if deaths occur in the first two years, and $2$ at the end of the second policy year if the life survives.

(d) Explain why your answer to part (c) is greater than your answer to part (b).

36. **[Structural Question]** You are given:

$$l_x = \sqrt{144 - x}$$

for $0 \leq x \leq 144$.

(a) Derive an expression for $k p_x$ in terms of $x$ and $k$ for $k \leq 144 - x$.

(b) $A_{50,3}^{1}$ at $i = 6\%$.

(c) $A_{50,3}^{1}$ at $i = 6\%$. 

37. [Structural Question] Prove the following recursion formulas:
(a) \[ A_{x \pi |}^1 = v q_x + v p_x A_{x+1, \pi -1}^1 \]
(b) \[ A_{x \pi |} = v q_x + v p_x A_{x+1, \pi -1} \]
(c) \[ (DA)_{x \pi |}^1 = v n q_x + v p_x (DA)_{x+1, \pi -1}^1 \]

38. [Structural Question] You are given the following select-and-ultimate life table:

<table>
<thead>
<tr>
<th>[x]</th>
<th>( l_{[x]} )</th>
<th>( l_{[x]+1} )</th>
<th>( l_{[x]+2} )</th>
<th>( l_{[x]+3} )</th>
<th>( l_{[x]+4} )</th>
<th>( x + 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[40]</td>
<td>100,000</td>
<td>99,899</td>
<td>99,724</td>
<td>99,520</td>
<td>99,288</td>
<td>44</td>
</tr>
<tr>
<td>[44]</td>
<td>99,120</td>
<td>98,964</td>
<td>98,726</td>
<td>98,429</td>
<td>98,067</td>
<td>48</td>
</tr>
</tbody>
</table>

(a) State the select period.
(b) Calculate \( 4 p_{44} \).
(c) Calculate \( 3|2 q_{[40]+3} \).
(d) Assuming \( i = 8\% \), calculate \( (IA)_{[40]|3}^1 \).

39. [Structural Question] You are given that \( \mu_x = \mu \) for all \( x \geq 0 \).

(a) By considering the fact that \( \overline{A}_x = \frac{\mu}{\mu + \delta} \) under the assumption above, prove the following:
(i) \( \overline{A}_x = \frac{\mu e^{-(\mu+\delta)n}}{\mu + \delta} \)
(ii) \( \overline{A}_{x\pi|}^1 = \frac{\mu (1 - e^{-(\mu+\delta)n})}{\mu + \delta} \)

(b) (i) Determine the value of \( \overline{A}_x \) when \( \delta = 0 \). Verbally explain your answer.
(ii) Determine the value of \( \overline{A}_{x\pi|} \) when \( \delta = 0 \). Verbally explain your answer.
(iii) Determine the value of \( \overline{A}_{x\pi|}^1 \) when \( \mu = 0 \). Verbally explain your answer.

(c) Using the results in part (a), or otherwise, find an expression for \( \text{Var}(Z) \), where \( Z \) is the present value random variable for an \( n \)-year term life insurance on \( (x) \) with a benefit of \$1 payable at the moment of death, in terms of \( \mu, \delta \) and \( n \).
(You are not required to simplify the expression.)
40. [Structural Question] You are given \( i = 0.03 \) and the following select-and ultimate table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( q_x )</th>
<th>( q_{x+1} )</th>
<th>( q_{x+2} )</th>
<th>( x + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>0.01</td>
<td>0.04</td>
<td>0.07</td>
<td>67</td>
</tr>
<tr>
<td>66</td>
<td>0.03</td>
<td>0.06</td>
<td>0.09</td>
<td>68</td>
</tr>
<tr>
<td>67</td>
<td>0.05</td>
<td>0.08</td>
<td>0.12</td>
<td>69</td>
</tr>
</tbody>
</table>

(a) Calculate \( l(66, 1) \).

(b) Calculate \( \bar{A}^1_{[65]+1.3} \).

(c) Explain verbally why \( \bar{A}^1_{[66],3} < \bar{A}^1_{[65]+1.3} \).
Solutions to Exercise 3

1. From Statement (i), we know that mortality follows De Moivre’s law with $\omega = 95$. By the shortcut formula discussed in Section 3.7,

$$
\overline{A}_{30} = \overline{a}_{65} = 1 - e^{-0.05 \times 65}/65 \times 0.05 = 0.2958.
$$

Alternatively, $T_{30}$ is uniformly distributed over $[0, 65)$. Hence, $f_{30}(t) = q_{30} \mu_{30+t} = 1/65$, and

$$
\overline{A}_{30} = \int_0^{65} e^{-0.05t} \frac{1}{65} dt = -\frac{20}{65} \left[ e^{-0.05t} \right]_0^{65} = 20(1 - e^{-3.25})/65 = 0.2958.
$$

2. Let $Z$ be the present value random variable for a whole life insurance of $1$ on $(25)$.

$$
E(Z) = \frac{\mu}{\mu + \delta} = \frac{0.06}{0.06 + 0.08} = \frac{3}{7}.
$$

$$
E(Z^2) = \frac{\mu}{\mu + 2\delta} = \frac{0.06}{0.06 + 0.16} = \frac{3}{11}.
$$

$$
Var(Z) = \frac{3}{11} - \left(\frac{3}{7}\right)^2 = 0.089054.
$$

Since $Z^* = 1000Z$, $Var(Z^*) = 1000^2 Var(Z) = 89054$.

3. Statement (i) implies mortality follows De Moivre’s law with $\omega = 100$. By the shortcut formula discussed in Section 3.7,

$$
\overline{A}_{20,30}^I = \frac{\overline{a}_{30}}{100 - 20} = 1 - e^{-0.06 \times 30}/80 \times 0.06 = 0.1739.
$$

Alternatively, since $T_{20}$ is uniformly distributed over $[0, 80)$. Hence $f_{20}(t) = q_{20} \mu_{20+t} = 1/80$ and

$$
\overline{A}_{20,30}^I = \int_0^{30} e^{-0.06t} \frac{1}{80} dt = -\frac{1}{80 \times 0.06} \left[ e^{-0.06t} \right]_0^{30} = \frac{1}{4.8} (1 - e^{-1.8}) = 0.1739.
$$

4. Statement (i) implies mortality follows De Moivre’s law with $\omega = 110$. This means that $T_{20}$ is uniformly distributed over $[0, 90)$. By the shortcut formula discussed in Section 3.7,

$$
10_0 A_{20} = 10 E_{20} \overline{A}_{30} = \frac{8}{9} e^{-0.05 \times 10} \times \overline{a}_{30}/80 = \frac{8}{9} e^{-0.5} \times 0.24542 = 0.1323.
$$

Alternatively, since $f_{20}(t) = q_{20} \mu_{20+t} = 1/90$, and

$$
10_0 A_{20} = \int_0^{90} e^{-0.05t} \frac{1}{90} dt = -\frac{20}{90} e^{-0.05t} \bigg|_0^{90} = -\frac{20}{90} (e^{-4.5} - e^{-0.5}) = 0.1323.
$$
5. Note that \( A_{50:56}^{1} = A_{50} - v^{6} A_{50} = A_{50} - v^{6} P_{50} A_{56} \).

From the table, \( A_{50} = 0.24905, A_{56} = 0.31733, \)
\( \delta P_{50} = l_{56} / l_{50} = 8,563,435 / 8,950,901 = 0.956712. \)

Hence, \( A_{50:56}^{1} = 0.24905 - \frac{0.956712}{1.06^{6}} \times 0.31733 = 0.0350. \)

6. Using a summation-type formula, we have
\[
A_{50:56}^{1} = vq_{50} + v^{2} p_{50} q_{51} + v^{3} p_{50} p_{51} q_{52}
\]
\[
= 0.05 + 0.95 \times 0.05 + 0.95 \times 0.95 \times 0.05
\]
\[
= 0.1297. \]

7. Note that \( A_{60} = A_{60:2}^{1} + 2 \cdot A_{60} = A_{60:2}^{1} + v^{2} p_{60} A_{62}. \) Therefore,
\[
0.630 = \frac{0.1}{1.05} + \frac{0.9 \times 0.1}{1.05^{2}} + \frac{0.9^{2}}{1.05^{3}} A_{62},
\]
which gives \( A_{62} = 0.617. \)

8. Note that
\[
A_{40:20}^{1} = A_{40:20}^{1} + A_{40:20}^{1}.
\]

From the last column of the Illustrative Life Table, we obtain
\[
A_{40:20}^{1} = 20 E_{40} = 0.27414,
\]
and
\[
A_{40:20}^{1} = A_{40} - 20E_{40}
\]
\[
= A_{40} - 20E_{40} A_{60}
\]
\[
= 0.16132 - 0.27414 \times 0.36913
\]
\[
= 0.06013
\]
Hence, \( A_{40:20}^{1} = 0.06013 + 0.27414 = 0.33427. \)

9. The present value random variable is given by
\[
Z = 1000 e^{0.02T} e^{-0.06T} = 1000 e^{-0.04T}. \]
This is just the same as the present value random variable for standard whole life insurance with a level benefit of 1000 and a force of interest \( \delta = 0.04. \)

Hence, \( \text{Var}(Z) \) is given by
\[
\text{Var}(Z) = 1000^2 (\overline{A}_{20} - \overline{A}_{20}^2) \\
= 1000^2 \left( \frac{\mu}{\mu + 2\delta} - \left( \frac{\mu}{\mu + \delta} \right)^2 \right) \\
= 1000^2 \left( \frac{0.06}{0.06 + 0.08} - \left( \frac{0.06}{0.06 + 0.04} \right)^2 \right) \\
= 68571.
\]

Alternatively, you may obtain the answer by calculating \(E(Z)\) and \(E(Z^2)\) with integration. But the alternative approach is much less efficient than using the shortcuts!

10. Using the recursion for an annually decreasing \(n\)-year term life insurance, we have

\[
(DA)_{40|15}^1 = 10vq_{40} + vp_{40}(DA)_{41|15}^1 \\
5.8 = 10 \times \frac{1}{1.05} \times 0.1 + \frac{1}{1.05} \times 0.9 \times (DA)_{41|15}^1 \\
(DA)_{41|15}^1 = 5.6556.
\]

11. The force of mortality is a constant for \(0 \leq t \leq 10\) and another constant for \(t > 10\).

We use

\[
\overline{A}_x = \overline{A}_{x+10} + v^{10} p_x \overline{A}_{x+10}.
\]

For \(t > 10\), \(\mu_{x+t} = 0.07\) and \(\delta = 0.06\). Hence,

\[
\overline{A}_{x+10} = \frac{0.07}{0.06 + 0.07} = \frac{7}{13}.
\]

For \(0 \leq t \leq 10\), \(\mu_t = 0.05\) and \(\delta = 0.03\). Hence, \(v^{10} p_x = e^{-0.03 \times 10} e^{-0.05 \times 10} = e^{-0.8} = 0.4493\), and

\[
\overline{A}_{x+10}^1 = \int_0^{10} e^{-0.03t} e^{-0.05t} 0.05 dt \\
= \frac{0.05}{0.08} (1 - e^{-0.08 \times 10}) = 0.3442.
\]

Finally,

\[
1000\overline{A}_x = 1000 \times \left( 0.3442 + 0.4493 \times \frac{7}{13} \right) = 586.13.
\]
12. First, we need to find \( q_{60} \). Using the recursion for whole life insurances, we have

\[
A_{60} = vq_{60} + vp_{60}A_{61}
\]

\[
0.585 = q_{60} + \frac{1-q_{60} \times 0.605}{1.05}
\]

\[
q_{60} = \frac{0.00925}{0.395} = 0.0234.
\]

Also, \( q_{61} = q_{60} = 0.0234 \). Using the recursion for whole life insurances again, we have

\[
A_{61} = vq_{61} + vp_{61}A_{62}
\]

\[
0.605 = \frac{0.0234 + 1-0.0234}{1.05} A_{62}
\]

\[
A_{62} = 0.627.
\]

13. Since \( \mu_x = \mu \) for all \( x \geq 0 \), we have \( e_x = \frac{1}{\mu} = 25 \). This gives \( \mu = 0.04 \). Also, we have

\[
A_x = \frac{\mu}{\mu + \delta} = \frac{0.04}{0.04 + 0.06} = 0.4,
\]

which gives \( \delta = 0.06 \).

We then use \( A_{x+10} = A_{x+10}^1 + A_{x+10}^1 \).

First,

\[
A_{x+10}^1 = v^{10} \cdot \mu x = e^{-\delta 	imes 10} e^{-\mu x 	imes 10} = e^{-0.4} e^{-0.6} = 0.36788.
\]

Second,

\[
A_{x+10} = A_{x} - 10 A_{x} + A_{x+10} = A_{x} - v^{10} p_{x} A_{x+10}.
\]

When \( \mu_x = \mu \) for all \( x \geq 0 \),

\[
A_x = A_{x+10} = \frac{\mu}{\mu + \delta}.
\]

Hence,

\[
A_{x+10} = 0.4 - 0.36788 	imes 0.4 = 0.25285.
\]

Finally,

\[
A_{x+10} = 0.25285 + 0.36788 = 0.62073.
\]

14. To solve this problem, you can look at the special case where the force of mortality is a constant \( \mu \) for all ages. This is because if the result holds in general, it must hold for constant force.

Under the old assumptions,
\[ A_x = \frac{\mu}{\mu + \delta} = \frac{\mu}{\mu + 0.06} = 0.6, \]

which gives \( \mu = 0.09. \)

Under the new assumptions, \( \mu = 0.09 + 0.03 = 0.12 \) and \( \delta = 0.06 - 0.03 = 0.03. \) Hence, the revised APV is given by

\[ A_x = \frac{0.12}{0.12 + 0.03} = 0.8. \]

Hence, the answer is (D).

Now here comes the solution to the general case, which you should read after studying Chapter 4. Under the old assumptions,

\[ p_x = \exp\left(-\int_0^t \mu_{x+u} \, du\right) \]

and

\[ a_x = \int_0^\infty e^{-\delta t} p_x \, dt = \frac{1 - A_x}{\delta} = \frac{1 - 0.6}{0.06} = \frac{20}{3}. \]

We use \( * \) to denote functions evaluated under the new assumptions. Then

\[ p_x^* = \exp\left(-\int_0^t (\mu_{x+u} + 0.03) \, du\right) = e^{-0.03t} \exp\left(-\int_0^t \mu_{x+u} \, du\right) = e^{-0.03t} p_x, \]

and

\[ a_x^* = \int_0^\infty e^{-(\delta - 0.03)t} (p_x^*) \, dt = \int_0^\infty e^{-\delta t} e^{0.03t} e^{-0.03t} p_x \, dt = \int_0^\infty e^{-\delta t} p_x \, dt = a_x. \]

Hence, \[ A_x^* = 1 - (\delta^*)(a_x^*) = 1 - 0.03 \times \frac{20}{3} = 0.8. \]

15. The present value random variable for this special insurance is given by

\[ Z = b_x v^T = e^{0.06T} e^{-0.08T} = e^{-0.02T}, \]

which is the same as that for a standard whole life insurance with a level benefit of 1 and a force of interest \( \delta = 0.02. \) Using the shortcuts for constant force of mortality at all ages, we have

\[
\text{Var}(Z) = \frac{\mu}{\mu + 2\delta} \left( \frac{\mu}{\mu + \delta} \right)^2 \\
= \frac{0.05}{0.05 + 2 \times 0.02} \left( \frac{0.05}{0.05 + 0.02} \right)^2 \\
= \frac{5}{9} - \left( \frac{5}{7} \right)^2 = 0.045
\]

Hence, the answer is (D).
16. We use the identity
\[ 20 \backslash A_x = v^{20} 20 p_x \backslash A_{x+20} \, . \]
First, we must find \( \mu \). Since the force of mortality is constant, we have
\[ e^\circ_x = \frac{1}{\mu} = 16, \]
which gives \( \mu = 0.0625 \).
Second, \( \delta = \ln(1.05) = 0.04879 \).
Finally,
\[ 20 \backslash A_x = v^{20} p_x \backslash A_{x+20} \]
\[ = e^{-20 \delta} e^{-20 \mu} \frac{\mu}{\mu + \delta} \]
\[ = e^{-20(0.04879+0.0625)} \frac{0.0625}{0.0625 + 0.04879} \]
\[ = 0.0606. \]
Hence, the answer is (B).

17. The single benefit premium is 1000 \( \backslash A_x \). If we write \( \backslash A_x = \backslash A_{x+10} + 10 \backslash A_x = \backslash A_{x+10} + v^{10} 10 p_x \backslash A_{x+10} \),
then \( \backslash A_{x+10} \) and \( v^{10} 10 p_x \) can be found using \( \delta = 0.04 \) and \( \mu = 0.06 \), while \( \backslash A_{x+10} \) can be found using \( \delta = 0.05 \) and \( \mu = 0.07 \).
We have
\[ \backslash A_{x+10} = \int_0^{10} e^{-0.04t} e^{-0.06t} 0.06dt \]
\[ = -0.06 \end{array} \]
\[ = 0.37927, \]
and
\[ v^{10} 10 p_x = e^{-0.04 \times 10} e^{-0.06 \times 10} = e^{-1}, \]
and
\[ \backslash A_{x+10} = \frac{0.07}{0.07 + 0.05} = \frac{7}{12} \text{ (using } \delta = 0.05 \text{ and } \mu = 0.07). \]
Finally, the single benefit premium is given by
\[ 1000 \backslash A_x = 1000 \left( 0.37927 + e^{-1} \frac{7}{12} \right) = 593.87. \]
Hence, the answer is (E).
18. We call this a mixture problem: 25% are smokers \((s)\), while 75% are nonsmokers \((ns)\).

Let \(Z\) be the present value random variable for a unit-benefit 2-year term insurance.

\[
E(Z \mid s) = vq_s^x + v^2 p_s^x q_{x+1}^s = \frac{0.1}{1.02} + \frac{0.9 \times 0.2}{1.02^2} = 0.2710.
\]

\[
E(Z \mid ns) = vq_{ns}^x + v^2 p_{ns}^x q_{x+1}^{ns} = \frac{0.05}{1.02} + \frac{0.95 \times 0.1}{1.02^2} = 0.1403.
\]

\[
A_{x\mid x}^i = E(Z) = 0.25E(Z \mid s) + 0.75E(Z \mid ns)
\]
\[
= 0.25 \times 0.2711 + 0.75 \times 0.1403
\]
\[
= 0.1730.
\]

As a result, 10,000 \(A_{x\mid x}^i\) = 1730. Hence the answer is (C).

19. The idea here is to find expressions for \(E(Z)\) and \(\text{Var}(Z)\), and then use the equation given in Statement (iii) to solve for \(q_x\).

Since, the force of mortality is constant, \(p_x = e^{-\mu}\), and \(n p_x = e^{-n \mu} = (e^{-\mu})^n = (p_x)^n\) for any value of \(n\). For a 15-year pure endowment of 1, we have

\[
E(Z) = v^{15} 15 p_x = (0.9)^{15} (p_x)^{15},
\]
\[
E(Z^2) = (v^{15})^2 15 p_x = (0.9)^{30} (p_x)^{15},
\]
\[
\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = (0.9)^{30} (p_x)^{15} - (0.9)^{30} (p_x)^{30} = (0.9)^{30} (p_x)^{15} (1 - (p_x)^{15}).
\]

Using the equation in Statement (iii), we have

\[
(0.9)^{30} (p_x)^{15} (1 - (p_x)^{15}) = 0.065 \times (0.9)^{15} (p_x)^{15}.
\]

Dividing both sides by \((0.9)^{15} (p_x)^{15}\), we obtain

\[
0.065 = (0.9)^{15} (1 - (p_x)^{15})
\]

\[
\Rightarrow (p_x)^{15} = 0.6843
\]

\[
\Rightarrow p_x = 0.975 \Rightarrow q_x = 0.025.
\]

Hence, the answer is (B).

20. First, note that \(i = 0.05\) (not 0.06), which means that we cannot get \(A_{63}\) from the Illustrative Life Table. However, we can still use the \(q_x\) values from the table, as they do not depend on the interest rate.

The contract premium is \((1.12)(10000)A_{63} = 5233\), which gives \(A_{63} = 0.4672\). Lee will invest the amount of 5233 at an annual rate of return \(r\) for two years, and he wants to use the accumulated amount \(5233(1 + r)^2\) to buy a whole life policy that is going to cost him \((1.12)(10000)A_{65}\) at the end of year 2 (i.e., at age 65). Our goal is to find \(r\).

Using the recursion for whole life insurances, we have
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\(A_{63} = vq_{63} + vp_{63} A_{64}\)
\(A_{64} = \frac{(1+i)A_{63} - q_{63}}{1 - q_{63}}\)
\[= \frac{1.05 \times 0.4672 - 0.01788}{1 - 0.01788} = 0.4813,\]

and
\(A_{64} = vq_{64} + vp_{64} A_{65}\)
\(A_{65} = \frac{(1+i)A_{64} - q_{64}}{1 - q_{64}}\)
\[= \frac{1.05 \times 0.4813 - 0.01952}{1 - 0.01952} = 0.4955.\]

At age 65, the insurance premium will be \(1.12 \times 10000 \times 0.4955 = 5550\). The required investment return, \(r\), is calculated as follows:

\[5233(1 + r)^2 = 5550 \implies 1 + r = 1.03 \implies r = 0.03.\]

Hence, the answer is (A).

21. Let \(F(n)\) be the percentage of light bulbs that fail in year \(n\). It is obvious that \(F(1) = 0.10\).

At the end of year 1, there are 90\% original (1-year-old) and 10\% new bulbs. Thus,

\[F(2) = 0.30 \times 0.90 + 0.10 \times 0.10 = 0.28.\]

At the end of year 2, there are \(0.70 \times 0.90 = 63\%\) original (2-year-old) bulbs, \(0.10 \times 0.90 = 9\%\) 1-year-old bulbs, and 28\% new bulbs. Thus,

\[F(3) = 0.63 \times 0.50 + 0.09 \times 0.30 + 0.28 \times 0.10 = 0.37.\]

The APV of replacement is given by

\[10000 \left(\frac{0.10}{1.05} + \frac{0.28}{1.05^2} + \frac{0.37}{1.05^3}\right) = 6688.26.\]

Hence, the answer is (A).

22. The APV of benefits can be written as

\[1000A_{45} - 500v_{20}A_{45} = 1000A_{45} - 500v_{20}p_{45}A_{65}.\]

From the Illustrative Life Table, we obtain \(1000A_{45} = 201.20\), \(1000A_{65} = 439.80\), and

\[v_{20}^2 p_{45} = 20E_{45} = 0.25634.\]

[You may look up the column \(20E\) to get \(20E_{45}\).]
The APV is given by $201.20 - 0.25634 \times 0.5 \times 439.80 = 144.83$. Hence, the answer is (B).

23. It is not obvious at first glance what the recursion does, but from the choices you can see that it is associated with some insurance benefit. We can rewrite the recursion formula as

$$u(k-1) = (-\alpha(k) + u(k)) \frac{1}{\beta(k)} = \left( \frac{q_{k-1}}{p_{k-1}} + u(k) \right) \frac{p_{k-1}}{(1+i)} = vq_{k-1} + vp_{k-1}u(k).$$

Starting at age 40, we get the first two steps:

$$u(40) = vq_{40} + vp_{40}u(41)$$
$$= vq_{40} + vp_{40}(vq_{41} + vp_{41}u(42))$$
$$= vq_{40} + v^2p_{40}q_{41} + v^2p_{40}u(42).$$

Continuing this pattern, we get

$$u(40) = vq_{40} + v^2p_{40}q_{41} + v^3p_{40}q_{42} + \ldots + v^{30}p_{40}q_{69} + v^{30}p_{40} \times 1.$$ [The final 1 in the equation above is due to the fact that $u(70) = 1$.]

Therefore, $u(40) = A_{40\overline{30}}$, (an endowment insurance), and the answer is (C).

24. The salesman will sell $N$ policies, where $N$ follows a binomial distribution with $E(N) = 10 \times 0.1 = 1$ and $\text{Var}(N) = 10 \times 0.1 \times 0.9 = 0.9$. We let $X_i$ be the present value random variable for the $i$th policy, $i = 1, \ldots, N$. Then,

$$S = \sum_{i=1}^{N} X_i.$$ Since the policies are identical, we let $E(X) = E(X_i)$ and $\text{Var}(X) = \text{Var}(X_i)$.

From the Illustrative Life Table, we obtain

$$E(X) = 1000A_{22} = 71.35,$$

$$E(X^2) = (1000^2) \times A_{22} = 15870,$$

$$\text{Var}(X) = 15870 - 71.35^2 = 10779.$$ To solve this problem, you need the conditional variance formula:

$$\text{Var}(S) = \text{Var}(E(S|N)) + E(\text{Var}(S|N))$$
$$= \text{Var}(N E(X)) + E(N \text{Var}(X))$$
$$= [E(X)]^2 \text{Var}(N) + E(N) \text{Var}(X)$$
$$= 71.35^2 \times 0.9 + 1 \times 10779$$
$$= 15361.$$ Hence, the answer is (E).
25. Here, $Z$ is the present value random variable for a whole life insurance with a benefit $b$ (instead of 1). We can calculate the first two moments of $Z$ as follows:

$$E(Z) = bA_x = b \frac{\mu}{\mu + \delta} = b \left( \frac{0.02}{0.02 + 0.04} \right) = b \frac{3}{3},$$

$$E(Z^2) = b^2 ( 2A_x ) = b^2 \frac{\mu}{\mu + 2\delta} = b^2 \frac{0.02}{0.02 + 0.08} = 0.2b^2.$$

So, $\text{Var}(Z)$ is given by

$$\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = 0.2b^2 - \left( b \frac{3}{3} \right)^2 = b^2 \left( 0.2 - \frac{1}{9} \right).$$

According to Statement (iii), the single benefit premium (i.e. $E(Z)$) is equal to $\text{Var}(Z)$. It follows that

$$b^2 \left( 0.2 - \frac{1}{9} \right) = b \frac{3}{3},$$

which gives $b = 3.75$. Hence, the answer is (E).

26. The single benefit premium for the annually increasing 10-year term life insurance on $(40)$ is $100000 \, (IA)_{40:10}^1$.

Statement (v) says that $100000 \, (IA)_{41:10}^1 = 16736$. We need to relate $(IA)_{40:10}^1$ and $(IA)_{41:10}^1$.

Using the identity for annually increasing $n$-year term insurances, we have

$$(IA)_{40:10}^1 = vq_{40} + v p_{40} (A_{41:10}^1 + (IA)_{41:10}^1).$$

If we could find $(IA)_{41:10}^1$, we could finish the problem using the Illustrative Life Table. We do this by noting that the expressions for $(IA)_{41:10}^1$ and $(IA)_{41:5}^1$ differ by only one final term:

$$(IA)_{41:10}^1 = vq_{41} + 2v^2 p_{41}q_{42} + \ldots + 9v^9 p_{41}q_{49} + 10v^{10} p_{41}q_{50};$$

$$(IA)_{41:5}^1 = vq_{41} + 2v^2 p_{41}q_{42} + \ldots + 9v^9 p_{41}q_{49}.$$

Thus,

$$(IA)_{41:10}^1 - (IA)_{41:5}^1 = 10v^{10} p_{41}q_{50} = \frac{10}{1.06^{10}} \times \frac{8950901}{9287264} \times 0.00592 = 0.03186,$$

which gives $(IA)_{41:5}^1 = 0.16736 - 0.03186 = 0.13550$.

Now we can use the Illustrative Life Table to get the rest of what we need.
\[ A_{41}^{1} = A_{41} - q_{41} A_{41} \]
\[ = A_{41} - v_{q_{41}} p_{41} A_{50} \]
\[ = 0.16869 - \frac{1}{1.06^9} \times \frac{8950901}{9287264} \times 0.24905 \]
\[ = 0.02662, \]

\[ q_{40} = 0.00278, \text{ and } p_{40} = 0.99722. \]

\[ (IA)^{1}_{40} = v_{q_{40}} + v_{p_{40}} \left( A_{41}^{1} + (IA)^{0}_{41} \right) \]
\[ = \frac{0.00278}{1.06} + \frac{0.99722}{1.06} (0.02662 + 0.13550) \]
\[ = 0.15514. \]

Hence, the answer is (D).

27. The initial fund \( F \) is 1000\( P \), where \( P \) is the single benefit premium (i.e., the APV) of the benefit for one person.

\[ P = 1000 v_{q_{30}} + 500 v^{2} p_{30} q_{31} \]
\[ = \frac{1000 \times 0.00153}{1.06} + \frac{500 \times 0.99847 \times 0.00161}{1.06^2} \]
\[ = 2.15875. \]

This gives \( F = 1000P = 2158.75. \)

We charge a single benefit premium per person. The total premium of \( F = 1000P \) is expected to be just enough to cover the benefit payments over the period of two years. Hence, the expected size of the fund (as projected at time 0) is 0.

The actual fund at the end of the second year is not necessarily 0, because realized mortality and interest rates may not be the same as the expected values. Given the actual experience, the fund values at different times are calculated as follows.

Beginning fund: 2158.75
Fund value at the end of year 1: 2158.75 \times 1.07 - 1000 = 1309.86
Fund value at the end of year 2: 1309.86 \times 1.069 - 500 = 900.24

Hence, the answer is (C).

28. For this special 10-year term life insurance, the present value random variable is

\[ Z = b_{10} v^{T} = e^{0.03T} e^{-0.08T} e^{-0.05T}, T = 10 \]

and is zero otherwise

This is just the present value random variable for a standard 10-year term life insurance with a level benefit of 1 and a force of interest \( \delta = 0.05 \).

Using the shortcuts for constant force of mortality, we have
29. From Statement (iii), we obtain

\( q_x = 0.02, \quad q_{x+1} = 0.04 \) and \( q_{x+2} = 0.06 \).

This gives \( Pr(K_x = 0) = 0.98 \times 0.04 = 0.0392 \),
\( Pr(K_x = 2) = 0.98 \times 0.06 = 0.056448 \).

\[
E(Z) = b_1vPr(K_x = 0) + b_2v^2 Pr(K_x = 1) + b_3v^3 Pr(K_x = 2)
= 300 \times 1.06^{-1} \times 0.02 + 350 \times 1.06^{-2} \times 0.0392 + 400 \times 1.06^{-3} \times 0.056448
= 36.829.
\]

\[
E(Z^2) = (b_1v)^2 Pr(K_x = 0) + (b_2v^2)^2 Pr(K_x = 1) + (b_3v^3)^2 Pr(K_x = 2)
= 300^2 \times 1.06^{-2} \times 0.02 + 350^2 \times 1.06^{-4} \times 0.0392 + 400^2 \times 1.06^{-6} \times 0.056448
= 11772.6.
\]

\[ Var(Z) = 11772.6 - 36.829^2 = 10416. \]

Hence, the answer is (C).

30. Since mortality is Gompertz, \( p_x = \exp\left(-\frac{Bc^x}{\ln c} (c^t - 1)\right) \).

Putting \( x = 50 \) and \( t = 1 \),
\[ p_{50} = \exp(-\frac{5 \times 10^{-6} \times 1.2^{50}}{\ln 1.2}) = e^{-0.049914}. \]

Putting \( x = 51 \) and \( t = 1 \),
\[ p_{51} = \exp(-\frac{5 \times 10^{-6} \times 1.2^{51}}{\ln 1.2}) = e^{-0.059897}. \]

So, \( A_{50\text{|}x} = vq_{50} + v^2p_{50}q_{51} = \frac{0.048689}{1.03} + \frac{0.951311 \times 0.0581385}{1.03^2} = 0.0994. \)

The answer is (E).

31. We have \( X = \begin{cases} 
7v^{T_{35}} & T_{35} < 35 \\
0 & T_{35} \geq 35
\end{cases} \), \( Y = \begin{cases} 
0 & T_{35} < 25 \\
4v^{T_{35}} & 25 \leq T_{35} < 35 \\
0 & T_{35} \geq 35 \end{cases} \). We need to compute \( E(XY) \).

By the definition of \( X \) and \( Y \), \( XY = \begin{cases} 
28v^{2T_{35}} & 25 \leq T_{35} < 35 \\
0 & T_{35} \geq 35 \end{cases} \), which equals
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\[
1.75 Y^2 = \begin{cases} 
0 & T_{35} < 25 \\
1.75(4v^{T_{35}})^2 & 25 \leq T_{35} < 35 \\
0 & T_{35} \geq 35 
\end{cases}
\]

As a result,
\[
E(XY) = 1.75E(Y^2) = 1.75[\text{Var}(Y) + E^2(Y)] = 1.75[0.1 + 0.12^2] = 0.2002,
\]
and Cov\((X, Y) = 0.2002 - 2.8 \times 0.12 = -0.1358.\) Finally,
\[
\text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y) = 5.76 - 2(0.1358) + 0.1 = 5.5884.
\]

32. We use the relation \(A^{i(12)}_{x\bar{z}} = \frac{i}{i^{(12)}} A^i_{x\bar{z}}\) under UDD.

For \(i = 0.1\), we have
\[
A^i_{x\bar{z}} = vq_x + v^2 p_x q_{x+1} = \frac{0.04}{1.1} + \frac{0.96 \times 0.08}{1.1^2} = 0.0998347.
\]

Also, \(i^{(12)} = 12(1.1^{1/12} - 1) = 0.09568969.\)

So, the answer is \(A^{i(12)}_{x\bar{z}} = \frac{i}{i^{(12)}} A^i_{x\bar{z}} = \frac{0.1}{0.09568969} \times 0.098347 = 0.10433,\) which is (A).

33. (a) \(A^i_{x\bar{z}} = vq_x + v^2 p_x q_{x+1} = \frac{0.1}{1.12} + \frac{0.9 \times 0.2}{1.12^2} = 0.2327806\)

\[
\overline{A^i_{x\bar{z}}} = \frac{i}{\delta} A^i_{x\bar{z}} = \frac{0.12}{\ln 1.12} \times 0.2327806 = 0.246484
\]

\[
2E_x = v^2 p_x = \frac{0.9 \times 0.8}{1.12^2} = 0.573980
\]

So, \(\overline{A^i_{x\bar{z}}} = 0.246484 + 0.573980 = 0.82046.\)

(b) We have \(i^{(3)} = 3 \times [(1 + i)^{1/3} - 1] = 0.1154965.\)

\[
A^{i^{(3)}}_{x\bar{z}} = \frac{i}{i^{(3)}} A^i_{x\bar{z}} = \frac{0.12}{0.1154965} \times 0.2327806 = 0.241857
\]

So, \(A^{(3)}_{x\bar{z}} = 0.241857 + 0.573980 = 0.81584.\)

(c) \(\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = 2 A^{(3)}_{x\bar{z}} - (A^{(3)}_{x\bar{z}})^2.\) When \(i = 0.12,\) we have
\[
\delta = \ln 1.12 = 0.1133287,
\]
and hence a double force of interest means \(2\delta = 0.22665737,\) and
\[
2i = e^{0.22665737} - 1 = 0.2544.
\]

The corresponding \(2i^{(3)}\) is
\[
2i^{(3)} = 3 \times [(1 + 2i)^{1/3} - 1] = 0.2354394.
\]

So,
\[ 2A_{x\mid 2}^{(3)} - (A_{x\mid 2}^{(3)})^2 = 2A_{x\mid 2}^{(3)} + \ddot{\gamma}E_x - 0.81584^2 \]
\[ = \frac{0.2544}{0.2354394} \left( \frac{0.1}{1.2544} + 0.9 \times 0.2 \right) + \frac{0.9 \times 0.8}{1.2544^2} - 0.81584^2 \]
\[ = 0.20974508 + 0.45747302 - 0.81584^2 \]
\[ = 0.0017232 \]

The standard deviation of \( Z \) is \( 0.0017232^{1/2} = 0.04151 \).

34. (a) We are given that the force of mortality of \( T \) is \( \mu_t = \frac{c}{60-t} \), where \( c \) is the constant of proportionality. This gives
\[ S(t) = \exp \left( -\int_0^t \frac{c}{60-s} \, ds \right) = \left( 1 - \frac{t}{60} \right)^c . \]

The expected lifetime is
\[ E(T) = \int_0^{60} p_0 \, dt = \int_0^{60} S(t) \, dt = \frac{60}{c+1} \left[ \left( 1 - \frac{t}{60} \right)^{c+1} \right]_0^{60} = \frac{60}{c+1} . \]

Setting the above to 20, we get \( c = 2 \). So, \( S(t) = \left( 1 - \frac{t}{60} \right)^2 \).

(b) \( \text{Pr}(Z = 0) = \text{Pr}(T > 10) = \left( 1 - \frac{10}{60} \right)^2 = \frac{25}{36} \)

(c) The density of \( T \) is \( p_0 \mu_t = \left( 1 - \frac{t}{60} \right)^2 \frac{2}{60-t} = \frac{60-t}{1800} \) for \( 0 < t < 60 \).

The EPV of the guarantee is
\[ E(Z) = \frac{1000000}{180000} \int_0^{10} e^{-0.05t} \left( 60-t \right) \, dt . \]

We note that
\[ \int_0^{10} e^{-0.05t} \, dt = 20(1 - e^{-0.5}) \]
and
\[ \int_0^{10} te^{-0.05t} \, dt = \left[ -20te^{-0.05t} \right]_0^{10} + 20 \int_0^{10} e^{-0.05t} \, dt \]
\[ = -200e^{-0.5} + 400(1 - e^{-0.5}) \]
\[ = 400 - 600e^{-0.5} \]

So,
\[ E(Z) = \frac{10000}{9} \left[ 60 \times 20(1 - e^{-0.5}) - 400 + 600e^{-0.5} \right] \]
\[ = \frac{8000000 - 6000000e^{-0.5}}{9} = 484,535.12 \]
(d) The definition of $Z$ is $Z = \begin{cases} 2000000e^{-0.05T} & T < 10 \\ 0 & T \geq 10 \end{cases}$.

For $0 < z \leq 2000000$, the distribution function of $Z$ is

$$\Pr(Z \leq z) = \Pr\{Z = 0 \text{ and } T \geq 10\} \text{ or } \{Z = 2000000e^{-0.05T} \leq z \text{ and } T < 10\}$$

$$= \Pr(T \geq 10 \text{ or } 10 > T > -20\ln\frac{z}{2000000})$$

$$= S(-20\ln\frac{z}{2000000})$$

$$= \left(1 + \frac{\ln(z/2000000)}{3}\right)^2.$$ 

Setting the above to 0.81:

$$\left(1 + \frac{\ln(z/2000000)}{3}\right)^2 = 0.81$$

$$\ln(z/2000000) = -0.3$$

$$z = 1481636.4$$

35. (a) $\text{APV} = v q_x = 1.04^{-1} \times 0.2 = 0.19231$.

(b) $\text{APV} = 3v^2 q_{x+1} = 3v^2 p_{x+1} q_{x+3} = 3(1.04)^{-3}(0.9)(0.95)(0.04) = 0.09121$.

(c) $\text{APV} = v q_{x+2} + v^2 p_{x+2} q_{x+3} + 2v^2 p_{x+2}$

$$= (1.04)^{-2}(0.05) + (1.04)^{-2}(0.95)(0.04) + 2(1.04)^{-2}(0.95)(0.96) = 1.76960.$$ 

(d) Although the benefit amount in part (b) is higher, the probability of getting the benefit of $3$ (i.e., dying within the third policy year) is very low. On the other hand, for the insurance in part (c), it is very likely (with a probability of 0.912) that the policyholder will get the survival benefit.

36. (a) $p_x = \frac{l_{x+k}}{l_x} = \sqrt{\frac{144-x-k}{144-x}}$.

(b) We shall use the following values:

$$p_{50} = \sqrt{\frac{93}{94}}, \quad p_{50} = \sqrt{\frac{92}{94}}, \quad p_{50} = \sqrt{\frac{91}{94}}, \quad p_{51} = \sqrt{\frac{92}{93}}, \quad p_{52} = \sqrt{\frac{91}{92}}$$

This gives

$$A_{50:3}^{(1)} = v q_{50} + v^2 p_{50} q_{51} + v^3 p_{50} q_{52}$$

$$= \frac{1}{1.06} \left(1 - \frac{93}{94}\right) + \frac{1}{1.06^2} \sqrt{\frac{93}{94}} \left(1 - \frac{92}{93}\right) + \frac{1}{1.06^3} \sqrt{\frac{92}{94}} \left(1 - \frac{91}{92}\right)$$

$$= 0.014330.$$
(c) \( A_{50:3}^{1} = v^3 p_{30} = \frac{1}{1.06^3} \sqrt[94]{91} = 0.826112 \).

Hence, \( A_{50:3} = 0.014330 + 0.826112 = 0.84044 \).

37. (a) \[ A_{x|n}^{1} = \sum_{k=0}^{n-1} v^{k+1} k p_x q_{x+k} \]

\[ = vq_x + \sum_{k=1}^{n-1} v^{k+1} p_x \times p_{x+1} \times q_{x+k} \]

\[ = vq_x + \sum_{j=0}^{n-2} v^{j+2} p_x \times p_{x+j} \times q_{x+j+1} \]

\[ = vq_x + v p_x \sum_{j=0}^{n-2} v^{j+1} p_{x+j} \times q_{x+j+1} \]

\[ = vq_x + v p_x A_{x+1|n-1}^{1} \]

(b) \[ A_{x|n} = A_{x|n}^{1} + v^n p_x \]

\[ = vq_x + v p_x A_{x+1|n-1}^{1} + v^n p_x \]

\[ = vq_x + v p_x (A_{x+1|n-1}^{1} + v^{n-1} p_x) \]

\[ = vq_x + v p_x A_{x+1|n-3}^{1} \]

(c) \[ (DA)_{x|n}^{1} = \sum_{k=0}^{n-1} (n-k) v^{k+1} k p_x q_{x+k} \]

\[ = v n q_x + \sum_{k=1}^{n-1} (n-k) v^{k+1} p_x \times p_{x+1} \times q_{x+k} \]

\[ = v n q_x + \sum_{j=0}^{n-2} (n-1-j) v^{j+2} p_x \times p_{x+j} \times q_{x+j+1} \]

\[ = v n q_x + v p_x \sum_{j=0}^{n-2} (n-1-j) v^{j+1} p_{x+j} \times q_{x+j+1} \]

\[ = v n q_x + v p_x (DA)_{x+1|n-3}^{1} \]

38. (a) 4 years.

(b) \[ 4 P_{44} = \frac{l_{48}}{l_{44}} = \frac{98067}{99288} = 0.987702. \]

(c) \[ \frac{32 q_{[40]+3}}{l_{[40]+3}} = \frac{l_{[40]+3+3} - l_{[40]+3+3+2}}{l_{[40]+3}} = \frac{l_{46}-l_{48}}{l_{40}+3} = \frac{98752 - 98067}{99520} = 0.006883. \]
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(d) \[ (IA)_{[40]}^{l} = \sum_{k=0}^{2} (k+1) \nu^{k+1} k p_{[40]} q_{[40]+k} \]
\[ = \sum_{k=0}^{2} (k+1) \frac{l_{[40]+k} - l_{[40]+k+1}}{l_{[40]}} \]
\[ = \nu(l_{[40]} - l_{[40]+1}) + 2 \nu^{2}(l_{[40]+1} - l_{[40]+2}) + 3 \nu^{3}(l_{[40]+2} - l_{[40]+3}) \]
\[ = \frac{0.008794124}{l_{[40]}}. \]

39. (a) \[ n \bar{A}_x = v^n n p_x \bar{A}_x = e^{-\delta n} e^{-\mu n} = \frac{\mu e^{-(\mu+\delta)n}}{\mu + \delta} \]

(ii) \[ \bar{A}_x^{l} = \bar{A}_x - n \bar{A}_x = \frac{\mu}{\mu + \delta} - \frac{\mu e^{-(\mu+\delta)n}}{\mu + \delta} = \frac{\mu(1 - e^{-(\mu+\delta)n})}{\mu + \delta} \]

(b) (i) When \( \delta = 0 \), \( \bar{A}_x = 1 \). No matter how long \( x \) lives, a benefit of $1 will be paid at a certain future time point (assuming that \( m \) is non-zero. The present value of $1 is always $1 when \( d = 0 \). Therefore, the expected present value must be $1.

(ii) When \( \delta = 0 \), \( \bar{A}_x^{l} = 1 - e^{-\mu n} \). Since \( \delta = 0 \), the present value of the death benefit of $1 must always be $1. For an \( n \)-year term life policy, the probability that the death benefit will be paid is \( \Pr(T_x < n) \). When \( \mu_x = \mu \) for all ages, \( T_x \) follows an exponential distribution and therefore \( \Pr(T_x < n) = 1 - e^{-\mu n} \). So the expected value is simply \( 1 - e^{-\mu n} \).

(iii) When \( \mu = 0 \), \( \bar{A}_x^{l} = 0 \). When \( \mu = 0 \), the life becomes immortal, so the death benefit will never be paid. Hence, the time-0 value of the benefit must be zero.

(c) \[ \text{Var}(Z) = 2 \bar{A}_x^{l} - (\bar{A}_x^{l})^2 = \frac{\mu(1 - e^{-(\mu+2\delta)n})}{\mu + 2\delta} - \left( \frac{\mu(1 - e^{-(\mu+\delta)n})}{\mu + \delta} \right)^2. \]

40. (a) \( l(66, 1) = 1 - q_{[66]+1}/q_{67} = 1 - 0.06/0.07 = 0.142857. \)

(b) \( \bar{A}_{[65]+1:3} = vq_{[65]+1} + v^2 p_{[65]+1} q_{67} + v^2 p_{[65]+1} q_{68} \]
\[ = vq_{[65]+1} + v^2(1 - q_{[65]+1})q_{67} + v^3(1 - q_{[65]+1})(1 - q_{67})q_{68} \]
\[ = \frac{1}{1.03} \times 0.04 + \frac{1}{1.03^2} \times (1 - 0.04) \times 0.07 + \frac{1}{1.03^3} \times (1 - 0.04) \times (1 - 0.07) \times 0.09 \]
\[ = 0.175711. \]

(c) Both \( \bar{A}_{[66]+1:3} \) and \( \bar{A}_{[65]+1:3} \) represents APV of a three-year term life insurance of $1 sold to a person age 66. However, the former was sold to a person who has just been selected
(has just passed a medical examination), whereas the latter was sold to a person who was selected one year ago (passed a medical examination one year ago). Because selection effect tapers off with time, the person who has just been selected should have a lighter mortality than the person who was selected one year ago. Consequently, we have $\bar{A}^1_{[60]:3} < \bar{A}^1_{[65]:1:3}$.